Séminaire Laurent Schwartz

EDP et applications

Année 2011-2012

Claudio Muñoz

L^2-stability of multi-solitons

Séminaire Laurent Schwartz — EDP et applications (2011-2012), Exposé n^{o} IV, 9 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2011-2012____A4_0>

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
UMR 7640 CNRS/École polytechnique
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

cedram

Exposé mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2011-2012
Exposé n^o IV, 1-9

L^2-STABILITY OF MULTI-SOLITONS

CLAUDIO MUÑOZ

1. Introduction

The aim of this note is to give a short review of our recent work (see [5]) with Miguel A. Alejo and Luis Vega, concerning the L^2-stability, and asymptotic stability, of the *N-soliton* of the Korteweg-de Vries (KdV) equation

$$u_t + (u_{xx} + u^2)_x = 0. \tag{1.1}$$

Here $u = u(t, x)$ is a real valued function, and $(t, x) \in \mathbb{R}^2$. This equation arises in Physics as a model of propagation of dispersive long waves, as was pointed out by Russel in 1834 [30]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [18]. This equation was studied in a numerical work by Fermi, Pasta and Ulam, and by Kruskal and Zabusky [13, 19].

From the mathematical point of view, equation (1.1) is an *integrable model* [2, 3, 20, 1], with infinitely many conservation laws. Moreover, since the Cauchy problem associated to (1.1) is locally well posed in $L^2(\mathbb{R})$ (cf. Bourgain [8]), each solution is indeed global in time thanks to the *Mass* conservation law

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t, x)dx = M[u](0). \tag{1.2}$$

On the other hand, equation (1.1) has solitary wave solutions referred as *solitons*, solutions of the form

$$u(t, x) = Q_c(x - ct), \quad Q_c(s) := cQ(\sqrt{c}s), \quad c > 0, \tag{1.3}$$

and

$$Q(s) := \frac{3}{1 + \cosh(s)}. \tag{1.4}$$

The study of perturbations of solitons or solitary waves lead to the introduction of the concepts of *orbital and asymptotic stability*. In particular, it is natural to expect that solitons are stable in the energy space $H^1(\mathbb{R})$. Indeed, H^1-stability of KdV solitons has been considered by Benjamin and Bona-Souganidis-Strauss in [6, 7]. On the other hand, the asymptotic stability has been studied e.g. in Pego-Weinstein and Martel-Merle [35, 22].

Concerning the more involved case of the sum of $N \geq 2$ decoupled solitons, stability and asymptotic stability results are very recent. First of all, recall that, as

---

Date: February, 2011.
2000 *Mathematics Subject Classification.* Primary 35Q51, 35Q53; Secondary 37K10, 37K40.
Key words and phrases. KdV equation, Gardner equation, integrability, multi-soliton, stability, asymptotic stability, Miura transform.
This is a résumé of a talk given at the Ecole Polytechnique, Séminaire Laurent Schwartz, January 2011.

a consequence of the integrability property, KdV allows the existence of solutions behaving, as time goes to infinity, as the sum of N decoupled solitons. These solutions are well-known in the literature and are called *N-solitons*, or generically *multi-solitons* [14]. Indeed, any N-soliton solution has the form $u(t,x) := U^{(N)}(t,x) := U^{(N)}(x; c_j, x_j - c_j t)$, where

$$\big\{U^{(N)}(x; c_j, y_j) \,:\, c_j > 0,\, y_j \in \mathbb{R},\, j = 1, \ldots, N\big\} \tag{1.5}$$

is a family of explicit N-soliton profiles (see e.g. Maddocks-Sachs [21], §3.1). In particular, this solution describes multiple soliton collisions; but since solitons for KdV equation interact in a linear fashion, there is no residual appearing after the collisions, even if the equation is nonlinear in nature. In other words,

$$\lim_{t\to\pm\infty} \big\|U^{(N)}(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - c_j t - x_j^{\pm})\big\|_{H^1(\mathbb{R})} = 0,$$

with $x_j^{\pm} \in \mathbb{R}$ depending on the set (c_k). This is also a consequence of the integrability property.

In [21], Maddocks and Sachs considered the $H^N(\mathbb{R})$-stability of the N-soliton solution of KdV, by using $N+1$ conservation laws. Next, in [27, 26], Martel, Merle and Tsai improved the preceding result by proving stability and asymptotic stability of the *sum of N solitons*, *well decoupled* at the initial time, in the energy space. Their proof also applies for general nonlinearities and not only for the integrable cases, provided they have stable solitons, in the sense of Weinstein [37]. Their approach is based on the construction of N *almost conserved quantities*, related to the mass of each solitary wave, and the total energy of the solution. As a consequence of the existence of N-soliton solutions for KdV, the above results can be extended to give a global stability property, improving the Maddock-Sachs results. See also [23, 24, 25] for global H^1-stability results in some non-integrable cases.

As far as we know, the unique stability result for KdV solitons, below $H^1(\mathbb{R})$, was proved by Merle and Vega in [28]. Precisely, in that work, the authors prove that solitons of (1.1) are L^2-stable, by using the *Miura transform*

$$M[v] := \frac{3}{\sqrt{2}} v_x - \frac{3}{2} v^2, \tag{1.6}$$

which links solutions of the *defocusing, modified KdV* equation,

$$v_t + (v_{xx} - v^3)_x = 0, \quad v = v(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2, \tag{1.7}$$

with solutions of the KdV equation (1.1). In particular, the image of the family of *kink* solutions $\varphi_c(t,x) := \sqrt{c}\tanh(\sqrt{c/2}(x + ct + x_0))$, $c > 0$, $x_0 \in \mathbb{R}$, of (1.7) under the transformation (1.6) is the soliton Q_c above described, up to a standard Galilean transformation (cf. [28]).

Let us describe in more detail the Merle-Vega's approach. First of all, from the fact that the image of mKdV kinks are KdV solitons, and using the fact that the soliton is a minimizer of a well-known H^1 functional, one can define the inverse of the Miura transform (1.6) in a small L^2-vicinity of the soliton Q_c, to obtain a small H^1 neighborhood of the kink solution. Since the kink solution of (1.7) is H^1-stable (see e.g. Zhidkov, Merle-Vega [39, 28]), by applying once again the Miura transform to the mKdV solution, and using a well-known unicity argument, the authors concluded the L^2-stability of the KdV soliton. The following figure describes the aforementioned approach:

$$\begin{array}{ccccc}
 & \text{KdV} & \xrightarrow[v_0=M^{-1}[u_0]]{\text{Miura}} & \text{mKdV} & \\
 & u_0 \sim_{L^2} Q_c & & v_0 \sim_{H^1} \text{kink} & \\
 & & & & \\
\begin{matrix} L^2\text{-KdV flow} \\ \text{(Bourgain)} \end{matrix} & \Big\downarrow t>0 & & \begin{matrix} H^1\text{-mKdV flow} \\ \text{(Merle-Vega)} \end{matrix} \Big\downarrow & \begin{matrix} H^1\text{-stability} \\ \text{(Zhidkov)} \end{matrix} \\
 & & & & \\
 & \begin{matrix} u(t)=\bar{u}(t) \\ u(0)=u_0 \end{matrix} & \xleftarrow[\bar{u}(t)=M[v](t)]{\text{Miura}} & \begin{matrix} v(t) \text{ stable} \\ v(0)=v_0 \end{matrix} &
\end{array} \tag{1.8}$$

Fig. 1: *The Merle-Vega's approach.*

The Merle-Vega's idea has been applied to different models describing several phenomena. A similar Miura transform is available for the KP II equation, a two-dimensional generalization of the KdV equation. Mizumachi and Tzvetkov have shown the stability of solitary waves of KdV, seen as solutions of KP II, under periodic transversal perturbations [32]. Finally, we recall the L^2-stability result for solitary waves of the cubic NLS equation proved by Mizumachi and Pelinovsky in [31]. Other applications of the Miura transform are local well and ill-posedness results (cf. [17, 10]).

A natural question to consider is the generalization of the Merle-Vega's result to the case of multi-soliton solutions. In [12] (see also [36]), the authors state that the Miura transform sends *multi-kink* solutions of (1.7) towards a well defined family of *multi-soliton* solutions of (1.1). However, we have found that multi-kinks are hard to manipulate, due to the continuous interaction of non-local terms (recall that a kink does not belong to $L^2(\mathbb{R})$).[1] Therefore we have followed a different approach.

Indeed, in [5] we considered a *Gardner transform* [29, 11], well-known in the mathematical and physical literature since the late sixties, and which links H^1-solutions of the Gardner equation

$$v_t + (v_{xx} + v^2 - \beta v^3)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad \beta > 0, \tag{1.9}$$

with L^2-solutions of the KdV equation (1.1). More specifically, given any $\beta > 0$ and $v(t) \in H^1(\mathbb{R})$, solution of the Gardner equation (1.9), the Gardner transform [11]

$$u(t) = M_\beta[v](t) := [v - \frac{3}{2}\sqrt{2\beta}v_x - \frac{3}{2}\beta v^2](t), \tag{1.10}$$

is an L^2-solution of KdV. Compared with the original Miura transform (1.6), it has an additional *linear* term which simplifies the proofs.

In addition, the Gardner equation is also an integrable model [11], with soliton solutions of the form

$$v(t,x) := Q_{c,\beta}(x - ct),$$

[1] See [34] for a recent proof of the stability of multi-kinks of mKdV.

and[2]

$$Q_{c,\beta}(s) := \frac{3c}{1+\rho\cosh(\sqrt{c}s)}, \quad \text{with} \quad \rho := (1-\frac{9}{2}\beta c)^{1/2}, \quad 0 < c < \frac{2}{9\beta}. \quad (1.11)$$

In particular, in the formal limit $\beta \to 0$, we recover the standard KdV soliton (1.3)-(1.4). On the other hand, the Cauchy problem associated to (1.9) is globally well-posed under initial data in the energy class $H^1(\mathbb{R})$ (cf. Kenig-Ponce-Vega [16]), thanks to the mass (1.2) and *energy* conservation laws.

We have been interested in the image of the family of solutions (1.11) under the aforementioned, Gardner transform. Surprisingly enough, it turns out that the resulting family is **nothing but** the KdV soliton family (1.3). Indeed, a direct computation shows that for the Gardner soliton solution (1.11), one has

$$\begin{aligned} M_\beta[Q_{c,\beta}](t) &= \big[Q_{c,\beta} - \frac{3}{2}\sqrt{2\beta}Q'_{c,\beta} - \frac{3}{2}\beta Q^2_{c,\beta}\big](x-ct) \\ &= Q_c(x-ct-\delta), \end{aligned} \quad (1.12)$$

with $\delta = \delta(c,\beta) > 0$ provided $\beta > 0$, and Q_c the KdV soliton solution (1.3). In other words, the Gardner transform (1.10) sends the Gardner soliton towards a slightly translated KdV soliton. This last fact formally suggests that multi-soliton solutions of the Gardner equation (1.9) are sent towards (or close enough to) multi-soliton solutions of the KdV model (1.1), as is done in [36] for the case of the Miura transform.

In [5], we profit of this property to improve the H^1-stability and asymptotic stability properties proved by Martel, Merle and Tsai in [27], and Martel and Merle [26], now in the case of L^2-perturbations of the KdV multi-solitons. In order to maintain the presentation the simplest possible, we prefer do not state the asymptotic stability result that we have obtained. See [5] for more details.

Theorem 1.1 (L^2-stability of the N-soliton, [5]). *Let $\delta > 0$, $N \geq 2$, $0 < c_1^0 < \ldots < c_N^0$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $u(t)$ be a solution of (1.1) such that*

$$\|u(0) - U^{(N)}(\cdot; c_j^0, -x_j^0)\|_{L^2(\mathbb{R})} \leq \alpha,$$

with U^N the N-soliton profile described in (1.5). Then there exist $x_j(t)$, $j = 1, \ldots, N$, such that

$$\sup_{t\in\mathbb{R}} \big\|u(t) - U^{(N)}(\cdot; c_j^0, -x_j(t))\big\|_{L^2(\mathbb{R})} \leq \delta. \quad (1.13)$$

The above result can be seen as a consequence of the stability of an initial datum close enough to the **sum of N decoupled solitons** of the KdV equation, and the uniform continuity of the KdV flow for L^2-data, see e.g. [27], Corollary 1.

Theorem 1.2 (L^2-stability of the sum of N solitons of KdV). *Let $N \geq 2$ and $0 < c_1^0 < c_2^0 < \ldots < c_N^0$. There exist parameters $\alpha_0, A_0, L, \gamma > 0$, such that the following holds. Consider $u_0 \in L^2(\mathbb{R})$, and assume that there exist $L > L_0$, $\alpha \in (0, \alpha_0)$ and $x_1^0 < x_2^0 < \ldots < x_N^0$, such that*

$$x_j^0 > x_{j-1}^0 + L, \quad \textit{with} \quad j = 2, \ldots, N,$$

[2]See e.g. [9, 33] and references therein for a more detailed description of solitons and integrability for the Gardner equation.

and

$$\|u_0 - R_0\|_{L^2(\mathbb{R})} \leq \alpha, \quad \text{with} \quad R_0 := \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j^0). \tag{1.14}$$

Then there exist $x_1(t), \dots x_N(t)$ such that the solution $u(t)$ of the Cauchy problem for the KdV equation (1.1), with initial data u_0, satisfies

$$\sup_{t\geq 0} \big\|u(t) - \sum_{j=1}^{N} Q_{c_j^0}(\cdot - x_j(t))\big\|_{L^2(\mathbb{R})} \leq A_0(\alpha + e^{-\gamma_0 L}). \tag{1.15}$$

In the next section we sketch the proof of this last result. As a final remark, we believe that using the methods developed in [5], the stability –under periodic transversal perturbations– of the KdV multi-soliton $U^{(N)}$, seen as a solution of the KP II equation constant in the y-variable, can be handled via a Gardner transform, improving the results by Mizumachi and Tzvetkov [32].

2. Proof of Theorem 1.2

Let $u_0 \in L^2(\mathbb{R})$ satisfying (1.14). Let us denote by $z_0 := u_0 - R_0$, such that $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$. We want to solve the nonlinear Ricatti equation

$$M_\beta[v_0] = u_0 = R_0 + z_0, \tag{2.1}$$

with M_β the Gardner transform given by (1.10). We will prove that provided α and β are small enough, and using a fixed point argument, instead of the original Merle-Vega's idea.

Proposition 2.1 (Local invertibility around R_0)**.** *There exists $\beta_0 > 0$ such that, for all $0 < \beta < \beta_0$, the following holds. There exist $K_0, L_0, \gamma_0, \alpha_0 > 0$ such that for all $0 < \alpha < \alpha_0$, $L > L_0$, and $\|z_0\|_{L^2(\mathbb{R})} \leq \alpha$, there exists a solution $v_0 \in H^1(\mathbb{R})$ of (2.1), such that*

$$\|v_0 - S_0\|_{H^1(\mathbb{R})} \leq K_0(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}), \tag{2.2}$$

with $S_0(x) := \sum_{j=1}^{N} Q_{c_j^0,\beta}(x - x_j^0 - \delta_j)$,

$$\delta_j = \delta_j(c_j^0) := (c_j^0)^{-1/2}\cosh^{-1}(\frac{1}{\rho_j}),^3 \quad \rho_j := (1 - \frac{9}{2}\beta c_j^0)^{1/2}, \quad j = 1, \dots, N, \tag{2.3}$$

and $Q_{c,\beta}$ being the soliton solution of the Gardner equation (1.9).

Let us sketch the proof of this result. Assume $\beta > 0$ small, such that $\delta_j = O(\beta)$, independent of c_j^0. Note also that $S_0 \in H^1(\mathbb{R})$ with $\|S_0\|_{H^1(\mathbb{R})} \leq K$, independent of β. Moreover, a direct computation, using (1.12) shows that

$$M_\beta[S_0](t) = R_0 + O_{L^2(\mathbb{R})}(\beta e^{-\gamma_0 L}), \tag{2.4}$$

for some $\gamma_0 > 0$, independent of β small. Then, we look for a solution $v_0 \in H^1(\mathbb{R})$ of (2.1), of the form $v_0 = S_0 + w_0$, and w_0 small in $H^1(\mathbb{R})$. In other words, w_0 has to solve the nonlinear equation

$$\mathcal{L}[w_0] = (R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2, \tag{2.5}$$

[3] We take the positive inverse.

with

$$\mathcal{L}[w_0] := -\frac{3}{2}\sqrt{2\beta}w_{0,x} + (1-3\beta S_0)w_0. \tag{2.6}$$

We can see $\mathcal{L}$ as a unbounded operator in $L^2(\mathbb{R})$, with dense domain $H^1(\mathbb{R})$. From standard energy estimates (see [5]), one has that for $\beta > 0$ small enough, any solution $w_0 \in H^1(\mathbb{R})$ of the linear problem

$$\mathcal{L}[w_0] = f, \quad f \in L^2(\mathbb{R}), \tag{2.7}$$

must satisfy, for some fixed constant $K_0 > 0$,

$$\|w_0\|_{H^1(\mathbb{R})} \leq \frac{K_0}{\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})}. \tag{2.8}$$

In order to prove the existence and uniqueness of a solution of (2.7), we use (2.8) and a fixed point approach, in the spirit of [38, 15]. See [5] for the details.

In what follows, let us denote by $T := \mathcal{L}^{-1} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ the resolvent operator above mentioned. Now, from (2.5), we want to solve the nonlinear problem

$$w_0 = T[w_0] = \mathcal{L}^{-1}\big[(R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2\big]. \tag{2.9}$$

In order to invoke, once again, a fixed point argument, we consider the ball

$$\mathcal{B} := \Big\{ w_0 \in H^1(\mathbb{R}) \;\Big|\; \|w_0\|_{H^1(\mathbb{R})} \leq 2K_0\big(\frac{\alpha}{\sqrt{\beta}} + e^{-\gamma_0 L}\big)\Big\},$$

with $K_0 > 0$ the constant from (2.8), and $\gamma_0 > 0$ given in (2.4). A direct argument shows that T is a contraction mapping from $\mathcal{B}$ into itself, provided β is small. The proof is now complete.

The above proposition allows to construct the inverse of the Gardner transform in a small L^2-vicinity of R_0. Now we follow a similar approach to that of [28], applied this time to the Gardner equation. In order to do this, recall the following stability result for Gardner solitons, proved by Martel and Merle in the general case of gKdV equations.

Proposition 2.2 (H^1-stability for Gardner solitons, [27, 26])**.** *Let $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ be such that*

$$\partial_c \int_{\mathbb{R}} Q_{c,\beta}^2\Big|_{c=c_j} > 0, \quad \textit{for all } j = 1,\ldots,N. \tag{2.10}$$

There exists $\tilde\alpha_0, \tilde A_0, \tilde L_0, \tilde\gamma > 0$ such that the following is true. Let $v_0 \in H^1(\mathbb{R})$, and assume that there exists $\tilde L > \tilde L_0$, $\tilde\alpha \in (0, \tilde\alpha_0)$ and $\tilde x_1^0 < \tilde x_2^0 < \ldots < \tilde x_N^0$, such that

$$\|v_0 - \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde x_j^0)\|_{H^1(\mathbb{R})} \leq \tilde\alpha, \tag{2.11}$$

$$\tilde x_j^0 > \tilde x_{j-1}^0 + \tilde L, \quad j = 2,\ldots,N. \tag{2.12}$$

Then there exists $\tilde x_1(t), \ldots \tilde x_N(t)$ such that the solution $v(t)$ of the Cauchy problem associated to (1.9), with initial data v_0, satisfies

$$v(t) = S(t) + w(t), \quad S(t) := \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot - \tilde x_j(t)),$$

and

$$\sup_{t\geq 0}\Big\{\|w(t)\|_{H^1(\mathbb{R})}+\sum_{j=1}^{N}|\tilde{x}_j'(t)-c_j|\Big\}\leq \tilde{A}_0(\tilde{\alpha}+e^{-\tilde{\gamma}\tilde{L}}). \tag{2.13}$$

We apply this property to the initial datum v_0 obtained in Proposition 2.1. After this point, the proof of Theorem 1.2 follows closely the ideas of [28], giving the desired result. We finish with the following diagram, which describes the approach we have followed.

$$\begin{array}{ccc}
\begin{array}{c}\text{KdV}\\ u_0\sim_{L^2} R_0\end{array} & \xrightarrow[v_0=M_\beta^{-1}[u_0]]{\text{Gardner}} & \begin{array}{c}\text{Gardner}\\ v_0\sim_{H^1} S_0\end{array}\\
\begin{array}{c}L^2\text{-KdV flow}\\ \text{(Bourgain)}\end{array}\Big\downarrow t>0 & & \begin{array}{c}H^1\text{-Gardner flow}\\ \text{(K-P-V)}\end{array}\Big\downarrow\begin{array}{c}H^1\text{-stability}\\ \text{(Martel-Merle)}\end{array}\\
u(t)=\bar{u}(t) & \xleftarrow[\bar{u}(t)=M_\beta[v](t)]{\text{Gardner}} & v(t)\ \text{stable}
\end{array}$$

Fig. 2: *The Gardner's approach.*

See [5] for a detailed proof.

Acknowdlegments. I would like to thank Y. Martel and F. Merle for several and useful comments, and M. A. Alejo and L. Vega for their hospitality at the University of Basque Country.

References

[1] M. Ablowitz, and P. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, London Mathematical Society Lecture Note Series, 149. Cambridge University Press, Cambridge, 1991.

[2] M. Ablowitz, D. Kaup, A. Newell, and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math. 53 (1974), no. 4, 249–315.

[3] M. Ablowitz, and H. Segur, *Solitons and the inverse scattering transform*, SIAM Studies in Applied Mathematics, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981. x+425 pp.

[4] J. Albert, J. Bona, and N. Nguyen, *On the stability of KdV multisolitons*, Differential Integral Equations **20** (2007), no. 8, 841–878.

[5] Alejo, Miguel A., Muñoz C., and Vega, Luis, *The Gardner equation and the L^2-stability of the N-soliton solution of the Korteweg–de Vries equation*, to appear in Transactions of the AMS (arXiv:1012.5290).

[6] T.B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London A **328**, (1972) 153–183.

[7] J.L. Bona, P. Souganidis and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London **411** (1987), 395–412.

[8] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV equation*, Geom. Funct. Anal. **3** (1993), no. 3, 209-262.

[9] K.W. Chow, R.H.J Grimshaw, and E. Ding, *Interactions of breathers and solitons in the extended Korteweg-de Vries equation*, Wave Motion **43** (2005) 158–166.

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T.Tao, *Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$*, J. Amer. Math. Soc. **16** (2003), no. 3, 705–749 (electronic).

[11] C.S. Gardner, M.D. Kruskal, and R. Miura, *Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Math. Phys. **9**, no. 8 (1968), 1204–1209.

[12] F. Gesztesy, and B. Simon, *Constructing solutions of the mKdV-equation*, J. Funct. Anal. **89** (1990), no. 1, 53–60.

[13] E. Fermi, J. Pasta and S. Ulam, *Studies of nonlinear problems I*, Los Alamos Report LA1940 (1955); reproduced in Nonlinear Wave Motion, A.C. Newell, ed., Am. Math. Soc., Providence, R. I., 1974, pp. 143–156.

[14] R. Hirota, *Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons*, Phys. Rev. Lett., **27** (1971), 1192–1194.

[15] C.E. Kenig, and Y. Martel, *Global well-posedness in the energy space for a modified KP II equation via the Miura transform*, Trans. Amer. Math. Soc. **358** no. 6, pp. 2447–2488.

[16] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46**, (1993) 527–620.

[17] C.E. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), no. 3, 617–633.

[18] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of stationary waves*, Philos. Mag. Ser. 5, **39** (1895), 422–443.

[19] M.D. Kruskal and N.J. Zabusky, *Interaction of "solitons" in a collisionless plasma and recurrence of initial states*, Phys. Rev. Lett. **15** (1965), 240–243.

[20] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**, (1968) 467–490.

[21] J.H. Maddocks, and R.L. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math. **46**, 867–901 (1993).

[22] Y. Martel and F. Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity **18** (2005) 55–80.

[23] Y. Martel and F. Merle, *Description of two soliton collision for the quartic gKdV equation*, preprint arXiv:0709.2672 (2007), to appear in Annals of Mathematics.

[24] Y. Martel and F. Merle, *Stability of two soliton collision for nonintegrable gKdV equations*, Comm. Math. Phys. **286** (2009), 39–79.

[25] Y. Martel and F. Merle, *Inelastic interaction of nearly equal solitons for the quartic gKdV equation*, Invent. Math. **183** (2011), no. 3, 563–648.

[26] Y. Martel, and F. Merle, *Asymptotic stability of solitons of the gKdV equations with general nonlinearity*, Math. Ann. **341** (2008), no. 2, 391–427.

[27] Y. Martel, F. Merle and T. P. Tsai, *Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations*, Comm. Math. Phys. **231** (2002) 347–373.

[28] F. Merle; and L. Vega, *L^2 stability of solitons for KdV equation*, Int. Math. Res. Not. 2003, no. 13, 735–753.

[29] R.M. Miura, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9**, no. 8 (1968), 1202–1204.

[30] R.M. Miura, *The Korteweg–de Vries equation: a survey of results*, SIAM Review **18**, (1976) 412–459.

[31] T. Mizumachi, and D. Pelinovsky, *Bäcklund transformation and L^2-stability of NLS solitons*, preprint.

[32] T. Mizumachi, and N. Tzvetkov, *Stability of the line soliton of the KP–II equation under periodic transverse perturbations*, preprint.

[33] C. Muñoz, *On the inelastic 2-soliton collision for gKdV equations with general nonlinearity*, Int. Math. Research Notices (2010) **2010** (9): 1624–1719.

[34] C. Muñoz, *The Gardner equation and the stability of multi-kink solutions of the mKdV equation*, preprint arXiv:1106.0648.

[35] R.L. Pego, and M.I. Weinstein, *Asymptotic stability of solitary waves*, Commun. Math. Phys. **164**, 305–349 (1994).

[36] B. Thaller, *The Dirac equation*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992. xviii+357 pp.

[37] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure. Appl. Math. **39**, (1986) 51—68.

[38] M.V. Wickerhauser, *Inverse scattering for the heat operator and evolutions in* $2+1$ *variables,* Comm. Math. Phys. **108** (1987), 67–89.

[39] P. E. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.

Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637, USA

E-mail address: cmunoz@math.uchicago.edu