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DYNAMICS OF THE FOCUSING CRITICAL WAVE EQUATION

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1. INTRODUCTION

This note concerns results by Duyckaerts, Kenig and Merle [13], Jia [23] and Duyckaerts, Jia, Kenig and Merle [12] on the focusing energy critical wave equation:

$$\text{(NLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u, \quad x \in \mathbb{R}^N \\ \vec{u}_{\restriction t=0} = (u_0, u_1) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N), \end{cases}$$

where $N \in \{3, 4, 5\}$ and $\vec{u} = (u, \partial_t u)$. It is locally well-posed in the scale invariant-space $\mathcal{H} = \dot{H}^1 \times L^2$, and has two conserved quantities: the energy

$$E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_x u(t)|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\partial_t u(t)|^2 - \frac{N-2}{2N} \int |u(t)|^{\frac{2N}{N-2}}$$

and the momentum

$$P(\vec{u}) = \int_{\mathbb{R}^N} \nabla_x u \partial_t u.$$

The equation admits the following transformations:

- if u is a solution, $\lambda > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$, $R \in \mathcal{O}_N(\mathbb{R})$, $\iota_0, \iota_1 \in \{\pm 1\}$, then

$$v(t, x) = \frac{\iota_0}{\lambda^{\frac{N}{2}-1}} u\left(\frac{\iota_1(t - t_0)}{\lambda}, \frac{R(x - x_0)}{\lambda}\right)$$

  is also a solution with the same energy.
- *Lorentz transformation.* If u is a global solution, $\mathbf{p} \in \mathbb{R}^N$ and $p = |\mathbf{p}| < 1$, then

$$u_{\mathbf{p}}(t, x) = u\left(\frac{t - \mathbf{p} \cdot x}{\sqrt{1 - |\mathbf{p}|^2}}, \left(-\frac{t}{\sqrt{1 - p^2}} + \frac{1}{p^2}\left(\frac{1}{\sqrt{1 - p^2}} - 1\right) \mathbf{p} \cdot x\right)\mathbf{p} + x\right)$$

  is also a solution (the Lorentz transform of u). Note that this last transformation mixes the space and time variables, which is problematic when the solution u is not globally defined. Unlike the preceeding transformations, it does not preserve the energy.

We are interested by the global dynamics of the equation in a nonperturbative regime, i.e. without size restriction on the data. We start by giving a few examples of solutions.

2. EXAMPLES OF SOLUTIONS

2.1. **Scattering solutions.** Let $T_+(u)$ be the maximal time of existence of u. By definition, u *scatters* to a linear solution when $T_+(u) = +\infty$ and there exists a solution u_L of $\partial_t^2 u_L - \Delta u_L = 0$ such that

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\mathcal{H}} = 0$$

In the defocusing case (when there is a minus sign in front of the nonlinearity in (NLW)), all solutions scatter: see Grillakis [17, 18], Shatah and Struwe [35, 36], Kapitanski [24],

Ginibre and Velo [16], Nakanishi [34], Bahouri and Shatah [2], Bahouri and Gérard [1], Tao [38].

In the focusing case, we have the following properties:

- Scattering for solutions with small initial data in the energy space.
- *Existence of wave operators*: if u_L is a solution of the linear wave equation, there exists a solution u of the nonlinear wave equation (NLW) such that $T_+(u) = +\infty$ and
$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\mathcal{H}} = 0.$$
- *Stability:* the set of scattering solutions is open in the energy topology.

2.2. Type I blow-up.

A solution u of (NLW) is said to be a *type I blow-up solution* when $T_+(u) < \infty$ and
$$\lim_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{H}} = +\infty.$$

Note that this condition is not automatic for solution blowing-up in finite time: we will see that there also exist solutions such that $T_+(u)$ is finite that remain bounded in the energy space close to $T_+(u)$.

Examples of type I blow-up solution can be constructed using the ODE $y'' = |y|^{\frac{4}{N-2}}y$, that has solution blowing-up in finite time. Consider:
$$Y(t) = \left(\frac{N(N-2)}{4} \right)^{\frac{N-2}{4}} \frac{1}{(T-t)^{\frac{N-2}{2}}},$$

and let $L > T$. Then by finite speed of propagation, any radial, finite energy solution of (NLW) such that
$$|x| < L \implies (u_0, u_1)(x) = (Y(0), Y'(0))$$

blows up in finite time, and one can prove that the blow-up is of type I.

The type I blow-up is conjectured to be stable, but there are not much theoretical results in this direction for equation (NLW). Much more is known for pseudo-conformally sub-critical wave equations (polynomial nonlinearity $|u|^{p-1}u$ in space dimension $N = 1$, or, $N \geq 2$ with $p < \frac{N+3}{N-1}$): see the works of Merle and Zaag e.g. [32, 33].

In the energy-critical case there are numerical evidences that generic blow-up solutions behave like $y_0(t)$ see Bizoń, Chmaj and Tabor [3]. The stability of y_0 in light cones, in the energy topology was proved by Donninger [8]; see also previous results in stronger topology by Donninger and Schörkhuber [11].

We next give examples of bounded, non-scattering solutions.

2.3. Solitons.

Solitons, or solitary waves, are well-localized solution of a dispersive equation travelling at a constant speed. In the case of equation (NLW), all solitons are constructed from stationary solutions:

$$(E) \qquad\qquad -\Delta Q = |Q|^{\frac{4}{N-2}}Q, \quad Q \in \dot{H}^1(\mathbb{R}^N).$$

There is a unique (up to scaling and sign change) radial solution of (E), which is also the least energy solution of (E) and is called the ground-state. It is given by the formula:
$$W = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)} \right)^{\frac{N}{2}-1}}.$$

The energy of W is a threshold for the dynamics see Kenig-Merle [25]. We refer to Krieger, Nakanishi and Schlag [27] for the dynamics around W.

There also exist solutions of (E) with arbitrary large energies. See the works of W.Y. Ding [7], and Del Pino, Musso, Pacard and Pistoia [6].

Applying a Lorentz transform to any solution of (E), one obtains a soliton of the following form:

$$Q_{\mathbf{p}}(t, x) = Q\left(\left(-\frac{t}{\sqrt{1-p^2}} + \frac{1}{p^2}\left(\frac{1}{\sqrt{1-p^2}} - 1\right)\mathbf{p} \cdot x\right)\mathbf{p} + x\right)$$

where $\mathbf{p} \in \mathbb{R}^N$ with $p = |\mathbf{p}| < 1$. Note that $Q_{\mathbf{p}}$ travels at speed $\mathbf{p}$:

$$Q_{\mathbf{p}}(t, x) = Q_{\mathbf{p}}(0, x - t\mathbf{p}).$$

The energy of $Q_{\mathbf{p}}$ is given by:

$$E(\vec{Q}_{\mathbf{p}}(0)) = \frac{1}{\sqrt{1-p^2}} E(Q, 0).$$

The speed of a soliton is strictly smaller than 1 (and its distance to 1 is controled by the energy), whereas solutions of the linear equation travel at speed exactly 1.

2.4. **Global, non-scattering solutions close to W.**
There are many examples of solutions that are global and non-scattering, but are not equal to solitons.

The easiest way to construct such an example is to use the unstable direction around W: the linearized operator at W

$$L_W = -\Delta - \frac{N+2}{N-2}|W|^{\frac{4}{N-2}}$$

admits exactly one negative eigenvalue, $-\omega^2$. Denoting by $\mathcal{Y}$ a corresponding eigenfunction, one can construct a one parameter family of solution with energy $E(W, 0)$ such that

$$W_a(t) = W + ae^{-\omega t}\mathcal{Y} + \mathcal{O}(e^{-2\omega t}), \quad t \to +\infty.$$

where $a \in \mathbb{R}$: see Duyckaerts and Merle [14].

More generally, there exist global solutions of the form:

$$\vec{u}(t) = \left(\frac{1}{\lambda(t)^{\frac{N-2}{2}}} W\left(\frac{x}{\lambda(t)}\right), 0\right) + \vec{v}_L(t) + o(1) \text{ in } \mathcal{H}, \text{ as } t \to +\infty,$$

v_L is a small solution of the linear wave equation and

- $\lambda(t) = 1$ Krieger and Schlag [28].
- $\lambda(t) = t^\alpha$, $\alpha \in \mathbb{R}$, α small Donninger and Krieger [10].

Open questions: are there solutions with other stationary profile than W? What are the $v_L(t)$ admissible?

2.5. **Type II blow-up solutions.**
A *type II blow-up solution* is a solution of (NLW) such that $T_+(u) < \infty$ and

$$\limsup_{t \to T_+(u)} \|\vec{u}(t)\|_{\mathcal{H}} < \infty.$$

All known examples are of the form

$$\vec{u}(t) = \left(\frac{1}{\lambda(t)^{\frac{N-2}{2}}} W\left(\frac{\cdot}{\lambda(t)}\right), 0\right) + (v_0, v_1), \quad t \to T_+,$$

where $(v_0, v_1) \in \mathcal{H}$ and:

- $N = 3$, $\lambda(t) = (T_+ - t)^\alpha$ and $\alpha > 1$ (Krieger, Schlag and Tataru [29], Krieger and Schlag [30]). The instability of these solutions was proved by Krieger and Nahas [26].
- $N = 5$, $\lambda(t) = (T_+ - t)^\alpha$ and $\alpha > 9$, Jendrej [21].
- $N = 4$, $\lambda(t) \approx (T_+ - t)e^{-\sqrt{|\log(T_+ - t)|}}$ and (v_0, v_1) is smooth (Hillairet and Raphaël [20]).
- $N = 5$, $\lambda(t) \approx (T_+ - t)^4$, (v_0, v_1) is *any* smooth solution with $v_0(0) > 0$ (Jendrej [21]).
- $N = 3$, $\lambda(t) = (T_+ - t)^\alpha \exp(\varepsilon_0 \sin(\log(t)))$, $\alpha > 4$ (Donninger, Huang, Krieger and Schlag [9]).

2.6. **Multi-solitons.** There also exist global solutions that behave asymptotically as a sum of decoupled solitons. Two examples are known:

- A radial solution, constructed by Jendrej [22], such that

$$\vec{u}(t, x) = \left(W(x) + \frac{1}{\lambda(t)^2} W\left(\frac{x}{\lambda(t)} \right), 0 \right) + o(1) \text{ in } \mathcal{H}, \text{ as } t \to +\infty,$$

  where $N = 6$, $\lambda(t) = \sqrt{4/5}e^{-\sqrt{5/4}t}$.
- Multi-solitons with more profiles, constructed by Martel and Merle [31]:

$$\vec{u}(t, x) = \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^{\frac{3}{2}}} \vec{W}_{\mathbf{p}_j} \left(\frac{t}{\lambda_j}, \frac{x - x_j}{\lambda_j} \right) + o(1), \quad t \to +\infty,$$

  where $N = 5$, $\iota_j \in \{\pm 1\}$, $\lambda_j > 0$, $x_j \in \mathbb{R}^5$, $|\mathbf{p}_j| < 1$ (collinears if $J \geq 3$) and

$$j \neq k \implies \mathbf{p}_j \neq \mathbf{p}_k.$$

Open question: does their exists analogous examples in the finite time-blow-up case? (see Côte and Zaag [5] for subcritical equations in one space dimension).

3. SOLITON RESOLUTION

In view of the preceding examples, one can make the following conjecture:

Conjecture 1. *Let u be a non scattering solution such that $T_+(u) = +\infty$. Then there exists $J \geq 1$, a linear wave v_L, solitary waves $Q^j_{\mathbf{p}_j}$, $j = 1 \ldots J$, and parameters $x_j(t) \in \mathbb{R}^N$, $\lambda_j(t) > 0$, such that*

$$(1) \qquad u(t) = v_L(t) + \sum_{j=1}^{J} \frac{1}{\lambda_j^{\frac{N-2}{2}}(t)} Q^j_{\mathbf{p}_j} \left(0, \frac{x - x_j(t)}{\lambda_j(t)} \right) + r(t)$$

where

- $\displaystyle\lim_{t \to +\infty} \|\vec{r}(t)\|_{\mathcal{H}} = 0$
- $\forall j, \quad \displaystyle\lim_{t \to +\infty} \frac{x_j(t)}{t} = \mathbf{p}_j, \quad \lim_{t \to +\infty} \frac{\lambda_j(t)}{t} = 0$
- $\forall j, k, \quad j \neq k \implies \displaystyle\lim_{t \to +\infty} \frac{|x_j(t) - x_k(t)|}{\lambda_j(t)} + \frac{\lambda_j(t)}{\lambda_k(t)} + \frac{\lambda_k(t)}{\lambda_j(t)} = +\infty.$

An analogous conjecture can be formulated for type II blow-up solutions.

Let us mention that the conjecture might hold in a slightly weaker form, for example taking into accounts other transformations of the solitons that the one appearing in (1) (space translation and scaling). Until recently, this type of result was only known for

sufficiently smooth and decaying solutions of completely integrable equations: see for example Eckhaus and Schuur [15] for KdV. In this note, I will present two results for equation (NLW), which is not a completely integrable equation: the full resolution into solitons for radial solutions in space dimension 3, due to Duyckaerts, Kenig and Merle, and a weaker version due to Duyckaerts, Jia, Kenig and Merle in a more general context.

3.1. **Radial case, space dimension 3.** The following theorem is proved in [13].

Theorem 1. *Assume $N = 3$. Let u be a* radial *solution of (NLW) such that $T_+(u) = +\infty$. Then there exists $J \geq 0$ and:*

- v_L *such that $\partial_t^2 v_L - \Delta v_L = 0$,*
- *signs $\iota_j \in \{\pm 1\}$, $j = 1 \ldots J$,*
- *parameters $\lambda_j(t)$, $0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t$,*

such that:

$$
(2) \qquad u(t) = v_L(t) + \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^{\frac{1}{2}}(t)} W\left(\frac{x}{\lambda_j(t)}\right) + r(t),
$$

where:

$$
\lim_{t \to +\infty} \|\vec{r}(t)\|_{\mathcal{H}} = 0.
$$

There is an analogous result for type II blow-up solutions.
The proof of Theorem 1 uses

- Finite speed of propagation.
- Concentration compactness arguments adapted to (NLW) (profile decomposition of Bahouri and Gérard [1]).
- A bound from below of the exterior energy for radial, nonstationary solutions of (NLW).

Let us insist on this last point, which is the main new ingredient of the proof. The following proposition might be seen as a characterization of the stationary solution W:

Proposition 1. *Assume $N = 3$. Let u be a global, radial, nonstationary solution of (NLW). Then there exist $r_0 > 0$ and $\eta > 0$ such that the following hold for all $t \geq 0$ or for all $t \leq 0$:*

$$
(3) \qquad \int_{|t|+r_0}^{+\infty} \left(|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2\right) dx \geq \eta,
$$

It is possible to prove, using in a crucial way the finite speed of propagation, that no profile in a profile decomposition of a sequence $\vec{u}(t_n)$ can satisfy the condition (3) (*channels of energy* method).

The proof of Proposition 1 is based on exterior energy estimates for radial solutions of the linear wave equation. It is specific to radial solutions, since it uses the fact that a radial linear wave travels at speed one in one of the two inward or outward directions. There is absolutely no hope to use the same strategy in a nonradial setting, although the idea to consider the energy outside wave cone is not completely useless in the nonradial context (see below).

3.2. **General case.** The next theorem is due to Jia, Kenig, Merle and myself (see [12] and also the work of Jia [23] for a weaker version in the finite time blow-up case).

Theorem 2. *Assume $N = 3, 4, 5$. Let u be a solution such that $T_+(u) = +\infty$ and*

$$\limsup_{t \to +\infty} \|\vec{u}(t)\|_{\mathcal{H}} < \infty.$$

Then there exist $t_n \to +\infty$, $J \geq 0$, a linear wave v_L, solitary waves $Q_{\mathbf{p}_j}^j$, $j = 1 \ldots J$, and parameters $x_{j,n} \in \mathbb{R}^N$, $\lambda_{j,n} > 0$, such that

$$(4) \quad \vec{u}(t_n) = \vec{v}_L(t_n)$$

$$+ \sum_{j=1}^{J} \left(\frac{1}{\lambda_{j,n}^{\frac{N-2}{2}}} Q_{\mathbf{p}_j}^j \left(0, \frac{x - x_{j,n}}{\lambda_{j,n}} \right), \frac{1}{\lambda_{j,n}^{\frac{N-2}{2}}} Q_{\mathbf{p}_j}^j \left(0, \frac{x - x_{j,n}}{\lambda_{j,n}} \right) \right) + (r_{0,n}, r_{1,n}),$$

where

- $\displaystyle\lim_{n \to +\infty} \|(r_{0,n}, r_{1,n})\|_{\mathcal{H}} = 0$

- $\displaystyle\forall j, \quad \lim_{n \to +\infty} \frac{x_{j,n}}{t_n} = \mathbf{p}_j, \quad \lim_{n \to +\infty} \frac{\lambda_{j,n}}{t_n} = 0$

- $\displaystyle\forall j, k, \quad j \neq k \implies \lim_{n \to +\infty} \frac{|x_{j,n} - x_{k,n}|}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} = +\infty.$

There is an analogous theorem for Type II Blow-up solutions.

Let us mention that the restriction of the dimension $N \leq 5$ is of mere convenience. We refer to Bulut, Czubak, Li, Pavlović and Zhang [4] for the technical tools needed to generalize Theorem 2 to higher dimensions.

The new ingredients in the proof of Theorem 2 are a monotonicity formula of Morawetz type, from [23], which is very much similar to the one used for energy-critical wave maps (see Grillakis [19], Tao [39], Sterbenz and Tataru [37]), and an exterior energy bound for very specific solutions of (NLW) (thus much less general than the one appearing in Proposition 1). The Morawetz-type estimate is as follows:

Lemma 1. *Let u be a non-scattering such that $T_+(u) = +\infty$ and*

$$\limsup_{t \to \infty} \|\vec{u}(t)\|_{\mathcal{H}} < \infty.$$

then, translating u in time if necessary, there exists $C > 0$ such that, for $0 < 10t_1 < t_2$,

$$\int_{t_1}^{t_2} \int_{|x|<t} \left(\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1 \right) \frac{u}{t} \right)^2 dx \frac{dt}{t} \leq C \log \left(\frac{t_2}{t_1} \right)^{\frac{N}{N+1}}.$$

Note that the term $\int_{|x|<t} \left(\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1 \right) \frac{u}{t} \right)^2 dx$ is bounded, up to a constant, by $\|\vec{u}(t)\|_{\mathcal{H}}$. Using the fact that u is bounded in the energy space, we would get a bound of the form

$$\int_{t_1}^{t_2} \int_{|x|<t} \left(\partial_t u + \frac{x}{t} \cdot \nabla u + \left(\frac{N}{2} - 1 \right) \frac{u}{t} \right)^2 dx \frac{dt}{t} \leq C \log \left(\frac{t_2}{t_1} \right).$$

The lemma is a gain compared to this trivial bound when t_2/t_1 is large. As a corollary, there exists $t_n \to +\infty$ such that

$$\lim_{n \to \infty} \int_{|x|<t_n} \left(\partial_t u(t_n) + \frac{x}{t_n} \cdot \nabla u(t_n) + \left(\frac{N}{2} - 1 \right) \frac{u(t_n)}{t_n} \right)^2 dx = 0.$$

The proof of the lemma consists in multiplying equation (NLW) by

$$((1 + \varepsilon^2)t^2 - |x|^2)^{-1/2} \left(x\nabla u + t\partial_t u + \left(\frac{N}{2} - 1 \right) u \right),$$

where ε is a small number, depending on t_1 and t_2, to be specified. The conclusion of the lemma is obtained after integration by parts in the domain:

$$\left\{ t_1 \le t \le t_2, \ |x| \le t \right\},$$

and a careful estimates of the boundary terms. A new difficulty compared to the wave maps problem is given by the boundary terms on the submanifold $\{|x| = t\}$. In the wave maps equations, this terms have a favorable sign and can be ignored. It is not the case when dealing when equation (NLW), where the boundary term given by the nonlinearity, essentially

(*)
$$\int_{\substack{|x|=t \\ t_1 \le t \le t_2}} |u|^{\frac{2N}{N-2}} d\sigma(t,x)$$

does not come with the good sign. To override this difficult, one proves (using small data theory, finite speed of propagation and Strichartz estimates) that the following Strichartz norm outside a wave cone is finite for large R:

$$\left(\int_0^{+\infty} \left(\int_{|x| \ge R+|t|} |u|^{2\frac{N+2}{N-2}} dx \right)^{1/2} dt \right)^{\frac{N-2}{N+2}}.$$

As a consequence, using the inequality

$$\left| \nabla_{t,x} |u|^{\frac{2N}{N-2}} \right| \lesssim |u|^{\frac{N+2}{N-2}} |\nabla_{t,x} u|,$$

we see that

$$\nabla_{t,x} |u|^{\frac{2N}{N-2}} \in L^1 \left(\{(t,x), \ t \ge 0, \ |x| \ge t + R\} \right),$$

and thus, by a standard trace theorem, and after a suitable time translation:

$$|u|^{\frac{2N}{N-2}} \in L^1 \left(\{t = |x|\} \right),$$

hence the control of the boundary term (*).

Using Lemma 1 and a classical monotonicity formula, one can prove (after a delicate analysis) that the expansion (4) holds for a well-chosen sequence of times t_n, with a weaker convergence to zero of the remainder, namely, for all $c < 1$,:

(5) $$\|r_{L,n}\|_{L_t^{\frac{N+2}{N-2}} L_x^{2\frac{N+2}{N-2}}} + \|(\nabla r_{0n}, r_{1n})\|_{L^2(\{x: |x|<ct_n, \text{ or } |x|>t_n\})}$$

$$+ \left\| r_{1n} + \frac{x}{t_n} \cdot \nabla r_{0n} \right\|_{L^2(|x| \le t_n)} + \|\nabla_T r_{0n}\|_{L^2(\mathbb{R}^N)} \to 0,$$

where ∇_T is the tangential part of the spatial derivative ∇, and $r_{L,n}$ is the solution of the linear wave equation with initial data $(r_{0,n}, r_{1,n})$. To conclude the proof, we need to show that the stronger convergence property:

(6) $$\lim_{n \to \infty} \|(\nabla r_{0n}, r_{1n})\|_{L^2(\mathbb{R}^N)} = 0.$$

This is done using again *channels of energy*. More precisely we prove, as a consequence of a classical virial identity, that for any sequence of solution of (NLW) with initial data $(r_{0,n}, r_{1,n})$ satisfying (5) but not (6) there exists a subsequence that satisfies an exterior

energy estimate similar to (3). Using similar argument than in the $3d$, radial case we then obtain a contradiction.

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