Séminaire Laurent Schwartz

EDP et applications

Année 2014-2015

Patrick Gérard and Sandrine Grellier

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Séminaire Laurent Schwartz — EDP et applications (2014-2015), Exposé n° XI, 20 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2014-2015____A11_0>

Séminaire Laurent-Schwartz — EDP et applications
Institut des hautes études scientifiques, 2014-2015
Exposé n^{o} XI, 1-20

ON THE GROWTH OF SOBOLEV NORMS FOR THE CUBIC SZEGŐ EQUATION

PATRICK GÉRARD AND SANDRINE GRELLIER

Abstract. We report on a recent result establishing that trajectories of the cubic Szegő equation in Sobolev spaces with high regularity are generically unbounded, and moreover that, on solutions generated by suitable bounded subsets of initial data, every polynomial bound in time fails for high Sobolev norms. The proof relies on an instability phenomenon for a new nonlinear Fourier transform describing explicitly the solutions to the initial value problem, which is inherited from the Lax pair structure enjoyed by the equation.

1. Introduction

The large time behavior of solutions to Hamiltonian partial differential equations is an important problem in mathematical physics. In the case of finite dimensional Hamiltonian systems, many features of the large time behavior of trajectories are described using the topology of the phase space. For a given infinite dimensional system, several natural phase spaces, with different topologies, can be chosen, and the large time properties may strongly depend on the choice of such topologies. For instance, it is known that the cubic defocusing Schrödinger equation

$$i\partial_t u + \Delta u = |u|^2 u$$

posed on a Riemannian manifold M of dimension $d = 1, 2, 3$ with sufficiently uniform properties at infinity, defines a global flow on the Sobolev spaces $H^s(M)$ for every $s \geq 1$ (see e.g. [4]). In this case, a typical large time behavior of interest is the boundedness of trajectories. On the energy space $H^1(M)$, the conservation of energy trivially implies that all the trajectories are bounded. On the other hand, the existence of unbounded trajectories in $H^s(M)$ for $s > 1$ is a long standing open problem [2], naturally connected to weak turbulence. In [3], [28], it was proved that dispersion properties imply a polynomial bound in time on H^s norms as time goes to infinity, but so far the optimality of this kind of bound has not been established or

2010 *Mathematics Subject Classification.* 35B15, 47B35, 37K15.

disproved.[1] Let us mention however that, for some special solutions of cubic NLS on $\mathbb{R}\times\mathbb{T}^2$, Hani–Pausader–Tzvetkov–Visciglia [17] succeeded in proving that H^s norms may not be bounded for s big enough, with a minoration of the type

$$\mathrm{e}^{c(\log\log t)^{1/2}},$$

on a sequence of times going to infinity. Natural model problems for studying the unboundedness of Sobolev norms seem to be those for which the calculation of solutions is the most explicit, namely integrable systems. Typical examples are the Korteweg de Vries equation [21], [19] and the one dimensional cubic nonlinear Schrödinger defocusing equation [31], [15]. However, in these cases, the set of conservation laws is known to control the whole regularity of the solution, so that all the trajectories of elements of $H^s(M)$ are bounded in $H^s(M)$ for every nonnegative integer s [29].

In order to investigate direct and inverse cascades between spacial scales in one space dimension, Majda, Mc Laughlin, Tabak and their collaborators introduced, in a series of papers starting with [22], — see also Zakharov, Guyenne, Pushkarev, Dias [32] — a class of Hamiltonian equations on $\mathbb{T}$ including

$$i\partial_t u - |D|^\alpha u = |u|^2 u\ ,\ D := \frac{1}{i}\partial_x\ ,\ \alpha > 0\ , \tag{1}$$

making numerical simulations suggesting weak turbulence effects for some values of α. For $\alpha > 1$, it is easy to prove that this equation is globally well–posed on $H^s(\mathbb{T})$ for $s \geq \frac{\alpha}{2}$, and that the $H^{\alpha/2}$ norm is uniformly bounded along the trajectories. Moreover, polynomial bounds on the H^s norms for big s can also be obtained in that case [29].

In the limit case $\alpha = 1$, Equation (1) is a nonlinear (half–) wave equation in one space dimension, which can be proved to be globally well-posed on H^s for $s \geq \frac{1}{2}$ (see [11]). However, compared to the case $\alpha \neq 1$, this equation is no more dispersive, which suggests that large time transition to high frequencies may be facilitated. In [11] — see also Pocovnicu [26]—, we proved that a Birkhoff normal form of this equation near the origin is given at first order by the following cubic Szegő equation,

$$i\partial_t u = \Pi(|u|^2 u)\ , \tag{2}$$

where Π denotes the Szegő projector on $L^2(\mathbb{T})$,

$$\widehat{\Pi u}(k) = \mathbf{1}_{k\geq 0}\hat{u}(k)\ .$$

[1]Except in the case $M = \mathbb{R}^d$, $d = 2, 3, 4$, where global finiteness of Strichartz norms has been established , leading to scattering theory [13], [27], [20], [6]. In this case, boundedness of high H^s norms follows. We are grateful to N. Tzvetkov for drawing our attention to this fact.

Notice that the range of Π is the closed subspace $L^2_+(\mathbb{T})$ consisting of L^2 functions on the unit circle which admit a holomorphic extension to the unit disc. Equation (2) has been introduced in [7], where global wellposedness was established on $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$ for every $s \geq \frac{1}{2}$, as well as boundedness of the $H^{1/2}$ norm along every trajectory. Moreover, we proved that this equation enjoys a Lax pair structure, giving rise to integrability properties studied in [7] and [8], as well as the following a priori estimate on H^s norms of solutions,

$$\|u(t)\|_{H^s} \leq C_s \mathrm{e}^{C_s t} \ , \ s > 1 \ , \tag{3}$$

where C_s is a uniform constant depending only on a bound of the H^s norm of the initial data. It turns out that this estimate is almost optimal, as shown by the following result.

Theorem 1.

(1) *For every $s > \frac{1}{2}$, the set of initial data in $H^s_+(\mathbb{T})$ leading to an unbounded trajectory in $H^s_+(\mathbb{T})$ is a dense G_δ–subset of $H^s_+(\mathbb{T})$.*

(2) *For every positive integer M, for every $s > \frac{1}{2}$, there exists a bounded subset B of $\mathcal{C}^\infty_+(\mathbb{T}) := \cap_s H^s_+(\mathbb{T})$ such that*

$$\sup_{u_0 \in B} \frac{\|u(t)\|_{H^s}}{(1+|t|)^M} \to \infty \ \textit{as} \ |t| \to \infty,$$

where u denotes the solution of (2) with $u(0) = u_0$.

The above theorem calls for several comments.

- It is in fact possible to find the same dense G_δ–subset of initial data leading to unbounded trajectories in H^s for every $s > \frac{1}{2}$. However, at this stage we cannot display yet an explicit example of such an initial datum. For example, we know that every rational function with no pole in the closed unit disc gives rise to a bounded trajectory in every H^s (see [10]). This fact is in contrast with the case of the cubic Szegő equation on the line, where special examples of unbounded trajectories in H^s, for every $s > \frac{1}{2}$, are displayed in [25].
- In the above result, the superpolynomial growth of Sobolev norms is stated for families of solutions. In fact, we expect that this superpolynomial growth also holds for generic initial data, and that this growth is indeed exponential, making estimate (3) optimal. An explicit example of such a behaviour was recently displayed by H. Xu in [30], for the special perturbation

$$i\partial_t u = \Pi(|u|^2 u) + \alpha \left(\int_{\mathbb{T}} u \right) \ , \ \alpha > 0 \ .$$

- A very natural open question is of course whether Theorem 1 holds for the half-wave equation (1) for $\alpha = 1$.

The proof of Theorem 1 relies on explicit formulae for solutions of Equation (2), which are non trivial consequences of the Lax pair structure and will be presented without proof in section 2. Then we sketch the proof of part 1) of the theorem in section 3, and finally give a complete proof of part 2) in section 4.

2. Explicit formulae

In this section, we list the main formulae solving the Cauchy problem for the cubic Szegő equation (2). The detailed proofs will appear in a forthcoming paper.

First we introduce some additional notation. Given a positive integer n, we set

$$\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n \ .$$

Given a nonnegative integer $d \geq 0$, we recall that a Blaschke product of degree d is a rational function on $\mathbb{C}$ of the form

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{d} \frac{z - p_j}{1 - \overline{p}_j z} \ , \ \psi \in \mathbb{T} \ , \ p_j \in \mathbb{D} \ .$$

Alternatively, Ψ can be written as

$$\Psi(z) = \mathrm{e}^{-i\psi} \frac{P(z)}{z^d \overline{P}\left(\frac{1}{z}\right)} \ ,$$

where $\psi \in \mathbb{T}$ is called the angle of Ψ and P is a monic polynomial of degree d with all its roots in $\mathbb{D}$. Such polynomials are called Schur polynomials. We denote by $\mathcal{B}_d$ the set of Blaschke products of degree d. It is a classical result — see *e.g.* [18] — that $\mathcal{B}_d$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2d}$.

Given a n-tuple $(d_1, \dots, d_n)$ of nonnegative integers, we set

$$\mathcal{S}_{d_1,\dots,d_n} := \Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r} \ ,$$

endowed with the natural product topology. Given a sequence $(d_r)_{r\geq 1}$ of nonnegative integers, we denote by $\mathcal{S}^{(2)}_{(d_r)}$ the set of pairs

$$((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \in \mathbb{R}^\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}$$

such that

$$s_1 > \cdots > s_n > \cdots > 0 \ , \ \sum_{r=1}^{\infty}(d_r+1)s_r^2 < \infty \ .$$

We also endow $\mathcal{S}^{(2)}_{(d_r)}$ with the natural topology.

Finally, we denote by $\mathcal{S}_n$ the union of all $\mathcal{S}_{d_1,\dots,d_n}$ over all the n-tuples $(d_1,\dots,d_n)$, and by $\mathcal{S}^{(2)}_\infty$ the union of all $\mathcal{S}^{(2)}_{(d_r)}$ over all the sequences $(d_r)_{r\geq 1}$. Given $(\mathbf{s},\mathbf{\Psi})\in\mathcal{S}_n$ and $z\in\mathbb{C}$, we define the matrix $\mathcal{C}(z):=\mathcal{C}(\mathbf{s},\mathbf{\Psi})(z)$ as follows. If $n=2q$, the coefficients of $\mathcal{C}(\mathbf{s},\mathbf{\Psi})(z)$ are given by

$$c_{jk}(z) := \frac{s_{2j-1}-s_{2k}z\Psi_{2k}(z)\Psi_{2j-1}(z)}{s_{2j-1}^2-s_{2k}^2} \ , \ j,k=1,\dots,q \ . \tag{4}$$

If $n=2q-1$, we use the same formula as above, with $s_{2q}=0$.

Theorem 2. *For every $n\geq 1$, for every $(\mathbf{s},\mathbf{\Psi})\in\mathcal{S}_n$, for every $z\in\overline{\mathbb{D}}$, the matrix $\mathcal{C}(\mathbf{s},\mathbf{\Psi})(z)$ is invertible. We set*

$$u(\mathbf{s},\mathbf{\Psi})(z) = \langle \mathcal{C}(z)^{-1}(\Psi_{2j-1}(z))_{1\leq j\leq q}, \mathbf{1}\rangle, \tag{5}$$

where

$$\mathbf{1} := \begin{pmatrix}1\\ .\\ .\\ .\\ 1\end{pmatrix} \ , \ \textit{and } \langle X,Y\rangle := \sum_{k=1}^{q} X_kY_k.$$

For every $(\mathbf{s},\mathbf{\Psi})\in\mathcal{S}^{(2)}_\infty$, the sequence $(u_q)_{q\geq 1}$ with

$$u_q := u((s_1,\dots,s_{2q}),(\Psi_1,\dots,\Psi_{2q})),$$

is strongly convergent in $H^{1/2}_+(\mathbb{S}^1)$. We denote its limit by $u(\mathbf{s},\mathbf{\Psi})$.

The mapping

$$(\mathbf{s},\mathbf{\Psi})\in\bigcup_{n=1}^{\infty}\mathcal{S}_n\cup\mathcal{S}^{(2)}_\infty \longmapsto u(\mathbf{s},\mathbf{\Psi})\in H^{1/2}_+\setminus\{0\}$$

is bijective. Furthermore, its restriction to every $\mathcal{S}_{(d_1,\dots,d_n)}$ and to $\mathcal{S}^{(2)}_{(d_r)}$ is a homeomorphism onto its range.

Finally, the solution at time t of equation (2) with initial data $u_0=u(\mathbf{s},\mathbf{\Psi})$ is $u(\mathbf{s},\mathbf{\Psi}(t))$, where

$$\Psi_r(t) = \mathrm{e}^{i(-1)^r s_r^2 t}\,\Psi_r \ .$$

Let us briefly explain how the above nonlinear Fourier transform is related to spectral analysis. If $u \in H^{1/2}_+(\mathbb{S}^1)$, recall (see [23], [24]) that the Hankel operator of symbol u is the operator $H_u : L^2_+(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$ defined by

$$H_u(h) = \Pi(u\overline{h}) \ .$$

It can be shown that H_u^2 is a positive selfadjoint trace class operator. If S is the shift operator defined by

$$Sh(z) = zh(z) \ ,$$

H_u satisfies

$$S^* H_u = H_u S = H_{S^* u} \ .$$

We denote by K_u this new Hankel operator. Let us say that a positive real number s is a singular value associated to u if s^2 is an eigenvalue of H_u^2 or K_u^2. The main point in Theorem 2 is that the list $s_1 > \cdots > s_r > \dots$ is the list of singular values associated to $u = u(\mathbf{s}, \mathbf{\Psi})$, and that the corresponding list $\Psi_1, \dots, \Psi_r, \dots$ describes the action of H_u and of K_u on the eigenspaces of H_u^2, K_u^2 respectively, making more precise a theorem of Adamyan-Arov-Krein about the structure of Schmidt pairs of Hankel operators [1]. In fact, denoting by u_j the orthogonal projection of u onto $\ker(H_u^2 - s_{2j-1}^2 I)$ and by u'_k the orthogonal projection of u onto $\ker(K_u^2 - s_{2k}^2 I)$, one can show that $u_j \neq 0$, $u'_k \neq 0$, and

$$\begin{aligned} \dim\ker(H_u^2 - s_{2j-1}^2 I) &= \deg \Psi_{2j-1} + 1, \quad \Psi_{2j-1}(z) H_u(u_j)(z) = s_{2j-1} u_j(z), \\ \dim\ker(K_u^2 - s_{2k}^2 I) &= \deg \Psi_{2k} + 1, \qquad K_u(u'_k)(z) = s_{2k} \Psi_{2k}(z) u'_k(z) \ . \end{aligned}$$

Remark 1. These formulae have an interesting consequence in the special case where the Blaschke products Ψ_r are all equal to 1. Indeed, from formula (5), the Fourier coefficients of u are real, so that H_u and K_u act as self adjoint operators on the real subspace of L^2_+ corresponding to real Fourier coefficients. Furthermore, from the above formulae, these operators are positive.

As a consequence of Theorem 2, we get inverse spectral theorems on Hankel operators, which generalize to singular values with arbitrary multiplicity the ones we had proved in [8] and [9] for simple singular values. Finally, the last assertion of Theorem 2 is a consequence of the following Lax pair identities,

$$\begin{aligned} \frac{dH_u}{dt} &= [B_u, H_u] \ , \ B_u(h) := \frac{i}{2} H_u^2 h - i\Pi(|u|^2 h) \ , \\ \frac{dK_u}{dt} &= [C_u, K_u] \ , \ C_u(h) := \frac{i}{2} K_u^2 h - i\Pi(|u|^2 h) \ . \end{aligned}$$

3. Large time instability

Using Theorem 2, the proof of part 1) of Theorem 1 heavily relies on a new phenomenon, which is the loss of continuity of the map $(\mathbf{s}, \mathbf{\Psi}) \mapsto u(\mathbf{s}, \mathbf{\Psi})$ as three consecutive s_r's are collapsing. More precisely, an inspection of formula (5) yields the following elementary lemma.

Lemma 1. *Given $(s_1, \dots, s_n, \delta) \in \Omega_{n+1}$, $(\Psi_1, \dots, \Psi_n) \in \mathcal{B}_{d_1} \times \dots \times \mathcal{B}_{d_n}$, and $\varepsilon > 0$ small enough, the family*

$$u^\varepsilon := u((s_1, \dots, s_n, \delta + \varepsilon, \delta, \delta - \varepsilon), (\Psi_1, \dots, \Psi_n, \mathrm{e}^{i\varphi_+}, \mathrm{e}^{i\theta}, \mathrm{e}^{i\varphi_-}))$$

converge as ε tends to 0, in every H^s space, to

$$u = u((s_1, \dots, s_n, \delta), (\Psi_1, \dots, \Psi_n, \Psi))$$

for some $\Psi \in \mathcal{B}_1$, except if $\varphi_+ = \varphi_- \mod 2\pi$. If $\varphi_+ = \varphi_- \mod 2\pi$, then u^ε is unbounded in H^s for every $s > \frac{1}{2}$.

The mechanism of unboundedness of H^s norms for $s > \frac{1}{2}$ in the case $\varphi_+ = \varphi_- \mod 2\pi$ is the convergence, as ε tends to 0, of a pole of the rational function u^ε to the unit circle. Using this lemma, we get the following long time instability result.

Proposition 1. *For every $u_0 \in C^\infty_+(\mathbb{T})$, there exists a sequence (u_0^n) converging to u_0 in $C^\infty_+(\mathbb{T})$ and a sequence (t_n) of times such that the solution u^n of (2) with $u^n(0) = u_0^n$ satisfies*

$$\forall s > \frac{1}{2} \ , \ \|u^n(t_n)\|_{H^s} \underset{n\to\infty}{\longrightarrow} \infty \ .$$

Proposition 1 is a generalization to every smooth datum u_0 of an instability phenomenon already displayed in [7], section 6, corollary 5, and revisited in [10], section 4, in the special case

$$u_0(z) = z \ .$$

This instability phenomenon was also discovered in [5] for the cubic NLS equation on $\mathbb{T}^2$, and made more precise, with polynomial estimates, in [14]. Indeed, we observed that, for every $\varepsilon > 0$, the solution u^ε of the cubic Szegő equation with the initial datum $u_0^\varepsilon(z) = z + \varepsilon$ satisfies, at time $t^\varepsilon \sim \frac{\pi}{2\varepsilon}$,

$$\forall s > \frac{1}{2} \ , \ \|u^\varepsilon(t^\varepsilon)\|_{H^s} \simeq (t^\varepsilon)^{2s-1} \ .$$

This divergence is due to the existence of a pole of the rational function $z \mapsto u^\varepsilon(t^\varepsilon, z)$ at distance $d \simeq \varepsilon^2$ of the unit circle.
Let us sketch the proof of Proposition 1, which combines Lemma 1 with the last part of Theorem 2. By Theorem 2 and a standard density argument, we may assume that u_0 has the form

$$u_0 = u((s_1, \dots, s_n), (\Psi_1, \dots, \Psi_n))$$

for some $((s_1,\dots,s_n),(\Psi_1,\dots,\Psi_n))\in\mathcal{S}_n$. Given $\delta>0$, Lemma 1 implies that

$$u_0^{\delta,\varepsilon}:=u((s_1,\dots,s_n,\delta+\varepsilon,\delta,\delta-\varepsilon),(\Psi_1,\dots,\Psi_n,1,1,-1))$$

converges in every H^s, as ε tends to 0, to $u((s_1,\dots,s_n,\delta),(\Psi_1,\dots,\Psi_n,\Psi))$ for some $\Psi\in\mathcal{B}_1$. Furthermore, it is easy to check that

$$u((s_1,\dots,s_n,\delta),(\Psi_1,\dots,\Psi_n,\Psi))\longrightarrow u_0$$

in every H^s as δ tends to 0. On the other hand, applying the last assertion of Theorem 2, the solution $u^{\delta,\varepsilon}$ with the initial datum $u_0^{\delta,\varepsilon}$ is given at time t by

$$u^{\delta,\varepsilon}(t)=u(((s_r)_{r\le n},\delta+\varepsilon,\delta,\delta-\varepsilon),((\mathrm{e}^{i(-1)^r ts_r^2}\Psi_r)_{r\le n},\mathrm{e}^{i\varphi_+(t)},\mathrm{e}^{i\theta(t)},\mathrm{e}^{i\varphi_-(t)}))\ ,$$

with

$$\varphi_+(t)=(-1)^{n+1}t(\delta+\varepsilon)^2\ ,\ \theta(t)=(-1)^n t\delta^2\ ,\ \varphi_-(t)=\pi+(-1)^{n+1}t(\delta-\varepsilon)^2\ .$$

Choosing

$$t^{\delta,\varepsilon}=\frac{\pi}{4\delta\varepsilon}\ ,$$

we observe that the angles $\varphi_+(t^{\delta,\varepsilon})$ and $\varphi_-(t^{\delta,\varepsilon})$ are the same modulo 2π, therefore, by Lemma 1, $u^{\delta,\varepsilon}(t^{\delta,\varepsilon})$ is unbounded in H^s for every $s>\frac12$ as ε tends to 0. The proof of Proposition 1 then follows by choosing

$$u_0^n:=u_0^{\delta_n,\varepsilon_n}\ ,\ t_n:=t^{\delta_n,\varepsilon_n}\ ,$$

with appropriate sequences $(\delta_n),(\varepsilon_n)$ satisfying $\varepsilon_n\ll\delta_n\ll 1$.

The first assertion of Theorem 1 is a consequence of Proposition 1 and of a Baire category argument. This argument comes back to Hani [16], who used it first for finding solutions with unbounded Sobolev norms for the totally resonant form of cubic NLS on $\mathbb{T}^2$, as investigated by [5] and [14]. Notice that the previously quoted result in [17] relies on [5] and [14], and on a remarkable modified scattering argument.

4. The failure of polynomial estimates for high Sobolev norms

In this section, we prove the super polynomial instability of the Sobolev norms for families of solutions of the Szegő cubic equation stated in the introduction. We recall this statement.

Theorem 3. *For every positive integer M, every $s>\frac12$, there exists a bounded subset B of $\mathcal{C}_+^\infty(\mathbb{S}^1)$ such that*

$$\sup_{u_0\in B}\frac{\|u(t)\|_{H^s}}{(1+|t|)^M}\to\infty\ \textit{as}\ |t|\to\infty$$

where $u(t)$ is the solution of (2) with $u(0)=u_0$.

Proof. This result is a consequence of the following proposition.

Proposition 2. *For every $N \geq 2$, every $\varepsilon > 0$, every $\xi := (\xi_j)_{1\leq j\leq N} \in \mathbb{R}^N$, $\eta := (\eta_k)_{1\leq k\leq N-1} \in \mathbb{R}^{N-1}$, such that*

$$\xi_1 > \eta_1 > \xi_2 > \eta_2 > \dots \eta_{N-1} > \xi_N,$$

we consider

$$u_\varepsilon(z) = \langle \mathcal{C}_\varepsilon^{-1}(z)(\mathbf{1}), \mathbf{1}\rangle$$

where $\mathcal{C}_\varepsilon(z) = (c_{\varepsilon,jk}(z))_{1\leq j,k\leq N}$, with

$$c_{\varepsilon,jk}(z) := \frac{1+\varepsilon\xi_j - z(1+\varepsilon\eta_k)}{(1+\varepsilon\xi_j)^2 - (1+\varepsilon\eta_k)^2} \ , \ 1 \leq k \leq N-1 \ ;$$

$$c_{\varepsilon,jN}(z) := \frac{1}{1+\varepsilon\xi_j} \ , \ 1 \leq j \leq N \ .$$

Then, there exists (ξ, η) such that

(1) $\|u_\varepsilon\|_{H^s} \geq \dfrac{C}{\varepsilon^{(N-1)(2s-1)}}$;

(2) *The family $\left(u_\varepsilon\left(\frac{1}{2\varepsilon}, \cdot\right)\right)$ is bounded in $\mathcal{C}_+^\infty(\mathbb{S}^1)$ as $\varepsilon \to 0$. Here $(u_\varepsilon(t), \cdot)$ denotes the solution of (2) with $u_\varepsilon(0) = u_\varepsilon$.*

Theorem 3 is a direct consequence of this proposition by considering $M = [(N-1)(2s-1)]$ and the family of initial data

$$B := \left\{ u_\varepsilon\left(\frac{1}{2\varepsilon}, \cdot\right) \ , \ \varepsilon > 0 \right\}. \qquad \square$$

The proof of proposition 2 requires several step. The first step consists in establishing that $\|u_\varepsilon\|_{H^s} \geq \dfrac{C}{\varepsilon^{(N-1)(2s-1)}}$ for a dense open set of choices of (ξ, η). This is based on the following lemma.

Lemma 2. *For a dense open set of (ξ, η), we have $(u_\varepsilon)'(1) \geq \dfrac{C}{\varepsilon^{2(N-1)}}$. Moreover, the poles of u_ε are simple,*

$$u_\varepsilon(z) = \alpha_0^\varepsilon + \sum_{j=1}^{N-1} \frac{\alpha_j^\varepsilon}{1 - p_j^\varepsilon z} \tag{6}$$

with $\alpha_0^\varepsilon, \alpha_j^\varepsilon > 0$, $0 < p_j^\varepsilon < 1$, and $p_j^\varepsilon \to 1$ as $\varepsilon \to 0$. Furthermore, $\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)$ is equivalent to $\varepsilon^{2(N-1)}$.

Let us first prove that the lemma implies that $\|u_\varepsilon\|_{H^s} \geq \dfrac{C}{\varepsilon^{(N-1)(2s-1)}}$. First, from (6) and the estimate on $(u_\varepsilon)'(1)$,

$$(u_\varepsilon)'(1) = \sum_{j=1}^{N-1} \frac{\alpha_j^\varepsilon p_j^\varepsilon}{(1-p_j^\varepsilon)^2} \geq \frac{C}{\varepsilon^{2(N-1)}}.$$

Hence, applying Cauchy-Schwarz, one gets

$$\begin{aligned}\frac{C}{\varepsilon^{2(N-1)}} &\le \sum_{j=1}^{N-1}\frac{\alpha_j^\varepsilon p_j^\varepsilon}{(1-p_j^\varepsilon)^2}\\ &\le \left(\sum_{j=1}^{N-1}\frac{(\alpha_j^\varepsilon)^2}{(1-p_j^\varepsilon)^{2s+1}}\right)^{1/2}\times\left(\sum_{j=1}^{N-1}\frac{(p_j^\varepsilon)^2}{(1-p_j^\varepsilon)^{3-2s}}\right)^{1/2}\\ &\le \left(\sum_{j=1}^{N-1}\frac{(\alpha_j^\varepsilon)^2}{(1-p_j^\varepsilon)^{2s+1}}\right)^{1/2}\times\frac{C}{\varepsilon^{(3-2s)(N-1)}}.\end{aligned}$$

Eventually,

$$\left(\sum_{j=1}^{N-1}\frac{(\alpha_j^\varepsilon)^2}{(1-p_j^\varepsilon)^{2s+1}}\right)^{1/2}\ge\frac{C}{\varepsilon^{(2s-1)(N-1)}}.$$

It remains to observe that

$$\|u_\varepsilon\|_{H^s}^2\ge C\sum_{n\ge 0}n^{2s}\left(\sum_{j=1}^{N-1}\alpha_j^\varepsilon(p_j^\varepsilon)^n\right)^2\ge C\sum_{j=1}^{N-1}\frac{(\alpha_j^\varepsilon)^2}{(1-p_j^\varepsilon)^{2s+1}}.$$

Proof. Let us turn to the proof of the lemma. We first estimate the derivative of u_ε at $z=1$. From the explicit formula of u_ε, we obtain

$$(u_\varepsilon)'(1)=\langle\mathcal{C}_\varepsilon(1)^{-1}\dot{\mathcal{C}}_\varepsilon\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1}),\mathbf{1}\rangle=\langle\dot{\mathcal{C}}_\varepsilon\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1}),{}^t\,\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\rangle \tag{7}$$

where $\dot{\mathcal{C}}_\varepsilon=(\dot c_{\varepsilon,jk})_{1\le j,k\le N}$, with

$$\dot c_{\varepsilon,jk}:=\frac{1+\varepsilon\eta_k}{(1+\varepsilon\xi_j)^2-(1+\varepsilon\eta_k)^2}\ ,\ 1\le k\le N-1\ ,\ \dot c_{\varepsilon,jN}:=0\ ,\ 1\le j\le N\ .$$

From the formula giving $\mathcal{C}_\varepsilon$, we have

$$\mathcal{C}_\varepsilon(1)=\left(\left(\frac{1}{2+\varepsilon(\xi_j+\eta_k)}\right)_{1\le j\le N;1\le k\le N-1},\left(\frac{1}{1+\varepsilon\xi_j}\right)_{1\le j\le N}\right)$$

which is a Cauchy matrix. Let us recall that a Cauchy matrix is a matrix of the form $\left(\frac{1}{a_j+b_k}\right)$. Its determinant is given by

$$\det\left(\frac{1}{a_j+b_k}\right)=\frac{\prod_{i<j}(a_i-a_j)\prod_{k<l}(b_k-b_l)}{\prod_{j,k}(a_j+b_k)}. \tag{8}$$

It allows to obtain

$$\left(\left(\frac{1}{a_j+b_k}\right)^{-1}\right)_{kj}=\left((-1)^{j+k}\frac{\lambda_j\mu_k}{a_j+b_k}\right)_{kj}$$

with

$$\lambda_j = \frac{\prod_l (a_j + b_l)}{\prod_{i<j}(a_i - a_j)\prod_{r>j}(a_j - a_r)}$$

and

$$\mu_k = \frac{\prod_l (a_l + b_k)}{\prod_{i<k}(b_i - b_k)\prod_{r>k}(b_k - b_r)}.$$

In the case of the matrix $\mathcal{C}_\varepsilon(\mathbf{1})$, we have

$$\lambda_j = \frac{2^{N-1}}{\varepsilon^{N-1}} \frac{(1+\varepsilon\xi_j)\prod_l (1+\varepsilon(\xi_j+\eta_l)/2)}{\xi'_j} \ , \ 1 \le j \le N \ ,$$

and

$$\mu_k = \frac{2^N}{\varepsilon^{N-2}} \frac{\prod_l (1+\varepsilon(\xi_l+\eta_k)/2)}{\eta'_k(1+\varepsilon\eta_k)}, \ k \le N-1, \ \mu_N = \frac{\prod_i (1+\varepsilon\xi_i)}{\prod_k (1+\varepsilon\eta_k)}$$

where we have set

$$\xi'_j := \prod_{i<j}(\xi_i - \xi_j)\prod_{r>j}(\xi_j - \xi_r), \ \eta'_k := \prod_{i<k}(\eta_i - \eta_k)\prod_{r>k}(\eta_k - \eta_r).$$

Eventually, if $k \le N-1$,

$$\begin{aligned}\Big((\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\Big)_k &= (-1)^k \mu_k \sum_{j=1}^N \frac{(-1)^j \lambda_j}{2+\varepsilon(\xi_j+\eta_k)} \\ &= (-1)^k \mu_k \frac{2^{N-2}}{\varepsilon^{N-1}} \sum_{j=1}^N (-1)^j \frac{(1+\varepsilon\xi_j)}{\xi'_j} \prod_{l\neq k}\left(1+\varepsilon\frac{\xi_j+\eta_l}{2}\right);\end{aligned}$$

and

$$\begin{aligned}\Big((\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\Big)_N &= (-1)^N \mu_N \sum_{j=1}^N \frac{(-1)^j \lambda_j}{1+\varepsilon\xi_j} \\ &= (-1)^N \mu_N \frac{2^{N-1}}{\varepsilon^{N-1}} \sum_{j=1}^N \frac{(-1)^j}{\xi'_j} \prod_l \left(1+\varepsilon\frac{\xi_j+\eta_l}{2}\right).\end{aligned}$$

In order to simplify these quantities, we use the following identities.

Lemma 3. *For any* $0 \le p \le N-1$,

$$\sum_{j=1}^N (-1)^j \frac{\xi_j^p}{\xi'_j} = \begin{cases} 0 & \textit{if } p < N-1 \\ -1 & \textit{if } p = N-1. \end{cases}$$

Proof. We view $\sum_{j=1}^N (-1)^j \frac{\xi_j^p}{\xi_j'}$ as a rational function of ξ_N denoted by $Q_p(\xi_N)$. Its poles are simple and equal to $\xi_1, \dots, \xi_{N-1}$. Identifying the residue at each of these poles, we get

$$\mathrm{Res}(Q_p; \xi_N = \xi_r) = \frac{(-1)^{r+1}\xi_r^p}{\prod_{i<r}(\xi_i - \xi_r)\prod_{N-1\geq j>r}(\xi_r - \xi_j)} + \frac{(-1)^{N+1}\xi_r^p}{\prod_{i<N, i\neq r}(\xi_i - \xi_r)} = 0\,.$$

Hence $Q_p(\xi_N)$ is in fact a polynomial. If $p < N-1$, it tends to 0 as ξ_N tends to ∞, therefore it is identically 0. If $p = N-1$, it is a constant, equal to its limit at infinity,

$$Q_p(\xi_N) = \frac{(-1)^N \xi_N^{N-1}}{(-1)^{N-1}\xi_N^{N-1}} = -1\ .$$

This completes the proof. ☐

Expanding in powers of ε the above formula giving $\left(\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\right)_k$, and using Lemma 3, we infer

$$\left(\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\right)_k = (-1)^{k+1}\mu_k\ ,\ \ k \leq N\ . \tag{9}$$

Let us compute ${}^t\mathcal{C}_\varepsilon^{-1}(1)(\mathbf{1})$ in a similar way. From the formula, we have

$$\begin{aligned}\left({}^t\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\right)_j &= (-1)^j\lambda_j\left(\sum_{k=1}^{N-1}\frac{(-1)^k\mu_k}{2+\varepsilon(\xi_j+\eta_k)} + \frac{(-1)^N\mu_N}{1+\varepsilon\xi_j}\right)\\ &=: (-1)^j\lambda_j\tilde{S}_j(\varepsilon).\end{aligned}$$

We expand in powers of ε and use again Lemma 3, changing N into $N-1$. We get

$$\begin{aligned}\tilde{S}_j(\varepsilon) &= \frac{2^{N-1}}{\varepsilon^{N-2}}\sum_{k=1}^{N-1}\frac{(-1)^k\prod_{i\neq j}\left(1+\varepsilon\frac{\xi_i+\eta_k}{2}\right)}{\eta_k'(1+\varepsilon\eta_k)} + (-1)^N + O(\varepsilon)\\ &= (-1)^N + O(\varepsilon)\\ &\quad + \frac{2^{N-1}}{\varepsilon^{N-2}}\sum_{k=1}^{N-1}(-1)^k\left(\varepsilon^{N-2}\sum_{p+q=N-2}(-1)^q C_{N-1}^p\frac{\eta_k^{p+q}}{2^p\eta_k'} + O(\varepsilon^{N-1})\right)\\ &= -1 + O(\varepsilon).\end{aligned}$$

As a consequence, we infer

$$\left({}^t\mathcal{C}_\varepsilon(1)^{-1}(\mathbf{1})\right)_j = (-1)^{j+1}\lambda_j(1+O(\varepsilon))\ . \tag{10}$$

Eventually, inserting (9) and (10) into the formula (7) of $(u_\varepsilon)'(1)$, it gives

$$\begin{aligned}(u_\varepsilon)'(1) &= \sum_{1\le j\le N}\sum_{1\le k\le N-1}\frac{1+\varepsilon\eta_k}{(1+\varepsilon\xi_j)^2-(1+\varepsilon\eta_k)^2}(-1)^{j+k}\lambda_j\mu_k(1+O(\varepsilon))\\ &= \frac{2^{2(N-1)}}{\varepsilon^{2(N-1)}}\Bigg(\sum_{1\le j\le N}\sum_{1\le k\le N-1}\frac{(-1)^{j+k}}{(\xi_j-\eta_k)\xi_j'\eta_k'}+O(\varepsilon)\Bigg).\end{aligned}$$

Considering

$$\sum_{1\le j,k\le N-1}\frac{(-1)^{j+k}}{(\xi_j-\eta_k)\xi_j'\eta_k'}$$

as a function of ξ_N, the pole η_{N-1} appears only once. Hence, this meromorphic function does not vanish on an open dense set of (ξ,η). This proves that

$$|(u_\varepsilon)'(1)|\ge\frac{C}{\varepsilon^{2(N-1)}}$$

for an open dense set of choices of (ξ,η).

Next we prove the second statement of Lemma 2. Thanks to remark 1, operators H_{u_ε} and K_{u_ε} act as positive self adjoint operators on the subspace of L^2_+ corresponding to real Fourier coefficients. As a consequence — see e.g. [12], Prop. 2.1 —, there exists a bounded positive measure μ on $[0,1[$ such that

$$\forall k\ge 0\ ,\ \hat u_\varepsilon(k)=\int_{[0,1[}t^k\,d\mu(t)\ .$$

On the other hand, H_{u_ε} has exactly N positive singular values, hence is an operator of finite rank N. By the identity $S^*H_{u_\varepsilon}=H_{u_\varepsilon}S$, S^* acts on the range of H_{u_ε}, therefore there exits a non trivial linear relation between the vectors $(S^*)^pu_\varepsilon\ ,\ p=0,\dots,N$. Equivalently, there exists non trivial coefficients c_p such that

$$\forall k\ge 0\ ,\ \sum_{p=0}^N c_p\hat u(k+p)=0\ .$$

This precisely means that there exists a non trivial polynomial P of degree N such that

$$\forall k\ge 0\ ,\ \int_{[0,1[}t^kP(t)\,d\mu(t)=0\ .$$

By the Weierstrass theorem, μ is therefore a positive linear combination of Dirac measures. This implies that the poles of u_ε are simple and that

$$u_\varepsilon(z)=\alpha_0^\varepsilon+\sum_{j=1}^{N-1}\frac{\alpha_j^\varepsilon}{1-p_j^\varepsilon z}$$

with $\alpha_j^\varepsilon > 0$, $0 < p_j^\varepsilon < 1$. In particular $(u_\varepsilon)'(1) > 0$ so that

$$(u_\varepsilon)'(1) \geq \frac{C}{\varepsilon^{2(N-1)}}.$$

Let us show that $p_j^\varepsilon \to 1$ as $\varepsilon \to 0$. We know that the denominator of $u_\varepsilon(z)$ is

$$P_\varepsilon(z) := \frac{\det \mathcal{C}_\varepsilon(z)}{\det \mathcal{C}_\varepsilon(0)} = \prod_{j=1}^{N-1}(1 - p_j^\varepsilon z),$$

therefore it suffices to prove that $P_\varepsilon(z) \simeq (1-z)^{N-1}$ as ε tends to 0. From the formula of $\mathcal{C}_\varepsilon(z)$,

$$(2\varepsilon)^{N-1} \det \mathcal{C}_\varepsilon(z) \to (1-z)^{N-1} \det\left(\left(\frac{1}{\xi_j - \eta_k}\right)_{1\leq j\leq N; 1\leq k\leq N-1}, \mathbf{1}\right)$$

and

$$(2\varepsilon)^{N-1} \det \mathcal{C}_\varepsilon(0) \to \det\left(\left(\frac{1}{\xi_j - \eta_k}\right)_{1\leq j\leq N; 1\leq k\leq N-1}, \mathbf{1}\right),$$

so the claim is proved.

Finally, we establish that

$$\min_{1\leq j\leq N-1}(1 - p_j^\varepsilon) \simeq \varepsilon^{2(N-1)}.$$

Let us first prove that $\min_{1\leq j\leq N-1}(1-p_j^\varepsilon) \leq C\varepsilon^{2(N-1)}$. From the estimate of $(u_\varepsilon)'(1)$,

$$\begin{aligned}
\frac{C}{\varepsilon^{2N-1}} \leq (u_\varepsilon)'(1) &= \sum \frac{\alpha_j^\varepsilon p_j^\varepsilon}{(1-p_j^\varepsilon)^2} \\
&\leq \frac{1}{\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)} \sum \frac{\alpha_j^\varepsilon p_j^\varepsilon}{(1-p_j^\varepsilon)} \\
&\leq \frac{u_\varepsilon(1)}{\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)} \leq \frac{\operatorname{tr}(H_{u_\varepsilon}) + \operatorname{tr}(K_{u_\varepsilon})}{\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)} \\
&\leq \frac{2N - 1 + O(\varepsilon)}{\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)}.
\end{aligned}$$

It gives the bound from above for $\min_{1\leq j\leq N-1}(1-p_j^\varepsilon)$. For the bound from below, we use formula (8) to obtain

$$\begin{aligned}
\prod_j (1-p_j^\varepsilon) = \frac{\det \mathcal{C}_\varepsilon(1)}{\det \mathcal{C}_\varepsilon(0)} &\simeq \varepsilon^{N-1} \det \mathcal{C}_\varepsilon(1) \\
&\simeq \varepsilon^{N-1} \det\left(\left(\frac{1}{2+\varepsilon(\xi_j+\eta_k)}\right)_{1\leq j\leq N; 1\leq k\leq N-1}, \left(\frac{1}{1+\varepsilon\xi_j}\right)_{1\leq j\leq N}\right) \\
&\simeq \varepsilon^{N(N-1)}.
\end{aligned}$$

On the other hand, as

$$-P'_\varepsilon(1)=\sum_j p_j^\varepsilon\prod_{i\neq j}(1-p_i^\varepsilon)\simeq\frac{\prod(1-p_i^\varepsilon)}{\min_{1\le j\le N}(1-p_j^\varepsilon)}\simeq\frac{\varepsilon^{N(N-1)}}{\min_{1\le j\le N}(1-p_j^\varepsilon)},$$

it suffices to bound $-P'_\varepsilon(1)$ from above. Using formula (8) again,

$$\begin{aligned}-\frac{d}{dz}(\det\mathcal{C}_\varepsilon(z))|_{z=1}=\sum_{k=1}^{N-1}\sum_{l=1}^{N}&\frac{(-1)^{k+l}(1+\varepsilon\eta_k)}{(1+\varepsilon\xi_j)^2-(1+\varepsilon\eta_k)^2}\\&\cdot\det\left(\left(\frac{1}{2+\varepsilon(\xi_j+\eta_m)}\right)_{j\neq l,m\neq k},\left(\frac{1}{1+\varepsilon\xi_j}\right)_{j\neq l}\right)\\&\le\varepsilon^{(N-2)^2-1}\end{aligned}$$

Eventually,

$$-P'_\varepsilon(1)\simeq-\varepsilon^{N-1}\frac{d}{dz}(\det\mathcal{C}_\varepsilon(z))|_{z=1}\le C\varepsilon^{(N-1)(N-2)}$$

so that

$$\min_{1\le j\le N-1}(1-p_j^\varepsilon)\ge C\varepsilon^{2(N-1)}.$$

This completes the proof of part 1 of Proposition 2. $\square$

We are left with proving the second part of Proposition 2. From Theorem 2, at time $t=\frac{1}{2\varepsilon}$

$$u_\varepsilon\left(\left(\frac{1}{2\varepsilon}\right),z\right)=\langle\tilde{\mathcal{C}}_\varepsilon(z)^{-1}(\Psi^\varepsilon_{\text{odd}}),\mathbf{1}\rangle$$

where $\Psi^\varepsilon_{\text{odd}}:=(\mathrm{e}^{-i\psi_j^\varepsilon(1/2\varepsilon)})_{1\le j\le N}$ with $\psi_j^\varepsilon(t):=(1+\varepsilon\xi_j)^2t$. Hence

$$\Psi^\varepsilon_{\text{odd}}=\mathrm{e}^{-i/2\varepsilon}(1+O(\varepsilon))\big(\mathrm{e}^{-i\xi_j}\big)_{1\le j\le N},$$

and, for $1\le j\le N$,

$$\tilde{\mathcal{C}}_\varepsilon(z)_{jk}=\frac{1+\varepsilon\xi_j-z(1+\varepsilon\eta_k)\mathrm{e}^{-i(\xi_j-\eta_k)}(1+O(\varepsilon))}{(1+\varepsilon\xi_j)^2-(1+\varepsilon\eta_k)^2}\ ,\ 1\le k\le N-1,$$

$$\tilde{\mathcal{C}}_\varepsilon(z)_{jN}=\frac{1}{1+\varepsilon\xi_j}.$$

Let $\tilde{P}_\varepsilon(z)=\frac{\det\tilde{\mathcal{C}}_\varepsilon(z)}{\det\tilde{\mathcal{C}}_\varepsilon(0)}$. From Theorem 2, $\tilde{P}_\varepsilon(z)$ has $(N-1)$ roots outside the closed unit disc $\overline{\mathbb{D}}$. Our aim is to prove that, for a suitable choice of (ξ,η), the distance of these roots to the boundary of $\overline{\mathbb{D}}$ is bounded from below. Define

$$P(\xi,\eta)(z):=\lim_{\varepsilon\to0}\tilde{P}_\varepsilon(z)\ .$$

Observe that

$$\begin{aligned} P(\xi,\eta)(z) &= c(\xi,\eta)\lim_{\varepsilon\to 0}\varepsilon^{N-1}\det\tilde{\mathcal{C}}_\varepsilon(z)\\ &= \tilde c(\xi,\eta)\det\left(\left(\frac{1-z\mathrm{e}^{-i(\xi_j-\eta_k)}}{\xi_j-\eta_k}\right)_{1\le j\le N;1\le k\le N-1},\mathbf{1}\right). \end{aligned}$$

Notice that the determinant in the right hand side is a polynomial in z of degree $N-1$ whose coefficient of z^{N-1} equals

$$(-1)^{N-1}\left(\prod_k \mathrm{e}^{i\eta_k}\right)\det\left(\left(\frac{\mathrm{e}^{-i\xi_j}}{\xi_j-\eta_k}\right)_{1\le j\le N;1\le k\le N-1},\mathbf{1}\right).$$

With the choice $\xi=\xi^*=(2\pi(N-j+1))_{1\le j\le N}$, this determinant is

$$(-1)^{N-1}\left(\prod_k \mathrm{e}^{i\eta_k}\right)\det\left(\left(\frac{1}{\xi_j-\eta_k}\right)_{1\le j\le N;1\le k\le N-1},\mathbf{1}\right)\neq 0\ .$$

The non vanishing of this determinant follows from developing with respect to the last column, and in view of the explicit formula for the Cauchy determinants. By continuity, this determinant remains non zero in a neighborhood of $\xi=\xi^*$. Furthermore, Theorem 2 tells us that the roots of $\tilde P_\varepsilon(z)$ are located outside the unit disc. Hence all the roots of $P(\xi,\eta)$ belong to $\{z,\ |z|\ge 1\}$. Furthermore,

$$P(\xi^*,\eta)(z)=\tilde c(\xi^*,\eta)\det\left(\left(\frac{1}{\xi_j-\eta_k}\right)_{1\le j\le N;1\le k\le N-1},\mathbf{1}\right)\prod_{k=1}^{N-1}(1-z\mathrm{e}^{i\eta_k}).$$

Hence, for a suitable choice of η so that $\eta_k\neq\eta_l\ (2\pi)$ for $k\neq l$, $P(\xi^*,\eta)$ has only simple zeroes which belong to the unit circle.

Lemma 4. *For any η such that $P(\xi^*,\eta)$ has only simple zeroes on the unit circle, there exists an open neighborhood of ξ^* such that, for every $\xi\neq\xi^*$ in this neighborhood, the zeroes of $P(\xi,\eta)$ are all outside $\overline{\mathbb{D}}$.*

Proof. Denote by

$$\{z_k(\xi,\eta);\ 1\le k\le N-1\}$$

the simple zeroes of $P(\xi,\eta)$, with $z_k(\xi^*,\eta)=\mathrm{e}^{-i\eta_k}$. The functions $\xi\mapsto z_k(\xi,\eta)$ are analytic and satisfy $|z_k(\xi,\eta)|^2\ge 1$ and $|z_k(\xi^*,\eta)|^2=1$. In particular the quadratic form $\xi\mapsto Q_k(\xi)$ associated to the Hessian of the function $\xi\mapsto|z_k(\xi,\eta)|^2$ is positive at any ξ^*. We want to prove that, for any k, Q_k is not identically 0 for η in a dense open set. It suffices to prove that the Laplacian of $\xi\mapsto|z_k(\xi,\eta)|^2$, which coincides with the trace of Q_k, is not identically zero. Let us compute this Laplacian.

$$\sum_{j=1}^N\frac{\partial^2}{\partial\xi_j^2}\left(\frac{|z_k|^2}{2}\right)_{|\xi=\xi^*}=\sum_{j=1}^N\left(\mathrm{Re}\left(\overline{z}_k\frac{\partial^2 z_k}{\partial\xi_j^2}\right)_{|\xi=\xi^*}+\left|\frac{\partial z_k}{\partial\xi_j}\right|^2_{|\xi=\xi^*}\right)\ .$$

Differentiating the equation $P(\xi,\eta)(z_k(\xi)) = 0$, we obtain

$$\frac{\partial z_k}{\partial \xi_j} = -\frac{\frac{\partial P}{\partial \xi_j}}{\frac{\partial P}{\partial z}}$$

and
$$\frac{\partial^2 z_k}{\partial \xi_j^2} = \frac{-\frac{\partial^2 P}{\partial \xi_j^2}}{\frac{\partial P}{\partial z}} + 2\frac{\frac{\partial^2 P}{\partial \xi_j \partial z}\frac{\partial P}{\partial \xi_j}}{\left(\frac{\partial P}{\partial z}\right)^2} - \frac{\left(\frac{\partial P}{\partial \xi_j}\right)^2 \frac{\partial^2 P}{\partial z^2}}{\left(\frac{\partial P}{\partial z}\right)^3}$$

Introduce the following quantities.

$$D_{jk} := \frac{1}{\xi_j - \eta_k}\det\left(\left(\frac{1}{\xi_r - \eta_l}\right)_{r\neq j, l\neq k}, \mathbf{1}\right),$$
$$D := \sum_j (-1)^{j+k} D_{jk},\ \zeta_k := \prod_{l\neq k}(1 - \mathrm{e}^{i(\eta_l - \eta_k)})$$
$$a_{lk} := -\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1-\mathrm{e}^{i(\eta_l-\eta_k)}} = \frac{\mathrm{e}^{i(\eta_l-\eta_k)/2}}{2i\sin(\eta_l-\eta_k)/2}.$$

Notice that $\mathrm{Re}(a_{lk}) = \frac{1}{2}$. Differentiating

$$P(\xi,\eta)(z) = \det\left(\left(\frac{1 - z\mathrm{e}^{-i(\xi_j - \eta_k)}}{\xi_j - \eta_k}\right)_{1\leq j\leq N; 1\leq k\leq N-1}, \mathbf{1}\right),$$

one gets, after some computations,

$$\frac{\partial P}{\partial \xi_j}_{|\xi=\xi^*, z=z_k} = i(-1)^{j+k} D_{jk}\zeta_k,\ \frac{\partial P}{\partial z}_{|\xi=\xi^*, z=z_k} = -\mathrm{e}^{i\eta_k}\zeta_k D$$
$$\frac{\partial^2 P}{\partial \xi_j^2}_{|\xi=\xi^*, z=z_k} = (-1)^{j+k}\zeta_k D_{jk}\left(1 - \frac{2i}{\xi_j - \eta_k}\right)$$
$$\frac{\partial^2 P}{\partial z^2}_{|\xi=\xi^*, z=z_k} = 2\sum_{l\neq k}\frac{\zeta_k \mathrm{e}^{i(\eta_l+\eta_k)}}{1-\mathrm{e}^{i(\eta_l-\eta_k)}} D = -2\zeta_k \mathrm{e}^{2i\eta_k}\sum_{l\neq k} a_{lk} D\ ,$$

and

$$\begin{aligned}\frac{\partial^2 P}{\partial \xi_j \partial z}_{|\xi=\xi^*, z=z_k} &= (-1)^{j+k}\zeta_k D_{jk}\mathrm{e}^{i\eta_k}\left(i + \frac{1}{\xi_j - \eta_k} - i\sum_{l\neq k}\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1-\mathrm{e}^{i(\eta_l-\eta_k)}}\right) \\ &\quad + \zeta_k \mathrm{e}^{i\eta_k}\sum_{l\neq k}(-1)^{j+l} D_{jl}\left(\frac{1}{\xi_j - \eta_l} - i\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1-\mathrm{e}^{i(\eta_l-\eta_k)}}\right) \\ &= \zeta_k \mathrm{e}^{i\eta_k}\left((-1)^{j+k} D_{jk}\left(i + \frac{1}{\xi_j - \eta_k} + i\sum_{l\neq k} a_{lk}\right)\right. \\ &\quad \left. + \sum_{l\neq k}(-1)^{j+l} D_{jl}\left(\frac{1}{\xi_j - \eta_l} + i a_{lk}\right)\right)\ .\end{aligned}$$

Hence, inserting these formulae to compute the Laplacian, we obtain

$$\begin{aligned}\mathrm{Re}\left(\overline{z}_k\frac{\partial^2 z_k}{\partial\xi_j^2}\right)_{|\xi=\xi^*} &= \mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\partial^2 z_k}{\partial\xi_j^2}\right)_{|\xi=\xi^*}\\ &= I_j+II_j+III_j\end{aligned}$$

with

$$\begin{aligned}I_j &= \mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{-\frac{\partial^2 P}{\partial\xi_j^2}}{\frac{\partial P}{\partial z}}\right)=(-1)^{j+k}\frac{D_{jk}}{D}\ ,\\ II_j &= 2\mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\frac{\partial^2 P}{\partial\xi_j\partial z}\frac{\partial P}{\partial\xi_j}}{\left(\frac{\partial P}{\partial z}\right)^2}\right)\end{aligned}$$

$$= 2\mathrm{Re}\left(\frac{i\big(D_{jk}^2\big(i+\frac{1}{\xi_j-\eta_k}+i\sum_{l\neq k}a_{lk}\big)+\sum_{l\neq k}(-1)^{k+l}D_{jk}D_{jl}\big(\frac{1}{\xi_j-\eta_l}+ia_{lk}\big)\big)}{D^2}\right)$$

$$=\frac{D_{jk}^2\big(-2-(N-2)\big)-\sum_{l\neq k}(-1)^{k+l}D_{jk}D_{jl}}{D^2}\ ,$$

$$III_j=-\mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\left(\frac{\partial P}{\partial\xi_j}\right)^2\frac{\partial^2 P}{\partial z^2}}{\left(\frac{\partial P}{\partial z}\right)^3}\right)=\mathrm{Re}\left(\frac{2\sum_{l\neq k}a_{lk}D_{jk}^2}{D^2}\right)=(N-2)\frac{D_{jk}^2}{D^2}.$$

Remark that

$$\sum_{l\neq k}(-1)^{j+l}D_{jl}=D-(-1)^{j+N}\det\left(\frac{1}{\xi_r-\eta_l}\right)_{r\neq j}.$$

Hence, it gives

$$\begin{aligned}\sum_{j=1}^N\frac{\partial^2}{\partial\xi_j^2}\left(\frac{|z_k|^2}{2}\right)_{|\xi=\xi^*} &= \sum_{j=1}^N\left(\sum_l(-1)^{k+l+1}\frac{D_{jk}D_{jl}}{D^2}+(-1)^{j+k}\frac{D_{jk}}{D}\right)\\ &=(-1)^{k+N}\frac{1}{D^2}\sum_j D_{jk}\det\left(\frac{1}{\xi_r-\eta_l}\right)_{r\neq j}\end{aligned}$$

It remains to check that

$$(-1)^{k+N}\sum_j D_{jk}\det\left(\frac{1}{\xi_r-\eta_l}\right)_{r\neq j}$$

is not identically zero. This follows from the fact that the limit as η_k tends to infinity of

$$(-1)^{k+N}\eta_k^2\sum_j D_{jk}\det\left(\frac{1}{\xi_r-\eta_l}\right)_{r\neq j}$$

equals

$$\sum_j \det\left(\left(\frac{1}{\xi_r - \eta_l}\right)_{r\neq j, l\neq k}, \mathbf{1}\right)^2$$

which is clearly not identically zero for a dense choice of (η, ξ). □

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Université Paris-Sud XI, Laboratoire de Mathématiques d'Orsay, CNRS, UMR 8628, et Institut Universitaire de France
E-mail address: `Patrick.Gerard@math.u-psud.fr`

Fédération Denis Poisson, MAPMO-UMR 6628, Département de Mathématiques, Université d'Orleans, 45067 Orléans Cedex 2, France
E-mail address: `Sandrine.Grellier@univ-orleans.fr`