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STABLE SELF-SIMILAR BLOW-UP DYNAMICS FOR SLIGHTLY L^2-SUPERCRITICAL GENERALIZED KDV EQUATIONS

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Abstract. In this paper we consider the slightly L^2-supercritical gKdV equations $\partial_t u + (u_{xx} + u|u|^{p-1})_x = 0$, with the nonlinearity $5 < p < 5 + \varepsilon$ and $0 < \varepsilon \ll 1$. We will prove the existence and stability of a blow-up dynamics with self-similar blow-up rate in the energy space H^1 and give a specific description of the formation of the singularity near the blow-up time.

1. INTRODUCTION

We consider the following gKdV equations:

$$\begin{cases} \partial_t u + (u_{xx} + u|u|^{p-1})_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{1.1}$$

with $1 \leq p < +\infty$.

This kind of problem appears in Physics, for example in the study of waves on shallow water (see [11]). These equations, with nonlinear Schrödinger equations, are considered as universal models for Hamiltonian systems in finite dimension.

From the result of C. E. Kenig, G. Ponce and L. Vega [8], (1.1) is locally well-posed in H^1 and thus for all $u_0 \in H^1$, there exists a maximal lifetime $0 < T \leq +\infty$ and a unique solution $u(t,x) \in C([0,T), H^1(\mathbb{R}))$ to (1.1). Moreover, we have: either $T = +\infty$ or $T < +\infty$ and $\lim_{t\to T} \|u_x(t)\|_{L^2} = +\infty$.

A solution with a finite maximal lifetime $0 < T < +\infty$, is called a blow-up solution. Numerical simulation for example [1] suggests that finite time blow up may occur for some initial data. But unlike the focusing nonlinear Schrödinger equations, there are no pseudo-conformal invariance or Virial's identity, which allow us to have explicit blow-up solution. Until recently, the blow dynamics of (1.1) for $p = 5$ has been established by Martel, Merle and Raphaël in a series of papers [19, 13, 14, 15, 16, 17, 18]. They prove the existence and stability of blow-up solutions to (1.1), and also give a specific description of the asymptotic behavior near the blow-up time.

In this paper, we will focus on the case with nonlinearity $5 < p < 5 + \delta$, where $\delta > 0$ is small enough. Numerical simulation in [4] suggests that there exists self-similar blow-up solution to (1.1) in this case. Here a self-similar solution is a solution of the following form:

$$u(t,x) \sim \frac{1}{(T-t)^{\frac{2}{3(p-1)}}} P\left(\frac{x}{(T-t)^{\frac{1}{3}}}\right).$$

We mention here that, in [15] Martel and Merle proved that there is no self-similar blow-up solution for the case $p = 5$ under certain conditions.

The main issue of this paper is to construct a blow up solution to (1.1) in the energy space H^1 with self-similar blow-up rate. Moreover, we will prove its stability and give a explicit description of the formation of singularity near the blow up time.

2. Preliminary

In this section we will recall some special feature of the Cauchy problem (1.1).

2.1. **Conservation laws.** The equation has two important conservation laws, i.e. mass and energy:

$$E(u(t)) := \frac{1}{2}\int |u_x(t)|^2 - \frac{1}{p+1}\int |u(t)|^{p+1} = E_0, \tag{2.1}$$

$$M(u(t)) := \int |u(t)|^2 = M_0. \tag{2.2}$$

2.2. **Scaling invariance.** Let $u(t)$ be a solution to (1.1), and $\lambda > 0$ be a positive constant. Then

$$u_\lambda(t) = \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right)$$

is still a solution to (1.1). Moreover, for all $\lambda > 0$, we have

$$\|u(0)\|_{\dot{H}^{\sigma_c}} = \|u_\lambda(0)\|_{\dot{H}^{\sigma_c}}, \quad \sigma_c = \frac{1}{2} - \frac{2}{p-1}.$$

Here

$$\dot{H}^s = \left\{ f \in \mathcal{S}' \middle| \|f\|^2_{\dot{H}^s} := \int |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi < +\infty \right\}.$$

If $\sigma_c < 0$ (or equivalently $p < 5$), the problem (1.1) is called (mass) subcritical. If $\sigma_=0$ or $p = 5$, the problem (1.1) is called (mass) critical. While if $\sigma_c > 0$ or $p > 5$, (1.1) is called (mass) supercritical.

From the energy and mass conservation laws, in the subcritical case, H^1 solutions are always global in time and uniformly bounded in H^1.

2.3. **Soliton solution.** The problem (1.1) has a special class of solution, i.e. called the soliton solutions. They are given by

$$u(t,x) = \frac{1}{\lambda_0^{\frac{2}{p-1}}} \mathcal{Q}_p\left(\frac{x - \lambda_0^{-2} t}{\lambda_0}\right),$$

where $\lambda_0 > 0$, $x_0 \in \mathbb{R}$ and $\mathcal{Q}_p$ is the unique radial nonnegative solution with exponential decay to the following elliptic equation:

$$\mathcal{Q}_p'' - \mathcal{Q}_p + \mathcal{Q}_p |\mathcal{Q}_p|^{p-1} = 0.$$

From [24], $\mathcal{Q}_p$ is the minimizer to the following functional:

$$J(f) := \min_{f \in H^1(\mathbb{R}), f \neq 0} \frac{\|f_x\|_{L^2}^{\frac{p-1}{2}} \|f\|_{L^2}^{\frac{p+3}{2}}}{\|f\|_{L^{p+1}}^{p+1}}.$$

A standard variation argument shows that for $p = 5$, if $\|u_0\|_{L^2} < \|\mathcal{Q}_5\|_{L^2}$, then

$$E(u_0) = \frac{1}{2}\int |\partial_x u_0|^2 - \frac{1}{6}\int |u_0|^6 \gtrsim \int |\partial_x u_0|^2.$$

Hence, the corresponding solution with initial data u_0 is global in time. In conclusion, for the critical problem, i.e. $p = 5$, a necessary condition for blow-up is that $\|u_0\|_{L^2} \geq \|Q_5\|_{L^2}$.

3. Main result

Now we can state the result of [12].

Theorem 3.1 (Existence and stability of a self-similar blow-up dynamics)**.** *There exists a $p^* > 5$ such that for all $p \in (5, p^*)$, there exist constants $\delta(p) > 0$ and $b^*(p) > 0$ with*

$$\lim_{p \to 5} \delta(p) = 0 \tag{3.1}$$

$$0 < c_0(p-5) \leq b^*(p) \leq C_0(p-5) \tag{3.2}$$

and a nonempty open subset $\mathcal{O}_p$ in H^1 such that the following holds. If $u_0 \in \mathcal{O}_p$, then the corresponding solution to (1.1) *blows up in finite time $0 < T < +\infty$, with the following dynamics : there exist geometrical parameters $(\lambda(t), x(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and an error term $\varepsilon(t)$ such that:*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} \left[Q_p + \varepsilon(t)\right]\left(\frac{x - x(t)}{\lambda(t)}\right) \tag{3.3}$$

with

$$\|\varepsilon_y(t)\|_{L^2} \leq \delta(p). \tag{3.4}$$

Moreover, we have:

(1) *The blow-up point converges at the blow-up time:*

$$x(t) \to x(T) \text{ as } t \to T, \tag{3.5}$$

(2) *The blow-up speed is self-similar:*

$$\forall t \in [0,T), \quad (1-\delta(p))\sqrt[3]{3b^*(p)} \leq \frac{\lambda(t)}{\sqrt[3]{T-t}} \leq (1+\delta(p))\sqrt[3]{3b^*(p)}. \tag{3.6}$$

(3) *The following convergence holds:*

$$\forall q \in [2, \frac{2}{1-2\sigma_c}), \quad u(t) \to u^* \text{ in } L^q \text{ as } t \to T. \tag{3.7}$$

(4) *The asymptotic profile u^* displays the following singular behavior:*

$$\big(1-\delta(p)\big)\int Q_p^2 \leq \frac{1}{R^{2\sigma_c}}\int_{|x-x(T)|<R} |u^*|^2 \leq \big(1+\delta(p)\big)\int Q_p^2. \tag{3.8}$$

for R small enough. In particular, we have for all $q \geq \frac{2}{1-2\sigma_c}$:

$$u^* \notin L^q.$$

Remark 3.2. Here the meaning of $q_c = \frac{2}{1-2\sigma_c}$ is given by the following Sobolev embedding:

$$\dot{H}^{\sigma_c} \hookrightarrow L^{q_c}.$$

That is, the asymptotic profile u^* is not in the critical space $\dot{H}^{\sigma_c}$, and the convergence (3.7) only exists in subcritical Lebesgue spaces.

Remark 3.3. It is easy to see from the L^2 conservation law that $\int |u^*|^2 = \int |u_0|^2$.

Remark 3.4. The conclusion here is almost the same to the Schrödinger case in [20]. But we need a totally differential strategy, due to the different structure of these two equations. Indeed, our strategy here is very close to the one in [16] for critical gKdV. But there are some significant difference between critical and supercritical equations. For example the singular dynamics for gKdV is located around some point $x(t)$, which always goes to infinity in finite time for critical equation[1]. While in this supercritical case, $x(t)$ converges to some finite point.

Remark 3.5. Theorem 3.1 is the first construction of blow-up solutions to the supercritical gKdV equations with initial data in H^1. This is a stable blow-up dynamics instead of a single blow-up solution. So it is not like the self-similar solution constructed by H. Koch in [9], though the construction in this paper relies deeply on H. Koch's work.

4. Outline of the proof

We will give in this subsection a brief insight of the proof of Theorem 3.1. We will first use the self-similar solution constructed by H. Koch in [9], to derive a finite dimensional dynamics, which fully describe the blow-up regime. Since we are considering the slightly supercritical case, it is helpful to view this equation as a perturbation of the critical equation in some sense. So we can use some critical techniques in our analysis, though they may have a totally different meaning in the supercritical case.

4.1. **Derivation of the law.** We are looking for a solution to (1.1) of the form:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} V_{b(t)}\left(\frac{x - x(t)}{\lambda(t)}\right), \tag{4.1}$$

and introduce the rescaled time:

$$\frac{ds}{dt} = \frac{1}{\lambda(t)^3}.$$

Then u is a solution to (1.1) if and only if V_b solves the following equation:

$$b_s \frac{\partial V_b}{\partial b} - \frac{\lambda_s}{\lambda} \Lambda V_b + (V_b'' - V_b + V_b|V_b|^{p-1})' = \left(\frac{x_s}{\lambda} - 1\right) V_b', \tag{4.2}$$

where Λ is the scaling operator:

$$\Lambda f = \frac{2}{p-1} f + y f'.$$

Similar to the Schrödinger case, the self-similar blow-up regime of (1.1) corresponds to the following finite dimensional dynamics:

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{x_s}{\lambda} = 1, \quad \frac{\lambda_s}{\lambda} = -b, \quad b_s = 0, \tag{4.3}$$

which, after integrating, leads to finite time blow-up for $b(0) > 0$ with:

$$\lambda(t) = c(u_0) \sqrt[3]{T - t}.$$

[1]This somehow explains why there is no self-similar blow-up solution for critical gKdV with initial data near soliton.

4.2. **Self-similar profile.** From the previous analysis, we can see it is very crucial to find a solution to the following ODE:

$$b\Lambda v + (v'' - v + v|v|^{p-1})' = 0. \tag{4.4}$$

This has been done by H. Koch in [9]. Indeed, Koch obtained a even larger range of solutions.

Proposition 4.1 (H. Koch [9])**.** *There exist $p^* > 5, b^* > 0$, such that there exist two smooth maps: $\gamma(b,p) : [0,b^*) \times [5,p^*) \to \mathbb{R}$, $v(b,p,y) : [0,b^*) \times [5,p^*) \times \mathbb{R} \to \mathbb{R}$, such that the following holds:*

(1) *The self-similar equation:*

$$b\big((1+\gamma(b,p))v + xv'\big) + (v'' - v + v|v|^{p-1})' = 0, \tag{4.5}$$

$$(v(b,p,\cdot), \mathcal{Q}_p'(\cdot)) = 0, \quad v(b,p,y) > 0. \tag{4.6}$$

(2) *For all $p \in [5,p^*)$, there exists a unique $b = b(p) \in [0,b^*)$ such that:*

$$\gamma(b(p),p) = -1 + \frac{2}{p-1}, \quad b(5) = 0, \tag{4.7}$$

Moreover,

$$\frac{db(p)}{dp}\Big|_{p=5} = \frac{\|\mathcal{Q}\|_{L^2}^2}{\|\mathcal{Q}\|_{L^1}^2} > 0, \tag{4.8}$$

$$\frac{\partial\gamma}{\partial b}\Big|_{b=b(p)} = -\frac{\|\mathcal{Q}_p\|_{L^1}^2}{8\|\mathcal{Q}_p\|_{L^2}^2} + O(|p-5|) < 0, \tag{4.9}$$

$$\frac{1}{2}\int |v_y(b(p),p,y))|^2 dy - \frac{1}{p+1}\int |v(b(p),p,y)|^{p+1} dy = 0. \tag{4.10}$$

(3) *$v(b,p,\cdot) \in \dot{H}^1 \cap L^{p+1}$, $v(b,p,\cdot) \notin L^2$ if $b > 0$ and $v(0,p,y) = \mathcal{Q}_p(y)$. Moreover,*

$$|\partial_y^k \partial_b^n v| \lesssim \begin{cases} e^{-\frac{1}{3b}}(1 + b^{-2/3}|1-by|)^{-1-\gamma-k} & \text{if } y > b^{-1}, \\ e^{-\frac{y}{10}} & \text{if } b^{-1} \geq y > 0, \\ \big|\partial_y^k \partial_b^n \big(b(1-by)^{-1-\gamma}\big)\big| + e^y & \text{if } y \leq 0. \end{cases} \tag{4.11}$$

Remark 4.2. From Proposition 4.1, for every $p \in (5, 5+\delta)$, $\delta > 0$ small enough, we can find a unique $b > 0$ and a function v solving (4.4). This gives us an explicit blow-up "solution" to (1.1) with self-similar blow-up rate. But this solution never belongs to the critical Sobolev space $\dot{H}^{\sigma_c}$ and hence H^1, due to a slowly decaying tail. Thus, we can't expect any stability result of this solution. Since a stability result usually requires us to work in a Cauchy space. In this situation, a natural Cauchy space is the energy space H^1.

But fortunately, we have the asymptotic behavior of the solution v, i.e. (4.11). Therefore we can choose a suitable approximation of v. More precisely, we fix a $p \in (5, 5+\delta)$, and let

$$b_c = b(p) \sim p - 5 > 0, \quad Q_b(y) = \chi(b_c y) v(v,p,y), \tag{4.12}$$

where χ is a smooth function such that $\chi(y) = 1$, if $|y| < 1$, $\chi(y) = 0$, if $|y| > 2$. Then from (4.11), we know that Q_b has exponential decay on the right. Moreover, Q_b belongs to H^1, whose H^1 norm is of size one. We will see that this approximated self-similar profile Q_b will lead to a stable self-similar blow-up dynamics.

4.3. **Decomposition of the flow.** Let us assume that the initial data is near the approximated profile Q_{b_c} up to scaling and translation, which is exactly the definition of the nonempty open subset $\mathcal{O}_p$ introduced in Theorem 3.1. Then from a standard implicit function argument introduced in [19, 15, 13, 16], we can find geometrical parameters $(\lambda(t), x(t), b(t))$ and error term $\varepsilon(t)$, such that the following decomposition holds:

$$u(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)}\left[Q_{b(t)} + \varepsilon(t)\right]\left(\frac{x-x(t)}{\lambda(t)}\right). \tag{4.13}$$

Moreover, the error term satisfies the following orthogonality condition:

$$(\varepsilon(t), \mathcal{Q}_p) = (\varepsilon(t), \Lambda\mathcal{Q}_p) = (\varepsilon(t), y\Lambda\mathcal{Q}_p) = 0. \tag{4.14}$$

4.4. **Modulation equation.** Differentiating the orthogonality condition (4.14), we will obtain the modulation equations of the parameters, which are:

$$\begin{aligned}
&\frac{\lambda_s}{\lambda} + b = O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H^1_{\mathrm{loc}}}),\\
&\frac{x_s}{\lambda} - 1 = O(b_c^{\frac{5}{2}} + \|\varepsilon\|_{H^1_{\mathrm{loc}}}),\\
&b_s + c_p(b-b_c)b_c = O(b_c^3 + b_c\|\varepsilon\|_{H^1_{\mathrm{loc}}}),\\
&s = \int_0^t \frac{1}{\lambda^3(\tau)}\,d\tau.
\end{aligned} \tag{4.15}$$

Our main task here is to control $\|\varepsilon\|_{H^1_{\mathrm{loc}}}$, which is done by a bootstrap argument. If such a control exists, we will see that (4.15) is just a small perturbation of the system (4.3), and has the same asymptotic behavior.

4.5. **Local H^1 control of the error term.** The key techniques in this paper are the monotonicity of energy and a dispersive control of $\|\varepsilon\|_{H^1_{\mathrm{loc}}}$.

In supercritical case any critical or subcritical norm of the error term ε cannot be controlled, for example $\|\varepsilon\|_{L^2}$ or even $\int_{y>0}\varepsilon^2$. This contrasts with the critical case, where $\|\varepsilon\|_{L^2}$ is small. And for the same reason, we can no longer use the L^1 control of ε on the right as Martel, Merle and Raphaël did in the critical case in [16].

Fortunately, we can still control $\|\varepsilon_y\|_{L^2}$. Moreover, from the energy conservation law and a localization argument, we can obtain an even better control of the L^2 norm of ε_y on the half-line $[\kappa B, +\infty)$:

$$\int_{y>\kappa B} \varepsilon_y^2 \lesssim b_c^{\frac{55}{7}}. \tag{4.16}$$

where $\kappa > 0$ is a small universal constant, and

$$B = b_c^{-\frac{1}{20}}.$$

Together with Gagliardo-Nirenberg inequality, we can give a good control of the localized L^2 norm of ε on the right:

$$\int_{\kappa B<y<2B^2} \varepsilon^2 \lesssim b_c^2. \tag{4.17}$$

Next, we construct a nonlinear functional:

$$\mathcal{F} = \int \left[\varepsilon_y^2\psi_B + \varepsilon^2\zeta_B - \frac{2}{p+1}\big(|\varepsilon + Q_b|^{p+1} - Q_b^{p+1} - (p+1)\varepsilon Q_b^p\big)\psi_B\right],$$

for well chosen functions (ψ_B, ζ_B), which are exponentially decaying to the left and bounded on the right. A similar functional was introduced in [16] for the critical equations, but they have a totally different meaning. Here the key point in this case is that we cannot control $\int_{y>0}\varepsilon^2$. We need to find a different way to control ε on the right. However, if we choose ζ such that it is compactly supported on the right, i.e. supp $\zeta \subset (-\infty, 2B^2]$, then for $y > 0$, only localized L^2 norm of ε appears in $\mathcal{F}$, which can be controlled by (4.17).

The most significant technique here is the *Lyapounov monotonicity*:

$$\frac{d\mathcal{F}}{ds} + \frac{1}{B}\|\varepsilon\|^2_{H^1_{\rm loc}} \lesssim b_c^{\frac{7}{2}}. \tag{4.18}$$

Moreover, the leading order term of $\mathcal{F}$ is:

$$\mathcal{F} \sim \int \left[\varepsilon_y^2\psi_B + \varepsilon^2\zeta_B - p\varepsilon^2 Q_b^{p-1}\psi_B\right],$$

which is coercive up to 3 bad direction. More precisely, we have:

Lemma 4.3. *There exists a $\kappa_0 > 0$ such that for all $f \in H^1$, there holds:*

$$\int f_x^2 + f^2 - p\mathcal{Q}_p^{p-1}f^2 \geq \kappa_0\|f\|^2_{H^1} - \frac{1}{\kappa_0}\left[(f,\mathcal{Q}_p)^2 + (f,\Lambda\mathcal{Q}_p)^2 + (f, y\Lambda\mathcal{Q}_p)^2\right].$$

From Lemma 4.3, orthogonality condition (4.14) and a standard localization argument, we have:

$$\mathcal{F} \gtrsim \|\varepsilon\|^2_{H^1_{\rm loc}}.$$

The above analysis shows that $\|\varepsilon\|_{H^1_{\rm loc}}$ (or equivalently $\mathcal{F}$) is almost decreasing with respect to $s \in [0, +\infty)$. Hence we have:

$$\|\varepsilon\|^2_{H^1_{\rm loc}} \lesssim b_c^{3+8\nu}, \tag{4.19}$$

where $\nu > 0$ is a small universal constant.

4.6. **Nonlinear estimate of the error term.** In the previous section, there are nonlinear term like

$$\int |\varepsilon|^{p+1}\zeta_B$$

evolved in the functional $\mathcal{F}$. So in order to estimate $\mathcal{F}$, a global L^{p_0} control of the error term ε is necessary, where $p_0 < p+1$. But from scaling, we cannot expect any global control of subcritical norm of ε, i.e. we must choose $p_0 > \frac{p-1}{2}$. Indeed,

$$p_0 = \frac{5}{2}$$

is enough for our analysis. To control $\|\varepsilon\|_{L^{p_0}}$, it is better to work in the original variable, i.e. we only need to find a control of

$$\tilde{u}(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)}\varepsilon\left(t, \frac{x - x(t)}{\lambda(t)}\right).$$

We write down the equation of $\tilde{u}$:

$$\partial_t\tilde{u} + \partial_{xxx}\tilde{u} = \big(f(\tilde{u})\big)_x + Q_S,$$

where $f(\tilde{u})$ is a nonlinear term of $\tilde{u}$, Q_S is the singular part of the original solution. It is easy to see that Q_S can be controlled by the modulation equations (4.15). Now, using the inhomogeneous Strichartz estimates introduced by Foschi in [5], we have:

$$\|\tilde{u}\|_{L^{p_0}} \leq \frac{b_c^{\frac{13}{28}}}{\lambda^{\frac{2}{p-1}-p_0}(t)},$$

or equivalently

$$\|\varepsilon\|_{L^{p_0}} \leq b_c^{\frac{13}{28}}, \tag{4.20}$$

which is exactly the required nonlinear estimate.

4.7. **End of the proof of Theorem 3.1.** From the control of the error term, we can see that the parameters $(\lambda(t), x(t), b(t))$ satisfies (4.3) up to some small perturbation. So they will have the same asymptotic behavior, which leads to the self-similar blow-up dynamics. This dynamics is also stable due to the openness of the initial data set $\mathcal{O}_p$.

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