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Abstract. We prove that an analog of the Scott-Vogelius finite elements are inf-sup stable on certain nondegenerate meshes for piecewise cubic velocity fields. We also characterize the divergence of the velocity space on such meshes. In addition, we show how such a characterization relates to the dimension of C^1 piecewise quartics on the same mesh.

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1. Introduction

In 1985 Scott and Vogelius [22] (see also [24]) presented a family of piecewise polynomial spaces in two dimensions that yield solutions to the Stokes equations with velocity approximations that are exactly divergence free. The velocity space consists of continuous piecewise polynomials of degree $k \geq 4$, and the pressure space is taken to be the divergence of the velocity space. Moreover, they proved stability of the method by establishing that the pair of spaces satisfy the so-called *inf-sup* condition assuming that the meshes are quasi-uniform and that the maximum mesh size is sufficiently small. In a recent paper [12] we gave an alternative proof of the inf-sup stability for $k \geq 4$ on more general meshes, assuming only that they are non-degenerate (shape regular). One key aspect in the proof is to use the stability of the $P^2 - P^0$ (or the Bernardi-Raugel [5]) finite element spaces. As a result the proof becomes significantly shorter. Here we utilize and extend the techniques to the case $k = 3$. The case $k = 3$ has been considered earlier [19].

One key concept in this paper is the notion of a *local interpolating vertex*. Roughly speaking, this is an interior vertex z such that for every finite element pressure p we can find a discrete velocity $\mathbf{v}$ in the finite element velocity space such that div $\mathbf{v}(z) = p(z)$ with support in the star of z and such that div $\mathbf{v}(\sigma) = 0$ for all other vertices. Moreover, we require that div $\mathbf{v}$ has zero mean on each triangle. We then show that if all interior vertices are local interpolating vertices then the inf-sup stability holds; see Theorem 6.3. We generalize this result to show that if some interior vertices are local interpolating and that there are acceptable paths from any other vertex to one of the local interpolating vertices then the inf-sup condition holds; see Theorem 7.6. In [12] we showed that all interior vertices are local interpolating vertices if piecewise quartic velocities or higher are used. In this article, we show that a generic interior vertex is local interpolating if piecewise cubics are used for the velocity space. In particular, we show that singular vertices and vertices with odd number of triangles touching it are local interpolating vertices. In the case that a non-singular interior vertex has an even number of triangles touching it then we give sufficient conditions for it to be a local interpolating vertex. Although a generic interior vertex is a local interpolating vertex, there are important meshes where no interior vertex is locally interpolating (e.g. three lines mesh).

Since the work of Scott and Vogelius [24, 22], there have been significant efforts to construct conforming finite elements that produce divergence fee approximations; [26, 8, 18, 10, 9, 19, 4, 11]. The thesis of Qin [19] has many intersting ideas including low-order elements. In particular, Qin [19] showed that low order elements can be stable on barycentric refinement in two dimensions; see also [4]. This was generalized by Zhang in three dimensions [25] and recently to arbitrary dimensions in [11]. Further mesh refinements were used by Christiansen and Hu [13]. Guzmán and Neilan use rational functions for the velocity basis functions [10, 9]. The importance of producing divergence free approximations is explained carefully in the review paper [14]; see also the recent paper [1].

It is known that the C^1 piecewise quartic space, which we denote by $\widehat{S}_h^4$, is related to the piecewise cubic, divergence free, Lagrange space. The dimension of $\widehat{S}_h^4$ has been computed [2]. In the last sections of this paper we use the onto-ness of the divergence operator on piecewise cubics to provide an alternate way to compute the dimension of $\widehat{S}_h^4$, on certain meshes. We also are able to compute the dimension of S_h^4, which are those functions in $\widehat{S}_h^4$ that vanish to second-order on the boundary, for certain meshes.

The paper is organized as follows. In the following section we begin with Preliminaries. In Section 3 we identify vertices at which we can interpolate pressure vertex values with the divergence of localized velocity fields. In Section 6 we prove the inf-sup stability for $k = 3$ under some restrictive assumptions on the mesh. In Section 7 we characterize the divergence of piecewise cubics on a broader class of meshes than considered in Section 6. In Section 8 we compare our the results with those of [19]. In Section 9 we relate our results to the dimension of C^1 piecewise quartics.

2. **Preliminaries**

We assume Ω is a polygonal domain in two dimensions. We let $\{\mathcal{T}_h\}_h$ be a non-degenerate (shape regular) family triangulation of Ω; see [6]. The set of vertices and the set of internal edges are denoted by

$$\mathcal{S}_h = \{x : x \text{ is a vertex of } \mathcal{T}_h\},$$
$$\mathcal{E}_h = \{e : e \text{ is an edge of } \mathcal{T}_h \text{ and } e \not\subset \partial\Omega\}.$$

We also let $\mathcal{S}_h^{\text{int}}$ and $\mathcal{S}_h^{\partial}$ denote all the interior vertices and boundary vertices, respectively.

Define the internal edges $\mathcal{E}_h(z)$ and triangles $\mathcal{T}_h(z)$ that have $z \in \mathcal{S}_h$ as a vertex via

$$\mathcal{E}_h(z) = \{e \in \mathcal{E}_h : z \text{ is a vertex of } e\}, \qquad \mathcal{T}_h(z) = \{T \in \mathcal{T}_h : z \text{ is a vertex of } T\}.$$

Finally, we define the star

$$\Omega_h(z) = \bigcup_{T \in \mathcal{T}_h(z)} T.$$

The diameter of this star is denoted by

$$h_z = \text{ diam } (\Omega_h(z)). \tag{2.1}$$

To define the pressure space we must define singular and non-singular vertices. Let $z \in \mathcal{S}_h$ and suppose that $\mathcal{T}_h(z) = \{T_1, T_2, \ldots T_N\}$, enumerated so that T_j, T_{j+1} share an edge for $j = 1, \ldots N - 1$ and T_N and T_1 share an edge in the case z is an interior vertex. If z is a boundary vertex then we enumerate the triangles such that T_1 and T_N have a boundary edge. Let θ_j denote the angle between the edges of T_j originating from z. Define

$$\Theta(z) = \begin{cases} \max\{|\sin(\theta_1 + \theta_2)|, \ldots, |\sin(\theta_{N-1} + \theta_N)|, |\sin(\theta_N + \theta_1)|\} & \text{if } z \text{ is an interior vertex} \\ \max\{|\sin(\theta_1 + \theta_2)|, \ldots, |\sin(\theta_{N-1} + \theta_N)|\} & \text{if } z \text{ is a boundary vertex} . \end{cases}$$

Definition 2.1. A vertex $z \in \mathcal{S}_h$ is a singular vertex if $\Theta(z) = 0$. It is non-singular if $\Theta(z) > 0$.
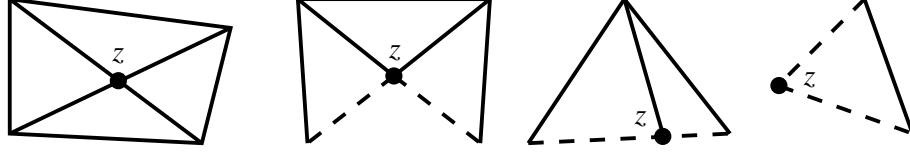
FIGURE 2.1. Example of singular vertices z. Dashed edges denote boundary edges.

This is equivalent to the original definition given in [16, 24].

We denote all the non-singular vertices by

$$\mathcal{S}_h^1 = \{x \in \mathcal{S}_h : x \text{ is non-singular, that is, } \Theta(x) > 0\},$$

and all singular vertices by $\mathcal{S}_h^2 = \mathcal{S}_h \backslash \mathcal{S}_h^1$. We also define, for $i = 1, 2$, $\mathcal{S}_h^{i,\text{int}} = \mathcal{S}_h^i \cap \mathcal{S}_h^{\text{int}}$, and $\mathcal{S}_h^{i,\partial} = \mathcal{S}_h^i \cap \mathcal{S}_h^\partial$.

Let q be a function such that $q|_T \in C(\overline{T})$ for all $T \in \mathcal{T}_h$. For each vertex $z \in \mathcal{S}_h^2$ define

$$A_h^z(q) = \sum_{j=1}^N (-1)^{j-1} q|_{T_j}(z). \tag{2.2}$$

The Scott-Vogelius finite element spaces for $k \geq 1$ are defined by

$$\mathbf{V}_h^k = \{v \in [C_0(\Omega)]^2 : v|_T \in [P^k(T)]^2, \forall T \in \mathcal{T}_h\},$$
$$Q_h^{k-1} = \{q \in L_0^2(\Omega) : q|_T \in P^{k-1}(T), \forall T \in \mathcal{T}_h, A_h^z(q) = 0 \text{ for all } z \in \mathcal{S}_h^2\}.$$

Here $P^k(T)$ is the space of polynomials of degree less than or equal to k defined on T. Also, $L_0^2(\Omega)$ denotes the subspace of L^2 of functions that have average zero on Ω.

We also make the following definition

$$\mathbf{V}_{h,0}^k = \{\mathbf{v} \in \mathbf{V}_h^k : \int_T \text{div } \mathbf{v} \, dx = 0, \quad \text{for all } T \in \mathcal{T}_h\}.$$

The definition of Q_h^{k-1} is based on the following result [21].

Lemma 2.2. *For $k \geq 1$, div $\mathbf{V}_h^k \subset Q_h^{k-1}$.*

The goal of this article is to prove the inf-sup stability of the pair $\mathbf{V}_h^k, Q_h^{k-1}$ for $k = 3$, for certain meshes.

Definition 2.3. The pair of spaces $\mathbf{V}_h^k, Q_h^{k-1}$ is inf-sup stable on a family of triangulations $\{\mathcal{T}_h\}_h$ if there exists $\beta > 0$ such that for all h

$$\beta \|q\|_{L^2(\Omega)} \leq \sup_{\mathbf{v} \in \mathbf{V}_h^k, \mathbf{v} \neq 0} \frac{\int_\Omega q \, \text{div } \mathbf{v} \, dx}{\|\mathbf{v}\|_{H^1(\Omega)}} \qquad \forall q \in Q_h^{k-1}. \tag{2.3}$$

2.1. **Notation for piecewise linears**

For every $z \in \mathcal{S}_h$ the function ψ_z is the continuous, piecewise linear function corresponding to the vertex z. That is, for every $y \in \mathcal{S}_h$

$$\psi_z(y) = \begin{cases} 1 & \text{if } y = z, \\ 0 & \text{if } y \neq z. \end{cases} \tag{2.4}$$
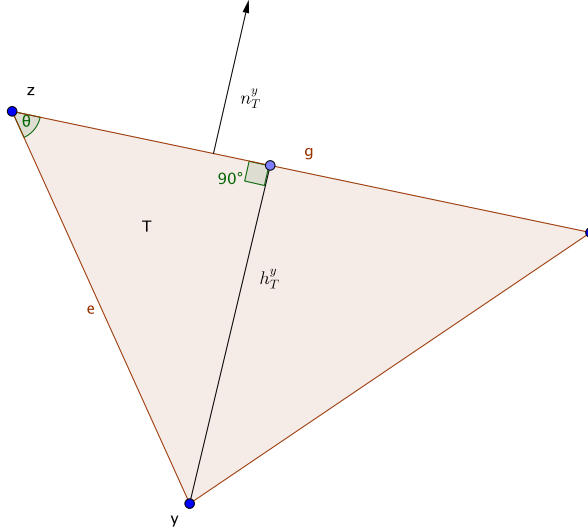
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Figure 2.2. Notation for single triangle

For $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$, where $e = \{z, y\}$, let $\mathbf{t}_e^z = |e|^{-1}(y - z)$ be a unit vector tangent to e, where $|e|$ denotes the length of the edge e. Then

$$\mathbf{t}_e^z \cdot \nabla \psi_y = \frac{1}{|e|} \text{ on } e. \tag{2.5}$$

More generally, suppose that $T \in \mathcal{T}_h(y)$ and let g be the edge of T that is opposite to y; see Figure 2.2. Let $\mathbf{n}_T^y$ be the unit normal vector to g that points out of T. Then

$$\nabla \psi_y|_T = -\frac{1}{h_T^y} \mathbf{n}_T^y, \tag{2.6}$$

where h_T^y is the distance of y to the line defined by the edge g. If z is another vertex of T and $e = \{z, y\}$ then

$$h_T^y = |e| \sin \theta, \tag{2.7}$$

where θ is the angle between the edges of T emanating from z.

2.2. **Preliminary stability results**

The following is a consequence of the stability of the Bernardi-Raugel [5] finite element spaces.

Proposition 2.4. *Let $k \geq 1$. There exists a constant α_1 such that for every $p \in Q_h^{k-1}$ there exists a $\mathbf{v} \in \mathbf{V}_h^2$ such that*

$$\int_T \operatorname{div} \mathbf{v} \, dx = \int_T p \, dx \quad \text{for all } T \in \mathcal{T}_h,$$

and

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \alpha_1 \|p\|_{L^2(\Omega)}.$$

The constant α_1 is independent of p and only depends on the shape regularity of the mesh and Ω.

The next result is a simple consequence of Lemma 2.5 of [24] and is based on a simple counting argument; also see [12] for a detailed proof.

Proposition 2.5. *Let $k \geq 1$. There exists a constant $\alpha_2 > 0$ such that for every $p \in Q_h^{k-1}$ with $p(z) = 0$ for all $z \in \mathcal{S}_h$ and $\int_T p \, dx = 0$ for all $T \in \mathcal{T}_h$ there exists $\mathbf{v} \in \mathbf{V}_h^k$ such that*

$$\operatorname{div} \mathbf{v} = p \ \ on \ \Omega,$$

and

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \alpha_2 \|p\|_{L^2(\Omega)}.$$

Using the above results we can prove inf-sup stability as long as we can interpolate pressure vertex values with the divergence of velocity fields. This is the subject of the next result.

Lemma 2.6. *Suppose there exists a constant $\alpha_3 > 0$ such that for every $p \in Q_h^2$ there exists a $\mathbf{v} \in \mathbf{V}_{h,0}^3$ satisfying*

$$(\operatorname{div} \mathbf{v} - p)(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \tag{2.8}$$

with the bound

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq \alpha_3 \|p\|_{L^2(\Omega)}.$$

Then, (2.3) holds for $k = 3$ with $\beta = \frac{1}{\alpha_1 + \alpha_3(1+\alpha_1) + \alpha_2(1+\alpha_3(1+\alpha_1))}$.

Proof. Let p be an arbitrary function in Q_h^2. First, we let $\mathbf{v}_1$ be from Proposition 2.4. We define $p_1 = p - \operatorname{div} \mathbf{v}_1$. Then, from our hypothesis let $\mathbf{v}_2 \in V_{h,0}^3$ be such that $(\operatorname{div}(\mathbf{v}_2) - p_1)(\sigma) = 0$ for all $\sigma \in \mathcal{S}_h$. Letting $p_2 = p_1 - \operatorname{div} \mathbf{v}_2$ we see that p_2 satisfies the hypotheses of Proposition 2.5 and we let $\mathbf{v}_3$ be the resulting vector field. Then, we set $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Then we see that

$$\operatorname{div} \mathbf{w} = p \quad \text{on } \Omega,$$

and

$$\|\mathbf{w}\|_{H^1(\Omega)} \leq (\alpha_1 + \alpha_3(1 + \alpha_1) + \alpha_2(1 + \alpha_3(1 + \alpha_1)))\|p\|_{L^2(\Omega)}. \tag{2.9}$$

Hence,

$$\|p\|_{L^2(\Omega)}^2 = \int_\Omega p \operatorname{div} \mathbf{w} \, dx \leq \|\mathbf{w}\|_{H^1(\Omega)} \sup_{\mathbf{v} \in \mathbf{V}_h^3, \mathbf{v} \neq 0} \frac{\int_\Omega p \operatorname{div} \mathbf{v} \, dx}{\|\mathbf{v}\|_{H^1(\Omega)}}.$$

The result now follows after applying (2.9). ∎

Hence, one way of proving inf-sup stability is two establish the hypothesis of Lemma 2.6. This is going to be the task of the next sections.

3. **Locally interpolating vertex values**

In this section we will identify vertices at which we can interpolate pressure vertex values with the divergence of velocity fields. We first define the local spaces

$$\mathbf{V}_h^k(z) = \{\mathbf{v} \in \mathbf{V}_h^k : \operatorname{supp} \mathbf{v} \subset \Omega_h(z)\},$$

$$\mathbf{V}_{h,0}^k(z) = \{v \in \mathbf{V}_h^k(z) : \int_T \operatorname{div} v \, dx = 0, \text{ for all } T \in \mathcal{T}_h(z)\},$$

$$\mathbf{V}_{h,00}^k(z) = \{v \in \mathbf{V}_{h,0}^k(z) : \operatorname{div} v(\sigma) = 0, \text{ for all } \sigma \in \mathcal{S}_h, \sigma \neq z\}.$$

Suppose that $z \in \mathcal{S}_h$ and that $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$, ordered as described following (2.1). Then in view of (2.2) we define

$$W(z) = \{(a_1, \ldots, a_N) \in \mathbb{R}^N : \text{ if } z \in \mathcal{S}_h^2, \text{ then } \sum_{j=1}^{N} (-1)^{j-1} a_j = 0\}.$$

Note that if $z \in \mathcal{S}_h^1$ is non-singular then $W(z) = \mathbb{R}^N$ and there is a constraint if z is singular.

Definition 3.1. Let $z \in \mathcal{S}_h$ and suppose that $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$. We say that z is a local interpolating vertex if for every $(a_1, \ldots, a_N) \in W(z)$ there exists a $\mathbf{v} \in V_{h,00}^3(z)$ such that

$$\operatorname{div} \mathbf{v}|_{T_j}(z) = a_j \quad \text{ for all } 1 \leq j \leq N. \tag{3.1}$$

If $z \in \mathcal{S}_h$ is a local interpolating vertex then, given $a = (a_1, \ldots, a_N) \in W(z)$ we define $M_a = \{\mathbf{v} \in V_{h,00}^3(z) : \mathbf{v} \text{ satisfying } (3.1)\}$. Also, we set

$$D_z = \max_{(a_1, \ldots, a_N) \in W(z)} \min_{\mathbf{v} \in M_a} \frac{\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))}}{\max_{1 \leq j \leq N} |a_j|}. \tag{3.2}$$

We let $\mathcal{L}_h$ be the collection of all local interpolating vertices. Then Definition 3.2 says that if $z \in \mathcal{L}_h$ then given $a \in W(z)$ there exists $\mathbf{v} \in V_{h,00}^3(z)$ satisfying (3.1) and

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))} \leq D_z \max_{1 \leq j \leq N} |a_j|.$$

In the next section we identify local interpolating vertices. It will be useful to define fundamental vector fields used in [12]. For every $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$ with $e = \{z, y\}$ define

$$\eta_e^z = \psi_z^2 \psi_y. \tag{3.3}$$

Let T_1 and T_2 be the two triangles that have e as an edge. Then we can easily verify the following (see also [12]):

$$\operatorname{supp} \eta_e^z \subset T_1 \cup T_2, \tag{3.4a}$$

$$\nabla \eta_e^z(\sigma) = 0 \quad \text{ for } \sigma \in \mathcal{S}_h \text{ and } \sigma \neq z. \tag{3.4b}$$

$$\|\nabla \eta_e^z\|_{L^\infty(T_1 \cup T_2)} \leq \frac{C}{|e|}. \tag{3.4c}$$

We then define the vector fields

$$\mathbf{w}_e^z = |e| \mathbf{t}_e^z \eta_e^z. \tag{3.5}$$
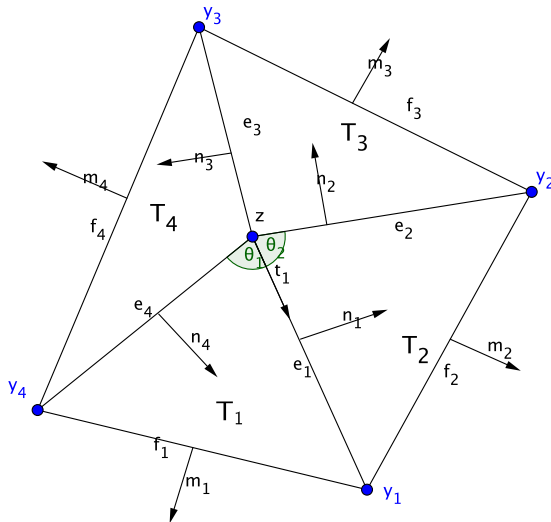
The following lemma is proved in [12].

Lemma 3.2. Let $z \in \mathcal{S}_h$ and $e \in \mathcal{E}_h(z)$ with $e = \{z, y\}$ and denote the two triangles that have e as an edge as T_1 and T_2. Let $\mathbf{w}_e^z$ be given by (3.5). Then

$$\mathbf{w}_e^z \in \mathbf{V}_{h,00}^3(z), \tag{3.6a}$$

$$\text{supp } \mathbf{w}_e^z \subset T_1 \cup T_2, \text{ and } \operatorname{div} \mathbf{w}_e^z|_{T_s}(z) = 1 \quad \text{for } s = 1, 2 \tag{3.6b}$$

$$\|\nabla \mathbf{w}_e^z\|_{L^\infty(T_1 \cup T_2)} \leq C. \tag{3.6c}$$

The constant C only depends on the shape regularity of T_1 and T_2.

FIGURE 3.1. Example of $\mathcal{T}_h(z)$ with $N = 4$.

3.1. Schematic for interior vertex

It will be useful to use the following notation for an interior vertex $z \in \mathcal{S}_h^{\mathrm{int}}$. We assume that $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$, enumerated as before so that T_j, T_{j+1} share an edge for $j = 0, \ldots N - 1$ and we identify T_0 as T_N (indices modulo N). For $1 \leq i \leq N$ we let e_i be the edge shared by T_i and T_{i+1} and e_N is the edge shared by T_N and T_1. We let $y_1, y_2, \ldots, y_N$ be the vertices such that $e_i = \{z, y_i\}$. We set $y_0 = z$. Also, $\mathbf{n}_i$ is unit-normal to e_i pointing out of T_i and θ_i is the angle formed by the two edges of T_i emanating from z. Moreover, let $\mathbf{t}_i$ be the unit-tangent vector to e_i pointing towards y_i. The edge opposite to z belonging to T_i is denoted by f_i. The unit-normal to f_i pointing out of T_i is denoted $\mathbf{m}_i$.

$$\mathbf{m}_i = |f_i|^{-1}(y_i - y_{i-1})^{\perp} = |y_i - y_{i-1}|^{-1}(y_i - y_{i-1})^{\perp}. \tag{3.7}$$

Finally, $h_i = \mathrm{dist}(z, f_i)$. The notation $\cdot^{\perp}$ denotes rotation by 90 degrees counter clockwise. See Figure 3.1 for an illustration.

Using this notation, we will use the shorthand notation $\psi_i = \psi_{y_i}$ for $0 \leq i \leq N$ and $\mathbf{w}_i = \mathbf{w}_{e_i}^z$ for $1 \leq i \leq N$.

3.2. Singular vertices are local interpolating vertices: $\mathcal{S}_h^2 \subset \mathcal{L}_h$

In this section we recall that all singular vertices are local interpolating vertices. The proof can be found in [12], but we recall some of the details.

Lemma 3.3. *It holds $\mathcal{S}_h^2 \subset \mathcal{L}_h$. Moreover, there exists a constant C_{sing} such that*

$$D_z \leq C_{sing} \qquad \text{for all } z \in \mathcal{S}_h^2,$$

where C_{sing} only depends on the shape regularity of the mesh.

Proof.

We only consider interior singular vertices for simplicity (the proof for boundary singular vertices is similar). Suppose that $z \in \mathcal{S}_h^{2,\text{int}}$ and we use the notation in Section 3.1. Note that $N = 4$. Let $a = (a_1, \ldots, a_4) \in W(z)$. First define $b_1 = a_1$ and inductively define

$$b_j = a_j - b_{j-1} \qquad \text{for } j = 2, 3.$$

Then define

$$\mathbf{v} = b_1 \mathbf{w}_1 + b_2 \mathbf{w}_2 + b_3 \mathbf{w}_3.$$

By (3.6a), $\mathbf{v} \in \mathbf{V}_{h,00}^3(z)$. Using (3.6b) we see that

$$\operatorname{div} \mathbf{v}|_{T_j}(z) = a_j \text{ for } 1 \leq j \leq 3.$$

We also have

$$\operatorname{div} \mathbf{v}|_{T_4}(z) = b_3 = a_3 - a_2 + a_1 = a_4,$$

where we used that $(a_1, a_2, a_3, a_4) \in W(z)$. Moreover, using (3.6c) we have

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))} \leq C (b_1 + b_2 + b_3) \leq C \max_{1 \leq j \leq 4} |a_j|,$$

where the constant C only depends on the shape regularity constant. ∎

3.3. Interior vertices with odd number of triangles are local interpolating vertices

In this section we prove that if $z \in \mathcal{S}_h^{\text{int}}$ and $\mathcal{T}_h(z)$ has an odd number of triangles then $z \in \mathcal{L}_h$.

Lemma 3.4. *Let $z \in \mathcal{S}_h^{int}$ with $\mathcal{T}_h(z) = \{T_1, \ldots, T_N\}$ and suppose that N is odd. Then $z \in \mathcal{L}_h$. Moreover, there exists a constant C_{odd} such that*

$$D_z \leq C_{odd}.$$

Here C_{odd} is a fixed constant that only depends on the shape regularity of the mesh.

Proof. We use the notation in Section 3.1. We start by defining some auxiliary functions. First, define

$$\mathbf{v}_1 = \frac{1}{2} \sum_{j=1}^N (-1)^{j-1} \mathbf{w}_j.$$

We see that $\mathbf{v}_1 \in \mathbf{V}_{h,00}^3(z)$, by (3.6a). Moreover, using (3.6b) we see that

$$\operatorname{div} \mathbf{v}_1|_{T_j}(z) = \delta_{1j} \text{ for } 1 \leq j \leq N,$$

where δ_{ij} is the Kronecker delta function. Note that here we used crucially that N is odd. Moreover, we have by (3.6c) that

$$\|\nabla \mathbf{v}_1\|_{L^\infty(\Omega_h(z))} \leq C,$$

where C only depends on the shape regularity. Next, we define inductively

$$\mathbf{v}_i = \mathbf{w}_i - \mathbf{v}_{i-1} \quad \text{for } 2 \leq i \leq N.$$

Then, we easily see that $\mathbf{v}_i \in \mathbf{V}_{h,00}^3(z)$ and

$$\operatorname{div} \mathbf{v}_i|_{T_j}(z) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq N, \tag{3.8}$$

and, furthermore,

$$\|\nabla \mathbf{v}_i\|_{L^\infty(\Omega_h(z))} \leq C \quad \text{for } 1 \leq i, j \leq N, \tag{3.9}$$

where C only depends on the shape regularity. Now given $a = (a_1, \ldots, a_N) \in W(z)$ we set

$$\mathbf{v} = \sum_{j=1}^N a_j \mathbf{v}_j.$$

Then, using (3.8) we get (3.1). Moreover, using (3.9) we get
$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))} \leq C_{\text{odd}} \max_{1 \leq j \leq N} |a_j|,$$
where C_{odd} is a fixed constant only depending on the shape regularity of the mesh. ∎

3.4. Interior vertices with even number of triangles

If $z \in \mathcal{S}_h^{\text{int}}$ is non-singular and has an even number of triangles containing it, then it is not necessarily the case that $z \in \mathcal{L}_h$. In this section we give sufficient conditions for $z \in \mathcal{L}_h$. We use the notation in Section 3.1. To do this, in addition to the vector fields $\mathbf{w}_i$, we will need other vector fields. We start with
$$\boldsymbol{\chi}_i = \frac{12}{|e_i|} \eta_{e_i}^z \mathbf{n}_i = \frac{12}{|e_i|} \psi_z^2 \psi_{y_i} \mathbf{n}_i \quad \text{for } 1 \leq i \leq N. \tag{3.10}$$
In the following lemma, the indices are calculated modulo N.

Lemma 3.5. *It holds, for* $1 \leq i \leq N$
$$supp(\boldsymbol{\chi}_i) \subset T_i \cup T_{i+1}, \tag{3.11a}$$
$$(\text{div } \boldsymbol{\chi}_i)(y_j) = 0 \quad \text{for } 1 \leq j \leq N, \tag{3.11b}$$
$$(\text{div } \boldsymbol{\chi}_i)|_{T_i}(z) = \frac{12 \cot(\theta_i)}{|e_i|^2}, \quad (\text{div } \boldsymbol{\chi}_i)|_{T_{i+1}}(z) = \frac{-12 \cot(\theta_{i+1})}{|e_i|^2}, \tag{3.11c}$$
$$\int_{T_i} \text{div } \boldsymbol{\chi}_i \, dx = 1, \quad \int_{T_{i+1}} \text{div } \boldsymbol{\chi}_i \, dx = -1, \tag{3.11d}$$
$$\|\nabla \boldsymbol{\chi}_i\|_{L^\infty(T_i \cup T_{i+1})} \leq \frac{C}{|e_i|^2}. \tag{3.11e}$$

Proof. It follows from the definition (3.3) of $\eta_{e_i}^z$ that (3.11a) and (3.11b) hold. A simple calculation using (2.6) and (2.7) shows that
$$\nabla \psi_{y_i}|_{T_i} = \frac{1}{\sin(\theta_i)|e_i|} \mathbf{n}_{i-1}.$$
Thus
$$(\text{div } \boldsymbol{\chi}_i)|_{T_i}(z) = \frac{12}{|e_i|} \psi_z^2(z) \nabla \psi_{y_i}|_{T_i} \cdot \mathbf{n}_i = \frac{12}{\sin(\theta_i)|e_i|^2} \mathbf{n}_{i-1} \cdot \mathbf{n}_i = \frac{12 \cot(\theta_i)}{|e_i|^2}.$$
Similarly, we can show that
$$(\text{div } \boldsymbol{\chi}_i)|_{T_{i+1}}(z) = \frac{-12 \cot(\theta_{i+1})}{|e_i|^2}.$$
To show (3.11d) we use integration by parts and use that $\eta_{e_i}^z$ vanishes on $\partial T_i \backslash e_i$ to get
$$\int_{T_i} \text{div } \boldsymbol{\chi}_i \, dx = 12 \int_{e_i} \eta_{e_i} \, ds = 1.$$
Similarly, we can show that
$$\int_{T_{i+1}} \text{div } \boldsymbol{\chi}_i \, dx = -1.$$
To prove (3.11e) we use the definition of $\boldsymbol{\chi}_i$ and the bound (3.4c). ∎

Note that $\boldsymbol{\chi}_i$ is not in $\mathbf{V}_{h,0}^3(z)$ by (3.11d). However, again using (3.11d), we see that
$$\boldsymbol{\chi} := \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 + \cdots + \boldsymbol{\chi}_N,$$

does belong to $\mathbf{V}_{h,0}^3(z)$. In fact, using (3.11b) we have that $\boldsymbol{\chi} \in \mathbf{V}_{h,00}^3(z)$. We collect it in the following result.

Lemma 3.6. *It holds that $\boldsymbol{\chi} \in \mathbf{V}_{h,00}^3(z)$ and*

$$(\operatorname{div} \boldsymbol{\chi})|_{T_j}(z) = 12 \cot(\theta_j)\Big(\frac{1}{|e_j|^2} - \frac{1}{|e_{j-1}|^2}\Big) \quad for\ 1 \le j \le N, \tag{3.12}$$

$$\|\nabla \boldsymbol{\chi}\|_{L^\infty(\Omega_h(z))} \le \frac{C}{h_z^2}. \tag{3.13}$$

The inequality (3.13) follows from (3.11e).

So far, we have $\mathbf{w}_1, \ldots, \mathbf{w}_N$ and $\boldsymbol{\chi}$ that belong to $\mathbf{V}_{h,00}^3(z)$. Next we describe two more functions that also belong to the space.

We let $\mathsf{E}_1 = [1,0]^t$ and $\mathsf{E}_2 = [0,1]^t$ be canonical directions. We then define

$$\tilde{\boldsymbol{\xi}}_i := \psi_z^2 \mathsf{E}_i \quad \text{for } i = 1, 2.$$

The following result can easily be proven.

Lemma 3.7. *It holds, for $i = 1, 2$*

$$\tilde{\boldsymbol{\xi}}_i \in \mathbf{V}_h^3(z) \tag{3.14a}$$

$$(\operatorname{div} \tilde{\boldsymbol{\xi}}_i)(y_j) = 0 \quad for\ 1 \le j \le N \tag{3.14b}$$

$$(\operatorname{div} \tilde{\boldsymbol{\xi}}_i)|_{T_j}(z) = \frac{3}{|T_j|} b_{ji} \quad for\ 1 \le j \le N \tag{3.14c}$$

$$\int_{T_j} \operatorname{div} \tilde{\boldsymbol{\xi}}_i \, dx = b_{ji} \quad for\ 1 \le j \le N, \tag{3.14d}$$

where $|T_j|$ denotes the area of T_j, and using (3.7) we get

$$b_{ji} = \frac{-|f_j|\mathbf{m}_j \cdot \mathsf{E}_i}{3} = \frac{-(y_j - y_{j-1})^\perp \cdot \mathsf{E}_i}{3} = \frac{-(y_j - y_{j-1}) \cdot \mathsf{E}_i^\perp}{3}, \tag{3.15}$$

where $\mathbf{v}^\perp$ denotes the rotation of $\mathbf{v}$ by 90 degrees counter clockwise. Moreover, the following bound holds

$$\|\nabla \tilde{\boldsymbol{\xi}}_i\|_{L^\infty(\Omega_h(z))} \le \frac{C}{h_z}. \tag{3.16}$$

Proof. From the definition ψ_z we have that (3.14a) and (3.14b) hold. To show (3.14c) we have

$$(\operatorname{div} \tilde{\boldsymbol{\xi}}_i)|_{T_j}(z) = 2\psi_z(z)\nabla\psi_z|_{T_j} \cdot \mathsf{E}_i = \frac{-2\mathbf{m}_j \cdot \mathsf{E}_i}{h_j} = \frac{-|f_j|\mathbf{m}_j \cdot \mathsf{E}_i}{|T_j|}.$$

In the last equation we used that $|T_j| = \frac{h_j|f_j|}{2}$. Using that $\operatorname{div} \tilde{\boldsymbol{\xi}}_i$ is linear in T_j and (3.14b), (3.14c) we have

$$\int_{T_j} \operatorname{div} \tilde{\boldsymbol{\xi}}_i \, dx = \frac{|T_j|}{3}(\operatorname{div} \tilde{\boldsymbol{\xi}}_i)|_{T_j}(z) = b_{ji}.$$

Finally, (3.16) follows by a simple computation. ∎

We note that $\tilde{\boldsymbol{\xi}}_i$ does not belong to $\mathbf{V}_{h,0}^3(z)$ by (3.14d). However, by integration by parts and using that $\tilde{\boldsymbol{\xi}}_i$ vanishes on $\partial\Omega_h(z)$ we do have that $\int_{\Omega_h(z)} \operatorname{div} \tilde{\boldsymbol{\xi}}_i dx = 0$ and hence

$$b_{1i} + b_{2i} + \cdots + b_{Ni} = 0 \quad \text{for } i = 1, 2. \tag{3.17}$$

This also follows by summing (3.15).

We can now correct $\tilde{\boldsymbol{\xi}}_i$ to make it belong to $\mathbf{V}^3_{h,00}(z)$. We define

$$\boldsymbol{\xi}_i := \tilde{\boldsymbol{\xi}}_i - c_{1i}\boldsymbol{\chi}_1 - c_{2i}\boldsymbol{\chi}_2 - \cdots - c_{N-1\,i}\boldsymbol{\chi}_{N-1} \quad \text{for } i = 1, 2,$$

where

$$c_{ji} = b_{1i} + b_{2i} + \cdots + b_{ji} = \frac{1}{3}(y_N - y_j) \cdot \mathsf{E}_i^\perp,$$

for $j = 1, \cdots, N$.

In the following result, indices are calculated mod N, and in particular $c_{0,i} = c_{N,i} = 0$.

Lemma 3.8. *It holds, for $i = 1, 2$, $\boldsymbol{\xi}_i \in \mathbf{V}^3_{h,00}(z)$ and*

$$\text{div}(\boldsymbol{\xi}_i)|_{T_j}(z) = \frac{3}{|T_j|}b_{ji} - 12\cot(\theta_j)\Big(\frac{c_{ji}}{|e_j|^2} - \frac{c_{j-1\,i}}{|e_{j-1}|^2}\Big) \quad \text{for } 1 \leq j \leq N, \tag{3.18}$$

$$\|\nabla\boldsymbol{\xi}_i\|_{L^\infty(\Omega_h(z))} \leq \frac{C}{h_z}. \tag{3.19}$$

The bound (3.19) follows from (3.16) and (3.11e).
We define for $i = 1, 2$

$$d_{ji} := (\text{div}\,\boldsymbol{\xi}_i)|_{T_j}(z) = \frac{3}{|T_j|}b_{ji} - 12\cot(\theta_j)\Big(\frac{c_{ji}}{|e_j|^2} - \frac{c_{j-1\,i}}{|e_{j-1}|^2}\Big) \quad \text{for } 1 \leq j \leq N, \tag{3.20}$$

and

$$d_{j0} := (\text{div}\,\boldsymbol{\chi})|_{T_j}(z) = \cot(\theta_j)\Big(\frac{1}{|e_j|^2} - \frac{1}{|e_{j-1}|^2}\Big) \quad \text{for } 1 \leq j \leq N. \tag{3.21}$$

Using (3.19) and (3.13), we note that for all $1 \leq j \leq N$

$$|d_{ji}| \leq \frac{C}{h_z^s} \quad \text{where } s = 1 \text{ if } i = 1, 2 \text{ and } s = 2 \text{ if } i = 0. \tag{3.22}$$

We can now prove the following important result.

Lemma 3.9. *Let $z \in \mathcal{S}_h^{1,int}$ with $\mathcal{T}_h(z) = \{T_1, \ldots, T_N\}$ with N even. Assume that for at least one $i = 0, 1, 2$*

$$\mathcal{D}_i := \sum_{j=1}^{N} (-1)^j d_{ji} \neq 0, \tag{3.23}$$

then $z \in \mathcal{L}_h$. Moreover, in this case, there exists a constant C depending only on the shape regularity constant such that

$$D_z \leq C \max_{0 \leq i \leq 2}\Big(1 + \frac{1}{|\mathcal{D}_i|h_z^s}\Big), \tag{3.24}$$

where $s = 1$ if $i = 1, 2$ and $s = 2$ if $i = 0$.

Proof. We set $\boldsymbol{\xi}_0 = \boldsymbol{\chi}$. Let i be such that (3.23) holds, We let $s_1 = 0$ and define inductively

$$s_j = d_{ji} - s_{j-1} \quad \text{for } 2 \leq j \leq N.$$

Define

$$\mathbf{v}_1 = \frac{-1}{\mathcal{D}_i}(\boldsymbol{\xi}_i - s_2\mathbf{w}_2 - \cdots - s_{N-1}\mathbf{w}_{N-1} - s_N\mathbf{w}_N).$$

We easily have that, using (3.20), (3.21)

$$\text{div}\,\mathbf{v}_1|_{T_j}(z) = \frac{-1}{\mathcal{D}_i}(d_{ji} - (s_{j-1} + s_j)) = 0 \quad \text{for } 2 \leq j \leq N,$$

and

$$\text{div}\,\mathbf{v}_1|_{T_1}(z) = \frac{-1}{\mathcal{D}_i}(d_{1i} - s_N).$$

We see that $s_N = \sum_{j=2}^{N}(-1)^j d_{ji}$, we have $d_{1i} - s_N = -\sum_{j=1}^{N}(-1)^j d_{ji} = -\mathcal{D}_i$. Hence,

$$\operatorname{div} \mathbf{v}_1|_{T_1}(z) = 1.$$

The following bound follows from (3.22), (3.19), (3.13) and (3.6c)

$$\|\nabla \mathbf{v}_1\|_{L^\infty(\Omega_h(z))} \leq \frac{C}{|\mathcal{D}_i| h_z^s}. \tag{3.25}$$

Then, we define inductively

$$\mathbf{v}_\ell = \mathbf{w}_\ell - \mathbf{v}_{\ell-1} \quad \text{for } 2 \leq \ell \leq N.$$

We then easily see that

$$\operatorname{div} \mathbf{v}_\ell|_{T_j}(z) = \delta_{\ell j} \quad \text{for } 1 \leq \ell, j \leq N. \tag{3.26}$$

Moreover, from (3.6c) and (3.25) we have

$$\|\nabla \mathbf{v}_\ell\|_{L^\infty(\Omega_h(z))} \leq C\left(1 + \frac{1}{|\mathcal{D}_i| h_z^s}\right).$$

For an arbitrary $a = (a_1, a_2, \ldots, a_N) \in W(z)$ we simply define

$$\mathbf{v} = \sum_{\ell=1}^{N} a_\ell \mathbf{v}_\ell.$$

Then, $\mathbf{v}$ satisfies (3.1) and hence $z \in \mathcal{L}_h$. By (3.4) we have

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))} \leq C\left(1 + \frac{1}{|\mathcal{D}_i| h_z^s}\right) \max_\ell |a_\ell|,$$

which proves (3.24). ∎

3.5. **Simplification of condition** (3.23)

In the case $i = 0$, we have

$$\mathcal{D}_0 = \sum_{i=1}^{N}(-1)^j d_{j0} = \sum_{i=1}^{N}(-1)^j \cot(\theta_j)\left(|e_j|^{-2} - |e_{j-1}|^{-2}\right)$$

$$= \sum_{i=1}^{N}(-1)^j\left(\cot(\theta_j) + \cot(\theta_{j+1})\right)|e_j|^{-2}. \tag{3.27}$$

Thus we see that generically this is nonzero, since the lengths $|e_j|$ can be chosen independently of the angles θ_j. For $i = 1, 2$, we have

$$\sum_{j=1}^{N}(-1)^j \frac{3}{|T_j|} b_{ji} = \sum_{j=1}^{N}(-1)^j \frac{1}{|T_j|}\left(y_{j-1} - y_j\right) \cdot \mathsf{E}_i^\perp$$

$$= -\sum_{j=1}^{N}(-1)^j\left(\frac{1}{|T_j|} + \frac{1}{|T_{j+1}|}\right) y_j \cdot \mathsf{E}_i^\perp \tag{3.28}$$

For $i = 1, 2$, we have

$$-12 \sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{c_{j,i}}{|e_j|^2} - \frac{c_{j-1,i}}{|e_{j-1}|^2} \Big) = -12 \sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{c_{j,i}}{|e_j|^2} - \frac{c_{j-1,i}}{|e_{j-1}|^2} \Big)$$

$$= 4 \sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{y_j - y_N}{|e_j|^2} - \frac{y_{j-1} - y_N}{|e_{j-1}|^2} \Big) \cdot \mathsf{E}_i^{\perp}$$

$$= 4 \sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{y_j}{|e_j|^2} - \frac{y_{j-1}}{|e_{j-1}|^2} \Big) \cdot \mathsf{E}_i^{\perp}$$

$$- 4 \Big(\sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{1}{|e_j|^2} - \frac{1}{|e_{j-1}|^2} \Big) \Big) y_N \cdot \mathsf{E}_i^{\perp} \qquad (3.29)$$

$$= 4 \sum_{j=1}^{N} (-1)^j \cot(\theta_j) \Big(\frac{y_j}{|e_j|^2} - \frac{y_{j-1}}{|e_{j-1}|^2} \Big) \cdot \mathsf{E}_i^{\perp}$$

$$- 4 \Big(\sum_{j=1}^{N} (-1)^j d_{j0} \Big) y_N \cdot \mathsf{E}_i^{\perp}$$

$$= 4 \sum_{j=1}^{N} (-1)^j \frac{\cot(\theta_j) + \cot(\theta_{j+1})}{|e_j|^2} y_j \cdot \mathsf{E}_i^{\perp} - 4 \mathcal{D}_0 \, y_N \cdot \mathsf{E}_i^{\perp}.$$

Therefore, for $i = 1, 2$,

$$\mathcal{D}_i = \sum_{j=1}^{N} (-1)^j \Big(4 \frac{\cot(\theta_j) + \cot(\theta_{j+1})}{|e_j|^2} - \Big(\frac{1}{|T_j|} + \frac{1}{|T_{j+1}|} \Big) \Big) y_j \cdot \mathsf{E}_i^{\perp} - 4 \mathcal{D}_0 \, y_N \cdot \mathsf{E}_i^{\perp}. \qquad (3.30)$$

4. Meshes where (3.23) fails to hold

Lemma 3.9 gives sufficient conditions for an interior vertex z with an even number of triangles to be a local interpolating vertex (i.e., $z \in \mathcal{L}_h$). We see that (3.23) is a mild constraint and that a generic vertex will satisfy (3.23), however, there are important examples of vertices which do not satisfy (3.23) and perhaps are not local interpolating vertices. Here we present some examples.

4.1. Regular N-gon with N even

Suppose that $\Omega_h(z)$ is a triangulated regular N-gon with N even. More precisely, we assume that $\Omega_h(z)$ is subdivided by N similar triangles, with edge lengths $|e_1| = |e_2| = \cdots = |e_N|$ and interior angles $\theta_1 = \theta_2 = \cdots = \theta_N$. Then we can show that (3.23) does not hold.

First of all, the condition on the edge lengths alone implies that $d_{j0} = 0$ for all $j = 1, \ldots, N$. Thus $\mathcal{D}_0 = 0$.

Now consider $i = 1, 2$. The vertices of the regular N-gon can be written as

$$y_k = h e^{\iota 2\pi k/N},$$

for some h. Here we make the the standard association $e^{\iota \theta}$ with the vector $[\cos\theta, \sin\theta]^t$. We conclude that, for N even,

$$h^{-1} \sum_{j=1}^{N} (-1)^j y_j = \sum_{j=1}^{N/2} e^{\iota 4\pi j/N} - \sum_{j=1}^{N/2} e^{\iota 2\pi(2j-1)/N} = \big(1 - e^{-\iota 2\pi/N} \big) \sum_{j=1}^{N/2} e^{\iota 4\pi k/N} = \mathbf{0}. \qquad (4.1)$$
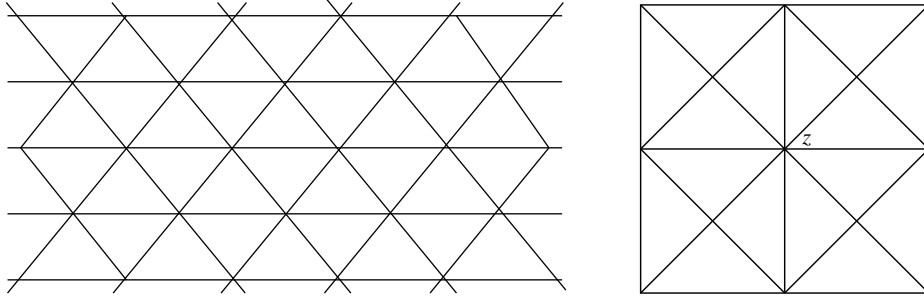
FIGURE 5.1. For the mesh on left no interior vertex satisfies (3.23). All interior vertices on the mesh on the right belong to $\mathcal{L}_h$.

Since all the triangles are the same, using (3.30) and that we showed $\mathcal{D}_0 = 0$ we have for $i = 1, 2$

$$\mathcal{D}_i = \left(\frac{8 \cot(\theta_1)}{|e_1|^2} - \frac{2}{|T_1|} \right) \sum_{j=1}^{N} (-1)^j y_j \cdot \mathsf{E}_i^{\perp} = 0,$$

where we used (4.1). Thus (3.23) fails for $i = 1, 2$ as well, and Lemma 3.9 cannot be used.

4.2. Three lines mesh

For the three-lines mesh generating a regular hexagonal pattern as shown on the left in Figure 5.1, the condition (3.23) also fails for $i = 0, 1, 2$ since each vertex is at the center of a regular hexagon.

5. Crossed triangles

Consider the mesh shown on the right in Figure 5.1. Half of the vertices are at the center of a regular 4-gon, but these are singular vertices, so these are all local interpolating vertices. For the other vertices, at the center of a non-regular 8-gon, we can argue as follows. Since the interior angles are all the same ($\pi/4$), and $\cot(\pi/4) = 1$, we have using (3.27)

$$\mathcal{D}_0 = 2 \sum_{j=1}^{8} (-1)^j \frac{1}{|e_j|^2}. \tag{5.1}$$

Let L be the length of the smallest edge: $L = \min \{|e_j| \,:\, j = 1, \ldots, 8\}$. Then the longest edge length is $\sqrt{2}L$, and the edge lengths $|e_j|$ alternate $L, \sqrt{2}L, L, \sqrt{2}L, \ldots$. Thus

$$\sum_{j=1}^{8} (-1)^j \frac{1}{|e_j|^2} = \pm \frac{2}{L^2},$$

depending on where we start the counting. Therefore condition (3.23) holds at these vertices, and thus Lemma 3.9 can be applied to conclude that these vertices are also local interpolating vertices. Thus all of the vertices in the mesh shown on the right in Figure 5.1 are in $\mathcal{L}_h$.

6. Inf-sup stability when all interior vertices belong to $\mathcal{L}_h$

In the previous section we have identified many local interpolating vertices. In particular, singular vertices and interior vertices with odd number of triangles are local interpolating vertices; see Lemmas 3.3 and 3.4. If z is an interior non-singular vertex with even number of triangles then Lemma 3.9

gives sufficient conditions for it to be a local interpolating vertex. None of the above examples address boundary vertices that are non-singular. In this section we will show how to interpolate at those vertices but in a non-local way. Then, using that result and assuming that $\mathcal{S}_h^{\text{int}} \subset \mathcal{L}_h$ we will prove inf-sup stability. In the next section we will address $\mathcal{S}_h^{\text{int}} \not\subset \mathcal{L}_h$.

For vertices that are not local interpolating vertices we can still interpolate there but with a side effect of polluting a neighboring vertex. In other words, the vector field will not belong to $\mathbf{V}_{h,00}^3(z)$. To do this, we will need to define a piecewise cubic function that has average zero on edges. For every $z \in \mathcal{S}_h^\partial$ and interior edge $e \in \mathcal{E}_h$ with $e = \{z, y\}$ we set

$$\kappa_e^z = \eta_e^z - \frac{1}{2}\psi_z\psi_y = \psi_z^2\psi_y - \frac{1}{2}\psi_z\psi_y. \tag{6.1}$$

The function κ_e^z will play the same role as γ_e^z in [12] but the difference is that the added term in κ_e^z is piecewise quadratic. Let $T_1, T_2 \in \mathcal{T}_h(z)$ be two triangles that have e as an edge, and let θ_i be the angle between the edges of T_i emanating from z.

Then we can easily verify the following:

$$\text{supp } \kappa_e^z \subset T_1 \cup T_2, \tag{6.2a}$$

$$(\nabla \kappa_e^z)(\sigma) = 0 \quad \text{for } \sigma \in \mathcal{S}_h, \ \sigma \neq z \text{ and } \sigma \neq y, \tag{6.2b}$$

$$\int_e \kappa_e^z \, ds = 0 \tag{6.2c}$$

$$\nabla\kappa_e^z|_{T_i}(z) = \frac{1}{2}\nabla\psi_y|_{T_i}, \quad \nabla\kappa_e^z|_{T_i}(y) = -\frac{1}{2}\nabla\psi_z|_{T_i} \quad \text{for } i = 1, 2. \tag{6.2d}$$

Using these functions we can prove the following result.

Lemma 6.1. *For every $p \in Q_h^2$ and $z \in \mathcal{S}_h^\partial$ there exists a $\mathbf{v} \in \mathbf{V}_{h,0}^3(z)$ such the following properties hold:*

$$\text{div } \mathbf{v}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h^\partial, \ \sigma \neq z, \tag{6.3a}$$

$$\text{div } \mathbf{v}|_T(z) = p|_T(z) \quad \text{for all } T \in \mathcal{T}_h(z). \tag{6.3b}$$

If z is non-singular

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_h(z))} \leq C\Big(\frac{1}{\Theta(z)} + 1\Big)\|p\|_{L^2(\Omega_h(z))}. \tag{6.4}$$

If z is singular

$$\|\nabla\mathbf{v}\|_{L^2(\Omega_h(z))} \leq C\|p\|_{L^2(\Omega_h(z))}. \tag{6.5}$$

The constant C only depends only on the shape regularity.

Proof. If z is singular, the result follows from Lemma 3.3, so now assume that z is non-singular. Enumerate the triangles such that T_1 and T_N each have a boundary edge, and T_j, T_{j+1} share an edge $e_j = \{z, y_j\}$, for $j = 1, \ldots N - 1$. Let θ_j denote the angle between the edges of T_j originating from z. Also let $\mathbf{n}_j$ be the normal to e_j out of T_j and $\mathbf{t}_j$ be tangent to e_j pointing away from z. Let $1 \leq s \leq N - 1$ be such that $|\sin(\theta_s + \theta_{s+1})| = \Theta(z)$. We will define vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_N$. We start by defining $\mathbf{v}_s$.

$$\mathbf{v}_s = \frac{2|e_s|\sin(\theta_s)}{\sin(\theta_s + \theta_{s+1})}\mathbf{t}_{s+1}\kappa_{e_s}^z.$$

Then, we can easily show that

$$\text{div } \mathbf{v}_s|_{T_i}(z) = \delta_{i,s} \text{ for } 1 \leq i \leq N,$$

where $\delta_{i,s}$ is the Kronecker δ. Indeed,

$$\text{div } \mathbf{v}_s|_{T_i}(z) = \frac{2|e_s|\sin(\theta_s)}{\sin(\theta_s + \theta_{s+1})}\mathbf{t}_{s+1} \cdot \nabla\kappa_{e_s}^z|_{T_i}(z) = \frac{|e_s|\sin(\theta_s)}{\sin(\theta_s + \theta_{s+1})}\mathbf{t}_{s+1} \cdot \nabla\psi_{y_s}|_{T_i}(z).$$

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The result follows after using (2.5) and (2.7) to calculate $\psi_{y_s}|_{T_i}$ and using basic trigonometry. Then we can define inductively for $s + 1 \le j \le N$.

$$\mathbf{v}_j = \mathbf{w}_{e_j}^z - \mathbf{v}_{j-1}.$$

Also, for $1 \le j \le s - 1$ we define

$$\mathbf{v}_j = \mathbf{w}_{e_j}^z - \mathbf{v}_{j+1}.$$

Hence, we have the following property:

$$\operatorname{div} \mathbf{v}_j|_{T_i}(s) = \delta_{i,j} \quad \text{for } 1 \le i, j \le N. \tag{6.6}$$

We then define

$$\mathbf{v} = \sum_{j=1}^{N} p|_{T_j}(z)\mathbf{v}_j.$$

The stated conditions on $\mathbf{v}$ are easily verified. ∎

We can then use the above lemma to prove the following result. We define

$$\Theta_{\min,\partial} = \min_{z \in \mathcal{S}_h^{1,\partial}} \Theta(z).$$

Lemma 6.2. *For every $p \in Q_h^2$ there exists a $\mathbf{v} \in \mathbf{V}_{h,0}^3$ such that*

$$(\operatorname{div} \mathbf{v} - p)(z) = 0 \quad \text{for all } z \in \mathcal{S}_h^{\partial}, \tag{6.7}$$

and

$$\|\nabla \mathbf{v}\|_{L^2(\Omega)} \le C_b \Big(\frac{1}{\Theta_{\min,\partial}} + 1 \Big) \|p\|_{L^2(\Omega)}.$$

The constant C_b only depends on the shape regularity.

Proof. Let $p \in Q_h^2$. For every $z \in \mathcal{S}_h^{\partial}$ let $\mathbf{v}^z$ be the vector field from Lemma 6.1 then we set

$$\mathbf{v} = \sum_{z \in \mathcal{S}_h^{\partial}} \mathbf{v}_z.$$

Then, it is easy to verify the conditoins on $\mathbf{v}$. ∎

We can now prove the main result of the section.

Theorem 6.3. *Assume that $\mathcal{S}_h^{int} \subset \mathcal{L}_h$. Then, for every $p \in Q_h^2$ there exists a $\mathbf{v} \in V_{h,0}^3$ satisfying*

$$(\operatorname{div} \mathbf{v} - p)(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \tag{6.8}$$

with the bound

$$\|\mathbf{v}\|_{H^1(\Omega)} \le C(1 + D)(1 + \frac{1}{\Theta_{\min,\partial}}) \|p\|_{L^2(\Omega)}.$$

where $D = \max_{z \in \mathcal{L}_h} D_z$.

Proof. Let $\mathbf{v}_1$ be from Lemma 6.2 and let $p_1 = p - \operatorname{div} \mathbf{v}_1$ and we note that $p_1 \in Q_h^2$ with p_1 vanishing on boundary vertices. Since $\mathcal{S}_h^{int} \subset \mathcal{L}_h$ for every $z \in \mathcal{S}_h^{int}$ there exists a $\mathbf{v}^z \in \mathbf{V}_{h,00}^3(z)$ such that
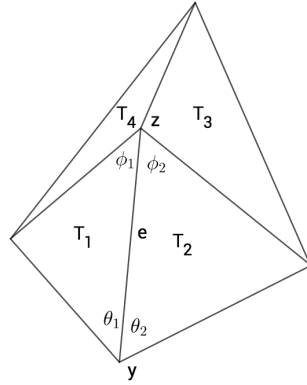
$$(\operatorname{div} \mathbf{v}^z - p_1)(z) = 0$$

with

$$\|\nabla \mathbf{v}^z\|_{L^\infty(\Omega_h(z))} \le D_z \max_{T \in \mathcal{T}_h(z)} |p_1|_T(z)|.$$

Using an inverse estimate we can show

$$\|\nabla \mathbf{v}^z\|_{L^2(\Omega_h(z))} \le C D_z \|p_1\|_{L^2(\Omega_h(z))}.$$

Figure 7.1. Illustration for Lemma 7.1, with $N = 4$.

If we set

$$\mathbf{v}_2 = \sum_{z \in \mathcal{S}_h^{\text{int}}} \mathbf{v}^z$$

and set $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, then the desired conditions on $\mathbf{v}$ are met. ∎

Using Lemma 2.6 we have the following corollary.

Corollary 6.4. *Assume that $\mathcal{S}_h^{int} \subset \mathcal{L}_h$, then the inf-sup condition holds for $Q_h^2 \times \mathbf{V}_h^3$ with constants given by Lemma 2.6 and Theorem 6.3.*

7. Inf-sup stability: the general case

As mentioned above for most meshes $\mathcal{S}_h^{\text{int}} \subset \mathcal{L}_h$ and hence by the previous section one can prove inf-sup stability. However, for very important meshes, such as the diagonal mesh, none of the interior vertices belong to $\mathcal{L}_h$ (i.e. $\mathcal{L}_h \cap \mathcal{S}_h^{\text{int}} = \emptyset$). However, if some interior nodes belong to $\mathcal{L}_h$ then it might hold that $\text{div}\,\mathbf{V}_h^3 = Q_h^2$ and we can give a bound for inf-sup constant. To do this we use a concept of a tree and paths. We consider the mesh as a graph and consider trees and paths that are subgraphs of the mesh. Precise statements are given below.

7.1. Paths and trees in a mesh

We will prove that if there is a tree of the mesh $\mathcal{T}_h$ with root in $\mathcal{L}_h$ satisfying certain mild conditions then we can interpolate a pressure on all the vertices of the tree. We start with some preliminary results. Let $T_1, T_2 \in \mathcal{T}_h(z)$ be two triangles that have e as an edge and let ϕ_i be the angle between the edges of T_i emanating from z. Then we define

$$M_e^z = \cot(\phi_1) + \cot(\phi_2). \tag{7.1}$$

Note that $M_e^z = 0$ if $\phi_1 + \phi_2 = \pi$.

Lemma 7.1. *Let $z, y \in \mathcal{S}_h$ and $e = \{z, y\} \in \mathcal{E}_h$ and suppose that $M_e^z \neq 0$. Let $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$ and suppose T_1 and T_2 are the two triangles that share e as an edge and let θ_i be the angles of T_i originating from y for $i = 1, 2$; see Figure 7.1. Let $a = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$ and define the alternating*

sum $s(a) = \sum_{j=1}^{N} (-1)^{j-1} a_j$. *Then, there exists a* $\mathbf{v} \in \mathbf{V}_{h,0}^3(z)$

$$\operatorname{div} \mathbf{v}|_{T_i}(z) = a_i \quad \text{for all } 1 \le i \le N \tag{7.2a}$$

$$\operatorname{div} \mathbf{v}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \ \sigma \ne z, \ \sigma \ne y, \tag{7.2b}$$

$$\operatorname{div} \mathbf{v}|_{T_1}(y) = -s(a)\frac{\cot(\theta_1)}{M_e^z}, \quad \operatorname{div} \mathbf{v}|_{T_2}(y) = s(a)\frac{\cot(\theta_2)}{M_e^z} \tag{7.2c}$$

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(z))} \le \frac{C}{|M_e^z|} \max_{1 \le i \le N} |a_i|. \tag{7.2d}$$

The constant C *depends only on the shape regularity.*

Proof. We prove the result in the case z is an interior vertex. The case z is a boundary vertex is similar. We use the notation from Section 3.1. We need to define some auxiliary vector fields. First let

$$\mathbf{r} = 2|e_1|\kappa_{e_1}^z \mathbf{n}_1,$$

where we recall that the definition of $\kappa_{e_1}^z$ is given in (6.1). Using, (6.2a) and (6.2c) it is easy to show that $\mathbf{r} \in \mathbf{V}_{h,0}^3(z)$. Then, using (6.2b) we have

$$\operatorname{div} \mathbf{r}|_{T_i}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \ \sigma \ne z, \ \sigma \ne y.$$

Also, using (6.2d), (2.6), (2.7) we can show that

$$\operatorname{div} \mathbf{r}|_{T_1}(z) = \cot(\phi_1), \quad \operatorname{div} \mathbf{r}|_{T_2}(z) = -\cot(\phi_2),$$

and

$$\operatorname{div} \mathbf{r}|_{T_1}(y) = -\cot(\theta_1), \quad \operatorname{div} \mathbf{r}|_{T_2}(z) = \cot(\theta_2).$$

If we let

$$\mathbf{v}_1 = \frac{1}{M_e^z}\big(\mathbf{r} + \cot(\phi_2)\mathbf{w}_1\big),$$

then using the properties of $\mathbf{w}_1$ (e.g. (3.6)) we have the following

$$\operatorname{div} \mathbf{v}_1|_{T_1}(z) = 1, \quad \operatorname{div} \mathbf{v}_1|_{T_2}(z) = 0,$$

$$\operatorname{div} \mathbf{v}_1|_{T_1}(y) = -\frac{\cot(\theta_1)}{M_e^z}, \quad \operatorname{div} \mathbf{v}_1|_{T_2}(y) = \frac{\cot(\theta_2)}{M_e^z},$$

and

$$\|\nabla \mathbf{v}_1\|_{L^\infty(\Omega_h(z))} \le \frac{C}{|M_e^z|},$$

where the constant C only depends on the shape regularity constant.

Next, we define inductively

$$\mathbf{v}_j = \mathbf{w}_{j-1} - \mathbf{v}_{j-1} \text{ for } 2 \le j \le N.$$

Using the properties of $\mathbf{v}_1$ just proved together with (3.6), we can show the following for all $1 \le j \le N$:

$$\mathbf{v}_j \in \mathbf{V}_{h,0}^3(z), \tag{7.3a}$$

$$\operatorname{div} \mathbf{v}_j(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \ \sigma \ne z, \ \sigma \ne y, \tag{7.3b}$$
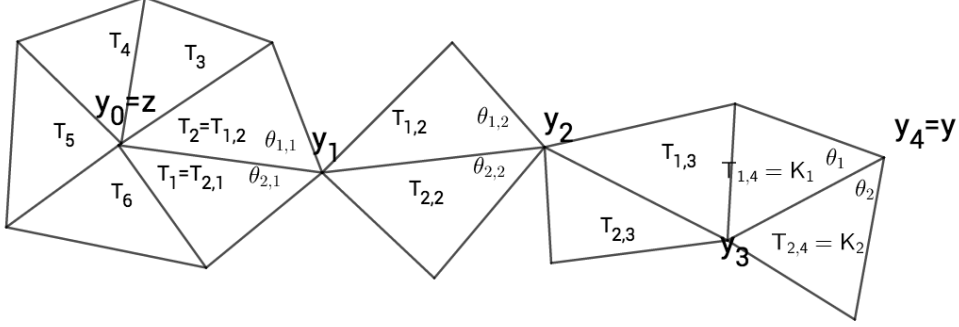
$$\operatorname{div} \mathbf{v}_j|_{T_i}(z) = \delta_{ij} \quad \text{for all } 1 \le i, j \le N, \tag{7.3c}$$

$$\|\nabla \mathbf{v}_j\|_{L^\infty(\Omega_h(z))} \le C\Big(1 + \frac{1}{|M_e^z|}\Big) \le \frac{C}{|M_e^z|}, \tag{7.3d}$$

where we used that $|M_e^z| \le c$ is bounded where the constant c depends on the shape regularity. By the definition of $\mathbf{v}_j$ and using that $\operatorname{div} \mathbf{w}_j(y) = 0$, we note that $\operatorname{div} \mathbf{v}_j(y) = (-1)^{j-1} \operatorname{div} \mathbf{v}_1(y)$. We now set

$$\mathbf{v} = \sum_{j=1}^{N} a_j \mathbf{v}_j.$$

FIGURE 7.2. Illustration of acceptable path, with $N = 6$, L=4.

We easily see conditions (7.2) hold. ∎

We can apply the previous result repeatedly to generalize the result for a path.

Definition 7.2. Given $z, y \in \mathcal{S}_h$ $P = \{y_0, y_1, \ldots, y_L\}$ is a path between $y_0 = z$, $y_L = y$ if $e_i = \{y_{i-1}, y_i\} \in \mathcal{E}_h$ and $y_i \neq y_j$ for $i \neq j$. We say that the path is *acceptable* if $M_{e_i}^{y_{i-1}} \neq 0$ for $1 \leq i \leq L$.

See Figure 7.2 for an illustration. For an acceptable path P as in Definition 7.2 we define for $1 \leq j \leq L - 1$

$$\tilde{\rho}_{z,y_j} := \frac{M_{e_1}^{y_1}}{M_{e_1}^{y_0}} \frac{M_{e_2}^{y_2}}{M_{e_2}^{y_1}} \cdots \frac{M_{e_j}^{y_j}}{M_{e_j}^{y_{j-1}}}.$$

We also let $\tilde{\rho}_{z,z} = 1$. Moreover, we define

$$\rho_{z,y_{j+1}} := \frac{\tilde{\rho}_{z,y_j}}{M_{e_{j+1}}^{y_j}}. \tag{7.4}$$

Finally, we let

$$\rho(P) = \max_{1 \leq j \leq L} |\rho_{z,y_j}|. \tag{7.5}$$

Also for any collection of vertices P we define $\mathcal{T}_h(P) = \{T \in \mathcal{T}_h : T \in \mathcal{T}_h(y) \text{ for some } y \in P\}$. We define $\Omega_h(P) = \bigcup_{T \in \mathcal{T}_h(P)} T$.

Lemma 7.3. *Suppose that $z, y \in \mathcal{S}_h$ and let $\mathcal{T}_h(z) = \{T_1, T_2, \ldots, T_N\}$. Let $P = \{y_0, y_1, \ldots, y_L\}$ with $y_0 = z$, $y_L = y$ be an acceptable path. Also, we denote by K_1 and K_2 the two triangles that share e_L and θ_i the corresponding angles. For every $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ there exists a $\mathbf{v} \in \mathbf{V}_{h,0}^3$ with support*

in $\Omega_h(P)$ such that

$$\operatorname{div} \mathbf{v}|_{T_i}(z) = a_i \quad \text{for all } 1 \le i \le N, \tag{7.6a}$$

$$\operatorname{div} \mathbf{v}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne z, \sigma \ne y, \tag{7.6b}$$

$$\operatorname{div} \mathbf{v}|_{K_1}(y) = \pm s(a)\tilde{\rho}_{z,y_{L-1}} \frac{\cot(\theta_1)}{M_{e_L}^{y_{L-1}}}, \quad \operatorname{div} \mathbf{v}|_{K_2}(y) = \mp s(a)\tilde{\rho}_{z,y_{L-1}} \frac{\cot(\theta_2)}{M_{e_L}^{y_{L-1}}} \tag{7.6c}$$

$$\operatorname{div} \mathbf{v}|_T(y) = 0 \quad \text{for all } T \in \mathcal{T}_h(y), T \ne K_1, T \ne K_2, \tag{7.6d}$$

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(P))} \le C\rho(P) \max_{1 \le i \le N} |a_i|. \tag{7.6e}$$

Here $s(a) = \sum_{j=1}^N (-1)^j a_j$. The constant C only depends on the shape regularity of the mesh.

Proof. Let $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ be given. We will use the following notation: e_j is the edge with vertices y_{j-1}, y_j. We assume that $\mathcal{T}_h(y_j) = \{T_{1,j}, T_{2,j}, \ldots, T_{N_j,j}\}$ and such that $T_{1,j}, T_{2,j}$ share e_j as a common edge. The corresponding angles are denoted by $\theta_{1,j}, \theta_{2,j}$. Note that $K_1 = T_{1,L}$ and $K_2 = T_{2,L}$. By Lemma 7.1 we have $\mathbf{r}_1 \in \mathbf{V}_{h,0}^3(y_0)$

$$\operatorname{div} \mathbf{r}_1|_{T_i}(y_0) = a_i \quad \text{for all } 1 \le i \le N \tag{7.7a}$$

$$\operatorname{div} \mathbf{r}_1(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne y_0, \sigma \ne y_1, \tag{7.7b}$$

$$\operatorname{div} \mathbf{r}_1|_{T_{1,1}}(y_1) = -s(a) \frac{\cot(\theta_{1,1})}{M_{e_1}^{y_0}}, \quad \operatorname{div} \mathbf{r}_1|_{T_{2,j}}(y_1) = s(a) \frac{\cot(\theta_{2,1})}{M_{e_1}^{y_0}} \tag{7.7c}$$

$$\operatorname{div} \mathbf{r}_1|_{T_{i,1}}(y_1) = 0 \quad \text{for all } 3 \le i \le N \tag{7.7d}$$

$$\|\nabla \mathbf{r}_1\|_{L^\infty(\Omega_h(y_0))} \le \frac{C}{|M_{e_1}^{y_0}|} \max_{1 \le i \le N} |a_i|. \tag{7.7e}$$

Now suppose that we have constructed $\mathbf{r}_j \in \mathbf{V}_{h,0}^3(y_{j-1})$ for $j = 2, \ldots, \ell$ with the following properties

$$(\operatorname{div} \mathbf{r}_j + \operatorname{div} \mathbf{r}_{j-1})(y_{j-1}) = 0, \tag{7.8a}$$

$$\operatorname{div} \mathbf{r}_j(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne y_{j-1}, \sigma \ne y_j, \tag{7.8b}$$

$$\operatorname{div} \mathbf{r}_j|_{T_{1,j}}(y_j) = \pm s(a)\tilde{\rho}_{z,y_{j-1}} \frac{\cot(\theta_{1,j})}{M_{e_j}^{y_{j-1}}}, \quad \operatorname{div} \mathbf{r}_j|_{T_{2,j}}(y_j) = \mp s(a)\tilde{\rho}_{z,y_{j-1}} \frac{\cot(\theta_{2,j})}{M_{e_j}^{y_{j-1}}} \tag{7.8c}$$

$$\operatorname{div} \mathbf{r}_j|_{T_{i,j}}(y_j) = 0 \quad \text{for all } 3 \le i \le N_j, \tag{7.8d}$$

$$\|\nabla \mathbf{r}_j\|_{L^\infty(\Omega_h(y_{j-1}))} \le C|\rho_{z,y_j}| \max_{1 \le i \le N} |a_i|. \tag{7.8e}$$

Setting $\tilde{a}_i = \operatorname{div} \mathbf{r}_\ell|_{T_{i,\ell}}(y_\ell)$ for $1 \le i \le N_\ell$ and using (7.8c) and (7.8d) we have

$$s(\tilde{a}) = \sum_{i=1}^{N_\ell} (-1)^{i-1}\tilde{a}_i = \tilde{a}_1 - \tilde{a}_2 = \pm s(a)\tilde{\rho}_{z,y_{\ell-1}} \frac{\cot(\theta_{1,\ell})}{M_{e_j}^{y_{\ell-1}}} \pm s(a)\tilde{\rho}_{z,y_{\ell-1}} \frac{\cot(\theta_{2,\ell})}{M_{e_\ell}^{y_{\ell-1}}},$$

$$= \pm s(a)\tilde{\rho}_{z,y_{\ell-1}} \frac{\cot(\theta_{1,\ell}) + \cot(\theta_{2,\ell})}{M_{e_j}^{y_{\ell-1}}} = \pm s(a)\tilde{\rho}_{z,y_{\ell-1}} \frac{M_{e_\ell}^{y_\ell}}{M_{e_\ell}^{y_{\ell-1}}} = \pm s(a)\tilde{\rho}_{z,y_\ell}. \tag{7.9}$$

Hence, using Lemma 7.1 we can find $\mathbf{r}_{\ell+1} \in \mathbf{V}_{h,0}^3(y_\ell)$ such that

$$\operatorname{div} \mathbf{r}_{\ell+1}|_{T_{i,\ell}}(y_\ell) = -\tilde{a}_i \quad \text{for all } 1 \le i \le N_\ell \tag{7.10a}$$

$$\operatorname{div} \mathbf{r}_{\ell+1}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne y_\ell, \sigma \ne y_{\ell+1}, \tag{7.10b}$$

$$\operatorname{div} \mathbf{r}_{\ell+1}|_{T_{1,\ell+1}}(y_{\ell+1}) = \pm s(\tilde{a}) \frac{\cot(\theta_{1,\ell+1})}{M_{e_{\ell+1}}^{y_\ell}}, \quad \operatorname{div} \mathbf{r}_{\ell+1}|_{T_{2,\ell+1}}(y_{\ell+1}) = \mp s(\tilde{a}) \frac{\cot(\theta_{2,\ell+1})}{M_{e_{\ell+1}}^{y_\ell}} \tag{7.10c}$$

$$\|\nabla \mathbf{r}_{\ell+1}\|_{L^\infty(\Omega_h(y_\ell))} \le \frac{C}{|M_{e_{\ell+1}}^{y_\ell}|} \max_{1 \le i \le N_\ell} |\tilde{a}_i| \le \frac{C}{|M_{e_{\ell+1}}^{y_\ell}|} |s(a)||\tilde{\rho}_{z,y_\ell}| = C|s(a)||\rho_{z,y_{\ell+1}}|. \tag{7.10d}$$

Hence, using (7.10) and (7.9) we get that (7.8) holds for $j = \ell + 1$ and hence by induction (7.8) holds for $2 \le j \le L$.

We let $\mathbf{v} = \mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_L$. We then easily see that (7.6a)-(7.6d) hold from (7.8) and (7.7). To prove (7.6e) we use that

$$\|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(P))} \le \max_{1 \le i \le L} \|\nabla \mathbf{v}\|_{L^\infty(\Omega_h(y_i))}.$$

Since finitely many $\{\mathbf{r}_j\}$ have support in $\Omega_h(y_i)$ the result follows from (7.8e) and (7.7e). ∎

The next task is to interpolate values on a tree of the mesh. We need some notation.

Definition 7.4. We say that $\mathcal{R}(r) := Y \times E$ with $Y = \bigcup_{0 \le j \le M} Y_j$ and $E = \bigcup_{1 \le j \le M} E_j$ is a tree of $\mathcal{T}_h$ with root r if the following hold

1) $Y_0 = \{r\}$,
2) for every $1 \le j \le M$, $Y_j \subset \mathcal{S}_h, E_j \subset \mathcal{E}_h$ and $|Y_j| = |E_j|$,
3) for every $y \in Y_j$ there is a unique $e \in E_j$ such that $e = \{y, z\}$ for some $z \in Y_{j-1}$,
4) $Y_j \cap Y_i = \emptyset$ for $i \ne j$.

If $y \in Y_j$ and $z \in Y_{j+s}$ for $s \ge 1$ and there is a path in $\mathcal{R}(r)$ connecting y to z then we say that z is a *descendant* of y and that y is an *ancestor* of z. If we let $y_0 = z$, $y_s = y$, by path we mean $P_{z,y} = \{y_0, y_1, \ldots, y_s\}$ such that $y_i \in Y_{j+s-i}$ for $0 \le i \le s$ such that $e_i = \{y_{i-1}, y_i\} \in E_{j+s-(i-1)}$ for $1 \le i \le s$. We let $\mathsf{D}(y)$ denote the set of all descendants of y and $\mathsf{A}(y)$ to be all the ancestors of y. We know that if $z \in Y$ then there is a unique path $P_{z,r}$ (which we denote by P_z) from z to the root r. We say that the tree $\mathcal{R}(r)$ is *acceptable* if P_z is acceptable for each $z \in Y$. Moreover, we define

$$\rho(\mathcal{R}(r)) = \max_{z \in Y} \rho(P_z), \tag{7.11}$$

where $\rho(P_z)$ is defined in (7.5).

We can now state the following result.

Lemma 7.5. *Let $\mathcal{R}(r) = Y \times E$ with $Y = \bigcup_{0 \le j \le M} Y_j$ and $E = \bigcup_{1 \le j \le M} E_j$ be an acceptable tree of $\mathcal{T}_h$ with root $r \in \mathcal{L}_h$. Then, for any $p \in Q_h^2$ there exist $\mathbf{v} \in \mathbf{V}_{h,0}^3$ such that*

$$\text{support}(\mathbf{v}) \subset \Omega_h(Y) \tag{7.12a}$$

$$(\text{div } \mathbf{v} - p)(\sigma) = 0 \quad \text{for all } \sigma \in Y \tag{7.12b}$$

$$\text{div } \mathbf{v}(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h \backslash Y. \tag{7.12c}$$

If in addition $\mathcal{T}_h$ is quasi-uniform the following bound holds

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_h(Y))} \le C\Big(1 + D_r\Big)\Big(1 + \Upsilon(\mathcal{R}(r))\rho(\mathcal{R}(r))\Big)\|p\|_{L^2(\Omega_h(Y))}, \tag{7.13}$$

where $\Upsilon(\mathcal{R}(r))^2 := \max_{z \in Y} \Big(\sum_{y \in \mathsf{A}(z)} |\mathsf{D}(y)|\Big)$. We recall that D_r is given in (3.2).

Proof. For every $z \in Y$ with $z \ne r$ there is a unique path $P_z \subset Y$ that connects z to r. By Lemma 7.3 we can find $\mathbf{v}_z \in \mathbf{V}_{h,0}^3$ such that

$$\text{support}(\mathbf{v}_z) \subset \Omega_h(P_z), \tag{7.14a}$$

$$(\text{div } \mathbf{v}_z - p)(z) = 0 \tag{7.14b}$$

$$\text{div } \mathbf{v}_z(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h, \sigma \ne z, \sigma \ne r, \tag{7.14c}$$

$$\|\nabla \mathbf{v}_z\|_{L^\infty(\Omega_h(P_z))} \le C \rho(P_z) \max_{T \in \mathcal{T}_h(z)} |p|_T(z)|, \tag{7.14d}$$

where C only depends on the shape regularity of the mesh.

Note that from (7.14d) and inverse estimates we get that for any $y \in \mathsf{A}(z)$, assuming $h_y \le C\, h_z$ (which holds if the mesh is quasi-uniform) we have

$$\|\nabla \mathbf{v}_z\|_{L^2(\Omega_h(y))} \le C\rho(P_z)\|p\|_{L^2(\Omega_h(z))}. \tag{7.15}$$

Then, we take

$$\mathbf{w} = \sum_{z \in Y, z \ne r} \mathbf{v}_z.$$

We then have the following properties of $\mathbf{w}$

$$(\operatorname{div} \mathbf{w} - p)(y) = 0 \qquad \text{for all } y \in Y \backslash \{r\} \tag{7.16}$$

$$\operatorname{support}(\mathbf{w}) \subset \Omega_h(Y) \tag{7.17}$$

$$\operatorname{div} \mathbf{w}(\sigma) = 0, \qquad \sigma \notin Y. \tag{7.18}$$

Since $\mathbf{r} \in \mathcal{L}_h$ (using Definition 3.1) we can find $\mathbf{r} \in \mathbf{V}^3_{h,00}(r)$ so that

$$(\operatorname{div} \mathbf{r} + \operatorname{div} \mathbf{w} - p)(r) = 0 \tag{7.19}$$

$$\|\nabla \mathbf{r}\|_{L^\infty(\Omega_h(r))} \le D_r \max_{T \in \mathcal{T}_h(r)} |(\operatorname{div} \mathbf{w} - p)|_T(r)|. \tag{7.20}$$

In this case, using inverse estimates, we can show that

$$\|\nabla \mathbf{r}\|_{L^2(\Omega_h(r))} \le C D_r (\|\nabla \mathbf{w}\|_{L^2(\Omega_h(r))} + \|p\|_{L^2(\Omega_h(r))}). \tag{7.21}$$

We next set $\mathbf{v} = \mathbf{w} + \mathbf{r}$ and we see that $\mathbf{v} \in \mathbf{V}^3_{h,0}$ and (7.12a), (7.12b) and (7.12c) hold.

To get the bound (7.13) we assume that $\mathcal{T}_h$ is quasi-uniform. Using the triangle inequality and (7.21)

$$\|\nabla \mathbf{v}\|_{L^2(\Omega_h(Y))} \le \|\nabla \mathbf{w}\|_{L^2(\Omega_h(Y))} + \|\nabla \mathbf{r}\|_{L^2(\Omega_h(Y))}$$

$$\le (1 + C\, D_r)(\|\nabla \mathbf{w}\|_{L^2(\Omega_h(Y))} + \|p\|_{L^2(\Omega_h(r))}).$$

Next, we estimate $\mathbf{w}$:

$$\|\nabla \mathbf{w}\|^2_{L^2(\Omega_h(Y))} \le \sum_{y \in Y} \|\nabla \mathbf{w}\|^2_{L^2(\Omega_h(y))}$$

$$\le \sum_{y \in Y} \| \sum_{z \in \mathsf{D}(y)} \nabla \mathbf{v}_z \|^2_{L^2(\Omega_h(y))} \qquad \text{by } (7.14a)$$

$$\le \sum_{y \in Y} (\sum_{z \in \mathsf{D}(y)} \|\nabla \mathbf{v}_z\|_{L^2(\Omega_h(y))})^2 \qquad \text{by the triangle inequality}$$

$$\le \sum_{y \in Y} |\mathsf{D}(y)| \sum_{z \in \mathsf{D}(y)} \|\nabla \mathbf{v}_z\|^2_{L^2(\Omega_h(y))} \qquad \text{by Hölder's inequality}$$

$$\le C\rho(\mathcal{R}(r))^2 \sum_{y \in Y} |\mathsf{D}(y)| \sum_{z \in \mathsf{D}(y)} \|p\|^2_{L^2(\Omega_h(z))} \quad \text{by } (7.15)$$

$$\le C\rho(\mathcal{R}(r))^2 \sum_{z \in Y} \|p\|^2_{L^2(\Omega_h(z))} \sum_{y \in \mathsf{A}(z)} |\mathsf{D}(y)| \quad \text{interchange summation}$$

$$\le 3\, C\rho(\mathcal{R}(r))^2 \Upsilon(\mathcal{R}(r))^2 \|p\|^2_{L^2(\Omega_h(Y))} \qquad \text{by definition of } \Upsilon(\mathcal{R}(r)).$$

Taking square roots we get

$$\|\nabla \mathbf{w}\|_{L^2(\Omega_h(Y))} \le C\, \rho(\mathcal{R}(r))\, \Upsilon(\mathcal{R}(r)) \|p\|_{L^2(\Omega_h(Y))}.$$

The result now follows. ∎

The next result follows immediately from the previous lemma.

Theorem 7.6. *Suppose that we have $\{r_1, \ldots, r_t\} \in \mathcal{L}_h$ and corresponding acceptable trees $\{\mathcal{R}(r_1), \mathcal{R}(r_2), \ldots, \mathcal{R}(r_t)\}$. If $\mathcal{R}(r_i) = Z_i \times E_i$ we require that $\bigcup_{i=1}^t Z_i = \mathcal{S}_h$ and $Z_i \cap Z_j = \emptyset$ for $i \ne j$.*

Then, for any $p \in Q_h^2$ there exist $\mathbf{v} \in \mathbf{V}_{h,0}^3$ such that

$$(\operatorname{div} \mathbf{v} - p)(\sigma) = 0 \quad \text{for all } \sigma \in \mathcal{S}_h$$

and if $\mathcal{T}_h$ is quasi-uniform

$$\|\nabla \mathbf{v}\|_{L^2(\Omega)} \leq C(1 + \bar{D})(1 + \bar{\Upsilon}\bar{\rho})\|p\|_{L^2(\Omega)}.$$

where

$$\bar{D} = \max_{1 \leq i \leq t} D_{r_i}, \quad \bar{\Upsilon} = \max_{1 \leq i \leq t} \Upsilon(\mathcal{R}(r_i)), \quad \bar{\rho} = \max_{1 \leq i \leq t} \rho(\mathcal{R}(r_i)). \tag{7.22}$$

Using Theorem 7.6 and Lemma 2.6 we have that the inf-sup condition (2.3) holds for $Q_h^2 \times \mathbf{V}_h^3$.

Corollary 7.7. *Assuming the hypothesis Theorem 7.6 then the estimate (2.3) holds for $Q_h^2 \times \mathbf{V}_h^3$ with constants given by Lemma 2.6 and Theorem 7.6.*

From Theorem 7.6 and Lemma 2.6 we deduce that if the hypotheses of Theorem 7.6 hold then $Q_h^2 \times \mathbf{V}_h^3$ is inf-sup stable (see Corollary 7.7) and in fact that $\operatorname{div} \mathbf{V}_h^3 = Q_h^2$. If we do not care about the inf-sup constant in (2.3) and only care if $\operatorname{div} \mathbf{V}_h^3 = Q_h^2$ then we can give weaker conditions. Inspecting the proof of Lemma 7.5 (and using Lemma 2.6), we can show the following.

Theorem 7.8. *. If for every $z \in \mathcal{S}_h$ there is exists an $r \in \mathcal{L}_h$ such that there is an acceptable path $P_{z,r}$ between them, then $\operatorname{div} \mathbf{V}_h^3 = Q_h^2$.*

Remark 7.9. The key to our analysis is using *local interpolating vertices*. There seems to be some parallels between these vertices and *confinable vertices* defined in [2], which they used to prove the dimension count of C^1 quartics on general meshes (see below). In particular, singular vertices and vertices with odd number of triangles are confinable vertices, and as we have shown are also local interpolating. In addition, they show that any vertex with four triangles is a confinable vertex, but we have not been able to show that they are local interpolating. It will be interesting to explore further the connections between local interpolating vertices and confinable vertices.

The other common idea between our paper and the one in [2] is the use of paths from vertices to vertices. Indeed, in Theorem 7.8 we use acceptable paths from non-local interpolating vertices to local interpolating vertices and in [2] they use paths from non-confinable vertices to confinable ones.

8. **Relationship to Qin's result**

Results concerning the pair of spaces V_h^k, Q_h^{k-1} were given in [19] for $k \leq 3$. Here we review the case $k = 3$. For the case $k = 1$, also see [20], and for the case $k = 2$, see [4]. Qin considered the mesh in Figure 9.3, which is called a Type I triangulation [15]. Of course, the upper-left and lower-right triangles are problematic, since the pressures will vanish at the corner vertices there. But more interestingly, Qin found an additional spurious pressure mode as indicated in Figure 8.1(a). We can relate this to the quantities $\mathcal{D}_i$ in (3.23) by computing them for this mesh, as indicated in Figure 8.1(b). There are only two angles in this mesh, $\pi/4$ and $\pi/2$, and $\cot(\pi/4) = 1$ and $\cot(\pi/2) = 0$. Similarly, the edge lengths are L and $L\sqrt{2}$, for some L. Thus the quantities d_{j0} in (3.21) are of the form $\pm A$ where $A = 1/2L^2$ for the $\pi/4$ angles, and 0 for the $\pi/2$ angles, as indicated in Figure 8.1(b). Computing the alternating sum of terms in (3.21), we get

$$\mathcal{D}_0 = \sum_{j=1}^{6} (-1)^j d_{j0} = A - (-A) + 0 - A + (-A) = 0. \tag{8.1}$$

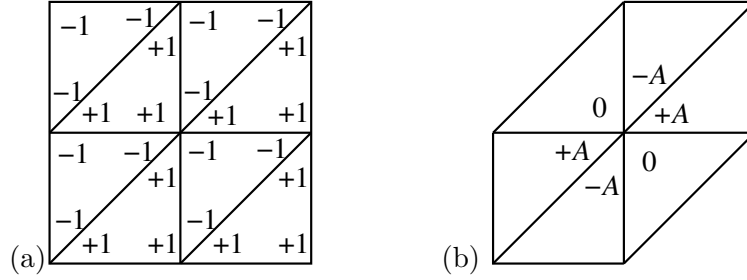Thus condition (3.23) is violated for $i = 0$ for all the interior vertices in Figure 9.3.

Figure 8.1. (a) A global spurious pressure mode on the mesh in Figure 9.3. (b) Computation of (8.1).

Now let us compute $\mathcal{D}_i$ for $i = 1, 2$. First, we note that the sequence of vertices y_k for a fixed interior vertex z

$$y_1 = L(1, 0) + z, \ y_2 = L(1, 1) + z, \ y_3 = L(0, 1) + z,$$
$$y_4 = L(-1, 0) + z, \ y_5 = L(-1, -1) + z, \ y_6 = L(0, -1) + z.$$

Thus,

$$\sum_{j=1}^{6} (-1)^j y_j = L(-1 + 1 - 1 + 1, 1 - 1 + 1 - 1) + z \sum_{j=1}^{6} (-1)^j = \mathbf{0}. \tag{8.2}$$

Letting $t_j = \cot(\theta_j) + \cot(\theta_{j+1})$, we easily can show

$$t_1 = 1, \ t_2 = 2, \ t_3 = 1, \ t_4 = 1, \ t_5 = 2, \ t_6 = 1.$$

Also, we have the sequence of values $|e_j|^{-2}$ are

$$|e_1|^{-2} = \frac{1}{L^2}, \ |e_2|^{-2} = \frac{1}{2L^2}, \ |e_3|^{-2} = \frac{1}{L^2}, \ |e_4|^{-2} = \frac{1}{L^2}, \ |e_5|^{-2} = \frac{1}{2L^2}, \ |e_6|^{-2} = \frac{1}{L^2}.$$

Hence, $\frac{(\cot(\theta_j) + \cot(\theta_{j+1}))}{|e_j|^2} = \frac{1}{L^2}$ for all j. Also, $|T_j| = |T_1|$ for all j. Hence, using (3.30), (8.1), and (8.2) we have for $i = 1, 2$

$$\mathcal{D}_i = \left(\frac{4}{L^2} - \frac{2}{|T_1|} \right) \sum_{j=1}^{n} (-1)^j y_j \cdot \mathsf{E}_i^\perp = 0.$$

Thus condtion (3.23) is violated for all $i = 0, 1, 2$ for all interior vertices in Figure 9.3. This suggests that the constraint (3.23) maybe required for inf-sup stability.

9. Strang's Dimension

For simplicity, let us assume that Ω is simply connected. Then the space $\mathbf{Z}_h^3 := \{ \mathbf{v} \in \mathbf{V}_h^3 : \operatorname{div} \mathbf{v} = 0 \}$ is the curl of the space S_h^4 of C^1 piecewise quartics on the same mesh, where the quartics must vanish to second order on the boundary:

$$\mathbf{Z}_h^3 = \nabla^\perp S_h^4,$$

where $\nabla^\perp \phi = (-\phi_y, \phi_x)$ is the two-dimensional curl operator. It is obvious that $\nabla^\perp S_h^4 \subset \mathbf{Z}_h^3$, and also $\nabla^\perp$ is injective on S_h^4 since the kernel of $\nabla^\perp$ consists of constants. Anything in $\mathbf{Z}_h^3$ is a curl of something, and it is not hard to show that this must be a C^1 piecewise polynomial. Thus $\dim \mathbf{Z}_h^3 = \dim \nabla^\perp S_h^4$.

The dimension of the space $\widehat{S}_h^4$ of C^1 piecewise quartics, without boundary conditions, is known [2] to be

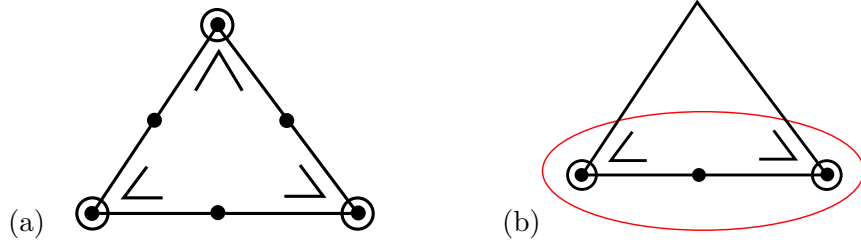$$\dim \widehat{S}_h^4 = E + 4V - V_0 + \sigma_i, \tag{9.1}$$

FIGURE 9.1. (a) Nodal basis for quartics. (b) Nodes that determine value and gradient on an edge.

where T is the number of triangles in $\mathcal{T}_h$, E (resp., E_0) is the number of edges (resp., interior edges) in $\mathcal{T}_h$, V (resp., V_0) is the number of vertices (resp., interior vertices) in $\mathcal{T}_h$, and σ_i is the number of singular interior vertices in $\mathcal{T}_h$.

The dimension formula (9.1) is essentially the one conjectured by Gil Strang [23], so we refer to this as the *Strang dimension* of $\widehat{S}_h^4$:

$$\mathbb{D}(\widehat{S}_h^4) = E + 4V - V_0 + \sigma_i. \tag{9.2}$$

For C^1 piecewise polynomials of degree $k \geq 5$, the Strang dimension was confirmed using an explicit basis [16]. But the Strang dimension for $k \leq 4$ is more complicated. However, what is known is that the derivation of Strang's conjecture [23] provides a lower bound [17]

$$\dim \widehat{S}_h^4 \geq \mathbb{D}(\widehat{S}_h^4). \tag{9.3}$$

We show how this lower bound can be used, together with the results developed here, to prove the dimension formula.

9.1. **Computing** $\dim S_h^4$

Now let us compute $\dim S_h^4 = \dim \mathbf{Z}_h^3$ under the assumption that the inf-sup condition holds. The space $\mathbf{V}_h^3$ can be described in terms of Lagrange nodes:

$$\dim \mathbf{V}_h^3 = 2(T + 2E_0 + V_0) = 2T + 4E_0 + 2V_0. \tag{9.4}$$

We have $\nabla \cdot \mathbf{V}_h^3 \subset Q_h^2$, where the latter space consists of mean-zero piecewise quadratics that satisfy the alternating condition $A_h^z(\operatorname{div} \mathbf{v}) = 0$ at singular vertices, where A_h^z is defined in (2.2):

$$\dim Q_h^2 = 6T - 1 - \sigma.$$

Definition 9.1. For any mesh $\mathcal{T}_h$, the number K defined by $K = \dim Q_h^2 - \dim \nabla \cdot \mathbf{V}_h^3$ is the number of spurious modes for a given mesh.

Thus $\nabla \cdot \mathbf{V}_h^3 = Q_h^2$ if and only if $K = 0$. Since

$$\dim \mathbf{V}_h^3 = \dim(\operatorname{image} \nabla \cdot \mathbf{V}_h^3) + \dim \mathbf{Z}_h^3,$$

we have more generally that

$$\begin{aligned}
\dim \mathbf{Z}_h^3 &= \dim \mathbf{V}_h^3 - \dim(\operatorname{image} \nabla \cdot \mathbf{V}_h^3) \\
&= \dim \mathbf{V}_h^3 - \dim Q_h^2 + K \\
&= 2T + 4E_0 + 2V_0 - 6T + 1 + \sigma + K \\
&= -4T + 4E_0 + 2V_0 + 1 + \sigma + K.
\end{aligned} \tag{9.5}$$

We have $3T = (E - E_0) + 2E_0 = E + E_0$. Thus

$$\dim \mathbf{Z}_h^3 = -T - (E + E_0) + 4E_0 + 2V_0 + 1 + \sigma + K. \tag{9.6}$$

By Euler's formula, $1 = T - E + V = T - E_0 + V_0 = 1$. Thus $E_0 - V_0 = T - 1$, and

$$\dim S_h^4 = \dim \mathbf{Z}_h^3 = (V_0 - E_0) - (E + E_0) + 4E_0 + 2V_0 + \sigma + K$$
$$= 2E_0 - E + 3V_0 + \sigma + K. \tag{9.7}$$

Technically, we actually have

$$\dim S_h^4 = \min\{0, 2E_0 - E + 3V_0 + \sigma + K\},$$

since the dimension can never be negative. There are cases where the number of boundary edges $E - E_0$ (which is the same as the number of boundary vertices) is larger than $E_0 + 3V_0 + \sigma$. For a domain consisting of only two triangles, $E_0 = 1$, $V_0 = 0$, $\sigma = 2$, and $E - E_0 = 4$, so the formula in (9.7) gives a negative number if $K = 0$. From now on, we assume that $E_0 - (E - E_0) + 3V_0 + \sigma \geq 0$ for $\mathcal{T}_h$.

Theorem 9.2. *Let $k = 3$ and suppose that $\mathcal{T}_h$ is any triangulation satisfying $E_0 + 3V_0 + \sigma \geq E - E_0$. Then*

$$\nabla \cdot \mathbf{V}_h^3 = Q_h^2$$

if and only if

$$\dim S_h^4 = 2E_0 - E + 3V_0 + \sigma.$$

More generally,

$$\dim \nabla \cdot \mathbf{V}_h^3 = \dim Q_h^2 - K$$

if and only if

$$\dim S_h^4 = 2E_0 - E + 3V_0 + \sigma + K,$$

where K is the number of spurious modes defined in Definition 9.1.

This is, as far as we know, the only known formula for $\dim S_h^4$, the space of C^1 piecewise quartics satisfying second-order boundary conditions. The number K in Theorem 9.2 is the number of spurious modes for the pair $(\mathbf{V}_h^3, \dim Q_h^2)$ for the Stokes system [19]. In the case of spurious modes ($K > 0$), we get a formula for $\dim S_h^4$ different from what we might expect, as we now explain.

9.2. **Computing** $\dim \widehat{S}_h^4$

Now let us relate the spaces S_h^4 and $\widehat{S}_h^4$ by imposing boundary conditions on $\widehat{S}_h^4$ to yield the space S_h^4. Using the approach pioneered by Strang [23], it is natural to conjecture that this involves simply imposing constraints on the boundary. For example, a C^1 piecewise quartic that vanishes to second order on $\partial\Omega$ must vanish to second order at each boundary vertex (3 constraints per boundary vertex). In addition, the value at one point on each boundary edge must vanish, as well as the normal derivative at two points on each boundary edge.

To see why this is the right number of constraints, we pick special nodal variables for quartics as indicated in Figure 9.1(a). These are

    (1) the value and gradient at each vertex,

    (2) the value at edge midpoints, and

    (3) the second-order cross derivatives $\partial_{e_i}\partial_{e_j}$ evaluated at the vertex ν_{ij} at the intersection of e_i and e_j, where the e_k's are the edges of the triangle.

More precisely, $\partial_{e_i}\phi(\nu_{ij})$ is defined as the directional derivative of ϕ in the direction of e_i away from ν_{ij}. These nodal variables are unisolvent for quartics, as follows. Vanishing of nodal variables of type (1) and (2) guarantee vanishing on each edge; these are the standard nodal variables for Hermite quartics.

Thus a quartic q with these nodal values zero is of the form $q = L_1 L_2 L_3 L$ where the non-trivial linear functions L_i vanish on e_i. But

$$\partial_{e_i} \partial_{e_j} q(\nu_{ij}) = (\partial_{e_i} L_j)(\partial_{e_j} L_i) L_k(\nu_{ij}) L(\nu_{ij}),$$

where $\{i, j, k\} = \{1, 2, 3\}$ and $L_k(\nu_{ij}) \neq 0$. Thus vanishing of the nodal variables of type (3) implies that $L \equiv 0$.

Moreover, similar arguments show that the nodal variables for q associated with a boundary edge, as indicated in Figure 9.1(b), determine q to second order on that edge. Thus satisfaction of second-order boundary conditions is guarenteed by setting these nodal values to zero. We can make this more precise by defining a map C from S_h^4 to $\mathbb{R}^{3(V-V_0)+3(E-E_0)}$ by evaluating the nodal values (1–3) at each boundary vertex and edge. Then

$$S_h^4 = \left\{ v \in \widehat{S}_h^4 \; : \; \mathsf{C} v = \mathbf{0} \right\}. \tag{9.8}$$

This means that the dimension of S_h^4 is less than the dimension of $\widetilde{S}_h^4$ by at most $3(V - V_0) + 3(E - E_0) = 6(E - E_0)$ (note $E - E_0 = V - V_0$). We will show that these conditions have (at least) one redundancy per singular boundary vertex. Thus

$$\dim S_h^4 \geq \dim \widehat{S}_h^4 - 6(E - E_0) + \sigma_b,$$

where σ_b is the number of singular boundary vertices. Therefore

$$\dim S_h^4 + 6(E - E_0) - \sigma_b \geq \dim \widehat{S}_h^4 \geq \mathbb{D}(\widehat{S}_h^4) = E + 4V - V_0 + \sigma_i. \tag{9.9}$$

Assume now that

$$\dim \nabla \cdot \mathbf{V}_h^3 = \dim Q_h^2 - K.$$

Using Theorem 9.2 and $E - E_0 = V - V_0$, we find

$$\begin{aligned}
\dim S_h^4 + 6(E - E_0) - \sigma_b &= E_0 - (E - E_0) + 3V_0 + \sigma + 6(E - E_0) - \sigma_b \\
&= E_0 + 3V_0 + \sigma_i + 5(E - E_0) = E_0 + 3V + \sigma_i + 2(E - E_0) \\
&= E + 3V + \sigma_i + (E - E_0) = E + 4V + \sigma_i - V_0.
\end{aligned} \tag{9.10}$$

Combining (9.9) and (9.10) proves the following result.

Theorem 9.3. *Suppose that $\mathcal{T}_h$ is a triangulation satisfying $E_0 + 3V_0 + \sigma \geq E - E_0$, and suppose that*

$$\dim \nabla \cdot \mathbf{V}_h^3 = \dim Q_h^2 - K$$

on this triangulation. If $K = 0$, the Strang dimension (9.1) is valid for $\widehat{S}_h^4$: $\dim \widehat{S}_h^4 = \mathbb{D}(\widehat{S}_h^4)$. Moreover, equality holds in (9.9), so the $6(E - E_0) - \sigma_b$ constraints are nonredundant. If $K > 0$, then there are K redundant constraints in (9.8) in addition to the ones at singular boundary vertices.

To complete the proof of Theorem 9.3, we need to verify the redundancy of constraints at singular boundary vertices. This occurs because the second-order cross derivatives $\partial_{e_i} \partial_{e_j}$ are linearly dependent at singular boundary vertices. For the case of a triangle with two boundary edges e_1 and e_2, the vanishing of the nodal variables of type (1) and (2) on e_1 and e_2 already imply vanishing on both edges, so necessarily $\partial_{e_1} \partial_{e_2}$ is already zero.

For the case where two triangles meet at a singular boundary vertex ν, see Figure 9.2(a). Then e_1 and e_3 are parallel, and thus

$$\partial_{e_1} \partial_{e_2} \phi(\nu) = -\partial_{e_3} \partial_{e_2} \phi(\nu) \tag{9.11}$$

for any C^1 piecewise quartic ϕ. Thus setting one of them to zero sets the other; they are redundant.

For the case where three triangles meet at a singular boundary vertex ν, see Figure 9.2(b). Equation (9.11) still holds, and in addition e_2 and e_4 are parallel, and thus

$$\partial_{e_3} \partial_{e_2} \phi(\nu) = -\partial_{e_3} \partial_{e_4} \phi(\nu)$$
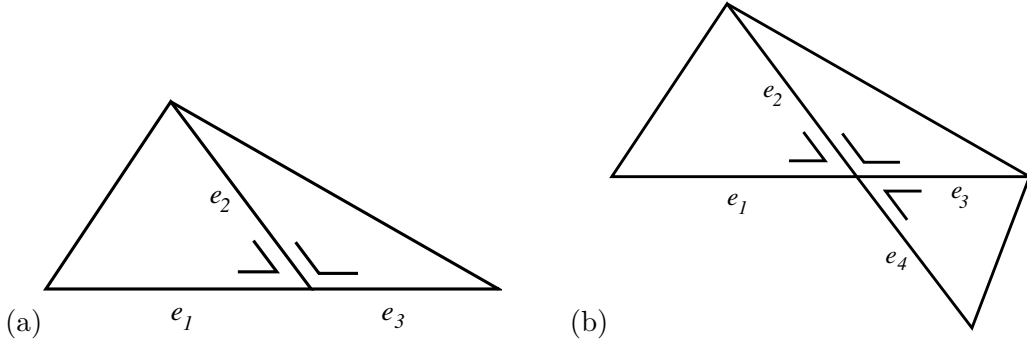
Figure 9.2. (a) Singular boundary vertex where two triangles meet. (b) Singular boundary vertex where three triangles meet.

for any C^1 piecewise quartic ϕ. Thus

$$\partial_{e_1}\partial_{e_2}\phi(\nu) = \partial_{e_3}\partial_{e_4}\phi(\nu),$$

and setting one of them to zero sets the other; they are redundant.

The arguments above mean that we can remove one degree of freedom per singular boundary vertex and define a new constraint map $\widetilde{\mathsf{C}}$, where

$$\widetilde{\mathsf{C}} : \widehat{S}_h^4 \to \mathbb{R}^{6(E-E_0)-\sigma_b},$$

and we have

$$S_h^4 = \left\{ v \in \widehat{S}_h^4 \ : \ \widetilde{\mathsf{C}}v = \mathbf{0} \right\}. \tag{9.12}$$

The number of spurious modes K is then the codiminsion of the image of $\widetilde{\mathsf{C}}$. This completes the proof of Theorem 9.3.

Since Theorem 7.6 allows us to prove the inf-sup condition for quite general meshes, this (9.1) for $\widehat{S}_h^4$ is correct for such meshes. In particular, if the hypothesis of Theorem 7.8 holds, then (9.1) But more strikingly, when the number of spurious modes K for cubic Lagrange elements for Stokes satisfies $K > 0$, there is a redundancy in the obvious constraints (9.12) defining second-order boundary conditions for C^1 piecewise quartics. Results of Qin show this can happen on well-behaved meshes.

9.3. **Connection to Qin's results**

There is a connection between Qin's results and dimension counting. Qin finds a spurious mode that indicates that div $\mathbf{V}_h^3 \neq Q_h^2$ on the right-traingle mesh in Figure 9.3. We conclude that the dimension of the space S_h^4 of C^1 quartics satisfying second-order boundary conditions on this mesh is at least one larger than the dimension for this space given in Theorem 9.2: $K \geq 1$. On the other hand, it is well known [15, 17] that the Strang dimension (9.1) is correct on Type I triangulations without boundary conditions. In view of (9.9), there is a further redundancy in the constraints (9.12) enforcing boundary conditions. (It should be noted that Qin proves $K = 1$ for the Type I triangulation.) Unfortunately, the dimension of splines in two dimensions satisfying boundary conditions has had only limited study [7, 3] so far.
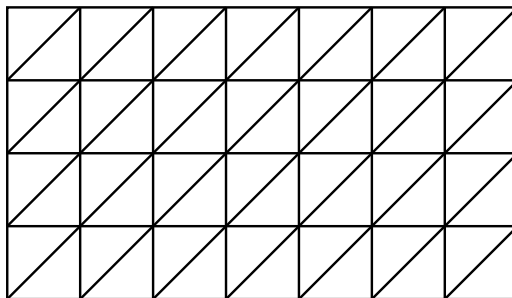
FIGURE 9.3. The Type I regular mesh studied in [19, Chapter 6]

References

[1] Naveed Ahmed, Alexander Linke, and Christian Merdon. Towards pressure-robust mixed methods for the incompressible navier–stokes equations. In *International Conference on Finite Volumes for Complex Applications*, pages 351–359. Springer, 2017.

[2] Peter Alfeld, Bruce Piper, and Larry L. Schumaker. An explicit basis for C^1 quartic bivariate splines. *SIAM Journal on Numerical Analysis*, 24(4):891–911, 1987.

[3] Le Anbo. On the dimension of spaces of pp functions with boundary conditions. *Approximation Theory and its Applications*, 5(4):19–29, 1989.

[4] Douglas N Arnold and Jinshui Qin. Quadratic velocity/linear pressure stokes elements. *Advances in computer methods for partial differential equations*, 7:28–34, 1992.

[5] Christine Bernardi and Genevieve Raugel. Analysis of some finite elements for the Stokes problem. *Mathematics of Computation*, pages 71–79, 1985.

[6] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15. Springer Science & Business Media, third edition, 2008.

[7] C.K. Chui and L.L. Schumaker. On spaces of piecewise polynomials with boundary conditions. II. Type-1 triangulations. In Zeev Ditzian, editor, *Second Edmonton Conference on Approximation Theory*, CMS Conf. Proc., 3. Amer. Math. Soc., Providence, R.I., 1983.

[8] Richard S Falk and Michael Neilan. Stokes complexes and the construction of stable finite elements with pointwise mass conservation. *SIAM Journal on Numerical Analysis*, 51(2):1308–1326, 2013.

[9] Johnny Guzmán and Michael Neilan. Conforming and divergence-free stokes elements in three dimensions. *IMA Journal of Numerical Analysis*, 34(4):1489–1508, 2014.

[10] Johnny Guzmán and Michael Neilan. Conforming and divergence-free stokes elements on general triangular meshes. *Mathematics of Computation*, 83(285):15–36, 2014.

[11] Johnny Guzman and Michael Neilan. Inf-sup stable finite elements on barycentric refinements producing divergence–free approximations in arbitrary dimensions. *arXiv preprint arXiv:1710.08044*, 2017.

[12] Johnny Guzmán and L. Ridgway Scott. The Scott-Vogelius finite elements revisited. *Mathematics of Computation*, to appear, 2017.

[13] S. Harald Christiansen and K. Hu. Generalized Finite Element Systems for smooth differential forms and Stokes problem. *ArXiv e-prints*, May 2016.

[14] Volker John, Alexander Linke, Christian Merdon, Michael Neilan, and Leo G Rebholz. On the divergence constraint in mixed finite element methods for incompressible flows. *SIAM Review*, 2016.

[15] Ming-Jun Lai and Larry L Schumaker. *Spline functions on triangulations*. Number 110 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2007.

[16] J. Morgan and L. R. Scott. A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$. *Math. Comp.*, 29:736–740, 1975.

[17] John Morgan and L. R. Scott. The dimension of the space of C^1 piecewise–polynomials. Research Report UH/MD 78, Dept. Math., Univ. Houston, 1990.

[18] Michael Neilan. Discrete and conforming smooth de rham complexes in three dimensions. *Mathematics of Computation*, 84(295):2059–2081, 2015.

[19] Jinshui Qin. *On the convergence of some low order mixed finite elements for incompressible fluids*. PhD thesis, Penn State, 1994.

[20] Jinshui Qin and Shangyou Zhang. Stability and approximability of the $P1$–$P0$ element for Stokes equations. *International journal for numerical methods in fluids*, 54(5):497–515, 2007.

[21] L. Ridgway Scott and Micheal Vogelius. Conforming finite element methods for incompressible and nearly incompressible continua. In *Large Scale Computations in Fluid Mechanics, B. E. Engquist, et al., eds.*, volume 22 (Part 2), pages 221–244. Providence: AMS, 1985.

[22] LR Scott and Michael Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO-Modélisation mathématique et analyse numérique*, 19(1):111–143, 1985.

[23] Gilbert Strang. Piecewise polynomials and the finite element method. *Bulletin of the American Mathematical Society*, 79(6):1128–1137, 1973.

[24] Michael Vogelius. A right-inverse for the divergence operator in spaces of piecewise polynomials. *Numerische Mathematik*, 41(1):19–37, 1983.

[25] Shangyou Zhang. A new family of stable mixed finite elements for the 3d stokes equations. *Mathematics of computation*, 74(250):543–554, 2005.

[26] Shangyou Zhang. Divergence-free finite elements on tetrahedral grids for $k \geq 6$. *Mathematics of Computation*, 80(274):669–695, 2011.