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Abstract

These notes are an introduction to the recent paper [7], about the well-posedness of the Prandtl equation. The difficulties and main ideas of the paper are described on a simpler linearized model.

1 Introduction

The general concern of these notes is the so-called *boundary layer* in fluid mechanics. It is related to the dynamics of high Reynolds number flows near a rigid wall. We recall that the *Reynolds number*, introduced by Osborne Reynolds [22] is the parameter $R = UL/\nu$, where U and L are the typical velocity and length scale of the fluid flow, whereas ν is the kinematic viscosity of the fluid. This parameter appears in the dimensionless form of the Navier-Stokes equation:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{R} \Delta \mathbf{u} = 0, \\ \text{div } \mathbf{u} = 0, \end{aligned} \tag{1.1}$$

for $t > 0$, and $\mathbf{x}$ in the fluid domain Ω. As usual, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ and $p = p(t, \mathbf{x})$ stand for the velocity field and pressure. When $R \gg 1$, it is tempting to describe the dynamics by the Euler system, that is with $R = \infty$. Convergence of the Navier-Stokes solutions to the Euler ones when R goes to infinity can be shown rigorously in many contexts, as long as the boundaries of the domain are neglected ($\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$). However, when Ω has a boundary, this asymptotics is far from clear. It is not even clear in two dimensions and for smooth data, although both Navier-Stokes and Euler have in this context global smooth solutions.

The problem comes from boundary conditions. For Euler, the natural one is no penetration: $\mathbf{u} \cdot n|_{\partial\Omega} = 0$, with n the normal vector. In particular, the tangential component of the velocity can be $O(1)$ at the boundary. On the contrary, for Navier-Stokes, the experimentally relevant boundary condition is no-slip: $\mathbf{u}|_{\partial\Omega} = 0$. Hence, the transition from $R < \infty$ to $R = \infty$ is formally associated to a jump of the tangential velocity. Concretely, as R is larger and larger, stronger and stronger velocity gradients develop near the boundary, in a thin zone called a boundary layer. The understanding of the concentration phenomenon in this layer is a big problem in fluid mechanics. The boundary layer prevents convergence of Navier-Stokes

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solutions to Euler ones in a strong topology, for instance in $L^\infty(H^1)$: see [12]. To determine if convergence holds in the energy space $L^\infty L^2$ is a big open issue.

To achieve a better understanding of the boundary layer, Ludwig Prandtl introduced in 1904 a model for the boundary layer [21]. This model has now become very classical, and can be found in any textbook of fluid mechanics. Let us consider $\Omega = \mathbb{R}^2_+$, and the small parameter $\varepsilon = 1/R$. The model of Prandtl is based on the idea of using different asymptotics for the solution $\mathbf{u} = \mathbf{u}^\varepsilon$ of (1.1), depending on the flow region under study:

1. Away from the boundary, where no concentration occurs, expansion of the Navier-Stokes solution is expected to be regular in ε. In particular, one should have

$$\mathbf{u}^\varepsilon \approx \mathbf{u}^0 = (u^0, v^0), \quad p^\varepsilon = p^0$$

where $(\mathbf{u}^0, p^0)$ is the Euler solution (starting from the same initial data).

2. However, in the boundary layer, there should be a fast variation along the normal variable. Prandtl suggests an asymptotics of the form (with $\mathbf{x} = (x, y)$) :

$$\mathbf{u}^\varepsilon \ \approx \ \left(u(t,x,y/\sqrt{\varepsilon}), \sqrt{\varepsilon} v(t,x,y/\sqrt{\varepsilon})\right), \quad p^\varepsilon \ \approx \ p(t,x,y/\sqrt{\varepsilon}).$$

Note that the Prandtl model involves the variable $Y = y/\sqrt{\varepsilon}$: the typical size of the layer is assumed to be $\sqrt{\varepsilon}$, which is natural with regards to the parabolic part $\partial_t \mathbf{u} - \varepsilon \Delta \mathbf{u}$ of (1.1).

Plugging this last Ansatz in (1.1) one finds at leading order the equations

$$\begin{aligned} \partial_t u + u \partial_x u + v \partial_Y u + \partial_x p - \partial_Y^2 u = 0, \\ \partial_Y p = 0, \\ \partial_x u + \partial_Y v = 0, \end{aligned} \tag{1.2}$$

set for (x, Y) in $\mathbb{R}^2_+$, together with the boundary conditions

$$\begin{aligned} & u(t,x,0) = v(t,x,0) = 0 \\ & \lim_{Y \to \infty} u(t,x,Y) = u_\infty(t,x) := u^0(t,x,0), \quad \lim_{Y \to \infty} p(t,x,Y) = p_\infty(t,x) := p^0(t,x,0). \end{aligned} \tag{1.3}$$

Note that the first line corresponds to the no-slip condition at the rigid wall, whereas the second line expresses a *matching* between the boundary layer solution (away from the wall) and the Euler solution (close to the wall). From (1.2b) and (1.3b), one has $p(t,x,Y) = p^\infty(t,x)$ for all Y, and the Prandtl system further simplifies. Denoting y instead of Y, we get

$$\begin{aligned} \partial_t u + u \partial_x u + v \partial_y u - \partial_y^2 u = -\partial_x p^\infty, \\ \partial_x u + \partial_y v = 0, \\ u(t,x,0) = v(t,x,0) = 0, \ \lim_{y \to \infty} u = u_\infty \end{aligned} \tag{1.4}$$

with initial data $u|_{t=0} = u_0$. This is the famous Prandtl system, set for (x, y) in $\mathbb{R}^2_+$. Compared to Navier-Stokes, its key features are the following:

- The pressure is no longer an unknown: it is determined by the Euler evolution, and therefore is a given source term in the equation for u.

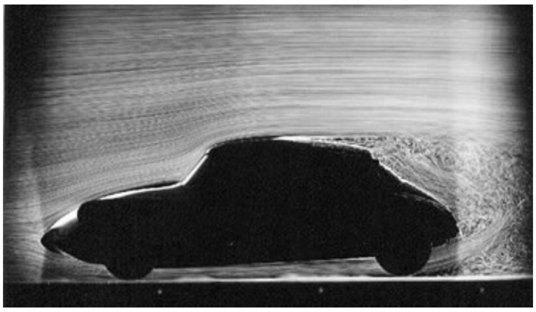

Figure 1: Typical picture of boundary layer separation.

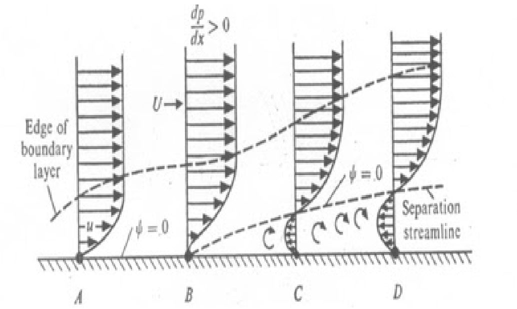

Figure 2: Velocity field in the boundary layer, under an adverse pressure gradient.

- There is no more evolution equation on v : the normal component is obtained in terms of u through integration in y of the divergence-free condition. Hence, the Prandtl system can be seen as a single evolution equation on the scalar unknown u. Note also that only the normal part of the diffusion term $(-\partial_y^2 u)$ appears in this equation.

Thus, the Prandtl model looks like a tempting alternative to Navier-Stokes in order to describe fluid flows near a rigid wall. However, the relevance of this model turns out to be a difficult issue. Indeed, from many experiments on flows around obstacles, it is known that many instabilities develop near the boundary, and may invalidate the Prandtl model.

One famous instability is the so called *boundary layer separation*: downstream from the leading edge of the obstacles, boundary layer flows may detach and penetrate inside the domain, see figure 1.

Physicists explain this separation by a loss of monotonicity of u in the y variable. More precisely, it is claimed that under an adverse pressure gradient ($\partial_x p > 0$), the boundary layer flow is slowed down, up to a point where monotonicity breaks down. From there, reverse flow and detachment modify completely the flow. See figure 1 or [10] for more details.

2 Former mathematical results

The instabilities evoked above make the mathematical analysis of the Prandtl equations very difficult. We focus here on the well-posedness theory for the Prandtl model (1.4). We shall not discuss wether or not the approximation of the exact Navier-Stokes solution by the Prandtl one is justified.

Until recently, there had been only two known settings for well-posedness:

1. The monotonic setting. For *data that are monotonic in y* (plus technical conditions), one can prove well-posedness locally in time [19], and even globally under a favorable pressure gradient ($\partial_x p \leq 0$) [24]. This type of results, mainly due to Oleinik, echoes the above discussion on boundary layer separation. They were obtained by using a tricky change of unknowns and variables, called Crocco transform. Recently, this monotonic setting was re-visited on the usual eulerian form of the Prandtl equation : see [1, 17]. We will come back below to article [17], by the second author and Tak-Kwong Wong.

2. The analytic setting. Without monotonicity, well-posedness was established locally in time, for *initial data that are analytic with respect to x*. We refer to [23, 16], and to the recent extensions [14, 13]. The assumption of analyticity can be understood as follows. By the divergence-free condition, one obtains $v = -\int_0^y \partial_x u$. Thus, the term $v\,\partial_y u$ in (1.4a) (seen as a functional of u) is first order in x. Moreover, it is not hyperbolic. For instance, let us consider the linearization of the Prandtl equation around a shear flow $\mathbf{u} = (U_s(y), 0)$:
$$\partial_t u + U_s \partial_x u + U_s' v - \partial_y^2 u = 0, \quad \partial_x u + \partial_y v = 0. \tag{2.1}$$
If we freeze the coefficients at some y_0 and compute the dispersion relation, we obtain the growth rate
$$\sigma(k_x, k_y) = U_s'(y_0)\frac{k_x}{k_y} - k_y^2$$
that increases linearly with the wavenumber k_x. This kind of growth rate is typical of well-posedness limited to the analytic setting.

However, as discussed in [11], this dispersion relation, formally obtained by freezing the coefficients, is misleading: for instance, the inviscid version of Prandtl (that is removing the $\partial_y^2 u$ term) is locally well-posed in C^k, through the method of characteristics. In the case of the full Prandtl system (1.4), the situation is even more complex, and was addressed a few years ago by the first author and Emmanuel Dormy in article [6]. This article contains a careful study of the linearized system (2.1), *in the case of a non-monotonic base flow U_s*:
$$\exists a, \quad U_s'(a) = 0, \quad U_s''(a) \neq 0.$$
In short, it is shown in [6], see also [3], that the linear system (2.1) admits approximate solutions with growth rate
$$\sigma(k_x) \sim \delta\sqrt{k_x}, \quad \delta > 0, \quad k_x \gg 1. \tag{2.2}$$
Let us stress that such growing solutions come from an interplay between the lack of monotonicity of U_s and the diffusion term $\partial_y^2 u$. They are therefore coherent with the well-posedness results obtained in the monotonic case, and for the inviscid Prandtl equation. As a result of their violent growth, one can prove that *the Prandtl equation is ill-posed in Sobolev spaces*, both linearly [6] and non-linearly [8].

Article [6] also contains numerical computations, performed on the linearized system (2.1). Starting from random initial data, one can compute the evolution, and determine the most unstable mode. The numerics show that this unstable mode is the one described analytically. In particular, the worst possible growth rate seems to be given by (2.2). This leaves room for local well-posedness below analytic regularity.

3 Gevrey well-posedness

This last remark was the basic motivation of our recent work [7]. Our goal was to confirm theoretically the instability rate (2.2) observed numerically, both at the linear level (2.1) and nonlinear level (1.4). In other words, we conjectured that the Prandtl equation is locally well-posed for data whose Fourier coefficients in x decay like $e^{-\sigma\sqrt{k_x}}$, for some $\sigma > 0$.

Note that this setting is intermediate between the analytic setting ($\sim e^{-\sigma k}$) and the Sobolev setting ($\sim k^{-s}$). Namely, it corresponds to *Gevrey data* in the x variable. In what follows, we always consider $x \in \mathbb{T}$, and $y \in \mathbb{R}_+$. In particular, the wave number k in x belongs to $\mathbb{Z}$. We recall the

Definition 1 *Let $m \geq 1$. The Gevrey space $G^m(\mathbb{T})$ is the set of functions f satisfying: $\exists C, \tau > 0$ such that*

$$|f^{(j)}(x)| \leq C\,\tau^{-j}(j!)^m, \quad \forall j \in \mathbb{N},\, x \in \mathbb{T}.$$

For a reminder on Gevrey spaces, we refer to the paper [5] by Foias and Temam as well as to the papers [15, 4, 20] where these spaces are used. Note that $m = 1$ corresponds to analytic functions, whereas for $m > 1$, $G^m(\mathbb{T})$ contains compactly supported functions. The connection with the behaviour of the Fourier coefficients is given by

Proposition 1 *$f \in G^m(\mathbb{T})$ if and only if it exists $C, \sigma > 0$ such that $|\hat{f}(k)| \leq Ce^{-\sigma|k|^{1/m}}$ for all k.*

Following the end of the last paragraph, one expects the Prandtl system (1.4) to be well-posed in a functional space of type G^2 in variable x. This conjecture is still open, but article [7] provides a close result : roughly, *we establish that the Prandtl system is well-posed in a space of type $G^{7/4}$ in x.*

To state a more precise but simpler result, we shall limit ourselves to the case where $u^\infty = p^\infty = 0$. We shall consider data $u^0 = u^0(x, y)$ that are of Gevrey class 7/4 in x, and Sobolev class with polynomial weight in y. One could as well consider exponential weights. Specifically, we define

$$H^s_\gamma := \{u = u(y), \quad (1+y)^{\gamma+k}\, u^{(k)} \in L^2(\mathbb{R}_+) \quad \forall\, k \leq s\}, \quad L^2_\gamma := H^0_\gamma,$$

and then

$$G^{7/4}_\tau(\mathbb{T}, H^s_\gamma) := \{u = u(x, y), \quad \sup_{x,k} \|\tau^k (k!)^{-m} \partial_x^k u(x, \cdot)\|_{H^s_\gamma} < +\infty\}. \tag{3.1}$$

We take the initial data to satisfy

$$u_0 \in G^{7/4}_{\tau_0}(\mathbb{T}, H^{s+1}_{\gamma-1}), \quad \omega_0 := \partial_y u_0 \in G^{7/4}_{\tau_0}(\mathbb{T}, H^s_\gamma) \tag{3.2}$$

for some τ_0. We make two more assumptions. The first one is technical, and related to the behavior of the data at infinity. Roughly, it states that the data ω_0 behaves like $(1+y)^{-\sigma}$ for some σ:

(H1) For $y \gg 1$, for all x, for all $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq 2$,

$$|\omega_0(x, y)| \geq \frac{\delta}{(1+y)^\sigma}, \quad |\partial^\alpha \omega_0(x, y)| \leq \frac{1}{\delta(1+y)^{\sigma+\alpha_2}}.$$

The second assumption expresses that we allow for non-monotonic data. More precisely, we assume that for each value of x, $u_0(x,\cdot)$ has one (non-degenerate) critical point:
(H2) $\omega_0(x,y) = 0$ *iff* $y = a_0(x), \quad a_0 > 0, \quad \partial_y \omega_0(x, a_0(x)) \neq 0.$

We can now state our main

Theorem 1 Let $\tau_0 > 0$, $s \gg 1$ even, $\sigma \geq \gamma + \frac{1}{2} \gg 1$. Let u_0 satisfying (3.2), $u_0|_{y=0} = 0$, as well as *(H1), (H2)*. There exists $T > 0$, $0 < \tau \leq \tau_0$ and a unique solution

$$u \in L^\infty\big(0,T; G^{7/4}_\tau(\mathbb{T};\, H^{s+1}_{\gamma-1})\big), \quad \omega = \partial_y u \in L^\infty\big(0,T; G^{7/4}_\tau(\mathbb{T};\, H^{s}_{\gamma})\big),$$

of (1.4), with initial data u_0.

This theorem expresses short time existence and uniqueness of a Gevrey solution. Note that the exponent τ in the definition of the functional space (3.1) deteriorates between time 0, for which $\tau = \tau_0$, and positive times, for which $\tau < \tau_0$. This decrease is natural in view of the exponential instability process taking place in the Prandtl evolution, namely the one described above, with growth rate (2.2).

Let us point out that the assumption (H2), namely the presence of a single curve $y = a_0(x)$ where $\omega_0(x,y) = 0$ could be relaxed, for instance we can assume the presence of a finite number of non degenerate critical points for each value of x. An interesting extension of our result (and also of [6]) would be to consider data that are monotonic in y for $x < x_0$, and with a curve of non-degenerate critical points for $x \geq x_0$ (more like in figure 1). See [13] for a result in a similar spirit.

4 Ideas of proof

To understand our proof of Theorem 1, it is better to consider the simple linearized equation (2.1). We suppose that U_s is smooth, and obeys assumptions analogue to (H1)-(H2) :

- $U_s(y) \sim -\dfrac{1}{(1+y)^\sigma}$ at infinity.
- $U_s'(a) = 0$ for a unique $a > 0$, and $U''(a) > 0$.

Now, as equation (2.1) has constant coefficients in x, we can use the Fourier transform. More precisely, we write

$$u(t,x,y) = e^{ikx}\hat{u}(t,y), \quad v(t,x,y) = ike^{ikx}\hat{v}(t,y).$$

We obtain, after removing the hats :

$$\begin{cases} (\partial_t + ikU_s)u \,+\, ikU_s' v - \partial_y^2 u = 0, \\ \qquad\qquad\qquad\qquad u + \partial_y v = 0. \end{cases} \tag{PL}$$

As often in fluid mechanics, it is also useful to write down the equation on vorticity. In this degenerate Prandtl context, vorticity is just $\omega = \partial_y u$, and satisfies

$$(\partial_t + ikU_s)\omega + ikvU_s'' - \partial_y^2 \omega = 0. \tag{PLω}$$

For such linearized equations, the point is to show local well-posedness in $G^{7/4}_\tau(\mathbb{T}, L^2_\gamma)$: higher Sobolev regularity does not add much trouble. By Proposition 1, this amounts to proving an estimate of the following type:

$$\|\omega(t)\|_{L^2_\gamma} \le C\, e^{\sigma k^{4/7} t}, \quad \text{for some } C, \sigma > 0. \tag{4.1}$$

The main problem is that standard L^2 energy estimates do not help: for instance, multiplying (PL) by $\overline{u}$ and integrating, one gets the annoying term $ik \int U_s' v \overline{u}$. This factor k is reminiscent of a first order term in x, and as the integral does not vanish, the L^2 norm could *a priori* grow like $e^{\sigma k t}$, which is much worse than (4.1). This difficulty appears for equation (PLω) as well.

Hence, the kinetic energy is not adapted to this system. *The point in paper [7] is to use smarter energy functionals, that allow to get rid of the bad term in v (with factor k).* These functionals are inspired from two tricky formal estimates, that we now present.

1. The first one is related to the vorticity equation (PLω) : instead of multiplying by $\overline{\omega}$, we multiply by $\frac{\overline{\omega}}{U_s''}$. We integrate, take the real part, and derive in this way the equality:

   $$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} \frac{|\omega|^2}{U_s''} + \int_{\mathbb{R}_+} \frac{|\partial_y \omega|^2}{U_s''} = k\mathcal{I}m \int_{\mathbb{R}_+} v\overline{\omega} \, + \, \dots$$

   where the dots refer to commutators. There, one can realize that

   $$\int_{\mathbb{R}_+} v\,\bar{\omega} = -\int_{\mathbb{R}_+} \partial_y v\, \bar{u} \, = \, \int_{\mathbb{R}_+} |u|^2,$$

   so that the bad term in v vanishes from the previous equality. If there was a uniform convexity property $U'' \ge \alpha > 0$, $E_\omega = \int_{\mathbb{R}_+} \frac{|\omega|^2}{U''}$ would define a positive energy, so that the cancellation would yield a good control : see [9] or [18] for a use of this idea in another context. However, in our boundary layer setting, this convexity assumption does not hold.

2. We then introduce another tricky energy estimate, inspired from [17]. Precisely, we multiply (PL) by $\frac{U_s''}{U_s'}$ and substract it from equation (PLω). Again, with this simple linear combination, one gets rid of the bad term in v (and bad factor k). One is left with an equation on a new quantity

   $$g = \omega - \frac{U_s''}{U_s'} u = U_s' \partial_y \frac{u}{U_s'}$$

   that reads

   $$(\partial_t + ikU_s) g - \partial_y^2 g = \dots$$

   where the dots still refer to commutators. Roughly, in the monotonic case ($U_s' > 0$), one can show that g is a good quantity, with $E_g = \|g\|^2_{L^2_\gamma}$ controlling $\|\omega\|_{L^2_\gamma}$. This can be used to show well-posedness in the monotonic case, see [17]. But for our non-monotonic data, it is still not enough.

With regards to the previous items and our assumptions, a natural idea is *to combine the above energies.* Close to the non-degenerate critical point, U is convex, and one can use an energy like E_ω. Away from the critical point, U is monotonic and one can rely on an energy like E_g. This idea is central in paper [7]. Concretely, we introduce:

- $E_1 \; = \; \displaystyle\int_{\mathbb{R}_+} \frac{\chi(\cdot - a)}{U_s''} |\omega|^2$, for χ a truncation near 0.
- $E_2 \; := \; \|\tilde{g}\|^2_{L^2_\gamma}, \quad \tilde{g} \; = \; (\psi U_s' + (1-\psi)) \; g, \quad \psi$ is a truncation, $\psi = 1$ on $[0, a+1]$.

One can show that

$$c(E_1 + E_2) \le \|\omega\|_{L^2_\gamma} \le C(E_1 + E_2)$$

so that these energies are relevant ones. Moreover, in view of the above computations, one can even expect to get rid of all powers of k in the estimates for E_1 and E_2 !

However, such expectation is too optimistic, as it would contradict ill-posedness of Prandtl in Sobolev spaces. Indeed, due to the various commutators involved in the calculations, powers of k re-emerge in the analysis. If treated too crudely, they yield again a growth like $e^{\sigma k t}$. This difficulty is overcome in [7], in the context of the full Prandtl equation. When translated to the linear equation (2.1), the point is the following: *one must carefully estimate the commutators, and derive inequalities of the form:*

$$\begin{aligned} \partial_t E_1 \; &\lesssim \; k \, E_1^{1/2} \, E_2^{1/2} + \dots \\ \partial_t E_2 \; &\lesssim \; E_1^{1/3} \, E_2^{2/3} + \dots \end{aligned} \tag{4.2}$$

If E_2 was replaced by E_1 in the first equation, one would obtain again an energy growth rate that scales like k. *But the crucial point is that E_1 and E_2 are mixed in these inequalities.* In particular, the right-hand side of the first inequality on E_1 contains E_2, whereas the second inequality on E_2 does not involve any factor k. Thus, one can somehow interpolate between the two inequalities : considering the anisotropic energy $E = E_1 + k^{6/7} E_2$, one gets

$$\partial_t E \; \le \; k^{4/7} \, E.$$

From there, (4.1) follows. Of course, the most substantial part of the work consists in the commutator estimates, that is deriving inequalities (4.2).

Besides these general ideas, the proof of Theorem 1 involves technical difficulties. Notably, one has to adapt the above reasoning to the nonlinear Prandtl equation. For instance, there is no more base flow $(U_s, 0)$: it is replaced by the solution itself, so that the energy functionals become non-quadratic. Also, one can not use so easily the Fourier transform, as the coefficients of the equation depend on x. We refer to [2] for a recent application of Gevrey spaces using the "Fourier" space characterization. Here, the x dependence requires the introduction of Gevrey type norms in the "physical" space. We refer to [7] for all necessary details.

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