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Frédéric Klopp

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Institut des hautes études scientifiques  
Le Bois-Marie • Route de Chartres  
F-91440 BURES-SUR-YVETTE  
<http://www.ihes.fr/>

Centre de mathématiques Laurent Schwartz  
UMR 7640 CNRS/École polytechnique  
F-91128 PALAISEAU CEDEX  
<http://www.math.polytechnique.fr/>

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## GLOBAL POISSONIAN BEHAVIOR OF THE EIGENVALUES AND LOCALIZATION CENTERS OF RANDOM OPERATORS IN THE LOCALIZED PHASE

FRÉDÉRIC KLOPP

ABSTRACT. In the present note, we review some recent results on the spectral statistics of random operators in the localized phase obtained in [12]. For a general class of random operators, we show that the family of the unfolded eigenvalues in the localization region considered jointly with the associated localization centers is asymptotically ergodic. This can be considered as a generalization of [10]. The benefit of the present approach is that one can vary the scaling of the unfolded eigenvalues covariantly with that of the localization centers. The convergence result then holds for all the scales that are asymptotically larger than the localization scale. We also provide a similar result that is localized in energy. Full proofs of the results presented here will be published elsewhere ([12]).

RÉSUMÉ. Dans cette note, nous passerons en revue les résultats récents sur l'ergodicité asymptotique des valeurs propres et des centres de localisation des opérateurs aléatoires dans la phase localisées, obtenus dans [12]. Ces résultats généralisent ceux de [10] en prenant en compte les centres de localisation. Plus précisément, on démontre que, pour une classe générale d'opérateurs aléatoires, dans la région de localisation, les couples constitués des valeurs propres "dépliées" et du centre de localisation associé sont asymptotiquement ergodiques. On démontre également un résultat similaire lorsqu'on se restreint à de petits intervalles d'énergie.

### 1. INTRODUCTION

On  $\ell^2(\mathbb{Z}^d)$ , consider the random Anderson model

$$H_\omega = -\Delta + \lambda V_\omega$$

where

- $-\Delta$  is the free discrete Laplace operator

$$(1.1) \quad (-\Delta u)_n = \sum_{|m-n|=1} u_m \quad \text{for } u = (u_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d);$$

- $V_\omega$  is the random potential

$$(1.2) \quad (V_\omega u)_n = \omega_n u_n \quad \text{for } u = (u_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d).$$

We assume that the random variables  $(\omega_n)_{n \in \mathbb{Z}^d}$  are independent identically distributed and that their common distribution admits a compactly supported bounded density, say  $g$ .

- The coupling constant  $\lambda$  is chosen positive.

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Let  $N$  be the integrated density of states of  $H_\omega$  (see e.g. [8]); it is a probability distribution on the essential spectrum, say,  $\Sigma$  of  $H_\omega$ .

For  $L \in \mathbb{N}$ , let  $\Lambda = \Lambda_L = [0, L]^d$  be a large box and  $|\Lambda| := \#\Lambda = (L+1)^d$  be its cardinality. Let  $H_\omega(\Lambda)$  be the operator  $H_\omega$  restricted to  $\Lambda$  with periodic boundary conditions. The notation  $|\Lambda| \rightarrow +\infty$  is a shorthand for considering  $\Lambda = \Lambda_L$  in the limit  $L \rightarrow +\infty$ . Let us denote the eigenvalues of  $H_\omega(\Lambda)$  ordered increasingly and repeated according to multiplicity by  $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \dots \leq E_{|\Lambda|}(\omega, \Lambda)$ .

For  $t \in [0, 1]$ , consider the following point process

$$(1.3) \quad \Xi(\omega, t, \Lambda) = \sum_{n=1}^{|\Lambda|} \delta_{|\Lambda|[N(E_n(\omega, \Lambda)) - t]}.$$

In [10], we have proved the asymptotic ergodicity of the unfolded eigenvalues in the large coupling regime, that is,

**Theorem 1.1** ([10]). *For sufficiently large coupling constant  $\lambda$ ,  $\omega$ -almost surely, when  $|\Lambda| \rightarrow +\infty$ , the probability law of the point process  $\Xi(\omega, \cdot, \Lambda)$  under the uniform distribution  $\mathbf{1}_{[0,1]}(t)dt$  converges to the law of the Poisson point process on the real line with intensity 1.*

In the present note, we present an extension of this result to the localization centers. To each eigenvalue  $E_n(\omega, \Lambda)$ , one can associate a localization center, say,  $x_n(\omega, \Lambda)$ : it is essentially defined by the fact that, if  $\varphi_n(\omega, \Lambda)$  is a normalized eigenfunction of  $H_\omega(\Lambda)$  associate to  $E_n(\omega, \Lambda)$ , then, for any  $\rho > 1$ , for  $L$  sufficiently large, one has

$$(1.4) \quad \|\varphi_n(\omega, \Lambda)\|_{\ell^2(\{|x-x_n(\omega, \Lambda)| \geq \log^\rho |\Lambda|\})} \leq e^{-\log^\rho |\Lambda|/2}.$$

So eigenfunctions (and, thus, eigenvalues) are only sensitive to the local environment values near its localization center. For large coupling constants  $\lambda$ , it is known that, with a good probability, such localization centers exist for all eigenfunctions (see e.g. [8] or section 2.2 and the discussion therein). Note that relation (1.4) does not define the point  $x_n(\varphi, \Lambda)$  uniquely; nevertheless, one can show that all such points are “close” to each other in the sense that they are all contained in a ball of radius of order  $\log |\Lambda|$ ; this should be compared to the side length of the cube  $|\Lambda|^{1/d}$ .

Fix  $\alpha > 1$  and an increasing sequence of scales  $\ell = (\ell_\Lambda)_\Lambda$  and  $\Lambda = \Lambda_L$  such that, one has

$$(1.5) \quad \log^\alpha L \leq \ell_\Lambda \leq L.$$

Assume moreover that the following limit exists

$$(1.6) \quad \lim_{L \rightarrow +\infty} L^{-1} \ell_\Lambda =: c \in [0, 1].$$

Let  $g_E : [0, 1] \rightarrow \mathbb{R}^+$  and  $g_X : [0, 1]^d \rightarrow \mathbb{R}^+$  be two probability densities. For a fixed configuration  $\omega$ , consider the point process

$$(1.7) \quad \Xi_\Lambda^2(e, x; \ell, \omega) = \sum_{j=1}^N \delta_{\ell_\Lambda^d [e_j(\omega, \Lambda) - e]} \otimes \delta_{(x_j(\omega, \Lambda) - Lx)/\ell_\Lambda}.$$

under the distribution of density  $g_E \otimes g_X$  over  $[0, 1]^{1+d}$ .

Our main result on the Anderson model is

**Theorem 1.2.** *For sufficiently large coupling constant  $\lambda$ ,  $\omega$ -almost surely, the probability law of the point process  $\Xi_\Lambda^2(\cdot, \cdot; \ell, \omega)$  under the distribution  $g_E(e)g_X(x) dx$  converges to the law of the Poisson point process with intensity 1 on  $\mathbb{R}^{1+d}$  if  $c = 0$  and on  $\mathbb{R} \times [-1/c, 1/c]^d$  if  $c > 0$ .*

Actually, Theorem 1.2 is a prototype of the general result we present below. Essentially, we prove that the claim in Theorem 1.2 holds in the localization region for any random Hamiltonian satisfying a Wegner and a Minami estimate (see assumptions (W) and (M) in section 2.1). To do so, we use the analysis made in [7, 10].

Let us note that, in Theorem 1.2, the minimal size of the coupling constant for the result to hold is independent of the sequence of scale  $(\ell_\Lambda)_\Lambda$ ;  $\lambda$  need only to be chosen large enough so that the whole spectrum of  $H_\omega$  be localized (see e.g. [8] and references therein).

To conclude this introduction, let us note that Theorem 1.1 follows from Theorem 1.2 when  $\ell_\Lambda = L$  and by integration in the variable  $x$  in the case when the distribution is uniform i.e.  $g_E \otimes g_X = \mathbf{1}_{[0,1]^{1+d}}$ .

It is interesting to note that the limiting law does not depend on the laws under which one considers the random process  $\Xi_\Lambda^2(\cdot, \cdot; \ell, \omega)$  (as long as those have a density). This can be understood as follows. Actually, the convergence of the process holds even if one restricts the process to a local energy interval (see Theorem 2.3). In this case, the process only feels the local behavior of the distribution  $g_E \otimes g_X$  under which it is considered. As one assumes the existence of a density, this local behavior is essentially that of the uniform distribution except for the normalization factor given by the density.

## 2. THE RESULTS

Consider  $H_\omega = H_0 + V_\omega$ , a  $\mathbb{Z}^d$ -ergodic random Schrödinger operator on  $\mathcal{H} = L^2(\mathbb{R}^d)$  or  $\ell^2(\mathbb{Z}^d)$  (see e.g. [14, 15]). Typically, the background potential  $H_0$  is the Laplacian  $-\Delta$ , possibly perturbed by a periodic potential. Magnetic fields can be considered as well; in particular, the Landau Hamiltonian is also admissible as a background Hamiltonian (see the discussions in the introductions of [7, 10] for more details). For the sake of simplicity, we assume that  $V_\omega$  is almost surely bounded; hence, almost surely,  $H_\omega$  have the same domain  $H^2(\mathbb{R}^d)$  or  $\ell^2(\mathbb{Z}^d)$ .

**2.1. The setting and the assumptions.** For  $\Lambda$ , a cube in either  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ , we let  $H_\omega(\Lambda)$  be the self-adjoint operator  $H_\omega$  restricted to  $\Lambda$  with periodic boundary conditions. As in [7], our analysis stays valid for Dirichlet boundary conditions.

Furthermore, we shall denote by  $\mathbf{1}_J(H)$  the spectral projector of the operator  $H$  on the energy interval  $J$ .  $\mathbb{E}(\cdot)$  denotes the expectation with respect to  $\omega$ .

Our first assumption will be an independence assumption for local Hamiltonians that are far away from each other, that is,

**(IAD):** There exists  $R_0 > 0$  such that for any two cubes  $\Lambda$  and  $\Lambda'$  such that  $\text{dist}(\Lambda, \Lambda') > R_0$ , the random Hamiltonians  $H_\omega(\Lambda)$  and  $H_\omega(\Lambda')$  are stochastically independent.

Let  $\Sigma$  be the almost sure spectrum of  $H_\omega$  (see e.g. [14]). Pick  $I$  a relatively compact open subset of  $\Sigma$ . Assume the following holds:

**(W):** a Wegner estimate holds in  $I$ , i.e. there exists  $C > 0$  such that, for  $J \subset I$ , and  $\Lambda$ , a cube in  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ , one has

$$(2.1) \quad \mathbb{E} [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda)))] \leq C|J||\Lambda|.$$

**(M):** a Minami estimate holds in  $I$ , i.e. there exists  $C > 0$  and  $\rho > 0$  such that, for  $J \subset I$ , and  $\Lambda$ , a cube in  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ , one has

$$(2.2) \quad \mathbb{E} [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) \cdot [\text{tr}(\mathbf{1}_J(H_\omega(\Lambda))) - 1]] \leq C(|J||\Lambda|)^{1+\rho}.$$

**Remark 2.1.** We refer to [10, Remark 1.2] for a description of the validity of assumptions (W) and (M). Let us just say here that (W) has been proved to hold for a wide range of models (see e.g. [9, 16]) whereas (M) is known only for a few models (see e.g. [2, 3, 6, 11] and the discussions and references therein).

The integrated density of states is defined as

$$(2.3) \quad N(E) := \lim_{|\Lambda| \rightarrow +\infty} \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ less than } E\}}{|\Lambda|}.$$

By (W),  $N(E)$  is the distribution function of a measure that is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ . Let  $\nu$  be the density of state of  $H_\omega$  i.e. the distributional derivative of  $N$ . In the sequel, for a set  $I$ , we will often write  $N(I)$  for the mass the measure  $\nu(E)dE$  puts on  $I$  i.e.

$$(2.4) \quad N(I) = \int_I \nu(E)dE.$$

Let us now describe what we call the localized regime in the introduction. For  $L \geq 1$ ,  $\Lambda_L$  denotes the cube  $[0, L]^d$  in either  $\mathbb{R}^d$  or  $\mathbb{Z}^d$ . In the sequel, we write  $\Lambda$  for  $\Lambda_L$  i.e.  $\Lambda = \Lambda_L$  and when we write  $|\Lambda| \rightarrow +\infty$ , we mean  $L \rightarrow +\infty$ .

Let  $\mathcal{H}_\Lambda$  be  $\ell^2(\Lambda \cap \mathbb{Z}^d)$  in the discrete case and  $L^2(\Lambda)$  in the continuous one. For a vector  $\varphi \in \mathcal{H}$ , we define

$$(2.5) \quad \|\varphi\|_x = \begin{cases} \|\mathbf{1}_{\Lambda(x)}\varphi\|_2 & \text{where } \Lambda(x) = \{y; |y-x| \leq 1/2\} & \text{if } \mathcal{H} = L^2(\mathbb{R}^d), \\ |\varphi(x)| & & \text{if } \mathcal{H} = \ell^2(\mathbb{Z}^d). \end{cases}$$

Let  $I$  be a compact interval. We assume that  $I$  lies in the region of complete localization (see e.g. [4, 5]) for which we use the following finite volume version. We assume that

**(Loc):** for all  $\xi \in (0, 1)$ , one has

$$(2.6) \quad \sup_{L>0} \sup_{\substack{f \subset I \\ |f| \leq 1}} \mathbb{E} \left( \sum_{\gamma \in \mathbb{Z}^d} e^{|\gamma|^\xi} \|\mathbf{1}_{\Lambda(0)}f(H_\omega(\Lambda_L))\mathbf{1}_{\Lambda(\gamma)}\|_2 \right) < +\infty.$$

For  $L \in \mathbb{N}$ , recall that  $\Lambda = \Lambda_L$  and that  $H_\omega(\Lambda)$  is the operator  $H_\omega$  restricted to  $\Lambda$  with periodic boundary conditions. The notation  $|\Lambda| \rightarrow +\infty$  is a shorthand for considering  $\Lambda = \Lambda_L$  in the limit  $L \rightarrow +\infty$ .

Finally, let  $E_1(\omega, \Lambda) \leq E_2(\omega, \Lambda) \leq \dots \leq E_N(\omega, \Lambda) \leq \dots$  denote the eigenvalues of  $H_\omega(\Lambda)$  ordered increasingly and repeated according to multiplicity.

**2.2. The localization centers.** To define the localization centers, we state two results that are proved in [7].

**Lemma 2.1** ([7]). *Under assumptions (W) and (Loc), for any  $p > 0$ , there exists  $q > 0$  such that, for  $L$  sufficiently large, with probability larger than  $1 - L^{-p}$ , if*

- (1)  $\varphi_{n,\omega}$  is a normalized eigenvector of  $H_\omega(\Lambda_L)$  associated to  $E_{n,\omega} \in I$ ,
- (2)  $x_n(\omega) \in \Lambda_L$  is a maximum of  $x \mapsto \|\varphi_{n,\omega}\|_x$  in  $\Lambda_L$ ,

then, for  $x \in \Lambda_L$ , one has

$$\|\varphi_{n,\omega}\|_x \leq L^q e^{-|x-x_n(\omega)|^\xi}.$$

If for an eigenfunction  $\varphi$  we define the set of localization centers as  $C(\varphi) = \{x \in \Lambda; \|\varphi\|_x = \max_{\gamma \in \Lambda} \|\varphi\|_\gamma\}$ , then, one easily proves

**Lemma 2.2** ([7]). *Pick  $I$  such that (Loc) holds in  $I$  for  $H_\omega$ . For any  $\xi \in (0, 1)$  and  $p > 0$ , there exists  $C_p > 0$ , such that, for  $L$  sufficiently large, with probability larger than  $1 - L^{-p}$ , if  $E_j(\omega, \Lambda) \in I$  then the diameter of  $C(\varphi_j(\omega, \Lambda))$  is less than  $C_p \log^{1/\xi} |\Lambda|$ .*

To define the localization center uniquely, we order  $C(\varphi_j(\omega, \Lambda))$  lexicographically and take the largest element.

If, in addition to (W) and (Loc), one assumes (M), then the eigenvalues of  $H_\omega(\Lambda)$  are almost surely simple. Thus, one can associate localization centers to an eigenvalue. To fix notation, for any  $j$ , let  $x_j(\omega, \Lambda)$  be the localization center associated to the eigenvalue  $E_j(\omega, \Lambda)$  (or the eigenfunction  $\varphi_j(\omega, \Lambda)$ ).

**2.3. The results.** We state our results in two cases. In the first case described in section 2.4, we consider a macroscopic energy interval i.e. the energy interval in which we study the eigenvalues is a fixed compact interval where all the above assumptions hold. In the second case described in section 2.5, the energy interval shrinks to a point but not too fast so as to contain enough eigenvalues that is asymptotically infinitely many eigenvalues.

For  $J = [a, b]$  a compact interval such that  $N(b) - N(a) = N(J) > 0$ . For  $H_\omega(\Lambda)$ , we define the renormalized (or unfolded) energy levels in  $J$  to be

$$(2.7) \quad e_j(\omega, \Lambda) = N_J(E_j(\omega, \Lambda))$$

where we have set

$$(2.8) \quad N_J(\cdot) := \frac{N(\cdot) - N(a)}{N(b) - N(a)}.$$

$N_J$  is the integrated density of states renormalized so as to be the distribution function of a probability measure on  $J$ .

To each renormalized energy  $e_j(\omega, \Lambda) = N_J(E_j(\omega, \Lambda))$ , we associate the localization center  $x_j(\omega, \Lambda)$  associated to  $E_j(\omega, \Lambda)$  in section 2.2.

**2.4. Macroscopic energy intervals.** Fix  $\alpha \in (0, 1]$  and an increasing sequence of scales  $\ell = (\ell_\Lambda)_\Lambda$  satisfying (1.5) for some  $\alpha > 1$  and (1.6).

Let  $g_E : [0, 1] \rightarrow \mathbb{R}^+$  and  $g_X : [0, 1]^d \rightarrow \mathbb{R}^+$  be two probability densities.

For  $J = [a, b]$  a compact interval such that  $N(b) - N(a) = N(J) > 0$  and a fixed configuration  $\omega$ , consider the point process  $\Xi_J^2(e, x; \omega, \ell, \Lambda)$  defined by

$$(2.9) \quad \Xi_J^2(e, x; \omega, \ell, \Lambda) = \sum_{j=1}^N \delta_{N(J)\ell_\Lambda^d [e_j(\omega, \Lambda) - e]} \otimes \delta_{(x_j(\omega, \Lambda) - Lx)/\ell_\Lambda}.$$

under the distribution of density  $g_E \otimes g_X$  over  $[0, 1] \times [0, 1]^d$  in  $(e, x)$ .

**Remark 2.2.** Note that, in the introduction, in the case of the discrete Anderson model, as  $N(+\infty) - N(-\infty) = 1$ , it is unnecessary to rescale the density of states if one considers the whole spectrum. This explains the difference between (2.9) and (1.7).

Our main result is

**Theorem 2.1.** *Assume (IAD), (W), (M) and (Loc) hold. Assume that  $J \subset I$ , the localization region, is such that  $N(J) > 0$ .*

*Then,  $\omega$ -almost surely, the probability law of the point process  $\Xi_J^2(\cdot, \cdot; \omega, \ell, \Lambda)$  under the distribution of density  $g_E \otimes g_X$  converges to the law of the Poisson point process with intensity 1 on  $\mathbb{R}^{1+d}$  if  $c = 0$  and on  $\mathbb{R} \times [-1/c, 1/c]^d$  if  $c > 0$ .*

First, let us note that Theorem 1.1 of [10] is an immediate consequence of Theorem 2.1. Indeed, it is obtained for the scale  $\ell_\Lambda = L$  by integrating  $\Xi_J^2(\cdot, \cdot; \omega, \ell, \Lambda)$  in  $x$  if one picks  $g_E \otimes g_X = \mathbf{1}_{[0,1]} \otimes \mathbf{1}_{[0,1]^d}$ .

It is well known that, for the discrete Anderson model at large coupling, the assumptions (W), (M) and (Loc) are satisfied on the whole spectrum (see e.g. [1, 8, 2]); thus, Theorem 1.2 is a consequence of Theorem 2.1.

As in [10], one can also study the statistics of the levels themselves i.e. before unfolding. One obtains

**Theorem 2.2.** *Assume (IAD), (W), (M) and (Loc) hold. Assume that  $J = [a, b] \subset I$  is a compact interval in the localization region satisfying  $N(J) > 0$ .*

*Define*

- *the probability density  $\nu_J := \frac{1}{N(J)} \nu(E) \mathbf{1}_J(E)$  where  $\nu = \frac{dN}{dE}$  is the density of states of  $H_\omega$ ;*
- *the point process*

$$(2.10) \quad \tilde{\Xi}_J^2(E, x; \omega, \ell, \Lambda) = \sum_{E_n(\omega, \Lambda) \in J} \delta_{\nu(E)\ell_\Lambda^d [E_n(\omega, \Lambda) - E]} \otimes \delta_{(x_j(\omega, \Lambda) - Lx)/\ell_\Lambda}.$$

*Then,  $\omega$ -almost surely, the probability law of the point process  $\tilde{\Xi}_J^2(\cdot, \cdot; \omega, \ell, \Lambda)$  under the distribution of density  $\nu_J(E) \otimes g_X(x)$  converges to the law of the Poisson point process on  $\mathbb{R}^{d+1}$  with intensity 1.*

Theorem 2.2 is deduced from Theorem 2.1 in the same way as Theorem 1.2 is deduced from [10, Theorem 1.1].

**2.5. Microscopic energy intervals.** One can also prove a version of Theorem 2.1 that is local in energy. In this case, one needs the density of states not to be too small on the energy interval under consideration; indeed, it is clear that a result of the type of Theorem 2.1 can only hold if the number of eigenvalues under consideration grows to  $\infty$  when  $|\Lambda|$  goes to  $\infty$ .

This is not the only restriction one meets for a result local in energy. For the same reasons as in the studies done in [7, 10], one needs that the weight the IDS  $N$  gives the energy interval under consideration be not too small compared with the length of this interval; the reason for this is the non optimality of the assumption (W) (see [6]).

One proves

**Theorem 2.3.** *Assume (IAD), (W), (M) and (Loc) hold. Pick  $E_0 \in I$ . Fix  $(I_\Lambda)_\Lambda$  a decreasing sequence of intervals such that  $\sup_{I_\Lambda} |x| \xrightarrow{|\Lambda| \rightarrow +\infty} 0$ .*

*Let us assume that*

$$(2.11) \quad \text{if } \ell' = o(L) \text{ then } \frac{|N(E_0 + I_{\Lambda_{L+\ell'}})|}{|N(E_0 + I_{\Lambda_L})|} \xrightarrow{L \rightarrow +\infty} 1.$$

*Then, there exists  $\tau = \tau(\rho)$  such that, if, for  $\Lambda$  large, one has*

$$(2.12) \quad |N(E_0 + I_\Lambda)| \cdot |I_\Lambda|^{-1-\tilde{\rho}} \geq 1 \quad \text{and} \quad |\Lambda|^\delta \cdot |N(E_0 + I_\Lambda)| \xrightarrow{|\Lambda| \rightarrow +\infty} +\infty$$

*for some  $\delta \in (0, 1)$  and  $\tilde{\rho} > 0$  satisfying*

$$(2.13) \quad \frac{\delta \tilde{\rho}}{1 + \tilde{\rho}} < \tau$$

*then,  $\omega$ -almost surely, the law of the point process  $\Xi_{E_0+I_\Lambda}^2(\cdot, \cdot; \omega, \ell, \Lambda)$  under the distribution of density  $g_E \otimes g_X$  on  $[0, 1]^{d+1}$  converges to the law of the Poisson point process with intensity 1 on  $\mathbb{R}^{1+d}$  if  $c = 0$  and on  $\mathbb{R} \times [-1/c, 1/c]^d$  if  $c > 0$ .*

The exponent  $\tau = \tau(g)$  can be computed explicitly. The first condition in (2.12) requires that  $N$  is not too flat at  $E_0$ . How flat it may be depends on the exponent  $\tilde{\rho}$ , thus, in part on the value of  $\tau$  if  $\delta$  is not less than  $\tau$ . Indeed, if  $\delta < \tau$ , then (2.13) is satisfied for any  $\tilde{\rho} > 0$  and actually, we can take  $\tilde{\rho} = +\infty$  i.e. drop the first condition in (2.12); note that this is what happens in the case of macroscopic intervals. If  $\delta \geq \tau$ , a condition on the flatness of  $N$  kicks in.

Condition (2.11) is necessary as we don't impose anything else on how the density of states of the intervals  $E_0 + I_\Lambda$  be have; they could oscillate which could presumably ruin convergence.

As a consequence of Theorem 2.3, using the results of [13], one can prove a counterpart of Theorem 2.2 for “large” enough neighborhoods of the point; here, “large” does not mean that the neighborhood needs to be large: it merely needs not to shrink too fast to 0 (see (2.12)).

### 3. A PRECISE DESCRIPTION OF THE EIGENVALUES AND LOCALIZATION CENTERS IN THE LOCALIZED REGIME

In this section, we state a number of results that are crucial for our derivation of Theorems 1.2, 2.1 and 2.3. Some of these results are taken



from [7]; others are extensions thereof and are proved in [12].

These results explain the nature of Theorems 2.1 and 2.3. They are also of interest on their own as they give a precise description of the eigenvalues and localization centers. We shall not explain how to use them to derive the results described in section 2.

**3.1. I.I.D approximations to the eigenvalues.** The second ingredient of our proof is a description of most of the eigenvalues of  $H_\omega(\Lambda)$  in some small interval, say,  $I_\Lambda$  in terms of i.i.d. random variables. These random variables are the eigenvalues of the restrictions of  $H_\omega(\Lambda)$  to much smaller disjoint cubes, the distribution of which we computed in Lemma 3.2. This description of the eigenvalues of  $H_\omega(\Lambda)$  holds with a large probability.

**3.1.1. Localization estimates and localization centers.** We first recall a result of [7] defining and describing localization centers, namely,

**Lemma 3.1** ([7]). *Under assumptions (W) and (Loc), for any  $p > 0$  and  $\xi \in (0, 1)$ , there exists  $q > 0$  such that, for  $L \geq 1$  large enough, with probability larger than  $1 - L^{-p}$ , if*

- (1)  $\varphi_{n,\omega}$  is a normalized eigenvector of  $H_\omega(\Lambda_L)$  associated to  $E_{n,\omega} \in I$ ,
- (2)  $x_n(\omega) \in \Lambda_L$  is a maximum of  $x \mapsto \|\varphi_{n,\omega}\|_x$  in  $\Lambda_L$ ,

then, for  $x \in \Lambda_L$ , one has

$$\|\varphi_{n,\omega}\|_x \leq L^q e^{-|x-x_n(\omega)|^\xi}$$

where  $\|\cdot\|_x$  is defined in (2.5).

Define  $C(\varphi) = \{x \in \Lambda; \|\varphi\|_x = \max_{\gamma \in \Lambda} \|\varphi\|_\gamma\}$  to be the set of localization centers for  $\varphi$ . Then, the diameter of  $C(\varphi_j(\omega, \Lambda))$  is less than  $q(\log |\Lambda|)^{1/\xi}$ .

We define localization centers in a unique way by ordering the set  $C(\varphi)$  lexicographically and take the supremum.

**3.1.2. An approximation theorem for eigenvalues.** The approximation theorem [7, Theorem 1.15] for the eigenvalues of  $H_\omega(\Lambda)$  that was used in [10] to prove the asymptotic ergodicity of the eigenvalues is not precise enough to obtain Theorems 1.2, 2.1 and 2.3. More precisely, the assumptions of [7, Theorem 1.15] impose restrictions on the length scales  $(\ell_\Lambda)_\Lambda$  that do not cover all the regimes where it is natural to expect Theorems 1.2, 2.1 and 2.3 to hold, that is, length scales  $\ell_\Lambda$  that are asymptotically larger than the size of the localization cube i.e  $\ell_\Lambda \gg \log L$  if assumption (Loc) holds with  $\xi = 1$  and  $\ell_\Lambda \gg (\log L)^{1/\xi}$  for some  $\xi < 1$  if assumption (Loc) holds with any  $\xi \in (0, 1)$ . The assumptions of [7, Theorem 1.15] impose that  $\ell_\Lambda \gg (\log L)^{d/\xi}$ .

Pick  $\xi \in (0, 1]$ ,  $R > 1$  large and  $\rho' \in (0, \rho)$  where  $\rho$  is defined in (M). For a cube  $\Lambda$ , consider an interval  $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$ . Set  $\ell'_\Lambda = (R \log |\Lambda|)^{\frac{1}{\xi}}$ . Let  $\alpha = (\alpha_\Lambda)_\Lambda$  be a positive sequence such that  $\alpha_\Lambda \rightarrow 0$  when  $|\Lambda| \rightarrow +\infty$ . We say that the sequence  $(I_\Lambda)_\Lambda$  is  $(\xi, R, \alpha)$ -admissible if, for any  $\Lambda$ , one has

$$(3.1) \quad |\Lambda|N(I_\Lambda) \geq 1, \quad N(I_\Lambda)|I_\Lambda| \geq \alpha_\Lambda, \quad N(I_\Lambda)(\ell'_\Lambda)^d \leq 1.$$

In the spirit of [7, Theorem 1.15], one proves

**Theorem 3.1** ([12]). *Assume (IAD), (W), (M) and (Loc) hold. Let  $\Lambda = \Lambda_L$  be the cube of center 0 and side length  $L$ .*

*Pick  $\beta \in (0, 1)$  and a sequence of intervals that is  $(\xi, R, \alpha)$ -admissible, say,  $(I_\Lambda)_\Lambda$  such that  $\ell'_\Lambda \ll \tilde{\ell}_\Lambda \ll L$  and  $N(I_\Lambda)\alpha_\Lambda^\beta \tilde{\ell}_\Lambda^d \rightarrow 0$  as  $|\Lambda| \rightarrow \infty$ .*

*There exists a decomposition of  $\Lambda_L$  into disjoint cubes of the form  $\Lambda_{\ell_\Lambda}(\gamma_j) := \gamma_j + [0, \ell_\Lambda]^d$ , where  $\ell_\Lambda = \tilde{\ell}_\Lambda(1 + \mathcal{O}(\tilde{\ell}_\Lambda/|\Lambda_L|)) = \tilde{\ell}_\Lambda(1 + o(1))$  such that*

- (1)  $\cup_j \Lambda_{\ell_\Lambda}(\gamma_j) \subset \Lambda_L$ ,
- (2)  $\text{dist}(\Lambda_{\ell_\Lambda}(\gamma_j), \Lambda_{\ell_\Lambda}(\gamma_k)) \geq \ell'_\Lambda$  if  $j \neq k$ ,
- (3)  $\text{dist}(\Lambda_{\ell_\Lambda}(\gamma_j), \partial\Lambda_L) \geq \ell'_\Lambda$ ,
- (4) if  $\ell_\Lambda \ll \ell'' \leq L$  then  $|\Lambda_{\ell''} \setminus [\Lambda_{\ell''} \cap \cup_j \Lambda_{\ell_\Lambda}(\gamma_j)]| \lesssim |\Lambda_{\ell''}| \ell'_\Lambda / \ell''$ ,

*such that, for any  $p > 0$ , for  $L$  sufficiently large (depending only on  $(\xi, R, \alpha, p)$  but not on the admissible sequence of intervals), there exists a set of configurations  $\mathcal{Z}_\Lambda$  such that*

- $\mathcal{Z}_\Lambda$  is large, namely,

$$(3.2) \quad \mathbb{P}(\mathcal{Z}_\Lambda) \geq 1 - |\Lambda|^{-p} - \exp\left(-c|I_\Lambda|^{1+\rho}|\Lambda|\ell_\Lambda^{d\rho}\right) - \exp\left(-c|\Lambda||I_\Lambda|\ell'_\Lambda\ell_\Lambda^{-1}\right)$$

- for  $\omega \in \mathcal{Z}_\Lambda$ , there exists  $|\Lambda|\ell_\Lambda^{-d}(1 + o(1))$  disjoint boxes  $\Lambda_{\ell_\Lambda}(\gamma_j)$  satisfying the properties:

- (1) the Hamiltonian  $H_\omega(\Lambda_{\ell_\Lambda}(\gamma_j))$  has at most one eigenvalue in  $I_\Lambda$ , say,  $E_n(\omega, \Lambda_{\ell_\Lambda}(\gamma_j))$ ;
- (2)  $\Lambda_{\ell_\Lambda}(\gamma_j)$  contains at most one center of localization, say  $x_{k_j}(\omega, L)$ , of an eigenvalue of  $H_\omega(\Lambda)$  in  $I_\Lambda$ , say  $E_{k_j}(\omega, \Lambda)$ ;
- (3)  $\Lambda_{\ell_\Lambda}(\gamma_j)$  contains a center  $x_{k_j}(\omega, \Lambda)$  if and only if

$$\sigma(H_\omega(\Lambda_{\ell_\Lambda}(\gamma_j))) \cap I_\Lambda \neq \emptyset;$$

*in which case, one has*

$$(3.3) \quad |E_{k_j}(\omega, \Lambda) - E_n(\omega, \Lambda_{\ell_\Lambda}(\gamma_j))| \leq |\Lambda|^{-R} \text{ and } \text{dist}(x_{k_j}(\omega, L), \Lambda_L \setminus \Lambda_{\ell_\Lambda}(\gamma_j)) \geq \ell'_\Lambda$$

*where we recall that  $\ell'_\Lambda = (R \log |\Lambda|)^{\frac{1}{\xi}}$ ;*

- the number of eigenvalues of  $H_\omega(\Lambda)$  that are not described above is bounded by

$$(3.4) \quad CN(I_\Lambda)|\Lambda| \left( [N(I_\Lambda)\ell_\Lambda^d]^\rho \alpha_\Lambda^{-1-\rho} + \alpha_\Lambda^{-1} \ell'_\Lambda \ell_\Lambda^{-1} \right);$$

*this number is  $o(N(I_\Lambda)|\Lambda|)$  provided*

$$(3.5) \quad \alpha_\Lambda^{-1} \ell'_\Lambda \ll \ell_\Lambda \ll \alpha_\Lambda^{(1+\rho)/\rho} N(I_\Lambda)^{-\frac{1}{d}}.$$

We note that the assumptions on  $(I_\Lambda)_\Lambda$  in Theorem 3.1 imply that  $|I_\Lambda| \rightarrow 0$  and  $N(I_\Lambda)$  must go to 0 faster than logarithmically in  $|\Lambda|$  (see the right hand side of (3.5)).

Let us now briefly explain how the length scale  $\ell_\Lambda \asymp \tilde{\ell}_\Lambda$  will be chosen in our analysis. We will pick the intervals  $I_\Lambda$  so that  $(\log |\Lambda|)^{-\alpha_+} \lesssim N(I_\Lambda) \lesssim (\log |\Lambda|)^{-\alpha_-}$  for some large positive  $\alpha_- < \alpha_+$ , and  $\ell_\Lambda \asymp N(I_\Lambda)^{-\nu}$ . The effect of taking  $\alpha_-$  large is to reduce the checking of the validity of (3.5)

to checking that  $\frac{\rho'}{1+\rho'} < \frac{\rho-\rho'}{d(1+\rho)(1+\rho')}$  which follow from the assumption  $\rho' \in [0, \rho/(1+(\rho+1)d))$ . The exponent  $\nu$  is then chosen so that

$$(3.6) \quad \frac{\rho'}{1+\rho'} < \nu < \frac{\rho-\rho'}{d(1+\rho)(1+\rho')}.$$

Note that the right hand side inequality in (3.5) implies that  $N(I_\Lambda)^{\frac{1}{1+\rho}} \tilde{\ell}_\Lambda^d \rightarrow 0$  as  $(\rho-\rho')/(1+\rho) < 1$ . With these choices, the bound (3.4) then becomes bounded by  $N(I_\Lambda)|\Lambda|(\log|\Lambda|)^{-\beta}$  for some  $\beta > 0$ .

**3.2. Distribution of the unfolded eigenvalues.** We now describe the distribution of the unfolded eigenvalues for the operator  $H_\omega$  in a small cube. Consider a cube  $\Lambda = \Lambda_\ell$  centered at 0 of side length  $\ell$ . Pick an interval  $I_\Lambda = [a_\Lambda, b_\Lambda] \subset I$  (i.e.  $I_\Lambda$  is contained in the localization region) for  $\ell$  sufficiently large.

Consider the following random variables:

- $X = X(\Lambda, I_\Lambda)$  is the Bernoulli random variable  
 $X = \mathbf{1}_{H_\omega(\Lambda)}$  has exactly one eigenvalue in  $I_\Lambda$  with localization center in  $\Lambda_{\ell-\ell'}$
- $\tilde{E} = \tilde{E}(\Lambda, I_\Lambda)$  is this eigenvalue conditioned on  $X = 1$ .

Let  $\tilde{\Xi}$  be the distribution function of  $\tilde{E}$ . We know

**Lemma 3.2** ([7]). *Assume (W), (M) and (Loc) hold. For  $\nu \in (0, 1)$ , one has*

$$(3.7) \quad |\mathbb{P}(X = 1) - N(I_\Lambda)|\Lambda|| \leq C \left( |\Lambda|^{1+\rho}|I_\Lambda|^{1+\rho} + |I_\Lambda|^{-C} e^{-\ell^\nu/C} + N(I_\Lambda)|\Lambda|\ell'\ell^{-1} + \ell^d e^{-(\ell')^\nu} \right)$$

By (W), the distribution function  $\tilde{\Xi}$  is Lipschitz continuous. Moreover, setting  $N(x, y) := (N(a_\Lambda + x|I_\Lambda) - N(a_\Lambda + y|I_\Lambda))|\Lambda|$ , one has

$$(3.8) \quad \left| (\tilde{\Xi}(x) - \tilde{\Xi}(y)) P(X = 1) - N(x, y, \Lambda) \right| \leq C \left( |x - y|^{1+\rho} |\Lambda|^{1+\rho} |I_\Lambda|^{1+\rho} + (|I_\Lambda||x - y|)^{-C} e^{-\ell^\nu/C} + N(x, y, \Lambda)\ell'\ell^{-1} + \ell^d e^{-(\ell')^\nu} \right).$$

This result is useful only when  $N(I_\Lambda)|\Lambda| \gg (|I_\Lambda||\Lambda|)^{1+\rho}$ . Thus, by (W), it requires that  $|I_\Lambda||\Lambda| \rightarrow 0$  as  $|\Lambda| \rightarrow +\infty$ . So, we apply this result to intervals much smaller (here, we measure intervals with the density of states) than the typical spacing between eigenvalues.

**3.3. A large deviation principle for the eigenvalue counting function.** Define the random numbers

$$(3.9) \quad N(I_\Lambda, \Lambda, \omega) := \#\{j; E_j(\omega, \Lambda) \in I_\Lambda\}.$$

Write  $I_\Lambda = [a_\Lambda, b_\Lambda]$  and recall that  $N(I_\Lambda) = N(b_\Lambda) - N(a_\Lambda)$  where  $N$  is the integrated density of states. Using Theorem 3.1 and standard large deviation estimates for i.i.d. random variables, one shows that  $N(I_\Lambda, \Lambda, \omega)$  satisfies a large deviation principle, namely,

**Theorem 3.2** ([7]). *Assume (IAD), (W), (M) and (Loc) hold. For any  $\tilde{\rho} \in (0, \rho)$  ( $\rho$  is defined in Assumption (M)) and  $\nu \in (0, 1)$ , there exists  $\delta > 0$  such that, if  $(I_\Lambda)_\Lambda$  is a sequence of compact intervals in the localization region  $I$  satisfying*

- $N(I_\Lambda) (\log |\Lambda|)^{1/\delta} \rightarrow 0$  as  $|\Lambda| \rightarrow +\infty$
- $N(I_\Lambda) |\Lambda|^{1-\nu} \rightarrow +\infty$  as  $|\Lambda| \rightarrow +\infty$
- $N(I_\Lambda) |I_\Lambda|^{-1-\tilde{\rho}} \rightarrow +\infty$  as  $|\Lambda| \rightarrow +\infty$ ,

*then, for  $|\Lambda|$  sufficiently large (depending on  $\tilde{\rho}$  and  $\nu$  but not on the specific sequence  $(I_\Lambda)_\Lambda$ ), one has*

$$(3.10) \quad \mathbb{P} \left( |N(I_\Lambda, \Lambda, \omega) - N(I_\Lambda)|\Lambda| \geq N(I_\Lambda) |\Lambda| (\log |\Lambda|)^{-\delta} \right) \leq e^{-(N(I_\Lambda) |\Lambda|)^\delta}.$$

Note that as  $J \mapsto N(J)$  is a measure, thus, additive, for  $J \subset I$  the region of localization, one may split  $J$  into intervals  $(J_k)_k$  such that  $N(J_k) \asymp (\log |\Lambda|)^{-1/\delta'}$ ,  $\delta' \in (0, \delta)$ , and sum the estimates given by Theorem 1.2 on each  $J_k$  to obtain that

$$\begin{aligned} \mathbb{P} \left( |N(J, \Lambda, \omega) - N(J)|\Lambda| \geq N(J) |\Lambda| (\log |\Lambda|)^{-\delta} \right) \\ \lesssim N(J) (\log |\Lambda|)^{1/\delta'} e^{-|\Lambda|^\delta (\log |\Lambda|)^{-\delta/\delta'}}. \end{aligned}$$

This gives a useful large deviation estimate for intervals of macroscopic size.

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(Frédéric Klopp) LAGA, U.M.R. 7539 C.N.R.S, INSTITUT GALILÉE, UNIVERSITÉ PARIS-NORD, 99 AVENUE J.-B. CLÉMENT, F-93430 VILLETANEUSE, FRANCE  
*E-mail address:* klopp@math.univ-paris13.fr