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Earnshaw's Theorem in Electrostatics and a Conditional Converse to Dirichlet's Theorem *

Jeffrey Rauch †

Abstract

For the dynamics $x'' = -\nabla_x V(x)$, an equilibrium point $\underline{x}$ are always unstable when on a neighborhood of $\underline{x}$ the non constant V satisfies $P(x, \partial)V = 0$ for a real second order elliptic P. The proof uses a result of Kozlov [6].

1 Introduction

This article presents an instability theorem for Newton's law

$$\frac{d^2x}{dt^2} = -\nabla_x V(x)\,, \qquad V \in C^\infty(|x| < r)\,, \qquad x \in \mathbb{R}^d\,. \tag{1.1}$$

The theorem implies, in particular, Earnshaw's Theorem [3] that equilibria in electrostatic fields are always unstable.

Example 1.1 *Here is an example of an electrostatic equilibrium for which the instability is easy to understand. Consider a point charge placed midway between two positive charges.*

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The middle position is an unstable equilibrium.
If the charge in the middle is positive, then if it starts at rest just above the middle, it will be repelled to infinity vertically. If the charge in the middle is negative a small displacement to the right has the particle accellerated to a finite time collision with the charge on the right. The nature of the instability depends on the sign of the test charge.

Earnshaw's Theorem is discussed in the very interesting §116 of Maxwell's Treatise [8]. There are two standard proofs. Maxwell's proof is based on the flawed idea that for small displacement from a stable equilibrium there must be a restoring force, see §3. The other proof observes that the potential energy is harmonic and therefore cannot have a local minimum, see §2. In those sections, examples show that the each of the two instability arguments are insufficient. However, the fact that V is harmonic is sufficient for instability applying a nontrivial result of Kozlov.
We prove that equilibria of convervative mechanical system whose potential energy is a solution of any real second order elliptic partial differential equation are all unstable.

Hypothesis. *Suppose that $r > 0$ and that*

$$P(x,\partial) \;:=\; \sum_{I,j=1}^{d} a_{ij}(x)\partial_i\partial_j \;+\; \sum_j b_j(x)\partial_j \;+\; c(x) \tag{1.2}$$

has real coefficients in $C^\infty(|x| < r)$.

Definition 1.1 *$P(x,\partial)$ is elliptic at $\underline{x}$ when*

$$\forall\, \xi \in \mathbb{R}^d \setminus 0, \qquad \sum_{i,j} a_{ij}(\underline{x})\,\xi_i\xi_j \;\neq\; 0\,.$$

Theorem 1.2 *Suppose that the non constant real valued $V \in C^\infty(|x| < r)$ satisfies $P(x,\partial)V = 0$ with P as above. If $\underline{x}$ is an equilibrium for $x'' = -\nabla V(x)$ and P is elliptic at $\underline{x}$ then the equilibrium is unstable.*

Since the electrostatic potential, discussed in §2.4 is harmonic, this implies Earnshaw's Theorem.

2 Dirichlet's Theorem and its converse

The phase space formulation of (1.1) is

$$\frac{dx}{dt} \;=\; v\,, \qquad \frac{dv}{dt} \;=\; -\nabla_x V(x)\,.$$

The equilibria in x, v-space are points $(\underline{x}, 0)$ with $\underline{x}$ a critical point of V.

2.1 Dirichlet's Theorem

Theorem 2.1 *If $r > 0$ and $V \in C^2(|x| < r)$ has a strict local minimum at 0 in the sense that $V(0) < V(x)$ for all $0 < |x| < r$ then 0 is a stable equilibrium.*

Sketch of proof. A minimum is a critical point, hence an equilbrium. Conservation of the energy $\mathcal{E}$ implies that the open sets

$$\mathcal{O}_\alpha \;:=\; \Big\{(x,v) \,:\, \mathcal{E} := \frac{|v|^2}{2} + V(x) < \alpha\Big\}$$

are invariant. For $0 < \alpha$ define $\mathcal{U}_\alpha$ to be the connected component of $\mathcal{O}_\alpha$ that contains $(0,0)$. For α decreasing to zero the $\mathcal{U}_\alpha$ are a neighborhood basis of the origin in $\mathbb{R}^{d+d}_{x,v}$. This implies stability. ■

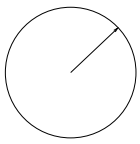
2.2 Converse Dirichlet is false

The assertion that critical points for which there are arbitrarily nearby points of lower potential energy are unstable is standard lore in physics. It is reasonable to think that if $\underline{x}$ is a critical point that is NOT a local minimum, then there will be orbits that accellerate away from $\underline{x}$ toward regions of lower potential energy.

Here is a classical counterexample. For $x \in \mathbb{R}$ define

$$V(x) \;:=\; e^{-1/|x|}\;\sin 1/|x|\,.$$

The graph of V oscillates rapidly between $\pm e^{-1/|x|}$. Two peaks of the graph are sketched. Orbits with energy $\mathcal{E} = V(x) + v^2/2 < a$ must lie in the set $\{V < a\} \cap \{v^2 < 2a\}$.



V increases as one moves radially
away from the equilibrium at the center

As $a \to 0$, $r(a) \to 0$.

An orbit with $\mathcal{E} < a$ that starts in $\{|x| < r\}$ stays in $|x| < r$. To escape it would hit $x = \pm r$ where $\mathcal{E} \geq V = a$.

Thus $\mathcal{U}_a := \{|x| < r(a)\,,\, v^2 < 2a\,,\, \mathcal{E} < a\}$ are flow invariant. They are a neighborhood basis for $(0,0)$. Proving stability *in spite of the fact that $V(0)$ is not a local minimum.*

2.3 Two easy instability theorems

There are two easy proofs of instability. In the case of $d = 1$ with $V = c\,x^m + \text{h.o.t.}$ with m odd and $c \neq 0$, one finds an unstable orbit (that is converges to the origin as $t \to -\infty$) using level curves of the energy.

In any dimension, if the hessian of V at an equilibrium has at least one strictly negative eigenvalue then there is instability by a theorem of Lyapunov. More generally, if the hessian has k negative eigenvalue, counting multiplicities, then there is an unstable manifold with dimension k.

2.4 Arnold's conjecture

In his 1976 talk on the Hilbert Problems, Arnold proposed his own problems. One is to prove that for $x'' = -\nabla V(x)$, *if $V \in C^\omega$ and $\underline{x}$ is an equilibrium that is not a local minimum, then the equilibrium is unstable.*

The conjecture is easy when $d = 1$. Taliaferro [10] proved the case $d = 2$ in 1980.

In electrostatics, a compactly supported finite signed measure ρ descibes the charges. The electric field is given by Coulomb's law

$$E \;=\; -\operatorname{grad}\Big(\rho * \frac{1}{|x|}\Big)\,.$$

The field E has a partial differential equations characterization as the unique solution of

$$\operatorname{curl} E = 0\,, \qquad \operatorname{div} E = 4\pi\rho\,, \qquad E = O(1/|x|^2) \;\text{ as }\; x \to \infty\,. \tag{2.1}$$

Since $\operatorname{curl} E = 0$ there is a potential function $V(x)$ so that $E = -\operatorname{grad} V$. In regions without charge,

$$\Delta V \;=\; \operatorname{div}\operatorname{grad} V \;=\; -\operatorname{div} E \;=\; 0\,, \quad \text{so} \quad V \in C^\omega\,.$$

In particular, E is smooth outside of the support of ρ. The $O(1/|x|^2)$ in (2.1) is in the classical sense $|E(x)| = O(1/|x|^2)$.

The maximum principal for harmonic functions implies that V cannot have a minimum. The lack of minima is one of the two standard proofs of instability. However not having a minimum is insufficient for instability. Since the electrostatic potential is real analytic, Arnold's conjecture if proved would imply Earnshaw's theorem.

3 Discussion of Maxwell's proof

3.1 Gauss' Theorem

The instability argument in §116 of Maxwell [8] uses Gauss' Theorem.

Theorem 3.1 *If B an open ball, whose boundary does not meet* $\operatorname{supp}\rho$*, then,*

$$\text{the flux of } E \text{ through } \partial B \;=\; 4\pi\Big(\text{total charge in } B\Big)\,.$$

Proof. $\int_{\partial B} E\cdot\mathbf{n}\,d\sigma = \int_B \operatorname{div} E\;dx = \int_B 4\pi\,d\rho\,.$ ■

Example 3.2 *If there are no charges in B, then the total flux through ∂B vanishes.*

3.2 Maxwell's argument

Maxwell asserts in §116 that *if $\underline{x}$ is a stable equilibrium for a positive test charge, then the field E at nearby $\underline{x}+\delta x$, must push back toward the equilibrium at $\underline{x}$.* In the next sentence he reveals that what he means by this is that the force F must satisfy $F(x+\delta x)\cdot\delta x<0$.

For a positive test charge, this asserts that $E(x+\delta x)\cdot\delta x<0$. This cannot be the case. If B is a small ball centered at $\underline{x}$, then taking δx with length equal to the radius yields

$$\int_{\partial B} E\cdot\mathbf{n}\;d\sigma \;=\; \int_{\partial B} \texttt{negative}\;d\sigma \;<\; 0\,.$$

Gauss' Theorem implies that there is a net negative charge in B violating the assumption that B is charge free. In the case of a negative test charge one finds that B must contain postive charges. The erroneous statement in this argument is in italics. We present a detailed discussion.

For the differential equation $x''=F(x)$, introduce two conditions,

$$(*)\quad \exists r>0,\ \forall x,\quad 0<|x-\underline{x}|<r \;\Rightarrow\; F(x)\cdot(x-\underline{x})<0\,,$$

and,

$$(**)\quad \forall r>0,\ \ \exists x,\ \ |x-\underline{x}|<r,\ \ \text{and}\ \ F(x)\cdot(x-\underline{x})>0\,.$$

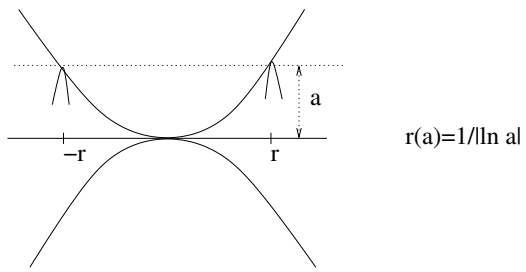
Maxwell asserts that stability $\Rightarrow$ $(*)$. Equivalently, the denial of $(*)$ implies instability.

$(**)$ is stronger than the denial of $(*)$. The denial has $F \cdot (x - \underline{x}) \geq$ instead of $>$. We will see that even the stronger condition $(**)$ does not imply instability. One could hope that $(*)$ implies stability.

Theorem 3.3 i. *Condition $(*)$ implies stability if F is conservative (***Defn.** *$F(x) = -\nabla V(x)$), but not in general.*

ii. *Condition $(**)$ does not imply instability, even in the conservative case.*

Proof. i. Denote by $V(x)$ the potential in the conservative case. $(*)$ implies that for $0 < |x - \underline{x}| < r$, $V(x)$ is strictly increasing on the ray $[0,1] \ni s \mapsto \underline{x} + s(x - \underline{x})$ connecting the center $\underline{x}$ to x.



Therefore V has a strict local minimum at $\underline{x}$. Dirichlet's Theorem implies that the equilibrium is stable.

An example shows that $(*)$ does not imply stability in the non conservative case. In $\mathbb{R}^2$, $\partial_\theta := x_2\partial_1 - x_1\partial_2$. Define

$$F(\varepsilon, x) := (x_2\,,\, -x_1) - \varepsilon\, x \;=\; \partial_\theta - \varepsilon x\,, \qquad 0 < \varepsilon << 1\,,$$

with equilibrium $\underline{x} = 0$. The linear system $x' = v$, $v' = F(0,x)$ has matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \text{with eigenvalues} \quad \frac{1 \pm i}{\sqrt{2}}\,,\ \frac{-1 \pm i}{\sqrt{2}}\,.$$

Since $F \cdot x = -\varepsilon\,|x|^2$, F satisfies $(*)$ for $\varepsilon > 0$. For $\varepsilon << 1$ the matrix has eigenvalues near $(1 \pm i)/\sqrt{2}$ so lying in the right half plane. This implies instability.

ii. Subsection 2.2 gives an example of a V with a stable equilibrium at 0 and so that $V(0)$ is not a local minimum. Thus there are nearby p with $V(p) < 0$. On the segment $[\underline{x}, p]$, V has decreased. This proves that the conservative system satisfies $(**)$ and nevertheless is stable. This example proves **ii.** ■

4 Proof of the instability Theorem 1.2

The proof combines two ingredients. The first is a theorem of Kozlov's and the second is a study of the Taylor series at an equilibrium of a potential satisfying $PV = 0$.

4.1 A theorem of Kozlov

In the direction of converse Dirichlet, Kozlov proved several important result. One of them is the following [6]. The differentiability hypothesis is radically reduced to in [11].

Theorem 4.1 *Suppose $V \in C^\infty$, 0 is a critical point, and, 0 is not a local minimum. Suppose that the Taylor expansion at 0 is*

$$V \;\sim\; V_m(x) \;+\; V_{m+1}(x) \;+\; \cdots$$

with V_j homogeneous of degree j. If the leading term V_m does not have a local minimum at 0, then the equilibrium $(0,0)$ is unstable.

4.1.1 An illustrative example of Kozlov's theorem

When $V = V_m$, Kozlov's Theorem is an elementary computation. Denote by p a point on the unit sphere $|x| = 1$ where V_m attains its minimum with $V(p) < 0$.
Orbits starting at rest on the ray from the origin through through p remain on that ray since the acceleration $-\nabla V_m$ must have no component perpendicular to p thanks to the minimality on the sphere.
At p the Euler homogeneity relation implies $\nabla V(p) = m\,V(p) := -b < 0\,.$

The orbit starting at rest at the point $\underline{a}\,p$ with $\underline{a} > 0$ is equal to $a(t)\,p$ where $a(t)$ is the unique solution of

$$a'' \;=\; b\,a^{m-1}\,, \qquad a(0) \;=\; \underline{a}\;, \quad a'(0) = 0\,.$$

It follows that $a(t)$ decreases to zero as $t \to -\infty$.
Kozlov's Theorem implies that such an orbit survives perturbation by higher order terms. The solution just computed is the leading term of an asymptotic expansion that describes an orbit that coverges to the equilibrium as $t \to -\infty$.

4.2 Critical points of solutions of elliptic equations

Theorem 4.2 *Suppose that the non constant $V \in C^\infty(|x| < r)$ satisfies*

$$V(0) = 0, \qquad \nabla_x V(0) = 0, \qquad \text{and,} \qquad P(x,\partial)V = 0$$

with P elliptic at 0. Then there is an $m \geq 2$ so that the Taylor expansion of V at 0 is

$$V \;\sim\; V_m(x) \;+\; \text{h.o.t.} \tag{4.1}$$

where V_m is a nonzero homogeneous polynomial of degree m that satisfies the elliptic equation

$$\sum_{i,j} a_{ij}(0)\partial_i\partial_j V_m \;=\; 0\,. \tag{4.2}$$

The critical point 0 is not a local minimum of V_m.

Proof. By the strong unique continuation theorem for scalar real second order elliptic partial differential equations, V is not infinitely flat at the origin. Thus V has a Taylor expansion (4.1) with $V_m \neq 0$.
Compute the leading order terms in the Taylor expansions to find

$$\begin{aligned} a_{ij}(x)\,\partial_i\partial_j V(x) \;&=\; a_{ij}(0)\,\partial_i\partial_j V_m(x) \;+\; O(|x|^{m-1}),\\ b_j(x)\,\partial_j V(x) \;&=\; b_j(0)\,\partial_j V_m(x) \;+\; O(|x|^m) \;=\; O(|x|^{m-1})\,,\\ c(x)\,V(x) \;&=\; c(0)\,V_m(x) \;+\; O(|x|^{m+1}) \;=\; O(|x|^m)\,. \end{aligned}$$

Therefore with leading term homogeneous of degree $m-2$,

$$PV(x) \;=\; \sum_{i,j} a_{ij}(0)\partial_i\partial_j V_m(x) \;+\; O(|x|^{m-1})\,.$$

With x fixed but arbitrary and $\varepsilon \to 0$,

$$\big(P(x,\partial)V\big)(\varepsilon x) \;=\; \varepsilon^{m-2}\sum_{i,j} a_{ij}(0)\partial_i\partial_j V_m(x) \;+\; O(\varepsilon^{m-1})\,.$$

The equation $PV = 0$ implies that the left hand side vanishes. Therefore

$$\sum_{i,j} a_{ij}(0)\partial_i\partial_j V_m(x) \;=\; O(\varepsilon).$$

Letting $\varepsilon \to 0$ proves (4.2).
The minimum principal for the elliptic operator $\sum_{i,j} a_{ij}(0)\partial_i\partial_j$ implies that if V_m has a local minimum, then V_m is constant. Since $V_m(0) = 0$, if V_m is constant it is identically zero. This contradicts the definition of V_m as the non zero leading term. ■

4.3 Proof of Theorem 1.2

Proof. Theorem 4.2 shows that any non constant potential $V(x)$ that satisfies a homogeneous second order elliptic equation satisfies the hypotheses of Kozlov's Theorem at all critical points. Therefore all such equilibria are unstable. ■

Summary and prospects. i. I think that it is more important that V satisfies a second order equation, than that V is real analytic. The differential equation imparts more structure.

iii. *There is more to the story.* There are further instability theorems stated in §116 of Maxwell's Treatise. One of these asserts that for a charged perfectly conducting body in an electrostatic field, there is no stable position of equilibrium. The proof of that theorem will appear in joint work with G. Allaire [1]. Maxwell also asserts the equilibria of a rigid body with fixed charges are also unstable. If the body is only allows translational motion Kozlov applies. The rotating case is work in progress.

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