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ON THE TRANSPORT OF GAUSSIAN MEASURES UNDER THE FLOW OF HAMILTONIAN PDES

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ABSTRACT. This manuscript is based on a talk given by the second author at the seminar Laurent Schwartz, École Polytechnique, Paris, on December 15, 2015.

1. INTRODUCTION

1.1. **Gaussian measures on Sobolev spaces.** On a finite dimensional Hilbert space, the standard Gaussian measure μ is defined by

$$d\mu = Z^{-1}e^{-\frac{1}{2}\|x\|^2}dx.$$

By drawing an analogy, one can consider Gaussian measures on Sobolev spaces $H^s(\mathbb{T})$, $s \in \mathbb{R}$. Given $s \in \mathbb{R}$, let μ_s be the Gaussian measure, formally defined by

$$d\mu_s(u) = Z_s^{-1}e^{-\frac{1}{2}\|u\|_{H^s}^2}du = Z_s^{-1}\prod_{n\in\mathbb{Z}}e^{-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2}d\widehat{u}_n.$$

This expression may suggest that μ_s is a probability measure on $H^s(\mathbb{T})$. In order to make sense of μ_s, however, we *need to enlarge* the space $H^s(\mathbb{T})$.

In fact, we can define μ_s in a rigorous manner by viewing it as the induced probability measure under the map:

$$\omega \in \Omega \longmapsto u^\omega(x) = \sum_{n\in\mathbb{Z}}\frac{g_n(\omega)}{\langle n\rangle^s}e^{inx},$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables and $\langle\,\cdot\,\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$. It is easy to see that $u^\omega \in L^2(\Omega; H^\sigma(\mathbb{T}))$ if and only if $\sigma < s - \frac{1}{2}$. Therefore, for the same range of σ, μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T})$ (but $\mu_s(H^{s-\frac{1}{2}}(\mathbb{T})) = 0$). The triplet (H^s, H^σ, μ_s) forms an abstract Wiener space. Note that we have the Brownian loop when $s = 1$.

In the following, we discuss transport properties of the Gaussian measure μ_s under various transformations. In particular, we study the transport properties of μ_s under several Hamiltonian PDE dynamics.

Before proceeding further, recall the following definition of quasi-invariant measures. Given a measure space (X, μ), we say that μ is *quasi-invariant* under a transformation $T : X \to X$ if the transported measure $T_*\mu = \mu \circ T^{-1}$ and μ are equivalent, i.e. mutually absolutely continuous with respect to each other.

1.2. **The Cameron-Martin theorem (1944).** Let $\sigma < s - \frac{1}{2}$ and fix $u \in H^\sigma(\mathbb{T})$. Consider the following translation map:

$$T_u : h \longmapsto h + u, \qquad h \in H^\sigma(\mathbb{T}). \tag{1.1}$$

Question: How does the transport of μ_s behave under this translation map?

Theorem 1.1 (Cameron-Martin [2]). *The Gaussian measure μ_s is quasi-invariant under* (1.1) *if and only if $u \in H^s(\mathbb{T})$.*

We briefly go over the proof of this theorem in the following. First, suppose that $u \in H^s(\mathbb{T})$. Note that

$$\|u + h\|_{H^s}^2 = \|u\|_{H^s}^2 + \|h\|_{H^s}^2 + 2(u, h)_s,$$

where $(\cdot, \cdot)_s$ denotes the scalar product on $H^s(\mathbb{T})$ given by $(u, h)_s = \mathrm{Re} \int_{\mathbb{T}} u\overline{h} \, dx$. Then, the absolute continuity of the image measure with respect to the original Gaussian measure μ_s follows once we show that $(u, h)_s$ is finite μ_s-almost surely.

The proof of the μ_s-almost sure finiteness of the scalar product $(u, h)_s$ reduces to check that for each $\{c_n\}_{n \in \mathbb{Z}} \in \ell^2$, we have

$$\sum_{n \in \mathbb{Z}} c_n g_n(\omega) < \infty, \quad \text{a.s. in } \omega, \tag{1.2}$$

which can be easily seen to be true.

Next, suppose that $u \notin H^s(\mathbb{T})$. Then, by Banach-Steinhaus theorem, there exists $v \in H^s(\mathbb{T})$ such that $(u, v)_s = \infty$. Define

$$A := \{w : (w, v)_s < \infty\}.$$

Then, arguing as before, it follows from (1.2) that $\mu_s(A) = 1$. Denote by ρ_s the image measure of μ_s under the map $T_u : h \mapsto h + u$. Let

$$B = \{w - u, w \in A\}.$$

On the one hand, we have $\rho_s(A) = \mu_s(B)$. On the other hand, we have $(h, v)_s = \infty$ for every $h \in B$. In particular, we have $B \subset A^c$ and hence $\mu_s(B) = 0$. This in turn implies that $\rho_s(A) = 0$. Therefore, the image measure $\rho_s = \mu_s \circ T_u^{-1}$ and the original Gaussian measure μ_s are mutually singular.

1.3. **Ramer's generalization of the Cameron-Martin theorem (1974).** Fix again $\sigma < s - \frac{1}{2}$. Consider the following nonlinear generalization of (1.1):

$$h \longmapsto h + F(h). \tag{1.3}$$

Note that while (1.1) is a translation by a deterministic element u, $F(h)$ in (1.3) may be nonlinear and depend on h. We have the following statement on the transport of μ_s under (1.3).

Theorem 1.2 (Ramer [9]). *The Gaussian measure μ_s is quasi-invariant under* (1.3) *if, for every $x \in H^\sigma(\mathbb{T})$,*

$$DF(x) : H^s \longrightarrow H^s$$

is a Hilbert-Schmidt map.

In the above statement, we only stated a crucial condition for quasi-invariance. See [9] for the precise statement. For $s > 1$, an example of a map satisfying the assumption of Ramer's theorem is

$$F(h) = (1 - \partial_x^2)^{-\frac{\alpha}{2}}(h^2), \quad \alpha > 1.$$

1.4. **Cruzeiro's generalization of the Cameron-Martin theorem (1983).** In [3], Cruzeiro considered a general evolution equation of the form:

$$\partial_t u = X(u), \tag{1.4}$$

where X is an infinite dimensional vector field. In particular, she proved the quasi-invariance of μ_s under the flow of (1.4) if we suppose the following exponential moment assumption:

$$\int_{H^\sigma(\mathbb{T})} e^{\mathrm{div}(X(u))} d\mu_s(u) < \infty. \tag{1.5}$$

Problem: How do we check (1.5) for concrete examples?

In the following, we consider several Hamiltonian PDE dynamics and study quasi-invariance properties of μ_s. Very roughly speaking, our results aim to verify assumptions of type (1.5) "in practice".

2. Quasi-invariance of μ_s for "integrable" PDEs

We first consider the KdV equation on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\partial_t u + \partial_x^3 u + \partial_x(u^2) = 0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}.$$

Let $s \geq 0$ be an integer. Then, the quasi-invariance of (the real valued version of) μ_s under the KdV flow follows from (i) Zhidkov [15] when $s \geq 2$, (ii) Bourgain [1] when $s = 1$, and (iii) Quastel-Valkó [8] and Oh [6] when $s = 0$.

For $s \geq 1$, a similar result holds for the "integrable" cubic NLS on $\mathbb{T}$ (cf. Zhidkov [14] for $s \geq 2$ and Bourgain [1] for $s = 1$):

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0$$

and the Benjamin-Ono (BO) equation on $\mathbb{T}$ (cf. Tzvetkov-Visciglia [11, 12] for $s \geq 2$ and for Deng-Tzvetkov-Visciglia [4] $s = 1$):

$$\partial_t u + H\partial_x^2 u + \partial_x(u^2) = 0,$$

where H denotes the Hilbert transform. The $s = 0$ case seems to be completely out of reach of the present techniques both for NLS and BO.

The key point for "integrable" equations is that one has infinitely many conservation laws of the following form:

$$E_s(u) = \frac{1}{2}\|u\|_{H^s}^2 + \text{lower order terms}, \ s = 0, 1, 2, 3, \ldots$$

Then, the work mentioned above shows that the weighted Gaussian measure ρ_s formally defined by

$$d\rho_s(u) = Z_s^{-1}\chi(u)e^{-\frac{1}{2}E_s(u)}du,$$

is absolutely continuous with respect to μ_s and *invariant* under the flow of the corresponding equation. Here, $\chi(u)$ is a suitable cut-off depending the previous conservation laws $E_0, \ldots, E_{s-1}$. Using the invariance of ρ_s (and varying the cut-off χ), one may conclude that μ_s is quasi-invariant.

Question : What about the non-integrable equations?

3. BBM type models

A first example of a non-integrable equation related to KdV is the BBM equation:

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x(u^2) = 0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}.$$

Theorem 3.1 ([10]). *Let $s \geq 1$ be an integer. Then, (the real valued version of) the Gaussian measure μ_s is quasi-invariant under the flow of the BBM equation.*

A similar result holds for the generalized (lower dispersion) BBM model:

$$\partial_t u + \partial_x u + \partial_t |D_x|^\gamma u + \partial_x(u^2) = 0, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}, \tag{3.1}$$

provided $\gamma > 4/3$ ($\gamma = 1$ corresponds to the Benjamin-Ono BBM model). Observe that only $1/3$-smoothing is needed to obtain the quasi-invariance, while the Cameron-Martin's result would require $1/2$-smoothing. Intuitively, in the context of (3.1), $\gamma = 2$ corresponds to the borderline case for Ramer's result and $\gamma = 3/2$ is the Cameron-Martin threshold.

4. The cubic fourth order NLS

In the following, we consider the fourth order NLS:

$$i\partial_t u = \partial_x^4 u + |u|^2 u, \qquad (x,t) \in \mathbb{T} \times \mathbb{R}. \tag{4.1}$$

The Cauchy problem (4.1) is globally well-posed in $H^\sigma(\mathbb{T})$ for $\sigma \geq 0$ and strongly ill-posed for $\sigma < 0$. Here is the main result of this exposé.

Theorem 4.1 ([7]). *Let $s > 3/4$. Then, the Gaussian measure μ_s is quasi-invariant under the flow of (4.1).*

We expect that the main line of our proof works in the (optimal) range $s > 1/2$. One would, however, require some important additional ideas.

In sharp contrast with the BBM models, there is no apparent smoothing in the equation (4.1). One can, however, exhibit smoothing effects after using some *gauge and normal form transformations.*

4.1. **The gauge transform.** The invariance of μ_s under the gauge transformation we define below is a key ingredient in our analysis. Given $t \in \mathbb{R}$, we define a gauge transformation $\mathcal{G}_t$ on $L^2(\mathbb{T})$ by setting

$$\mathcal{G}_t[f] := e^{\pi^{-1}it \int_{\mathbb{T}} |f(y)|^2 dy} f.$$

The gauge transform $\mathcal{G}_t$ is invertible with inverse $\mathcal{G}_{-t}$.

Let $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ be a solution to the fourth order NLS (4.1). Define $\widetilde{u}$ by

$$\widetilde{u}(t) := \mathcal{G}_t[u(t)].$$

Then, it follows from the the mass conservation that $\widetilde{u}$ is a solution to the following renormalized fourth order NLS:

$$i\partial_t \widetilde{u} = \partial_x^4 \widetilde{u} + \left(|\widetilde{u}|^2 - \pi^{-1} \int_{\mathbb{T}} |\widetilde{u}|^2 dx \right) \widetilde{u}.$$

The last equation behaves better than the fourth order NLS (4.1) with respect to resonant interactions.

4.2. **Filtering the free evolution.** Define the interaction representation v of $\widetilde{u}$ by

$$v(t) = S(-t)\widetilde{u}(t), \qquad S(t) = e^{-it\partial_x^4}.$$

For simplicity of notations, we use v_n to denote the Fourier coefficient of v. Then, we obtain the following equation for $\{v_n\}_{n \in \mathbb{Z}}$:

$$\partial_t v_n = -i \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n, \tag{4.2}$$

where the phase function $\phi(\bar{n})$ and the plane $\Gamma(n)$ are given by

$$\phi(\bar{n}) = n_1^4 - n_2^4 + n_3^4 - n^4,$$

$$\Gamma(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3 \text{ and } n_1, n_3 \neq n\}.$$

On $\Gamma(n)$, we have the following key factorization of $\phi(\bar{n})$:

$$\phi(\bar{n}) = (n_1 - n_2)(n_1 - n)\big(n_1^2 + n_2^2 + n_3^2 + n^2 + 2(n_1 + n_3)^2\big).$$

4.3. **Global analysis.** We shall perform our PDE analysis on the equation (4.2) for $\{v_n\}_{n \in \mathbb{Z}}$. Let us denote by $\Psi(t)$ the solution map of (4.2), sending initial data at time 0 to solutions at time t. Then, by letting $\Phi(t)$ denote the flow map of fourth order NLS (4.1) (the one we study), we have

$$\Phi(t) = \mathcal{G}_{-t} \circ S(t) \circ \Psi(t).$$

Therefore, we can reduce the quasi-invariance issue for $\Phi(t)$ to the study of $\Psi(t)$ thanks to the following important proposition.

Proposition 4.2. *Let $s > 1/2$. For every $t \in \mathbb{R}$, the Gaussian measure μ_s is invariant under the transformations $S(t)$ and $\mathcal{G}_t$.*

The invariance under $S(t)$ follows from the invariance of complex Gaussian random variables under rotations. The invariance under $\mathcal{G}_t$ is more intricate. One has the following elementary, yet remarkable statement.

Lemma 4.3. *Given a complex-valued mean-zero Gaussian random variable g with variance σ, i.e. $g \in \mathcal{N}_{\mathbb{C}}(0, \sigma)$, let $Tg = e^{it|g|^2}g$ for some $t \in \mathbb{R}$. Then, $Tg \in \mathcal{N}_{\mathbb{C}}(0, \sigma)$.*

With this statement in hand, one may exploit the *independence* and once again *the invariance of complex Gaussian random variables under rotations* to complete the proof of the invariance of μ_s under $\mathcal{G}_t$.

Let us remark that when $s = 1$, one may deduce the invariance of μ_1 under $\mathcal{G}_t$ by invoking the properties of the Brownian loop under conformal mappings. See for example [5].

4.4. **The case $s > 1$.** We first sketch the proof of Theorem 4.1 when $s > 1$. In this case, by performing a direct normal form analysis on the equation (4.2) for $\{v_n\}_{n \in \mathbb{Z}}$, we show that $\Psi(t)$ satisfy the assumptions of the Ramer result (Theorem 1.2).

By writing (4.2) in an integral form, we have

$$v_n(t) = v_n(0) - i \int_0^t \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t'} v_{n_1} \overline{v_{n_2}} v_{n_3}(t') dt' + i \int_0^t |v_n|^2 v_n(t') dt'.$$

Let $\mathcal{N}(v)(n, t)$ and $\mathcal{R}(v)(n, t)$ be the second and the third terms on the right hand-side, respectively. For $s > 1/2$, we have the bound

$$\|\mathcal{N}(v)(t)\|_{H^{s+2}} \lesssim \|v(0)\|_{H^s}^3 + \|v(t)\|_{H^s}^3 + |t| \sup_{t' \in [0,t]} \|v(t')\|_{H^s}^5. \tag{4.3}$$

Moreover, for $s \geq 0$, we have

$$\|\mathcal{R}(v)(t)\|_{H^{3s}} \lesssim |t| \sup_{t' \in [0,t]} \|v(t)\|_{H^s}^3.$$

In order to obtain the bound (4.3) for $\mathcal{N}(v)(t)$, we perform integration by parts and write

$$\mathcal{N}(v)(n, t) = \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t'}}{\phi(\bar{n})} v_{n_1}(t') \overline{v_{n_2}(t')} v_{n_3}(t') \bigg|_{t'=0}^{t} - \sum_{\Gamma(n)} \int_0^t \frac{e^{-i\phi(\bar{n})t'}}{\phi(\bar{n})} \partial_t (v_{n_1} \overline{v_{n_2}} v_{n_3})(t') dt'.$$

Then, we use the equation (4.2) to express $\partial_t(v_{n_1} \overline{v_{n_2}} v_{n_3})$ as quinti-linear terms in v.

We point out that the regularity restriction $s > 1$ comes from the bound on the linearization of the resonant term

$$i|v_n|^2 v_n,$$

requiring $s + \frac{1}{2}+ \leq 3\big(s - (\frac{1}{2}+)\big)$, i.e. $s > 1$.

5. The case $s \in (3/4, 1]$

In this section, we sketch the argument for $s \in (3/4, 1]$. In this case, we use modified energies (in the spirit of the so-called I-method) instead of direct analysis of the equation.

By the equation (4.2) for $\{v_n\}_{n \in \mathbb{Z}}$, we have

$$\frac{d}{dt}\|v(t)\|_{H^s}^2 = -2 \operatorname{Re} i \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} e^{-i\phi(\bar{n})t} \langle n \rangle^{2s} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n}.$$

We can write the right-hand side as the difference of

$$2 \operatorname{Re} \frac{d}{dt} \left[\sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n} \right] \tag{5.1}$$

and

$$2 \operatorname{Re} \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} \partial_t (v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n}).$$

This leads us to define a modified energy $E_t(v)$ as

$$E_t(v) = \|v\|_{H^s}^2 + R_t(v),$$

where $R_t(v)$ is defined according to (5.1), namely

$$R_t(v) = -2 \operatorname{Re} \sum_{n \in \mathbb{Z}} \sum_{\Gamma(n)} \frac{e^{-i\phi(\bar{n})t}}{\phi(\bar{n})} \langle n \rangle^{2s} v_{n_1} \overline{v_{n_2}} v_{n_3} \overline{v_n}.$$

5.1. **Local analysis: the energy estimate.** Consider the following truncated form of (4.2):

$$\partial_t v_n = \mathbf{1}_{|n| \leq N} \left\{ -i \sum_{\Gamma_N(n)} e^{-i\phi(\bar{n})t} v_{n_1} \overline{v_{n_2}} v_{n_3} + i|v_n|^2 v_n \right\}, \tag{5.2}$$

where $\Gamma_N(n)$ is defined by

$$\Gamma_N(n) = \Gamma(n) \cap \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : |n_j| \leq N\}.$$

Then, we have the following crucial energy estimate on the modified energy E_t, uniformly in $N \in \mathbb{N}$ and $t \in \mathbb{R}$.

Proposition 5.1. *Let $s > 3/4$ and let $\mathbf{P}_{\leq N}$ be the projection onto the frequencies $\{|n| \leq N\}$. Then, for any sufficiently small $\varepsilon > 0$, there exist $\theta > 0$ and $C > 0$ such that*

$$\left| \frac{d}{dt} E_t(\mathbf{P}_{\leq N} v) \right| \leq C \|v\|_{L^2}^{4+\theta} \|v\|_{H^{s-\frac{1}{2}-\varepsilon}}^{2-\theta}$$

for all $N \in \mathbb{N}$, $t \in \mathbb{R}$, and any solution v to (5.2).

The proof of this proposition relies on some elementary number theory (divisor counting argument). The restriction $s > 3/4$ appears in estimating $\partial_t v_n$ in ℓ_n^∞ via (5.2). Indeed, the regularity restriction $s > 3/4$ in Theorem 4.1 comes from this proposition. In order to go further, one needs to make additional normal form reductions and obtain an improved energy estimate.

5.2. **The modified measures.** Given $N \in \mathbb{N}$, $r > 0$, and $t \in \mathbb{R}$, define $F_{N,r,t}(v)$ and $F_{r,t}(v)$ by

$$F_{N,r,t}(v) = \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R_t(\mathbf{P}_{\leq N} v)} \quad \text{and} \quad F_{r,t}(v) = \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R_t(v)}.$$

We would like to construct probability measures $\rho_{s,N,r}$ and $\rho_{s,r}$ of the form:

$$d\rho_{s,N,r} = Z_{s,N,r}^{-1} F_{N,r,t} d\mu_s \quad \text{and} \quad d\rho_{s,r} = Z_{s,r}^{-1} F_{r,t} d\mu_s.$$

The next statement shows that it is indeed possible.

Proposition 5.2. *Let $s > \frac{1}{2}$, $r > 0$, and $t \in \mathbb{R}$. Then, $F_{N,r,t}(v) \in L^p(\mu_s)$ for any $p \geq 1$ with a uniform bound in N, depending only on $p \geq 1$ and $r > 0$. Moreover, for any finite $p \geq 1$, $F_{N,r,t}(v)$ converges to $F_{r,t}(v)$ in $L^p(\mu_s)$ as $N \to \infty$.*

The proof of this proposition is very close to the construction of the Gibbs measures for NLS by Bourgain [1].

5.3. **Global analysis: a change of variable formula.** Let $\Psi_N(t, \tau)$ be the solution map of the truncated equation (5.2), sending data at time τ to solutions at time t. By definition, we have

$$\rho_{s,N,r}(\Psi_N(t,\tau)(A)) = Z_{s,N,r}^{-1} \int_{\Psi_N(t,\tau)(A)} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}R_t(\mathbf{P}_{\leq N} v)} d\mu_s(v)$$

for any measurable set $A \subset L^2(\mathbb{T})$. The following change-of-variable formula plays an important role in our analysis.

Proposition 5.3. *Let $s > \frac{1}{2}$, $N \in \mathbb{N}$, and $r > 0$. Then, we have*

$$\rho_{s,N,r}(\Psi_N(t,\tau)(A)) = \widehat{Z}_{s,N,r}^{-1} \int_A \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}E_t(\mathbf{P}_{\leq N}\Psi_N(t,\tau)(v))} dL_N \otimes d\mu_{s,N}^{\perp}$$

for any $t, \tau \in \mathbb{R}$ and any measurable set $A \subset L^2(\mathbb{T})$.

In order to prove this proposition, we write

$$d\rho_{s,N,r} = \widehat{Z}_{s,N,r}^{-1} \mathbf{1}_{\{\|v\|_{L^2} \leq r\}} e^{-\frac{1}{2}E_t(\mathbf{P}_{\leq N}v)} dL_N \otimes d\mu_{s,N}^{\perp},$$

where $dL_N = \prod_{|n| \leq N} d\widehat{u}_n$ denotes the Lebesgue measure on $\mathbb{C}^{2N+1}$. Then, we need to check that the corresponding vector field is "divergence free" (inherited from the Hamiltonian structure of (5.2)).

5.4. **The measure evolution property.** Combining the local and global analysis performed in the previous subsections, we can establish the following growth bound on the weighted Gaussian measure $\rho_{s,N,r}$.

Proposition 5.4. *Let $s > \frac{3}{4}$. There exists $0 \leq \beta < 1$ such that, given $r > 0$, there exists $C > 0$ such that*

$$\frac{d}{dt}\rho_{s,N,r}(\Psi_N(t,0)(A)) \leq Cp^{\beta}\big\{\rho_{s,N,r}(\Psi_N(t,0)(A))\big\}^{1-\frac{1}{p}}$$

for any $p \geq 2$, any $N \in \mathbb{N}$, any $t \in \mathbb{R}$, and any measurable set $A \subset L^2(\mathbb{T})$.

As in the work [11] on invariant measures for the Benjamin-Ono equation, we write

$$\frac{d}{dt}\rho_{s,N,r}(\Psi_N(t,0)(A))\Big|_{t=t_0} = \frac{d}{dt}\rho_{s,N,r}\big(\Psi_N(t_0+t,t_0)(\Psi_N(t_0,0)(A))\big)\Big|_{t=0}.$$

Then, we can apply the change-of-variable formula (Proposition 5.3), the energy estimate (Proposition 5.1), and some basic Gaussian estimates to conclude the proof.

By applying a Yudovich-type argument [13], this measure evolution property implies the following statement.

Proposition 5.5. *Let $s > \frac{3}{4}$. Then, given $t \in \mathbb{R}$, $r > 0$, and $\delta > 0$, there exists $C = C(t,r,\delta) > 0$ such that*

$$\rho_{s,N,r}(\Psi_N(t,0)(A)) \leq C(\rho_{s,N,r}(A))^{1-\delta}$$

for any $N \in \mathbb{N}$ and any measurable set $A \subset L^2(\mathbb{T})$.

Finally, by establishing an approximation property of (4.2) by the truncated flow (5.2) (local PDE analysis) and applying some soft measure theoretic argument, we can take the limits as $N \to \infty$ and $r \to \infty$, yielding the desired statement. This concludes the sketch of the proof of Theorem 4.1.

6. Final remarks

(1) The argument presented above establishes mutual absolute continuity of the transported measure $\Phi(t)_*\mu_s$ and the original Gaussian measure μ_s. Our argument, however, does not tell us much about the Radon-Nikodym derivative of $\Phi(t)_*\mu_s$ with respect to μ_s. It would be interesting to study more about the the resulting Radon-Nikodym derivatives. In particular,

can we establish quantitative versions of the quasi-invariance to prove new bounds on the growth of higher Sobolev norms of solutions in a probabilistic manner?

(2) So far, the quasi-invariance of the Gaussian measures is known for a handful of Hamiltonian PDEs. It would be of interest to decide how much of the obtained results can be extended to other Hamiltonian PDEs. Moreover, it would be very intriguing to find a situation where we can prove that the transported measure is singular with respect to the initial Gaussian measure and describe its measure evolution. For example, what can we say about the (non-integrable) quintic NLS on $\mathbb{T}$:

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0?$$

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