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RECENT RESULTS ON STATIONARY CRITICAL KIRCHHOFF SYSTEMS IN CLOSED MANIFOLDS

EMMANUEL HEBEY AND PIERRE-DAMIEN THIZY

Abstract. We report on results we recently obtained in Hebey and Thizy [11, 12] for critical stationary Kirchhoff systems in closed manifolds. Let (M^n, g) be a closed n-manifold, $n \geq 3$. The critical Kirchhoff systems we consider are written as

$$\Big(a + b\sum_{j=1}^{p}\int_M |\nabla u_j|^2 dv_g\Big)\Delta_g u_i + \sum_{j=1}^{p} A_{ij}u_j = |U|^{2^\star-2}\,u_i$$

for all $i = 1, \dots, p$, where Δ_g is the Laplace-Beltrami operator, A is a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p\times p$ matrices with real entries, the A_{ij}'s are the components of A, $U = (u_1, \dots, u_p)$, $|U| : M \to \mathbb{R}$ is the Euclidean norm of U, $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and we require that $u_i \geq 0$ in M for all $i = 1, \dots, p$. We discuss the two following issues in this text: the question of the existence of nontrivial solutions to our systems, together with the dual question of getting nonexistence results in parallel to our existence results, and the question of the stability of our systems which measures how much the equations are robust with respect to variations of their natural parameters a, b, and A.

The Kirchhoff equation was proposed in 1883 by Kirchhoff [13] as an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. It was written as

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0\ ,$$

where L is the length of the string, h is the area of the cross-section, E is the young modulus (elastic modulus) of the material, ρ is the mass density, and P_0 is the initial tension. Almost one century later, Jacques Louis Lions [16] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with external force term which was written as

$$\frac{\partial^2 u}{\partial t^2} + \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x, u)\ ,$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace-Beltrami Euclidean Laplacian. We investigate in this paper the stationary version of this equation, in the case of closed manifolds, and when f is a pure power nonlinearity. We choose the pure power nonlinearity to be critical, and shift to the more involved case of a multi-valued version of the equation. The combination of the nonlocal aspects inherent to the Kirchhoff equation and of the multi-valued character of systems lead to surprising effects such as the energy and bubbling control we can get for the equations, or such as the nonexistence results we can prove when $p \geq 3$ and which exhibit the importance of the cooperativeness assumption in the existence results we obtain by passing through the subcritical equations and using the compactness theory attached to bubbling control we just mentioned.The 3-dimensional case of our equations has been investigated in Hebey and Thizy [11]. The n-dimensional case, $n \geq 4$, was investigated in Hebey and Thizy [12].

In what follows we let (M^n, g) be a closed Riemannian n-manifold with $n \geq 3$, $p \in \mathbb{N}^\star$ be a nonzero integer, $a, b > 0$ be positive real numbers, and $A : M \to M^p_s(\mathbb{R})$ be a C^1-map from M into the space $M^p_s(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. The Kirchhoff system of p equations we investigate in this paper is written as

$$\Big(a + b\sum_{j=1}^p \int_M |\nabla u_j|^2 dv_g\Big)\Delta_g u_i + \sum_{j=1}^p A_{ij}u_j = |U|^{2^\star - 2}\, u_i \tag{0.1}$$

for all $i = 1, \ldots, p$, where $\Delta_g = -\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator, the A_{ij}'s are the components of A, $U = (u_1, \ldots, u_p)$, $|U| : M \to \mathbb{R}$ is given by

$$|U| = \sqrt{\sum_{j=1}^p u_j^2}\,,$$

$2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and we require that $u_i \geq 0$ in M for all $i = 1, \ldots, p$. We address several questions in this paper such as the question of the existence or nonexistence of solutions, and the question of the compactness associated to (0.1). As a general remark, elliptic regularity theory applies so that any H^1-solution to a system like (0.1) is also a strong solution of class C^2 of the system. Because of this remark, solutions in this text are strong C^2-solutions.

1. Bounded energy

Let $a, b > 0$ be positive real numbers, and $A : M \to M^p_s(\mathbb{R})$ be a C^1-map. We consider perturbations of our Kirchhoff system (0.1) where we allow asymptotically-critical subcritical nonlinearities. These are written as

$$\Big(a_\alpha + b_\alpha\sum_{j=1}^p \int_M |\nabla u_j|^2 dv_g\Big)\Delta_g u_i + \sum_{j=1}^p A^\alpha_{ij}u_j = |U|^{p_\alpha - 2}\, u_i \tag{1.1}$$

for all $i = 1, \ldots, p$, where $(p_\alpha)_\alpha$ is a sequence of numbers in $(2, 2^\star]$ such that $p_\alpha \to 2^\star$ as $\alpha \to +\infty$, $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ are two sequences of positive real numbers converging to a and b, and $(A_\alpha)_\alpha$ is a sequence of C^1-maps $A_\alpha : M \to M^p_s(\mathbb{R})$ converging C^1 to A. A sequence $(U_\alpha)_\alpha$ of p-maps is naturally said to be a sequence of nonnegative solutions of (1.1) if U_α has nonnegative components and solves the α-equation (1.1) for all α. We let the H^1-norm of a p-map U be given by $\|U\|_{H^1} = \sum_i \|u_i\|_{H^1}$, where the u_i's are the components of U, and $\|\cdot\|_{H^1}$ is the usual H^1-norm for functions given by $\|u\|^2_{H^1} = \int_M (|\nabla u|^2 + u^2) dv_g$. One of the main effect of the nonlinear aspect of the Kirchhoff equations is that it produces bounded energy. This statement, see Theorem 1.1 below, is true in almost all dimensions. We let

$$S = \frac{n(n-2)\omega_n^{2/n}}{4} \tag{1.2}$$

be the sharp constant in the Euclidean Sobolev inequality $S\|u\|^2_{L^{2^\star}} \leq \|\nabla u\|^2_{L^2}$, where ω_n is the volume of the unit n-sphere.

Theorem 1.1 (Bounded Energy; Hebey and Thizy [11, 12])**.** *Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 3$, $p \in \mathbb{N}^\star$ be a nonzero integer, $a, b > 0$ be positive real numbers, and $A : M \to M^p_s(\mathbb{R})$ be a C^1-map from M into the space $M^p_s(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. When $n = 4$ we assume either that the scalar curvature S_g of g is positive or that $bS^{2^\star/2} > 1$, where S is as in (1.2). Then there exists $C > 0$ such that*

$$\|U_\alpha\|_{H^1} \leq C \tag{1.3}$$

for all α, all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (1.1)*, and all perturbed systems* (1.1)*, where $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converge to a and b, $(p_\alpha)_\alpha$ is a sequence of numbers in $(2, 2^\star]$ converging to $2^\star$ and $(A_\alpha)_\alpha$ converges C^1 to A.*

The 4-dimensional case in Theorem 1.1 turns out to be very special. The two following remarks from Hebey and Thizy [12] comment on this special case. Remark 1.1 shows that the condition $bS^{2^\star/2} > 1$ can be generalized when we restrict ourselves to purely critical perturbations (for which, by definition, $p_\alpha = 2^\star$ for all α). Remark 1.2 shows that the two extra 4-dimensional conditions requiring that either $S_g > 0$ in M, or $bS^{2^\star/2} > 1$, are necessary. The notation $\lambda\mathbb{N}^\star$ for $\lambda \in \mathbb{R}$ refers to the set consisting of the λn's for $n \geq 1$ integer.

Remark 1.1 (Bounded energy for almost all b)**.** *Let (M^4, g) be a closed Riemannian 4-manifold, $p \in \mathbb{N}^\star$ be a nonzero integer, $a, b > 0$ be positive real numbers, and $A : M \to M_s^p(\mathbb{R})$ be a C^1-map. Assume that $\frac{1}{b} \notin S^{2^\star/2}\mathbb{N}$, where S is as in* (1.2)*. Then there exists $C > 0$ such that*

$$\|U_\alpha\|_{H^1} \leq C$$

for all α, all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (1.1)*, and all perturbed systems* (1.1) *in the purely critical case for which $p_\alpha = 2^\star$ for all α, where $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converge to a and b, and $(A_\alpha)_\alpha$ converges C^1 to A.*

Concerning the sharpness of the 4-dimensional conditions we also get that the following holds true. The result here is built on the Pistoia and Vétois [18] bubble construction for Schrödinger equations.

Remark 1.2 (Sharpness of the 4-dimensional conditions)**.** *Let (M^4, g) be a closed Riemannian 4-manifold and $a > 0$ be a positive real number. Assume that the scalar curvature S_g of g is nonpositive in M. Then, there exist $A : M \to M_s^p(\mathbb{R})$ a C^1-map, $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ sequences of positive real numbers converging to a and $b = S^{-2^\star/2}$, where S is as in* (1.2)*, $(A_\alpha)_\alpha$ a sequence converging C^1 to A, $(p_\alpha)_\alpha$ a sequence of real numbers in $(2, 2^\star]$ converging to $2^\star$ as $\alpha \to +\infty$, and $(U_\alpha)_\alpha$ a sequence of nonnegative solutions of* (1.1) *such that $\|U_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$.*

Remark 1.2 clearly shows that we cannot expect to have (1.3) if we do not contradict the assumptions $S_g \leq 0$ and $bS^2 = 1$. For the moment it is still an open question to know whether or not we can get a similar remark in the purely critical case of (1.1).

2. Bubbling Control and stability

Bounded energy is a necessary condition for applying the H^1-theory for blow-up. When dealing with sequences $(U_\alpha)_\alpha$ of solutions of (1.1) of bounded energy, and more generally with Palais-Smale sequences, the H^1-theory as developed by Struwe [21] applies. For such sequences, see Druet, Hebey and Vétois [8] or Thizy [22], there holds that, up to passing to a subsequence,

$$U_\alpha = U_\infty + \sum_{i=1}^{k} c_i K_\alpha^{1/(p_\alpha - 2)} \mathcal{B}_\alpha^i + \mathcal{R}_\alpha \tag{2.1}$$

for some $k \in \mathbb{N}$, where $U_\infty : M \to \mathbb{R}^p$ is the weak limit in H^1 (or the strong limit in L^2) of the U_α's, K_α is given by $K_\alpha = a_\alpha + b_\alpha \sum_{i=1}^p \int_M |\nabla u_{i,\alpha}|^2 dv_g$, $\mathcal{R}_\alpha \to 0$ in H^1 as $\alpha \to +\infty$, and the $(\mathcal{B}_\alpha^i)_\alpha$'s are vector bubbles given by

$$\mathcal{B}_\alpha^i(x) = \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha}, x)^2}{n(n-2)}} \right)^{(n-2)/2} \Lambda_i \tag{2.2}$$

for all $x \in M$ and all α, where $(x_{i,\alpha})_\alpha$ is a converging sequence of points in M, $(\mu_{i,\alpha})_\alpha$ is a sequence of positive real numbers converging to 0, Λ_i is a unit vector in $\mathbb{R}^p$ with nonnegative components, and d_g is the Riemannian distance. In this H^1-decomposition (2.1), the c_i's are positive real numbers in $[1, +\infty)$ which satisfy that $c_i = 1$ in the purely critical case. The vector bubbles (2.2) are built on the Druet, Hebey and Vétois [8] extension to systems of the classification result in Caffarelli, Gidas and Spruck [3]. We define $\mathcal{N}(U_\alpha)$ to be the maximum integer k we can have in (2.1) for a subsequence of a H^1-bounded sequence $(U_\alpha)_\alpha$ of nonnegative solutions of (1.1). Another very surprising effect of the nonlocal aspect of the Kirchhoff equations is that we do get a control on $\mathcal{N}(U_\alpha)$. By definition an operator like $\Delta_g + \frac{1}{a}A$ is said to be coercive (or positive) if its energy controls the square of the H^1-norm. We say that $-A$ is cooperative if $A_{ij} \leq 0$ for all $i \neq j$ in $\{1, \ldots, p\}$. The following theorem holds true.

Theorem 2.1 (Bubbling Control; Hebey and Thizy [11, 12])**.** *Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 3$, $p \in \mathbb{N}^\star$ be a nonzero integer, $a, b > 0$ be positive real numbers, and $A : M \to M^p_s(\mathbb{R})$ be a C^1-map from M into the space $M^p_s(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries. Then:*

(Bd1) *if $n = 3$, $\Delta_g + \frac{1}{a}A$ is coercive, $-A$ is cooperative, and $A \leq C\Lambda_g Id_p$ in M in the sense of bilinear forms for some positive constant $C > 0$ and some positive function $\Lambda_g : M \to (0, +\infty)$ with the property that $\Delta_g + \Lambda_g$ has positive mass, then $a + bS^{3/2}\sqrt{C}\mathcal{N}(U_\alpha) \leq C$ for all α,*

(Bd2) *if $n \geq 4$, then $b\mathcal{N}(U_\alpha)S^{n/2}a^{(n-4)/2} \leq \frac{2}{n-2}\left(\frac{n-4}{n-2}\right)^{(n-4)/2}$ for all α, where we adopt the convention that the right hand side in this equation is 1 if $n = 4$,*

(Bd3) *if $n \geq 4$, $S_g > 0$ in M, and $A \leq \frac{n-2}{4(n-1)}CS_g$ in M in the sense of bilinear forms for some positive constant $C > 0$, where S_g is the scalar curvature of g, then $abS^{n/2}a^{(n-4)/2}\mathcal{N}(U_\alpha) \leq (C - a)^+$ for all α,*

where $\mathcal{N}(U_\alpha)$ is the maximum number of bubbles we can have in the H^1-decomposition (2.1) of sequences of nonnegative solutions of (1.1), S is the sharp Sobolev constant as in (1.2), S_g is the scalar curvature of g, and the above three statements hold true for all sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of positive real numbers converging to a and b, all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M^p_s(\mathbb{R})$ converging C^1 to A, all sequences $(p_\alpha)_\alpha$ of real numbers in $(2, 2^\star]$ converging to $2^\star$, and all H^1-bounded sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (1.1).

Given $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ two sequences of positive real numbers, and $(A_\alpha)_\alpha$ a sequence of C^1-maps $A_\alpha : M \to M^p_s(\mathbb{R})$, we now consider the purely critical perturbations

$$\Big(a_\alpha + b_\alpha \sum_{j=1}^p \int_M |\nabla u_j|^2 dv_g\Big)\Delta_g u_i + \sum_{j=1}^p A^\alpha_{ij} u_j = |U|^{2^\star - 2} u_i \qquad (2.3)$$

for all $i = 1, \ldots, p$, where $A_\alpha = (A^\alpha_{ij})_{i,j=1,\ldots,p}$. Following the terminology in Hebey [10], we say that (0.1) is:

(i) *bounded and stable* if for any sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converging to a and b, any sequence $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M^p_s(\mathbb{R})$ converging C^1 to A, and any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of (2.3), a subsequence of the U_α's converge in C^2 to a nonnegative solution U of (0.1).

(ii) *analytically stable* if the bounded stability convergence holds for the less general category of sequences $(U_\alpha)_\alpha$ of solutions of (2.3) which are bounded in H^1.

In the setting of stationary Schrödinger equations, see Hebey [10], there are equations which are analytically stable but not bounded and stable. In the present setting, as a direct consequence of Theorem 1.1, and if we forget about the special case of dimension 4, the two notions of bounded stability and analytic stability coincide. This is one of the main feature of these Kirchhoff equations. We define the constant $C^\star(a,b)$ by

$$\begin{aligned} C^\star(a,b) &= a+\frac{1}{2}b^2S^3+\frac{1}{2}bS^{3/2}\sqrt{4a+b^2S^3} \text{ if } n=3\ , \\ C^\star(a,b) &= \frac{(n-2)a}{4(n-1)}\left(1+bS^{n/2}a^{(n-4)/2}\right) \text{ if } n\geq 4\ , \end{aligned} \tag{2.4}$$

where S is the sharp constant given by (1.2). As a by product of Theorems 1.1 and 2.1, we do get that $(St1)$, $(St2)$ and $(St4)$ in the following stability theorem hold true. Point $(St3)$ follows from independent different arguments.

Theorem 2.2 (Stability; Hebey and Thizy [11, 12])**.** *Let (M^n,g) be a closed Riemannian n-manifold with $n\geq 3$, $p\in\mathbb{N}^\star$ be a nonzero integer, $a,b>0$ be positive real numbers, and $A:M\to M^p_s(\mathbb{R})$ be a C^1-map from M into the space $M^p_s(\mathbb{R})$ of symmetric $p\times p$ matrices with real entries. The following propositions hold true:*

(St1) *if $n=3$, $\Delta_g+\frac{1}{a}A$ is coercive, $-A$ is cooperative, and $A(x)<C^\star(a,b)\Lambda_g(x)Id_p$ for all $x\in M$, in the sense of bilinear forms, where $\Lambda_g:M\to(0,+\infty)$ is such that $\Delta_g+\Lambda_g$ has positive mass, then* (0.1) *is bounded and stable,*

(St2) *if a and b satisfy that $bS^{n/2}a^{(n-4)/2}>\frac{2}{n-2}\left(\frac{n-4}{n-2}\right)^{(n-4)/2}$ when $n\geq 5$, and that $bS^{2^\star/2}>1$ when $n=4$, then* (0.1) *is bounded and stable,*

(St3) *if $n\geq 4$, $A(x)$ is definite positive for all x, and $S_g\leq 0$ in M, then the Kirchhoff system* (0.1) *is analytically stable when $n=4$, and bounded and stable when $n\geq 5$ and $n\neq 6$.*

(St4) *if $n\geq 4$, $S_g>0$ in M, and $A(x)<C^\star(a,b)S_g(x)Id_p$ for all $x\in M$, in the sense of bilinear forms, then the Kirchhoff system* (0.1) *is bounded and stable,*

where, in the above statements, Id_p is the identity $p\times p$ matrix, $C^\star(a,b)$ is given by (2.4)*, and S_g is the scalar curvature of g.*

3. The diagonal geometric case

We assume in this section that A is given by the geometric diagonal model

$$A\equiv\frac{n-2}{4(n-1)}S_g\mathrm{Id}_p \tag{3.1}$$

which extends to the vector case the geometric potential of the conformal Laplacian. Assuming that $S_g>0$ everywhere in M, and that $n=4$ or 5, building on Theorem 1.1 and the theory developed in Druet and Hebey [6], we state below a surprising theorem where only very specific values of a and b can lead to the instability of (0.1). As above, the notation $\lambda\mathbb{N}^\star$ for $\lambda\in\mathbb{R}$ refers to the set consisting of the λn's for $n\geq 1$ integer. The main result in this section is stated as follows.

Theorem 3.1 (Stability for almost all a and b; Hebey and Thizy [12])**.** *Let (M^n,g) be a closed Riemannian n-manifold with positive scalar curvature of dimension $n=4$ or 5, $p\in\mathbb{N}^\star$ be a nonzero integer, $a,b>0$ be positive real numbers, and $A:M\to M^p_s(\mathbb{R})$ be given by the geometric diagonal model* (3.1)*. Assume that*

$$\frac{1-a}{b}\notin S^{n/2}\mathbb{N}^\star\ ,$$

where S is the sharp constant in the Sobolev inequality as in (1.2)*. Then the Kirchhoff system* (0.1) *is bounded and stable.*

The result is even slightly more precise and we get informations on the case where $\frac{1-a}{b} = k_0 S^{n/2}$ for $k_0 \geq 1$ integer. In that case, there might be that there are sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of real numbers converging to a and b, sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (2.3) which blow up as $\alpha \to +\infty$. By Theorem 1.1 they all have bounded energy. As an additional information that we get from the proof of Theorem 3.1, there holds that the sole number of bubbles that such sequences can have in their H^1-decompositions (2.1) is $k = k_0$. Also we easily get from the blow-up examples in Esposito-Pistoia-Vétois [9] that such sequences do exist in various contexts when $\frac{1-a}{b} = S^{n/2}$ (i.e. $k_0 = 1$). Many other examples can be constructed in higher dimensions. A non exhaustive list of blow-up solutions associated with stationary Schrödinger equations is given by Brendle [1], Brendle and Marques [2], Chen, Wei and Yan [4], Druet and Hebey [5], Esposito, Pistoia and Vétois [9], Hebey [10], Hebey and Vaugon [14], and Robert and Vétois [20].

4. Existence and nonexistence results

In order to end the presentation of our results we briefly discuss the question of the existence and nonexistence of solutions of (0.1). By hand constructions, based on the stationary Schrödinger equation, can be developed. A more general approach, based on the above compactness results, can be developed as well. First we very briefly comment on the by hand constructions. These are interesting since they provide several results related to the existence of blowing-up sequences of solutions for the Kirchhoff systems. More precisely, by building on the 1-dimensional Schrödinger theory, we easily get existence of nonnegative nontrivial solutions for systems like (0.1). Suppose for instance that $n = 4$. Let $v > 0$ solve the equation

$$\Delta_g v + hv = v^{2^\star - 1} \tag{4.1}$$

in M. Let $b > 0$ be such that $b \int_M |\nabla v|^2 dv_g < 1$. Then the multi-valued map $U = \left(\frac{1}{\sqrt{p}}u, \dots, \frac{1}{\sqrt{p}}u\right)$ solves (0.1) as soon as A and u satisfy

$$\begin{aligned} \sum_{j=1}^{p} A_{ij} &= \left(a + \frac{ab \int_M |\nabla v|^2 dv_g}{1 - b \int_M |\nabla v|^2 dv_g}\right) h \ , \\ u &= \sqrt{a + \frac{ab \int_M |\nabla v|^2 dv_g}{1 - b \int_M |\nabla v|^2 dv_g}}\ v \ . \end{aligned} \tag{4.2}$$

In these examples, of course, A and U are very special. More general A are handled in the result below. The result follows from both variational arguments based on the mountain pass lemma that we develop in the subcritical case, leading to the existence of solutions to the subcritical systems, and then from the compactness associated to Theorem 2.1 in order to make the subcritical solutions converge to critical ones.

Theorem 4.1 (Existence; Hebey and Thizy [11, 12])**.** *Let (M^n, g) be a closed Riemannian n-manifold with $n \geq 3$, $p \in \mathbb{N}^\star$ be a nonzero integer, $a, b > 0$ be positive real numbers, and $A : M \to M_s^p(\mathbb{R})$ be a C^1-map from M into the space $M_s^p(\mathbb{R})$ of symmetric $p \times p$ matrices with real entries such that $\Delta_g + \frac{1}{a}A$ is coercive and $-A$ is cooperative. The Kirchhoff system* (0.1) *possesses a nonnegative nontrivial solution in each of the following cases:*

(Ex1) $n = 3$, *and* $A(x) < C^\star(a,b)\Lambda_g(x)Id_p$ *for all* $x \in M$, *in the sense of bilinear forms, where* $\Lambda_g : M \to (0,+\infty)$ *is such that* $\Delta_g + \Lambda_g$ *has positive mass,*

(Ex2) a *and* b *satisfy that* $bS^{n/2}a^{(n-4)/2} > \frac{2}{n-2}\left(\frac{n-4}{n-2}\right)^{(n-4)/2}$ *when* $n \geq 5$, *and that* $bS^{2^\star/2} > 1$ *when* $n = 4$,

(Ex3) $n \geq 4$, $S_g > 0$ *in* M, *and* $A(x) < C^\star(a,b)S_g(x)Id_p$ *for all* $x \in M$, *in the sense of bilinear forms,*

where Id_p *is the identity* $p \times p$ *matrix,* $C^\star(a,b)$ *is given by* (2.4), *and* S_g *is the scalar curvature of* g.

The system nature of our equations make that we can also prove nonexistence results. In the following we let $A \in M_s^p(\mathbb{R})$ and assume that A is positive definite with no nonnegative nontrivial eigenvectors. An example of such a matrix is given by

$$A = \frac{1}{42}\begin{pmatrix} 80 & 22 & -26 \\ 22 & 110 & -4 \\ -26 & -4 & 62 \end{pmatrix} \tag{4.3}$$

which has $1, 2, 3$ as eigenvalues, and has $(-4, 1, -5)$, $(-1, 1, 1)$, and $(-2, -3, 1)$ as corresponding eigenvectors. We need here $p \geq 3$. The following nonexistence result can be proved. A higher dimensional version of the theorem can be seen in Hebey and Thizy [12].

Theorem 4.2 (Nonexistence; Hebey and Thizy [11])**.** *Let* (M^3, g) *be a closed Riemannian* 3*-manifold,* $p \in \mathbb{N}^\star$ *be a nonzero integer, and* $A \in M_s^p(\mathbb{R})$ *be such that* A *is positive definite but does not possess nonnegative nontrivial eigenvectors. Then there exists* $K \gg 1$ *such that for any positive real numbers* a *and* b *satisfying that* $a + b \geq K$, (0.1) *does not possess nonnegative nontrivial solutions.*

Theorem 4.2 acts in contrast to Theorem 4.1. The tow conditions $a + b \geq K$ and $A < C^\star(a,b)\Lambda_g \mathrm{Id}_p$ both improve as a and b increase. Then a key point in Theorem 4.2 is that we do not assume that $-A$ is cooperative. By the Perron-Frobenius theorem, if $A \in M_s^p(\mathbb{R})$ is definite positive and $-A$ is cooperative, then A has a positive eigenvalue with a nonnegative eigenvector.

5. The black boxes

The above results are mainly built on three theorems for stationary critical Schrödinger systems. The first theorem is in the spirit of Druet, Hebey and Vétois [8], but we need there to get a more general statement where we allow asymptotically-critical subcritical nonlinearities. Concerning that extension it should be noted that passing from critical to asymptotically-critical subcritical nonlinearities can be invasive, as shown by the discussion below. The $(n, p) = (3, 1)$ case of our first theorem goes back to Li and Zhu [15]. The second and third theorems we need are already available in the literature and were proved in Druet and Hebey [6].

The three theorems we just mentioned in the above discussion are concerned with the stationary critical Schrödinger system

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}u_j = |U|^{2^\star - 2}\, u_i \,, \tag{5.1}$$

where $A : M \to M_s^p(\mathbb{R})$ is a C^1-map. Then we can consider asymptotically-critical subcritical perturbations, or purely critical perturbations of (5.1). In the case of the

first theorem of this section we consider perturbations of (5.1) in the asymptotically-critical subcritical regime. These are given by

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^{\alpha} u_j = |U|^{p_\alpha - 2} u_i \ , \tag{5.2}$$

where $(A_\alpha)_\alpha$ is a sequence of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and the p_α's are such that $p_\alpha \in (2, 2^\star]$ for all α and $p_\alpha \to 2^\star$ as $\alpha \to +\infty$. Theorem 5.1 below is related to the notion of bounded stability. Its proof goes through a delicate one bubble blow-up analysis of (5.2).

Theorem 5.1 (Bounded Stability)**.** *Let (M^n, g) be a closed Riemannian n-manifold, $n \geq 3$, $p \geq 1$ be an integer, and $A : M \to M_s^p(\mathbb{R})$ be a C^1-map satisfying that*

$$A < \frac{n-2}{4(n-1)} \Phi Id_p \tag{5.3}$$

in M in the sense of bilinear forms, where $\Phi \equiv \Lambda_g$ if $n = 3$, $\Phi \equiv S_g$ if $n \geq 4$, S_g is the scalar curvature of g, and Λ_g is such that $\Delta_g + \Lambda_g$ has positive mass. When $n = 3$, we also assume that $\Delta_g + A$ is coercive and that $-A$ is cooperative. Then, for any $\theta \in (0, 1)$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, all sequences $(p_\alpha)_\alpha$ in $(2, 2^\star]$ converging to $2^\star$, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (5.2). In particular, for any sequence $(A_\alpha)_\alpha$ of C^1-maps converging C^1 to A, any sequence $(p_\alpha)_\alpha$ in $(2, 2^\star]$ converging to $2^\star$, and any sequence $(U_\alpha)_\alpha$ of nonnegative solutions of (5.2), a subsequence of the U_α's converge in C^2 to a solution U_∞ of (5.1).

Of course, (5.3) implies that the potentials in Theorem 5.1 are small with respect to the geometry of the ambient manifold. The second result we need was proved in Druet and Hebey [6]. It deals with sequences of solutions which are bounded in H^1, it restricts itself to the purely critical case, but it allows large potentials. The restriction to bounded energy in Theorem 5.2 can be shown to be necessary by the examples in Chen, Wei and Yan [4]. We briefly comment here on the restriction to the purely critical case in Theorem 5.2. We consider perturbations of (5.1) in the sole pure critical regime given by

$$\Delta_g u_i + \sum_{j=1}^{p} A_{ij}^{\alpha} u_j = |U|^{2^\star - 2} u_i \ , \tag{5.4}$$

where $(A_\alpha)_\alpha$ is a sequence of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A. The restriction to (5.4) might seem odd, or even artificial. It turns out that it is also necessary. Indeed, by the blow-up examples in Micheletti, Pistoia and Vétois [17], Pistoia and Vétois [18], and Robert and Vétois [19], letting U_α be of the form

$$U_\alpha = \left(\frac{1}{\sqrt{p}} u_\alpha, \dots, \frac{1}{\sqrt{p}} u_\alpha \right) \ ,$$

and the A_α's satisfy $\sum_{j=1}^{p} A_{ij}^{\alpha} = h_\alpha$ for all α, where u_α and h_α are given by the above references, we get several examples of positively curved manifolds, and of equations like (5.2) with $p_\alpha < 2^\star$ for all α and $p_\alpha \to 2^\star$ as $\alpha \to +\infty$, such that $A_\alpha \to A$ in C^1 and

$$A > \frac{n-2}{4(n-1)} S_g \mathrm{Id}_p$$

in M in the sense of bilinear forms, and such that the equations (5.2) possess sequences $(U_\alpha)_\alpha$ of positive p-maps which blow up as $\alpha \to +\infty$. In this situation, the assumptions (H) and (H') in Theorem 5.2 are satisfied. However, blow-up occurs. This clearly shows that the restriction to the purely critical case in Theorem 5.2 is

also necessary. In what follows we let the matrix potential $A_L : M \to M_s^p(\mathbb{R})$ be given by

$$A_L(x) = A(x) - \frac{n-2}{4(n-1)} S_g(x) \mathrm{Id}_p$$

for all $x \in M$, where S_g is the scalar curvature of g, and Id_p is the identity $p \times p$ matrix. Theorem 5.2 is stated as follows.

Theorem 5.2 (Druet and Hebey [6])**.** *Let (M^n, g) be a closed Riemannian n-manifold, $n \geq 4$, $p \geq 1$ be an integer, and $A : M \to M_p^s(\mathbb{R})$ be a C^1-map satisfying that*

(H) $Ker(\Delta_g + A) \bigcap L^2(M, Vect_+(\mathbb{R}^p)) = \{0\}$,

(H') for any $x \in M$, and any $k \in \{1, \dots, p\}$, there does not exist an orthonormal family $(e_1, \dots, e_k)$ of isotropic vectors for $A_L(x)$ with nonnegative components such that $A_L(x).(V_k) \subset V_k$, where $V_k = Span(e_1, \dots, e_k)$ is the k-dimensional subspace of $\mathbb{R}^p$ with basis $(e_1, \dots, e_k)$.

Then, for any $\theta \in (0,1)$, and any $\Lambda > 0$, there exists $C > 0$ such that $\|U_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(A_\alpha)_\alpha$ of C^1-maps $A_\alpha : M \to M_s^p(\mathbb{R})$ converging C^1 to A, and all sequences $(U_\alpha)_\alpha$ of nonnegative solutions of (5.4) such that $\|U_\alpha\|_{H^1} \leq \Lambda$ for all α. In particular, for any sequence $(A_\alpha)_\alpha$ of C^1-maps converging C^1 to A, and any H^1-bounded sequence $(U_\alpha)_\alpha$ of nonnegative solutions of (5.4), a subsequence of the U_α's converge in C^2 to a solution U_∞ of (5.1).

The blow-up analysis behind Theorem 5.2 is a multi-bubble analysis. We start with the H^1-decomposition and get at once the entire collection of blow-up points. The C^0-theory in Druet, Hebey and Robert [7] (see also Druet and Hebey [6]) makes that we control our sequence $(U_\alpha)_\alpha$ of solutions in terms of the leading parts in their H^1-developments (like if the rest in the H^1-theory was zero). In particular we get sharp pointwise asymptotics for the U_α's. We conclude using an exterior Pohozaev identity and controlling Druet's notion of the range of interaction of bubbles. We use here a slightly weaker version of Theorem 5.2 where we ask that $\Delta_g + A$ is coercive, which implies (H), and that A_L does not possess isotropic vectors, which implies (H'). In the process of proving Theorem 5.2 it is established in Druet and Hebey [6], see also Hebey [10], that the pointwise limit of blowing-up sequences of nonnegative solutions of equations like (5.4) which are bounded in H^1 has to be zero when $n = 4, 5$.

Theorem 5.3 (Druet and Hebey [6])**.** *Let (M^n, g) be a closed Riemannian n-manifold of dimension $n = 4, 5$, $p \geq 1$ be an integer, $(A_\alpha)_\alpha$ be a sequence of C^1-maps $A_\alpha : M \to M_p^s(\mathbb{R})$ converging in C^1, and $(U_\alpha)_\alpha$ be a bounded sequence in H^1 of nonnegative solutions of (5.4). If the U_α's blow up, namely if $\|U_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$, then, up to passing to a subsequence, $U_\alpha \to 0$ a.e. in M.*

Theorem 5.3 plays an important role in the proof of Theorem 3.1. Of course the pointwise limit in Theorem 5.3 coincides with the weak limit in H^1 or the strong limit in L^2. Theorem 5.3 is also true when $n = 3$ but, as shown in Druet and Hebey [6], it stops to hold true when $n = 6$.

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