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THE RESOLUTION OF THE BOUNDED L^2 CURVATURE CONJECTURE IN GENERAL RELATIVITY

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Abstract. This paper reports on the recent proof of the bounded L^2 curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound of the volume radius of the corresponding initial data set.

1. Introduction

This paper reports on the recent proof of the bounded L^2 curvature conjecture. More precisely we show that the time of existence of a classical solution to the Einstein-vacuum equations depends only on the L^2-norm of the curvature and a lower bound of the volume radius of the corresponding initial data set.

The entire proof of the conjecture is contained in the sequence of papers [20] [33] [34] [35] [36] [37].

1.1. Initial value problem for the Einstein vacuum equations.

We consider the Einstein vacuum equations (EVE),

$$\mathbf{Ric}_{\alpha\beta} = 0 \tag{1}$$

where $\mathbf{Ric}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathcal{M}, \mathbf{g})$. (1) corresponds to an evolution problem. An initial data set consists of a three dimensional manifold Σ_0 together with a Riemannian metric g and a symmetric 2-tensor k on Σ_0. For a given initial data set (Σ_0, g, k), the Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (1) and an embedding of Σ_0 in $\mathcal{M}$ such that the metric induced by $\mathbf{g}$ on Σ_0 coincides with g and the 2-tensor k is the second fundamental form of the embedding.

Remark 1.1. *Since physically one should not be able to distinguished between different coordinates systems, a solution of the Cauchy problem can be unique only modulo a diffeomorphism.*

The equations (1) are overdetermined and the initial data set (Σ_0, g, k) has to satisfy the following compatibility conditions known as the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{tr} k = 0, \\ R_{scal} - |k|^2 + (\mathrm{tr} k)^2 = 0, \end{cases} \tag{2}$$

where the covariant derivative ∇ is defined with respect to the metric g, R_{scal} is the scalar curvature of g, and $\mathrm{tr}k$ is the trace of k with respect to the metric g.

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In this paper we restrict ourselves to asymptotically flat initial data sets with one end. More precisely, (Σ_0, g, k) is such that Σ_0 minus a compact set is diffeomorphic to $\mathbb{R}^3$ minus a ball, with $g_{ij} - \delta_{ij}$ and k_{ij} satisfying a suitable rate of fall off at infinity in this coordinates system.

The first local existence and uniqueness result for (EVE) was established by Y.C. Bruhat, see [3], with the help of wave coordinates which allowed her to cast the Einstein vacuum equations in the form of a system of nonlinear wave equations to which one can apply[1] the standard theory of nonlinear hyperbolic systems. The optimal, classical[2] result states the following.

Theorem 1.2 (Classical local existence [7] [8])**.** *Let* (Σ_0, g, k) *be an initial data set for the Einstein vacuum equations* (1)*. Assume that* Σ_0 *can be covered by a locally finite system of coordinate charts, related to each other by* C^1 *diffeomorphisms, such that* $(g,\, k) \in H^s_{loc}(\Sigma_0) \times H^{s-1}_{loc}(\Sigma_0)$ *with* $s > 5/2$*. Then there exists a unique*[3] *(up to an isometry) globally hyperbolic development* $(\mathcal{M}, \mathbf{g})$*, verifying* (1)*, for which* Σ_0 *is a Cauchy hypersurface*[4]*.*

Our goal will be to lower the assumptions of the previous theorem on the regularity of the initial data set. To motivate our result, let us first emphasize in the next section why pushing for rough solutions is a main theme in nonlinear evolution PDEs.

1.2. **The quest for rough solutions in nonlinear evolution PDEs.**

1.2.1. *First examples.* To illustrate the role played by rough solutions in nonlinear evolution PDEs, let us consider a nonlinear evolution equation possessing a conserved quantity which is positive definite and in particular controls a norm in a certain functional space. We refer to the conserved quantity as the energy, its associated functional space as the energy space, and its associated norm as the energy norm. One can then classify such evolution equations into three categories[5]

(1) One can prove a local existence result with a time of existence which depends only on the energy norm of the initial data. This case is referred as *energy subcritical.*
(2) One can prove a local existence result for initial data in the energy space, but with a time of existence which does not only depend on the energy norm of the initial data (i.e. there is no uniform lower bound on the size of the time interval of existence for initial data with a given energy norm). This case is referred as *energy critical.*
(3) One can not prove a local existence result for initial data in the energy space. This case is referred as *energy supercritical.*

As the energy supercritical case as it is still vastly open, we will focus on the two other cases. In the energy subcritical case, one can pile up time intervals of existence provided by the local existence result which are all of the same size since they only depend on the energy norm of the data which is conserved. One infers global existence for any initial data. A nice illustration of

[1]The original proof in [3] relied on representation formulas, following an approach pioneered by Sobolev, see [29].

[2]Based only on energy estimates and classical Sobolev inequalities.

[3]The original proof in [7], [8] actually requires one more derivative for the uniqueness. The fact that uniqueness holds at the same level of regularity than the existence has been obtained in [25].

[4]That is any past directed, in-extendable causal curve in $\mathcal{M}$ intersects Σ_0.

[5]One usually defines the criticality relative to the behavior of the energy under some notion of scaling. Here, we instead classify the equation with respect to an ability to prove local existence results. These two classifications agree in most cases and our choice only aims at simplifying the exposition.

this procedure is provided by [11] in the case of the classical Yang-Mills equations in dimension 1+3, where the energy space is the Sobolev space H^1.

In the energy critical case, the conjecture is that global existence holds for any data below the energy of the first nontrivial stationary solution or solitary wave. This conjecture has been proved in a large number of cases over the last thirty years. A spectacular achievement of this method is the recent proof of the conjecture in the case of the 2+1 wave map problem in [38], [31], [32] and [21].

A key step in the large data results mentioned above, both for the energy subcritical and energy critical cases, is a local well-posedness result at the level of the energy space, which is typically a low regularity well-posedness result. In the next section, we discuss another example of a nonlinear evolution partial differential equation for which making sense of rough solutions plays a fundamental role.

1.2.2. *The proof of the weak cosmic censorship in spherical symmetry.* In this section, we briefly discuss the influential proof by Christodoulou [6] of the weak cosmic censorship conjecture for the Einstein equations coupled to a scalar field in spherical symmetry[6].

Let us first recall the weak cosmic censorship conjecture of Penrose. The starting point of this conjecture is the existence of space times containing singularities, the most famous example being the Schwarzschild space-time which is spherically symmetric and contains a singularity at $r = 0$. Now, the existence of space-times containing a singularity could be considered as an undesirable feature from the point of view of physics. To come to term with such space-times, Penrose formulated the celebrated weak cosmic censorship conjecture[7].

Conjecture 1.3 (Weak cosmic censorship). *For generic asymptotically flat initial data set, singularities are hidden by a black hole.*

In view of this conjecture, singularities are acceptable as they not visible by an observer at infinity. At the moment, it is still an open problem in general, but the conjecture has been proved in the case of spherical symmetry in [6] for the Einstein equations coupled to a scalar field. This seminal work relies on the rough well posedness result of [5]. This well posedness result - which involves regularity assumptions at the level of a weighted bounded variation (BV) norm - allows in particular to make sense of solutions with a jump along the backward null cone from the singularity. This jump turns out to be essential in generating arbitrarily small perturbations of a given solution containing a singularity which are still strong enough to cover the singularity with a black hole[8].

The result in [6] provides thus yet another example of the importance of making sense of rough solutions for nonlinear evolution PDEs. It motivates our main result, which concerns well-posedness of rough solutions for the Einstein equations in the absence of symmetry. Now, BV norms are only adapted to hyperbolic problems in 1+1 dimension (and hence to spherical symmetry). This will force us to abandon BV norms and to instead measure the regularity

[6]Due to Birkhoff's theorem, the Einstein vacuum equations are non dynamical in spherical symmetry. To obtain a dynamical problem and yet retain the advantage of working in spherical symmetry, one adds a matter field to the right-hand side of the Einstein equations, a scalar field being the simplest possibility. For the sake of simplicity, we do not explicitly write down the equations in this case.

[7]He also introduced the strong cosmic censorship conjecture which despite its name is independent of the weak cosmic censorship conjecture.

[8]In the proof, the singularity is actually covered by a trapped region which, as it turns out, is enough.

of our solution using L^2 based norms which are the only norms which are propagated by the evolution in higher dimensions.

1.3. The resolution of the bounded L^2 curvature conjecture.

1.3.1. *The bounded L^2 curvature conjecture.* In this section, we consider the problem of going beyond the classical local existence result stated in Theorem 1.2. To make the discussion more tangible it is worthwhile to recall the form of the Einstein vacuum equations in the wave gauge. Assuming given coordinates x^α, verifying[9]

$$\square_{\mathbf{g}} x^\alpha = 0, \ \alpha = 0, \dots, 3, \tag{3}$$

the metric coefficients $\mathbf{g}_{\alpha\beta} = \mathbf{g}(\partial_\alpha, \partial_\beta)$, with respect to these coordinates, satisfy the system of quasilinear wave equations[10],

$$\square_{\mathbf{g}} \mathbf{g}_{\alpha\beta} = F_{\alpha\beta}(\mathbf{g}, \partial \mathbf{g}) \tag{4}$$

where $F_{\alpha\beta}$ are quadratic functions of $\partial \mathbf{g}$, i.e. the first order derivatives of $\mathbf{g}$ with respect to the coordinates x^α. In the harmonic coordinates, the wave operator on the curved background $\mathbf{g}$ is given by $\square_{\mathbf{g}} = \mathbf{g}^{\mu\nu}\partial_\mu\partial_\nu$. Equation (4) is obtained by expressing the Ricci tensor $\mathbf{Ric}_{\alpha\beta}$ in terms of the components of $\mathbf{g}$ and its first and second order derivatives. Conversely, to verify that the solution of (4) yields a solution to the Einstein vacuum equations (1), one has additionally to show that the coordinates conditions (3) propagate. This holds for solutions of (4), as was observed by Choquet-Bruhat, provided these coordinates conditions are satisfied initially on Σ_0 and (Σ_0, g, k) satisfies the constraint equations (2).

In a first approximation we may compare (4) with the semilinear wave equation,

$$\square \phi = F(\phi, \partial \phi) \tag{5}$$

with F quadratic in $\partial\phi$. Using standard energy estimates - i.e. differentiating (5) $s-1$ times, multiplying it with $\partial_t \partial^{s-1}\phi$, integrating by parts and using Gronwall's lemma - one obtains the following control for the Sobolev norm H^s of ϕ

$$\|\phi(t)\|_s \lesssim \|\phi(0)\|_s \exp\left(C_s \int_0^t \|\partial\phi(\tau)\|_{L^\infty} d\tau \right). \tag{6}$$

The classical exponent $s > 3/2 + 1$ arises simply from the Sobolev embedding of H^r, $r > 3/2$ into L^∞.

To go beyond the classical exponent, see [26], one has to replace Sobolev inequalities with Strichartz estimates of, roughly, the following type,

$$\left(\int_0^t \|\partial\phi(\tau)\|^2_{L^\infty} d\tau \right)^{1/2} \lesssim C \left(\|\partial\phi(0)\|_{H^{1+\epsilon}} + \int_0^t \|\square\phi(\tau)\|_{H^{1+\epsilon}} \right)$$

where $\epsilon > 0$ can be chosen arbitrarily small. This leads to a gain of $1/2$ derivatives, i.e. we can prove well-posedness for equations of type (5) for any exponent $s > 2$.

[9] $\square_{\mathbf{g}}$ is the covariant wave operator $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$.

[10] Nonlinear wave equations are either semilinear or quasilinear according to whether the higher order terms - here the terms containing second order derivatives - are linear or nonlinear.

The same type of improvement in the case of quasilinear wave equations requires a highly non-trivial extension of such estimates for wave operators with non-smooth coefficients. The first improved regularity results for quasilinear wave equations of the type,

$$g^{\mu\nu}(\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi) \tag{7}$$

with $g^{\mu\nu}(\phi)$ a non-linear perturbation of the Minkowski metric $m^{\mu\nu}$, are due to [1], [2], and [40], [41] and [13]. The best known results for equations of type (4) were obtained in [14] and [28]. According to them one can lower the Sobolev exponent $s > 5/2$ in Theorem 1.2 to $s > 2$. It turns out, see [22], that these results are sharp in the general class of quasilinear wave equations of type (4). However, the Einstein equations enjoy a special structure, and it was conjectured in [12] that one can obtain a well-posedness result at the level of $s = 2$[11].

Conjecture 1.4 (Bounded L^2 curvature conjecture)**.** *The Einstein- vacuum equations admit local Cauchy developments for initial data sets (Σ_0, g, k) with locally finite L^2 curvature and locally finite L^2 norm of the first covariant derivatives of k*[12].

1.3.2. *The bounded L^2 curvature theorem.* In this section, we state our main result which gives a positive answer to the above conjecture. We assume the space-time $(\mathcal{M}, \mathbf{g})$ to be foliated by the level surfaces Σ_t of a time function t. Let T denote the unit normal to Σ_t, and let k the the second fundamental form of Σ_t, i.e. $k_{ab} = -\mathbf{g}(\mathbf{D}_a T, e_b)$, where $e_a, a = 1, 2, 3$ denotes an arbitrary frame on Σ_t and $\mathbf{D}_a T = \mathbf{D}_{e_a} T$. We assume that the Σ_t foliation is maximal, i.e. we have

$$g^{ab}k_{ab} = 0 \tag{8}$$

where g is the induced metric on Σ_t.

We also recall below the definition of the volume radius on a general Riemannian manifold M.

Definition 1.5. *Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{vol}(p, r)$ at a point $p \in M$ and scales $\leq r$ is defined by*

$$r_{vol}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r^3},$$

with $|B_r|$ the volume of B_r relative to the metric on M. The volume radius $r_{vol}(M, r)$ of M on scales $\leq r$ is the infimum of $r_{vol}(p, r)$ over all points $p \in M$.

Our main result is the following.

Theorem 1.6 (Main theorem)**.** *Let $(\mathcal{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum equations* (1) *together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Assume that the initial slice (Σ_0, g, k) is such that the Ricci curvature $Ric \in L^2(\Sigma_0)$, $\nabla k \in L^2(\Sigma_0)$, and Σ_0 has a strictly positive volume radius on scales ≤ 1, i.e. $r_{vol}(\Sigma_0, 1) > 0$.*

[11] The curvature tensor of g and the first order derivatives of the second fundamental form k are both at the level of two derivatives of $\mathbf{g}$. Thus, Conjecture 1.4 is at the level of two derivatives of $\mathbf{g}$ in L^2 which indeed corresponds to the case $s = 2$.

[12] As we shall see, from the precise theorem stated below, other weaker conditions, such as a lower bound on the volume radius, are needed.

(1) L^2 ***regularity.*** *There exists a time*

$$T = T(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

and a constant

$$C = C(\|Ric\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{vol}(\Sigma_0, 1)) > 0$$

such that the following control holds on $0 \le t \le T$*:*

$$\|\mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \le C,\ \|\nabla k\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \le C \text{ and } \inf_{0\le t\le T} r_{vol}(\Sigma_t, 1) \ge \frac{1}{C}.$$

(2) ***Higher regularity.*** *Within the same time interval as in part (1) we also have the higher derivative estimates*[13]*,*

$$\sum_{|\alpha|\le m} \|\mathbf{D}^{(\alpha)}\mathbf{R}\|_{L^\infty_{[0,T]}L^2(\Sigma_t)} \le C_m \sum_{|i|\le m}\left[\|\nabla^{(i)}Ric\|_{L^2(\Sigma_0)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma_0)}\right], \tag{9}$$

where C_m *depends only on the previous* C *and* m*.*

Let us comment on Theorem 1.6.

(1) As mentioned in the previous section, the well posedness result of [28] in H^s with $s > 2$ is sharp for general quasilinear wave equations of type (4). To do better, one needs to take into account the so called *null structure*, i.e. the special nonlinear structure of the Einstein equations. In particular, Theorem 1.6 is the first well posedness result in which the full structure of the quasilinear hyperbolic system, not just its principal part, plays a crucial role.
(2) The assumptions of Theorem 1.6 concern the L^2 norm of the curvature tensor of g and of the first covariant derivatives of the second fundamental form k which are all invariant in the sense that these objects can be defined without reference to any coordinates system[14]. This allows, when working in the framework of the solutions constructed in Theorem 1.6, to retain an essential property of the Einstein equations, namely the freedom to pick a coordinates system (see Remark 1.1).
(3) The part of Theorem 1.6 dealing with the propagation of higher order regularity provides a continuation argument for the Einstein equations; that is the space-time constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and the first covariant derivatives of its second fundamental form remain L^2- bounded along the leaves of the foliation. In fact, Theorem 1.6 implies the break-down criterion previously obtained in [19] and improved in [24], [42]. Furthermore, this break-down criterion involves only invariant assumptions, and hence provides information on true singularities (as opposed to coordinates singularities).
(4) One may wonder whether the solutions constructed in Theorem 1.6 are as rough as possible. To discuss this issue, observe that the light cones of a Lorentzian space-time $(\mathcal{M}, \mathbf{g})$ can be obtained as the level hypersurfaces of a solution u to the Eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$

[13]Assuming that the initial has more regularity so that the right-hand side of (9) makes sense.

[14]Note that this is not the case for instance of the result in [14] where one has to choose a fixed coordinates system with respect to which the metric coefficients are in H^s for $s > 2$.

Now, a byproduct of the proof of Theorem 1.6 is the fact that L^2 bounds on the curvature is the minimum requirement to control solutions u to the Eikonal equation (see Remark 4.1), and hence to make sense of light cones. As far as light cones are fundamental objects in Lorentzian space-times, it is reasonable to expect their control to be necessary to make sense of solutions to the Einstein equations. For this reason, we conjecture that Theorem 1.6 is optimal.

The entire proof of Theorem 1.6 is contained in the sequence of papers [20] [33] [34] [35] [36] [37]. In the rest of this paper, we discuss the general strategy of the proof as well as the main steps.

2. Sketch of the proof of the main theorem

2.1. **Strategy of the proof.** As mentioned earlier, the well posedness result of [28] in H^s for $s > 2$ is sharp for general quasilinear wave equations of type (4). To do better one needs to take into account the special structure of the Einstein equations and rely on a class of estimates which go beyond Strichartz estimates, namely the so called bilinear estimates. In the case of semilinear wave equations, such as Wave Maps, Maxwell-Klein-Gordon and Yang-Mills, the first results which make use of bilinear estimates go back to [9], [10], [11]. In the particular case of the Yang-Mills equation the main observation was that, after the choice of a special gauge (Coulomb gauge), the most dangerous nonlinear terms exhibit a special, null structure so that the system reduces to the following schematic form

$$\square\phi = Q_{ij}(\phi, \nabla^{-1}\phi) + \nabla^{-1}(Q_{ij}(\phi, \phi)) + l.o.t, \tag{10}$$

where ϕ is vector valued[15] and Q_{ij} is the null form given by

$$Q_{ij}(\phi, \psi) = \partial_i\phi\partial_j\psi - \partial_i\psi\partial_j\phi, \ i, j = 1, 2, 3, \tag{11}$$

for which one can apply the bilinear estimates derived in [9]. With the help of these estimates one was able to derive a well posedness result, in the flat $1 + 3$ dimensional Minkowski space, for the exponent $s = 1$[16].

To carry out a similar program in the case of the Einstein equations one would need, at the very least to

1. Exhibit the null structure, i.e. provide a coordinate condition, relative to which the Einstein vacuum equations verify an appropriate version of the null condition.
2. Exploit the null structure, i.e. prove bilinear estimates for the null quadratic terms appearing in the previous step.

Concerning the coordinate condition, let us first mention that it is a priori not at all clear what it should be, or even if there is one for that matter.

Remark 2.1. *The only known structural condition related to the classical null condition, called the weak null condition* [23]*, tied to wave coordinates, fails the test. Indeed, the following simple system in Minkowski space*

$$\square\phi = 0, \qquad \square\psi = \phi \cdot \Delta\phi$$

[15]Note that (10) is a system. In particular, the schematic notation $Q_{ij}(\phi, \phi)$ should be understood as being a linear combination of terms of the type $Q_{ij}(\phi^m, \phi^l)$ where ϕ^l and ϕ^m denote components of the vector valued function ϕ.

[16]This corresponds precisely to the $s = 2$ exponent in the case of the Einstein-vacuum equations.

verifies the weak null condition and yet, according to [22]*, it is ill posed for* $s = 2$*. Coordinate conditions, such as spatial harmonic*[17]*, also do not seem to work.*

We rely instead on a Coulomb type condition, for orthonormal frames, adapted to a maximal foliation. Such a gauge condition appears naturally if we adopt a Yang-Mills description of the Einstein field equations using Cartan's formalism of moving frames, see [4]. It is important to note nevertheless that it is not at all a priori clear that such a choice would do the job. Indeed, the null form nature of the Yang-Mills equations in the Coulomb gauge is only revealed once we commute the resulting equations with the projection operator $\mathcal{P}$ on the divergence free vectorfields. Such an operation is natural in that case, since $\mathcal{P}$ commutes with the flat d'Alembertian. In the case of the Einstein equations, however, the corresponding commutator term $[\square_{\mathbf{g}}, \mathcal{P}]$ generates[18] a whole host of new terms and it is quite a miracle that they can all be treated by an extended version of bilinear estimates.

Concerning bilinear estimates, let us mention that these types of estimates where only available for the wave operator on the Minkowski space-time. This forces us to find an appropriate geometric framework to extend these estimates to the wave operator on a curved space-time. To this end, we need to rely on a plane wave representation - a parametrix - for solutions of the wave equation on a curved background. Moreover, this parametrix, unlike in the flat case, is only an approximate solution of the wave equation. In other words, when applying the wave operator to the parametrix, we obtain an error term which needs to be controlled.

Furthermore, there is another ingredient needed to establish bilinear estimates on a curved space-time. Numerous bilinear estimates need to be derived, and it turns out that the proof of several of these estimates reduces to sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix.

Finally, the above discussion leads to the following four steps which constitute the basic strategy of our main theorem

- **A**. Exhibit the null structure by recasting the Einstein vacuum equations as a quasilinear Yang-Mills theory[19].
- **B**. Prove appropriate bilinear estimates for solutions ϕ to the scalar wave equation on a curved space-time $\square_{\mathbf{g}}\phi = 0$.
- **C**. Construct an effective progressive wave representation Φ_F (parametrix) for solutions to the scalar linear wave equation $\square_{\mathbf{g}}\phi = F$, derive appropriate bounds for both the parametrix and the corresponding error term $E = F - \square_{\mathbf{g}}\Phi_F$ and use them to derive the desired bilinear estimates.
- **D**. Prove sharp $L^4(\mathcal{M})$ Strichartz estimates for a localized version of the parametrix of step C.

[17] Maximal foliation together with spatial harmonic coordinates on the leaves of the foliation would be the coordinate condition closest in spirit to the Coulomb gauge.

[18] Note also that additional error terms are generated by projecting the equations on the components of the frame.

[19] The classical Yang-Mills equations are semilinear, i.e. they are defined on a given (Lorentzian) background. Here, we recast the Einstein vacuum equations as a Yang-Mills theory on the background $(\mathcal{M}, \mathbf{g})$ solution to (1). As the background is not given but instead the unknown of the problem itself, we obtain a quasilinear analog of the Yang-Mills equations.

While Step A is purely algebraic, Steps B, C and D all require to establish estimates. The main difficulty is to implement these steps using only hypothetical L^2 bounds for the space-time curvature tensor, consistent with the statement of our main theorem. To achieve this, we crucially need to exploit the null structure of the equations at every stage in the proof.

In the rest of the paper, we comment on each of these steps. We start with Step A in the next section. We then show how to conclude the proof of the main theorem when assuming Steps B, C and D. Finally, we discuss Steps B, C and D.

2.2. **The Yang-Mills formalism (Step A).** We cast the Einstein-vacuum equations in a Yang-Mills form which corresponds to step A in the strategy outlined above. This relies on the Cartan formalism of moving frames. The idea is to give up on a choice of coordinates and instead express the Einstein vacuum equations in terms of the connection 1-forms associated to moving orthonormal frames, i.e. vectorfields e_α, which verify,

$$\mathbf{g}(e_\alpha, e_\beta) = \mathbf{m}_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1).$$

The connection 1-forms (they are to be interpreted as 1-forms with respect to the external index μ with values in the Lie algebra of $so(3,1)$), defined by the formulas,

$$(\mathbf{A}_\mu)_{\alpha\beta} = \mathbf{g}(\mathbf{D}_\mu e_\beta, e_\alpha) \tag{12}$$

verify the equations,

$$\mathbf{D}^\mu \mathbf{F}_{\mu\nu} + [\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}] = 0 \tag{13}$$

where, denoting $(\mathbf{F}_{\mu\nu})_{\alpha\beta} := \mathbf{R}_{\alpha\beta\mu\nu}$,

$$(\mathbf{F}_{\mu\nu})_{\alpha\beta} = \big(\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu]\big)_{\alpha\beta}. \tag{14}$$

In other words we can interpret the curvature tensor as the curvature of the $so(3,1)$-valued connection 1-form $\mathbf{A}$. Note also that the covariant derivatives are taken only with respect to the *external indices* μ, ν and do not affect the *internal indices* α, β. We can rewrite (13) in the form,

$$\Box_{\mathbf{g}} \mathbf{A}_\nu - \mathbf{D}_\nu(\mathbf{D}^\mu \mathbf{A}_\mu) = \mathbf{J}_\nu(\mathbf{A}, \mathbf{D}\mathbf{A}) \tag{15}$$

where,

$$\mathbf{J}_\nu = \mathbf{D}^\mu([\mathbf{A}_\mu, \mathbf{A}_\nu]) - [\mathbf{A}_\mu, \mathbf{F}_{\mu\nu}].$$

Observe that the equations (13)-(14) look just like the Yang-Mills equations on a fixed Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ except, of course, that in our case $\mathbf{A}$ and $\mathbf{g}$ are not independent but rather connected by (12), reflecting the quasilinear structure of the Einstein equations. Just as in the case of [9], which establishes the well-posedness of the Yang-Mills equation in Minkowski space in the energy norm (i.e. $s = 1$), we rely in an essential manner on a Coulomb type gauge condition. More precisely, we take e_0 to be the future unit normal to the Σ_t foliation and choose e_1, e_2, e_3 an orthonormal basis to Σ_t, in such a way that we have, essentially $\mathrm{d}iv\, A = \nabla^i A_i = 0$, where A is the spatial component of $\mathbf{A}$. It turns out that A_0 satisfies an elliptic equation while each component $A_i = \mathbf{g}(\mathbf{A}, e_i)$, $i = 1, 2, 3$ verifies an equation of the form,

$$\Box_{\mathbf{g}} A_i = -\partial_i(\partial_0 A_0) + A^j \partial_j A_i + A^j \partial_i A_j + \text{ l.o.t.} \tag{16}$$

with l.o.t. denoting nonlinear terms which can be treated by more elementary techniques (including non sharp Strichartz estimates).

2.3. **The proof of the bounded L^2 curvature Theorem.** In this section, assuming that step B holds - which corresponds to having appropriate bilinear estimates at our disposal - we conclude the proof Theorem 1.6. Now, to be in position to use these bilinear estimates, we first need to reduce the problem to a wave equation. In view of (16), we thus need to eliminate $\partial_i(\partial_0 A_0)$. To this end, we need to project (16) onto divergence free vectorfields with the help of a non-local operator which we denote by $\mathcal{P}$. In the case of the flat Yang-Mills equations, treated in [9], this leads to an equation of the form,

$$\Box A_i = \mathcal{P}(A^j \partial_j A_i) + \mathcal{P}(A^j \partial_i A_j) + \text{ l.o.t.}$$

where both terms on the right exhibit the null structure[20]. In our case however, the operator $\mathcal{P}$ does not commute with $\Box_{\mathbf{g}}$. It turns out, fortunately, that the terms generated by commutation can still be estimated by an extended class of bilinear estimates which includes contractions with the curvature tensor. Thus, we obtain in the end schematically for A_i

$$\Box_{\mathbf{g}} A_i = \text{null forms} + \text{ l.o.t.}, \tag{17}$$

where up to (cubic) lower order terms, the quadratic terms exhibit the null structure.

We are now in position to conclude the proof of our main theorem. Recall that the A_i are connection coefficients, and hence the curvature is a the level of one derivative of the A_i (see (14)). In particular, controlling the curvature tensor in L^2 corresponds to the control of first order derivatives of A_i in L^2. In other words, we need to run the energy estimate for the wave equation (17). In the case of the standard wave equation on the Minkowski space-time, the energy estimate is based on the usual timelike Killing vectorfield ∂_t. In our case, the corresponding vectorfield $e_0 = T$ (the future unit normal to Σ_t) is not Killing. This leads to another class of trilinear error terms. That is to say, to control the energy estimates for the wave equation (17) we need trilinear estimates, while to control the null forms in the right-hand side we need bilinear estimates. Assuming these bilinear and trilinear estimates hold, we finally control first order derivatives of A_i in L^2 and hence the curvature tensor in L^2. This concludes the proof of Theorem 1.6.

The rest of the paper is organized as follows. In section 3, we discuss the derivation of the bilinear estimates[21] which corresponds to Step B. This derivation relies on Step C and Step D which we discuss respectively in section 4 and 5.

3. Bilinear estimates (Step B)

3.1. **The plane wave representation on a curved space-time.** In order to establish bilinear estimates on a curved space-time, we need to rely on a plane wave representation formula[22] for solutions of scalar wave equations,

$$\Box_{\mathbf{g}} \phi = 0.$$

To build such a plane wave representation, consider a plane wave

$$e^{i\lambda\ {}^{\omega}u(t,x)},\ \lambda \in [0, +\infty),\ \omega \in \mathbb{S}^2,$$

[20] This corresponds to (10) where the null structure manifests itself in the presence of the null form Q_{ij} in the right-hand side.

[21] As we have seen, trilinear estimates have to be derived as well, but we skip this part for the sake of simplicity.

[22] We follow the proof of the bilinear estimates outlined in [15] which differs substantially from that of [9] and is reminiscent of the null frame space strategy used by Tataru in his fundamental paper [39].

with λ and ω parameters corresponding to Fourier variables in $\mathbb{R}^3$ in spherical coordinates. We compute

$$\square_{\mathbf{g}}(e^{i\lambda\ {}^\omega u}) = (-\lambda^2\mathbf{g}^{\alpha\beta}\partial_\alpha({}^\omega u)\partial_\beta({}^\omega u) + i\lambda\square_{\mathbf{g}}({}^\omega u))e^{i\lambda\ {}^\omega u}.$$

The first term turns out to be the most dangerous one[23]. This motivates to choose ${}^\omega u$ solution to the Eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha({}^\omega u)\partial_\beta({}^\omega u) = 0,$$

in which case we obtain

$$\square_{\mathbf{g}}(e^{i\lambda\ {}^\omega u}) = i\lambda\square_{\mathbf{g}}({}^\omega u)e^{i\lambda\ {}^\omega u}.$$

This yields in general an approximate solution to $\square_{\mathbf{g}}(\phi) = 0$. We then superpose these plane waves to obtain a full plane wave representation.

In the particular case of the standard wave equation on the Minkowski space-time, we recover the well-known plane wave representation which is an exact solution[24]. We have

$$\phi = \sum_\pm \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^\omega u_\pm(t,x)} f_\pm(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{18}$$

where ${}^\omega u_\pm(t,x) = \pm t + x\cdot\omega$ and $f_\pm$ can be explicitly computed in terms of the Fourier transform of the initial data set $(\phi(0,.), \partial_t\phi(0,.))$ of ϕ. In the general case, we superpose the basic plane waves as in the right-hand side of (18), and choose ${}^\omega u_\pm$ solution of the Eikonal equation with the following asymptotic behavior on Σ_0

$${}^\omega u_\pm(0,x) \sim x\cdot\omega \text{ when } |x| \to +\infty.$$

This asymptotic behavior is necessary to be able to generate any initial data of the wave equation.

In view of the previous paragraph, we consider the following representation formula[25]

$$\phi_f(t,x) = \int_{\mathbb{S}^2}\int_0^{\infty} e^{i\lambda\ {}^\omega u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{19}$$

where f represents schematically the initial data[26], and where ${}^\omega u$ is a solution of the eikonal equation[27],

$$\mathbf{g}^{\alpha\beta}\partial_\alpha({}^\omega u)\,\partial_\beta({}^\omega u) = 0, \tag{20}$$

with the following asymptotic behavior on Σ_0

$${}^\omega u(0,x) \sim x\cdot\omega \text{ when } |x| \to +\infty.$$

Remark 3.1. (20) *is a nonlinear transport equation. Hence, ${}^\omega u$ needs to be prescribed not only at infinity on Σ_0 as explained above, but everywhere on Σ_0. This choice of ${}^\omega u$ on Σ_0 turns out to be delicate and is discussed in section 4 (see Step C1 and related subsequent comments).*

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[23] λ should be understood as a Fourier variable corresponding to a derivative in physical space. The λ^2 term hence costs 2 derivatives while the wave equation only recovers one. Thus, this term is problematic as it induces a derivative loss.

[24] This is special to the flat case. In the general case, we only obtain an approximate solution.

[25] (19) actually corresponds to the representation formula for a half-wave. The full representation formula corresponds to the sum of two half-waves as in (18). Since the bilinear estimates are identical for both half waves, we only consider one of them for simplicity.

[26] Here f is in fact at the level of the Fourier transform of the initial data and the norm $\|\lambda f\|_{L^2(\mathbb{R}^3)}$ corresponds, roughly, to the H^1 norm of the data .

[27] As we have seen above, we have ${}^\omega u(t,x) = \pm t + x\cdot\omega$ in the flat Minkowski space.

Remark 3.2. *Note that* (19) *is a parametrix for a scalar wave equation. The lack of a good parametrix for a tensorial wave equation forces us to develop a strategy based on writing the main equation in components relative to a frame, i.e. instead of dealing with the tensorial wave equation* (15) *directly, we consider the system of scalar wave equations* (16)*. Unlike the flat case, this "scalarization" procedure produces several terms which are potentially dangerous, and it is fortunate, as in yet another manifestation of a hidden null structure of the Einstein equations, that they can still be controlled by the use of an extended*[28] *class of bilinear estimates.*

3.2. **Bilinear estimates on a curved space-time.** The bilinear estimates all involve after some reductions the null form Q_{ij} introduced in (11). Let us briefly explain how the structure of Q_{ij} is exploited to derive these estimates. For simplicity, we focus on two specific bilinear estimates[29].

The first example of a bilinear estimate on a curved space-time aims at controlling the $L^2(\mathcal{M})$ norm of the null form $Q_{ij}(\phi_f, \psi)$, where ϕ_f is given by (19). We compute

$$\begin{aligned} Q_{ij}(\phi_f, \psi) &= Q_{ij}\left(\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^{\omega}u(t,x)} f(\lambda\omega)\lambda^2 d\lambda d\omega, \psi\right) \\ &= \int_{\mathbb{S}^2}\int_0^{+\infty} Q_{ij}(e^{i\lambda\ {}^{\omega}u(t,x)}, \psi) f(\lambda\omega)\lambda^2 d\lambda d\omega \\ &= i\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^{\omega}u(t,x)} Q_{ij}({}^{\omega}u, \psi) f(\lambda\omega)\lambda^3 d\lambda d\omega. \end{aligned}$$

Now we have

$$Q_{ij}({}^{\omega}u, \psi) = \partial_i({}^{\omega}u)\partial_j\psi - \partial_j({}^{\omega}u)\partial_i\psi. \tag{21}$$

The fundamental observation which ultimately allows us to derive a bilinear estimate in this case is the fact that the structure of Q_{ij} is such that

$$Q_{ij}({}^{\omega}u, \psi) \text{ is tangent to the level hypersurfaces of } {}^{\omega}u, \tag{22}$$

as can be seen from (21).

The second example of a bilinear estimate on a curved space-time aims at controlling the following expression

$$\|\nabla^{-1}(Q_{ij}(\phi_{f_1}, \phi_{f_2}))\|_{L^2(\mathcal{M})}.$$

First, we decompose ϕ_{f_1} and ϕ_{f_2} in dyadic frequencies according to

$$\phi_f = \sum \phi_{f,p},\ \phi_{f,p} = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\ {}^{\omega}u(t,x)} \psi(2^{-p}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega$$

where $\lambda \sim 2^p$ on the support of $\psi(2^{-p}\lambda)$. We infer

$$\|\nabla^{-1}(Q_{ij}(\phi_{f_1}, \phi_{f_2}))\|_{L^2(\mathcal{M})} \lesssim \sum_{p\geq q}\|\nabla^{-1}(Q_{ij}(\phi_{f_1,p}, \phi_{f_2,q}))\|_{L^2(\mathcal{M})}.$$

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[28]involving contractions between the Riemann curvature tensor and derivatives of solutions of scalar wave equations.

[29]The two examples of bilinear estimates discussed here have both an analog in the semilinear case. Indeed, they correspond to estimating the $L^2_{t,x}$ norm of each of the term in the right-hand side of (10).

It is at this stage that we use the null structure of Q_{ij} by noticing that

$$Q_{ij}(\phi,\psi) = \partial_i(\phi\partial_j\psi) - \partial_j(\phi\partial_i\psi) = \partial_j(\psi\partial_i\phi) - \partial_i(\psi\partial_j\phi) \tag{23}$$

so that we may choose which derivative we factorize. We choose to factorize the derivative corresponding to the highest frequency which yields

$$\begin{aligned}\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})} &\lesssim \sum_{p\geq q}\|\nabla^{-1}\partial(\phi_{f_1,p}\partial\phi_{f_2,q})\|_{L^2(\mathcal{M})}\\ &\lesssim \sum_{p\geq q}\|\phi_{f_1,p}\|_{L^4(\mathcal{M})}\|\partial\phi_{f_2,q}\|_{L^4(\mathcal{M})}.\end{aligned}$$

The last ingredient is the sharp $L^4(\mathcal{M})$ Strichartz of Step D (see section 5) which finally yields

$$\begin{aligned}\|\nabla^{-1}(Q_{ij}(\phi_{f_1},\phi_{f_2}))\|_{L^2(\mathcal{M})} &\lesssim \sum_{p\geq q}2^{-\frac{|p-q|}{2}}\|\lambda f_{1,p}\|_{L^2(\mathbb{R}^3)}\|\lambda f_{2,q}\|_{L^2(\mathbb{R}^3)}\\ &\lesssim \|\lambda f_1\|_{L^2(\mathbb{R}^3)}\|\lambda f_2\|_{L^2(\mathbb{R}^3)}\end{aligned}$$

and concludes the proof of the second example of bilinear estimate.

Remark 3.3. *The null structure of Q_{ij} is exploited differently in the two examples of bilinear estimates presented above as can be seen by comparing* (22) *and* (23)*.*

4. Control of the parametrix (step C)

To prove the bilinear and trilinear estimates of Step B, we need in particular to control the parametrix given by (19). To this end, it turns out that it suffices to control the parametrix at initial time (i.e. restricted to the initial slice Σ_0)

$$\phi_f(0,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(0,x)}f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{24}$$

as well as the error term[30] corresponding to (19)

$$Ef(t,x) = \square_{\mathbf{g}}\phi_f(t,x) = i\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)}\square_{\mathbf{g}}({}^\omega u)f(\lambda\omega)\lambda^3 d\lambda d\omega. \tag{25}$$

This requires the following four sub steps

- **C1** *Make an appropriate choice for the equation satisfied by ${}^\omega u(0,x)$ on Σ_0, and control the geometry of the foliation of Σ_0 by the level surfaces of ${}^\omega u(0,x)$.*
- **C2** *Prove that the parametrix at $t=0$ given by* (24)[31] *is bounded in $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma_0))$ using the estimates for ${}^\omega u(0,x)$ obtained in* **C1***.*
- **C3** *Control the geometry of the foliation of $\mathcal{M}$ given by the level hypersurfaces of ${}^\omega u$.*
- **C4** *Prove that the error term* (25) *satisfies the estimate $\|Ef\|_{L^2(\mathcal{M})}\lesssim\|\lambda f\|_{L^2(\mathbb{R}^3)}$ using the estimates for ${}^\omega u$ and $\square_{\mathbf{g}}({}^\omega u)$ proved in* **C3***.*

[30] Note that ϕ_f is an exact solution of $\square_{\mathbf{g}}\phi = 0$ only if $\square_{\mathbf{g}}({}^\omega u) = 0$. Hence, ϕ_f is an exact solution only in flat space.

[31] (24) only corresponds to the value at $t=0$ of a half wave parametrix. The full parametrix at initial data is the sum of two half waves as in (18). Step C2 actually corresponds to proving that the parametrix at $t=0$ generates any initial data to the wave equation $\square_{\mathbf{g}}\phi = 0$ with a suitable control of the corresponding $f_\pm$. We have chosen to provide a more restricted statement of Step C2 to simplify the exposition.

To achieve Step C3 and Step C4, we need, at the very least, to control $\square_{\mathbf{g}}({}^{\omega}u)$ in L^∞. This issue was first addressed in the sequence of papers [16] [17] [18] where an L^∞ bound for $\square_{\mathbf{g}}({}^{\omega}u)$ was established, depending only on the L^2 norm of the curvature flux along null hypersurfaces. The proof required an interplay between both geometric and analytic techniques and had all the appearances of being sharp, i.e. we don't expect an L^∞ bound for $\square_{\mathbf{g}}({}^{\omega}u)$ which requires bounds on less than two derivatives in L^2 for the metric[32].

Remark 4.1. *It turns out, as a byproduct of the proof of Step C3, that the radius of injectivity of the level hypersurfaces of* ${}^{\omega}u$ *is controlled by the* L^∞ *norm of* $\square_{\mathbf{g}}({}^{\omega}u)$*. Furthermore, this control appears to be sharp. In other words, we expect to loose control over the radius of injectivity in the absence of this bound. Hence, in view of the discussion above,* L^2 *bounds on the curvature tensor appear to be minimal for the control of the Eikonal equation.*

To obtain the L^2 bound for the Fourier integral operator E defined in (25), we need, of course, to go beyond uniform estimates for $\square_{\mathbf{g}}({}^{\omega}u)$. The classical L^2 bounds for Fourier integral operators of the form (25) are not at all economical in terms of the number of integration by parts which are needed. In our case the total number of such integration by parts is limited by the regularity properties of the function $\square_{\mathbf{g}}({}^{\omega}u)$. To get an L^2 bound for the parametrix at initial time (24) and the error term (25) within such restrictive regularity properties we need, in particular:

- In Step C1 and Step C3, a precise control of derivatives of ${}^{\omega}u$ and $\square_{\mathbf{g}}({}^{\omega}u)$ with respect to both ω as well as with respect to various directional derivatives[33]. To get optimal control we need, in particular, a very careful construction of the initial condition for ${}^{\omega}u$ on Σ_0 and then sharp space-time estimates of Ricci coefficients, and their derivatives, associated to the foliation induced by ${}^{\omega}u$.
- In Step C2 and Step C4, a careful decompositions of the Fourier integral operators (24) and (25) in both λ and ω, similar to the first and second dyadic decomposition in harmonic analysis, see [30], as well as a third decomposition, which in the case of (25) is done with respect to the space-time variables relying on the geometric Littlewood-Paley theory developed in [18].

Below, we make further comments on Steps C1-C4:

(1) *The choice of* ${}^{\omega}u(0,x)$ *on* Σ_0 *in Step C1.* Let us note that the typical choice ${}^{\omega}u(0,x) = x\cdot\omega$ in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework. Instead, we need to find a geometric definition of ${}^{\omega}u(0,x)$. A natural choice would be that $u = {}^{\omega}u$ verifies
$$\square_{\mathbf{g}} u = 0 \text{ on } \Sigma_0$$
which by a simple computation turns out to be the following simple variant of the minimal surface equation[34]
$$\mathrm{d}iv\left(\frac{\nabla u}{|\nabla u|}\right) = k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

[32]classically, this requires, at the very least, the control of $\mathbf{R}$ in L^∞.

[33]Taking into account the different behavior in tangential and transversal directions with respect to the level surfaces of ${}^{\omega}u$.

[34]In the time symmetric case $k=0$, this is exactly the minimal surface equation.

Unfortunately, this choice does not allow us to have enough control of the derivatives of u in the normal direction to the level surfaces of u. This forces us to look for an alternate equation for u:

$$\mathrm{d}iv\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

This equation turns out to be parabolic in the normal direction to the level surfaces of u, and allows us to obtain the desired regularity in Step C1. A closer inspection reveals its relation to the mean curvature flow on Σ_0.

(2) *How to achieve Step C3.* The regularity obtained in Step C1, together with null transport equations tied to the eikonal equation, elliptic systems of Hodge type, the geometric Littlewood-Paley theory of [18], sharp trace theorems, and an extensive use of the structure of the Einstein equations, allows us to propagate the regularity on Σ_0 to the space-time, thus achieving Step C3.

(3) *The regularity with respect to ω in Steps C1 and C3.* The regularity with respect to x for u is clearly limited as a consequence of the fact that we only assume L^2 bounds on $\mathbf{R}$. On the other hand, $\mathbf{R}$ is independent of the parameter ω, and one might infer that u is smooth with respect to ω. Surprisingly, this is not at all the case. Indeed, the regularity in x obtained for u in Steps C1 and C3 is better in directions tangent to the level hypersurfaces of u. Now, the ω derivatives of the tangential directions have non zero normal components. Thus, when differentiating the structure equations with respect to ω, tangential derivatives to the level surfaces of u are transformed to non tangential derivatives which in turn severely limits the regularity in ω obtained in Steps C1 and C3.

(4) *How to achieve Steps C2 and C4.* The classical arguments for proving L^2 bounds for Fourier operators are based either on a TT^* argument, or a T^*T argument, which requires several integration by parts either with respect to x for T^*T, of with respect to (λ, ω) for TT^*. Both methods would fail by far within the regularity for u obtained in Step C1 and Step C3. This forces us to design a method which allows to take advantage both of the regularity in x and ω. This is achieved using in particular the following ingredients
- Geometric integrations by parts taking full advantage of the better regularity properties in directions tangent to the level hypersurfaces of u.
- The standard first and second dyadic decomposition in frequency space, with respect to both size and angle (see [30]), an additional decomposition in physical space relying on the geometric Littlewood-Paley projections of [18] for Step C4, as well as another decomposition involving frequency and angle for Step C2.

Even with these precautions, at several places in the proof, one encounters log-divergences which have to be tackled by ad-hoc techniques, taking full advantage of the null structure of the Einstein equations.

5. Sharp $L^4(\mathcal{M})$ Strichartz estimates (Step D)

Recall that the parametrix constructed in Step C is also used to prove sharp $L^4(\mathcal{M})$ Strichartz estimates. Indeed the proof of several bilinear estimates of Step B reduces to the proof of sharp $L^4(\mathcal{M})$ Strichartz estimates for the parametrix (19) with λ localized in a dyadic shell (see section 3.2).

More precisely, let $j \geq 0$, and let ψ a smooth function on $\mathbb{R}^3$ supported in

$$\frac{1}{2} \leq |\xi| \leq 2.$$

Let $\phi_{f,j}$ the parametrix (19) with a additional frequency localization $\lambda \sim 2^j$

$$\phi_{f,j}(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda\ {}^\omega u(t,x)}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{26}$$

We will need the sharp[35] $L^4(\mathcal{M})$ Strichartz estimate

$$\|\phi_{f,j}\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{27}$$

The standard procedure for proving[36] (27) is based on a TT^* argument which reduces it to an L^∞ estimate for an oscillatory integral with a phase involving ${}^\omega u$. This is then achieved by the method of stationary phase which requires quite a few integrations by parts. In fact the standard argument would require, at the very least[37], that the phase function $u = {}^\omega u$ verifies,

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega^2 u \in L^\infty. \tag{28}$$

This level of regularity is, unfortunately, incompatible with the regularity properties of solutions to our eikonal equation (20). In fact, based on the estimates for ${}^\omega u$ derived in step C3, we are only allowed to assume

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega u \in L^\infty. \tag{29}$$

We are thus forced to follow an alternative approach[38] to the stationary phase method inspired by [27] and [28] .

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[35]Note in particular that the corresponding estimate in the flat case is sharp.

[36]Note that the procedure we describe would prove not only (27) but the full range of mixed Strichartz estimates.

[37]The regularity (28) is necessary to make sense of the change of variables involved in the stationary phase method.

[38] We refer to the approach based on the overlap estimates for wave packets derived in [27] and [28] in the context of Strichartz estimates respectively for $C^{1,1}$ and $H^{2+\epsilon}$ metrics. Note however that our approach does not require a wave packet decomposition.

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