Séminaire Laurent Schwartz

EDP et applications

Année 2011-2012

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Abstract: *The aim of this talk is to present some recent existence results about quasi-periodic solutions for PDEs like nonlinear wave and Schrödinger equations in $\mathbb{T}^d$, $d \geq 2$, and the 1-d derivative wave equation. The proofs are based on both Nash-Moser implicit function theorems and KAM theory.*[1]

Keywords: KAM for PDE, Nash-Moser Theory, Quasi-Periodic Solutions, Small Divisors, Nonlinear Schrödinger and wave equation, Infinite Dimensional Hamiltonian Systems.

2000AMS subject classification: 35Q55, 37K55, 37K50.

1 Introduction

In the last years important mathematical progresses have been achieved in the study of evolutionary Partial Differential Equations (PDEs), like the nonlinear Schrödinger (NLS) and wave (NLW) equations, adopting the "dynamical systems philosophy", focusing, in particular, on the search of invariant tori of the phase space filled by periodic and quasi-periodic solutions.

A natural setting concerns the bifurcation of quasi-periodic solutions close to linearly stable (elliptic) equilibria of a PDE. The main difficulty for the existence proof is the presence of arbitrarily "*small divisors*" in the perturbative expansion series of the expected solutions. Such small divisors arise by complex resonance phenomena between the normal mode frequencies of the system.

The main strategies which have been developed to overcome the small divisors difficulty are based on quadratic iterative scheme like:

1. KAM (Kolmogorov-Nash-Moser) theory,

2. Newton-Nash-Moser implicit function theorems.

The KAM approach consists in generating iteratively a sequence of transformations of the phase space which bring the Hamiltonian system into a normal form with an invariant torus at the origin. This iterative procedure requires, at each step, to invert the so called linear "homological equations". In the usual KAM scheme the normal form has constant coefficients (reducibility), hence the homological equations have *constant coefficients* and can be solved by Fourier series imposing the "second order Melnikov" non-resonance conditions. The final KAM torus is linearly stable.

This scheme was effectively implemented by Kuksin [24] and Wayne [31] to prove the existence of quasi-periodic solutions for one dimensional (1-d) NLW and NLS equations. These pioneering results were limited to Dirichlet boundary conditions because the eigenvalues of ∂_{xx} had to be simple. Actually, the required second order Melnikov non resonance conditions are violated in presence of multiple normal frequencies (because differences of normal frequencies appear), for example, already for periodic boundary conditions (two eigenvalues of ∂_{xx} are equals).

Then a more direct bifurcation approach was proposed by Craig and Wayne [18] for 1-d NLS and NLW with periodic boundary conditions. After a Lyapunov-Schmidt decomposition, the search of the invariant torus is reduced to solve a functional equation in scales of Banach spaces, by some Newton-Nash-Moser implicit function theorem.

The main advantage of this approach is to require only the so called "first order Melnikov" non-resonance conditions for solving the linearized equations (homological equations) at each step of the

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[1]Text of a conference held the 14 February 2012 at the Séminaire Laurent Schwartz, IHES, Paris.

iteration. These conditions are essentially the minimal assumptions, and, in particular, do not involve differences of normal frequencies. Translated in the KAM language this corresponds to allow a non-constant coefficients normal form around the torus. The main difficulty of this strategy is that the homological equations are PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. Hence it is hard to estimate their inverses in high norms. Craig-Wayne [18] solved this problem for periodic solutions of 1-d analytic NLS and NLW and Bourgain [11] also for quasi-periodic solutions.

At present, the theory for 1-d NLS and NLW with semilinear nonlinearities has been sufficiently understood (see e.g. [25]), but much work remains about quasi-periodic solutions of PDEs

1. in higher space dimensions, e.g. $x \in \mathbb{T}^d$, $d \geq 2$,

2. with nonlinearities containing derivatives.

We shall first present in section 2 some new existence result of quasi-periodic solutions for NLW (and NLS) on $\mathbb{T}^d$, $d \geq 2$, via Nash-Moser theory, and, then, for 1-d Hamiltonian derivative wave equations via KAM theory (section 3).

2   PDEs in higher space dimension

The main difficulties for PDEs in higher space dimensions are that

1. the eigenvalues of $-\Delta + V(x)$ appear in clusters of unbounded sizes,

2. the eigenfunctions are, in general, "not localized with respect to the exponentials".

Roughly speaking, property 2 means that, if we expand the eigenfunctions of $-\Delta + V(x)$ with respect to the exponentials, the Fourier coefficients rapidly converge to zero. This property always holds in 1 space dimension (see [18]) but may fail for $d \geq 2$, see [13]. This problem has been often bypassed considering pseudo-differential PDEs where the multiplicative potential $V(x)$ is substituted by a "convolution potential" $V * (e^{ij \cdot x}) := m_j e^{ij \cdot x}$, $m_j \in \mathbb{R}$, $j \in \mathbb{Z}^d$ (which acts diagonally on the exponentials). The scalars m_j are called the "Fourier multipliers" and play the role of "external parameters".

The Newton-Nash-Moser approach is, in principle, very useful to overcome problem 1, because it requires only the first order Melnikov non-resonance conditions and therefore does not exclude multiplicity of normal frequencies. Actually, developing this perspective, Bourgain [13], [15] was able to prove the existence of quasi-solutions for NLW and NLS with Fourier multipliers on $\mathbb{T}^d$, $d \geq 2$.

More recently, also the KAM approach has been extended by Eliasson-Kuksin [20] for NLS on $\mathbb{T}^d$ with Fourier multipliers. The key issue is to control more accurately the perturbed frequencies after the KAM iteration and, in this way, the difference of the normal frequencies, verifying the second order Melnikov non-resonance conditions. A related technique is developed in section 3. We refer also to Procesi-Procesi [29] and Wang [30] for completely resonant NLS. For the NLW equation on $\mathbb{T}^d$ still no reducibility results are available.

2.1   Quasi-periodic solutions of NLW on $\mathbb{T}^d$

As a model equation we consider d-dimensional nonlinear wave equations like

$$u_{tt} - \Delta u + V(x)u = \varepsilon f(\omega t, x, u), \quad x \in \mathbb{T}^d, \ \varepsilon > 0, \tag{2.1}$$

where the multiplicative potential V is in $C^q(\mathbb{T}^d; \mathbb{R})$, the nonlinearity is quasiperiodic-in-time with a non-resonant frequency vector $\omega \in \mathbb{R}^\nu$ (see (2.5), (2.6)), and

$$f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R}) \tag{2.2}$$

for some $q \in \mathbb{N}$ large enough (fixed in Theorem 2.1).

Concerning the potential we suppose that

$$\mathrm{Ker}(-\Delta + V(x)) = 0 \,. \tag{2.3}$$

In (2.1) we use only *one* external parameter, namely the length of the frequency vector (time scaling). More precisely we assume that the frequency vector ω is co-linear with a fixed vector $\bar{\omega} \in \mathbb{R}^\nu$,

$$\omega = \lambda\bar{\omega}\,, \quad \lambda \in \Lambda := [1/2, 3/2] \subset \mathbb{R}, \quad |\bar{\omega}| \leq 1\,, \tag{2.4}$$

where $\bar{\omega}$ is Diophantine, namely for some $\gamma_0 \in (0,1)$,

$$|\bar{\omega} \cdot l| \geq \frac{\gamma_0}{|l|^\nu}\,, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\}\,, \tag{2.5}$$

and

$$\Big| \sum_{1 \leq i \leq j \leq \nu} \bar{\omega}_i\bar{\omega}_j p_{ij} \Big| \geq \frac{\gamma_0}{|p|^{\tau_0}}\,, \quad \forall p \in \mathbb{Z}^{\frac{\nu(\nu+1)}{2}} \setminus \{0\}\,. \tag{2.6}$$

There exists $\bar{\omega}$ satisfying (2.5) and (2.6) at least for $\tau_0 > \nu(\nu+1) - 1$ and γ_0 small. For definiteness we fix $\tau_0 := \nu(\nu+1)$.

Remark 2.1. *Condition* (2.6) *is required for NLW (not for NLS), see the end of the section.*

The dynamics of the linear wave equation

$$u_{tt} - \Delta u + V(x)u = 0 \tag{2.7}$$

is well understood. The eigenfunctions of

$$(-\Delta + V(x))\psi_j(x) = \mu_j\psi_j(x)$$

form a Hilbert basis in $L^2(\mathbb{T}^d)$ and the eigenvalues $\mu_j \to +\infty$ as $j \to +\infty$. By assumption (2.3) all the eigenvalues μ_j are different from 0. We list them in non-decreasing order

$$\mu_1 \leq \ldots \leq \mu_{n^-} < 0 < \mu_{n^-+1} \leq \ldots \tag{2.8}$$

where n^- denotes the number of negative eigenvalues (counted with multiplicity).

All the solutions of (2.7) are the linear superpositions of normal mode oscillations, namely

$$u(t,x) = \sum_{j=1}^{n^-}(\beta_j^- e^{-\sqrt{|\mu_j|}t} + \beta_j^+ e^{\sqrt{|\mu_j|}t})\psi_j(x) + \sum_{j \geq n^-+1} \mathrm{Re}(a_j e^{\mathrm{i}\sqrt{\mu_j}t})\,\psi_j(x)\,, \ \beta_j^\pm \in \mathbb{R}\,, a_j \in \mathbb{C}\,.$$

The first n^- eigenfunctions correspond to hyperbolic directions where the dynamics is attractive/repulsive. The other infinitely many eigenfunctions correspond to elliptic directions.

- QUESTION: *for ε small enough, do there exist quasi-periodic solutions $u(\omega t, x)$ of the nonlinear wave equation (2.1) for positive measure sets of $\lambda \in [1/2, 3/2]$?*

Note that, if $f(\varphi, x, 0) \not\equiv 0$ then $u = 0$ is not a solution of (2.1) for $\varepsilon \neq 0$.

The above question amounts to look for $(2\pi)^{d+\nu}$-periodic solutions $u(\varphi, x)$ of

$$(\omega \cdot \partial_\varphi)^2 u - \Delta u + V(x)u = \varepsilon f(\varphi, x, u) \tag{2.9}$$

in the Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{R}) \ := \ \Big\{ u(\varphi, x) := \sum_{(l,j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d} u_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} : \|u\|_s^2 := \sum_{i \in \mathbb{Z}^{\nu+d}} |u_i|^2 \langle i \rangle^{2s} < +\infty\,,$$

$$u_{-i} = \overline{u_i}\,, \ \text{where}\ i := (l,j)\,, \ \langle i \rangle := \max(|l|, |j|, 1) \Big\} \tag{2.10}$$

for some $(\nu + d)/2 < s \leq q$.

The above question turns into a bifurcation problem for equation (2.9) from the trivial solution $(u, \varepsilon) = (0, 0)$. The main difficulty is that the unperturbed linear operator

$$(\omega \cdot \partial_\varphi)^2 - \Delta + V(x)$$

possesses arbitrarily small eigenvalues $-(\omega \cdot l)^2 + \mu_j$, called "small divisors". As a consequence, its inverse operator, if any, is unbounded and the standard implicit function theorem can not be applied.

The following theorem is proved in [8] by a Nash-Moser implicit function iterative scheme.

Theorem 2.1. ([8]) *Assume* (2.5)-(2.6).
EXISTENCE: *There are* $s := s(d, \nu)$, $q := q(d, \nu) \in \mathbb{N}$, *such that:* $\forall f \in C^q$, $\forall V \in C^q$ *satisfying* (2.3), $\forall \varepsilon \in [0, \varepsilon_0)$ *small enough, there is a map*

$$u(\varepsilon, \cdot) \in C^1(\Lambda; H^s) \quad with \quad \sup_{\lambda \in \Lambda} \|u(\varepsilon, \lambda)\|_s \to 0 \text{ as } \varepsilon \to 0, \tag{2.11}$$

and a Cantor like set $\mathcal{C}_\varepsilon \subset \Lambda := [1/2, 3/2]$ *of asymptotically full Lebesgue measure, i.e.*

$$|\mathcal{C}_\varepsilon| \to 1 \quad as \quad \varepsilon \to 0, \tag{2.12}$$

such that, $\forall \lambda \in \mathcal{C}_\varepsilon$, $u(\varepsilon, \lambda)$ *is a solution of (2.9) with* $\omega = \lambda \bar{\omega}$.
REGULARITY: *If* V, f *are of class* C^∞ *then* $u(\varepsilon, \lambda) \in C^\infty(\mathbb{T}^d \times \mathbb{T}^\nu; \mathbb{R})$.

An analogous result holds for the Hamiltonian NLS equation

$$iu_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, u, \bar{u}), \quad x \in \mathbb{T}^d,$$

see [6]-[7]. Condition (2.6) is not required for NLS.

Remark 2.2. *It is clear that the existence of quasi-periodic solutions for just a Cantor like set of parameters* $\mathcal{C}_\varepsilon$ *is not a technical restriction! In a complementary region, chaotic motions and Arnold diffusion phenomena shall occur. In some sense Theorem 2.1 is complementary to the results in [16].*

The novelties of Theorem 2.1 are that we prove the existence of quasi-periodic solutions with:

1. finitely *differentiable* nonlinearities, see (2.2),

2. a *multiplicative* (finitely differentiable) potential $V(x)$,

3. a *pre-assigned* (diophantine) direction of the tangential frequencies, see (2.4).

1. **Finitely differentiable PDEs.** Theorem 2.1 (and the analogous in [6] for NLS) confirms the natural conjecture about the persistence of quasi-periodic solutions for Hamiltonian PDEs into a setting of finitely many derivatives. Actually almost all the previous literature was valid for analytic nonlinearities (actually polynomials in [13], [15]). The nonlinearity in Theorem 2.1, as well as the potential, is sufficiently many times differentiable, depending on the dimension and the number of the frequencies. Of course we can not expect the existence of quasi-periodic solutions under too weak regularity assumptions: for finite dimensional systems, it has been rigorously proved that, if the vector field is not sufficiently smooth, then all the invariant tori could be destroyed and only discontinuous Aubry-Mather invariant sets survive.

2. **Multiplicative potential.** Theorem 2.1 (and the anologous for NLS in [6]) is the first existence result of quasi-periodic solutions with a multiplicative potential $V(x)$ on $\mathbb{T}^d$, $d \geq 2$. We never exploit properties of "localizations" of the eigenfunctions of $-\Delta + V(x)$ with respect to the exponentials, that actually might *not* be true, see [13]. Along the multiscale analysis we use the exponential basis which diagonalizes $-\Delta + m$ where m is the average of $V(x)$ and not the eigenfunctions of $-\Delta + V(x)$. Further properties of the eigenfunctions of the Laplacian with a periodic potential seem to be unavoidable to prove also reducibility.

3. **Parameter dependence.** For finite dimensional systems, the existence of quasi-periodic solutions with tangential frequencies constrained along a fixed direction has been proved by Eliasson [19] and Bourgain [12]. The main difficulty clearly relies in satisfying the Melnikov non-resonance conditions, required at each step of the iterative process, using only *one* parameter. Bourgain raised in [12] the question if a similar result holds true also for infinite dimensional Hamiltonian systems. This has been recently proved in [4] for 1-dimensional PDEs, verifying the second order Melnikov non-resonance conditions of KAM theory. Theorem 2.1 answers positively to Bourgain's question also for PDEs in higher space dimension.

4. **PDEs defined on more general manifolds.** Finally we mention that the previous results (valid for PDEs on flat tori $\mathbb{T}^d$) should generalize to NLS and NLW defined on more general manifolds. The dynamics of a PDE on a compact Riemannian manifold strongly depends on its geometry, via the properties of the eigenvalues and the eigenfunctions of the Laplace-Beltrami operator. In [9]-[10] we proved the existence of periodic solutions of NLS and NLW defined on compact Zoll manifolds (i.e. spheres), Lie groups and homogeneous spaces. In these cases, the eigenvalues are highly degenerate and only weak properties of localization of the eigenfunctions hold. Interestingly, many tools in [9]-[10] resemble the Birkhoff normal form techniques developed by Bambusi, Delort, Grebért, Szeftel [2] for PDEs on spheres and Zoll manifolds.

Ideas of proof. Theorem 2.1 is proved by a Nash-Moser iteration. The main step concerns the invertibility of (any finite dimensional restriction of) the linearized operator

$$\mathcal{L}(u) := \mathcal{L}(\omega, \varepsilon, u) := L_\omega - \varepsilon g(\varphi, x) \tag{2.13}$$

where

$$L_\omega := (\omega \cdot \partial_\varphi)^2 - \Delta + V(x) \qquad \text{and} \qquad g(\varphi, x) := (\partial_u f)(\varphi, x, u) \,, \tag{2.14}$$

obtained linearizing (2.9) at a non zero approximate solution u (defined iteratively along the scheme). The function g depends also on λ through u.

Remark 2.3. *The main difficulty is that $\mathcal{L}(u)$ has non constant coefficients in both (φ, x) and then Fourier decomposition does not apply. Moreover, since L_ω^{-1} is unbounded, the zero-order perturbative term $\varepsilon g(\varphi, x)$ acts as a "singular perturbation" of L_ω.*

We decompose the multiplicative potential as $V(x) = m + V_0(x)$ where m is the average of $V(x)$ and $V_0(x)$ has zero mean value. Then we write

$$L_\omega = D_\omega + V_0(x) \qquad \text{where} \qquad D_\omega := (\omega \cdot \partial_\varphi)^2 - \Delta + m \tag{2.15}$$

has constant coefficients. In the Fourier basis $(e^{\mathrm{i}(l \cdot \varphi + j \cdot x)})$, the operator $\mathcal{L}(u)$ is represented by the infinite dimensional self-adjoint matrix

$$A(\omega) := A(\omega, \varepsilon, u) := D + T$$

where

$$D := \mathrm{diag}_{(l,j) \in \mathbb{Z}^\nu \times \mathbb{Z}^d} - (\omega \cdot l)^2 + \|j\|^2 + m := \mathrm{diag}_{i \in \mathbb{Z}^b} \delta_i \,,$$

$$\|j\|^2 := j_1^2 + \ldots + j_d^2 \,, \quad i := (l,j) \in \mathbb{Z}^b := \mathbb{Z}^\nu \times \mathbb{Z}^d \,, \quad \delta_i := -(\omega \cdot l)^2 + \|j\|^2 + m \tag{2.16}$$

and

$$T := T_2 - \varepsilon T_1 \,, \quad T := (T_i^{i'})_{i,i' \in \mathbb{Z}^b} \,, \quad T_i^{i'} := (V_0)_{j-j'} - \varepsilon g_{i-i'} \tag{2.17}$$

represents the multiplication operator by $V_0(x) - \varepsilon g(\varphi, x)$. The off-diagonal matrix T is *Töplitz*, namely $T_i^{i'}$ depends only on the difference of the indices $i - i'$, and, since the functions $g, V \in H^s$, then $T_i^{i'} \to 0$ as $|i - i'| \to \infty$ at a polynomial rate. In other words, T is "polynomially localized close to the diagonal".

Remark 2.4. *Since $g(\varphi, x)$ is real valued, the operator $\mathcal{L}(u)$ is self-adjoint, and the eigenvalues of all its finite dimensional restrictions vary smoothly with respect to the one dimensional parameter $\lambda \in [1/2, 3/2]$.*

We construct inductively better and better approximate solutions

$$u_n \in H_n := \left\{ u \in H^s \; : \; u = \sum_{|(l,j)| \leq N_n} u_{l,j} \, e^{\mathrm{i}(l\cdot\varphi + j\cdot x)} \right\}$$

of the NLW equation (2.9), solving by a Nash-Moser iteration, the "truncated" equations

$$(P_n) \qquad\qquad P_n\Big(L_\omega u - \varepsilon f(u)\Big) = 0\,, \quad u \in H_n\,,$$

where $P_n : H^s \to H_n$ denote the orthogonal projectors onto H_n and $N_n := N_0^{2^n}$. The P_n are *smoothing* operators.

The main step is to prove that the finite dimensional matrices $\mathcal{L}_n := \mathcal{L}_n(u_{n-1}) := P_n \mathcal{L}(u_{n-1})_{|H_n}$ are invertible for "most" parameters $\lambda \in \Lambda$ and satisfy the interpolation estimates

$$\|\mathcal{L}_n^{-1} h\|_s \leq C(s)(N_n^{\tau'+\delta s}\|h\|_{s_0} + N_n^{\tau'+\delta s_0}\|h\|_s)\,, \quad \forall s \geq s_0\,, \tag{2.18}$$

which are sufficient for the Nash-Moser convergence. Note that the exponent $\tau' + \delta s$ in (2.18) grows with s, unlike the usual Nash-Moser theory where the "tame" exponent is s-independent. Actually the conditions (2.18) are *optimal* for the convergence, as a famous counter-example of Lojasiewicz-Zehnder [27] shows: if $\delta = 1$ the Nash-Moser iterative scheme does not converge.

L^2**-bounds.** The first step is to show that, for "most" parameters $\lambda \in \Lambda$, the eigenvalues of $\mathcal{L}_n$ are in modulus bounded from below by $O(N_n^{-\tau})$ (1th *order Melnikov non-resonance conditions*) and so

$$\|\mathcal{L}_n^{-1}\|_0 = O(N_n^\tau)\,. \tag{2.19}$$

The proof is based on an eigenvalue variation argument. We explain it in the simplest case that $-\Delta + V(x) \geq \beta_0 I > 0$ is positive definite. Dividing $\mathcal{L}_n$ by λ^2, and setting $\xi := 1/\lambda^2$, we observe that the derivative with respect to ξ satisfies

$$\partial_\xi(\xi\mathcal{L}_n) = P_n(-\Delta + V(x))_{|H_n} + O(\varepsilon\|T_1\|_0 + \varepsilon\|\partial_\lambda T_1\|_0) \geq \frac{\beta_0}{2}\,,$$

for ε small, i.e. it is positive definite. So, the eigenvalues $\mu_{l,j}(\xi, \varepsilon)$ (which depend C^1-smoothly on ξ for fixed ε, see remark 2.4) of the self-adjoint matrix $\xi\mathcal{L}_n$ satisfy

$$\partial_\xi \mu_{l,j}(\xi, \varepsilon) \geq \frac{\beta_0}{2}\,, \quad \forall |(l,j)| \leq N_n\,,$$

which easily implies (2.19) except in a set of λ's of measure $O(N_n^{-\tau+d+\nu})$.

Remark 2.5. *The above excision of the parameters λ is the origin of the Cantor set $\mathcal{C}_\varepsilon$ in Theorem 2.1.*

Tame estimate for $\mathcal{L}_n^{-1}$. The L^2-estimate (2.19) alone implies only a bound like (2.18) with $\delta = 1$. In order to prove the sublinear decay (2.18) for the Green functions we have to exploit (mild) "separation properties" of the small divisors: not all the eigenvalues of $\mathcal{L}_n$ are $O(N_n^{-\tau})$ small. We have to worry only about the SINGULAR sites (l, j) such that

$$| - (\omega \cdot l)^2 + \|j\|^2 + m| \leq \rho\,. \tag{2.20}$$

Heuristically, the key is to show that, as $\rho \to 0$ the singular sites become "more and more rare", decomposing is separated huge clusters. This is the hard part of the analysis. This is obtained by a multiscale procedure, assuming the non-resonance condition

$$\left| n + \sum_{1 \leq i \leq j \leq \nu} p_{ij}\omega_i\omega_j \right| \geq \frac{\gamma}{1 + |p|^{\tau_0}}\,, \; \forall (n, p) \in \mathbb{Z}^{1 + \frac{\nu(\nu+1)}{2}} \setminus \{0\}, \; \gamma > 0\,, \tag{2.21}$$

which is satisfied by $\omega = \lambda\bar\omega$ for most $\lambda \in \Lambda$ (thanks to (2.6)). A condition like (2.21) is necessary because the singular sites are integer points near a cone, see (2.20), and not a paraboloid like for NLS. Then it is necessary to assume an irrationality condition on the "slopes" of this cone. Assumption (2.21) is weaker than in [15]. It seems to be the weakest possible.

3 KAM theory for 1-d derivative NLW

KAM theory for PDEs with nonlinearities containing derivatives has been first extended by Kuksin [25] and Kappeler-Pöschel [23] for KdV equations, and, for the $1d$-derivative NLS (DNLS) and Benjiamin-Ono equations, by Liu-Yuan [26]. The key idea of these results is again to provide only a non-reducible normal form around the torus. However, in this cases, the homological equations with non-constant coefficients are only *scalar* (not an infinite system as in the previous section). We remark that the KAM proof is more delicate for DNLS and Benjiamin-Ono, because these equations are less "dispersive" than KdV, i.e. the eigenvalues of the principal part of the differential operator grow only quadratically at infinity, and not cubically as for KdV. As a consequence of this difficulty, the quasi-periodic solutions in [25], [23] are analytic, in [26], only C^∞. Actually, for the applicability of these KAM schemes, the more dispersive the equation is, the more derivatives in the nonlinearity can be supported. The limit case of the derivative nonlinear wave equation (DNLW) -which is not dispersive at all- is excluded by these approaches.

In the paper [12] (which proves the existence of quasi-periodic solutions for semilinear $1d$-NLS and NLW), Bourgain claims, in the last remark, that his analysis works also for the Hamiltonian "derivation" wave equation

$$y_{tt} - y_{xx} + g(x)y = \left(- \frac{d^2}{dx^2} \right)^{1/2} F(x,y) \,,$$

see also [14], page 81. Unfortunately no details are given. However, Bourgain [14] provided a detailed proof of the existence of periodic solutions for the non-Hamiltonian equation

$$y_{tt} - y_{xx} + \mathrm{m}y + y_t^2 = 0 \,, \quad \mathrm{m} \neq 0 \,,$$

(for $\mathrm{m} = 0$ it is easy to see that non trivial periodic solutions do not exist).

These kind of problems have been then reconsidered by Craig in [17] for more general Hamiltonian derivative wave equations like

$$y_{tt} - y_{xx} + g(x)y + f(x, D^\beta y) = 0 \,, \quad x \in \mathbb{T} \,,$$

where $g(x) \geq 0$ and D is the first order pseudo-differential operator $D := \sqrt{-\partial_{xx} + g(x)}$. The perturbative analysis of Craig-Wayne [18] for the search of periodic solutions works when $\beta < 1$. The main reason is that the wave equation vector field gains one derivative and then the nonlinear term $f(D^\beta u)$ has a strictly weaker effect on the dynamics for $\beta < 1$. The case $\beta = 1$ is left as an open problem. Actually, in this case, the small divisors problem for periodic solutions has the same level of difficulty of quasi-periodic solutions with 2 frequencies.

The next theorem extends KAM theory to deal with the Hamiltonian derivative wave equation

$$y_{tt} - y_{xx} + \mathrm{m}y + f(Dy) = 0 \,, \quad \mathrm{m} > 0 \,, \quad D := \sqrt{-\partial_{xx} + \mathrm{m}} \,, \quad x \in \mathbb{T} \,, \qquad (3.1)$$

with real analytic nonlinearities

$$f(s) = as^3 + \sum_{k \geq 5} f_k s^k \,, \quad a \neq 0 \,. \qquad (3.2)$$

Theorem 3.1. ([5]) *For all* $\mathrm{m} > 0$, *for every choice of the tangential sites*

$$\mathcal{I} := \{ \mathrm{j}_1, \ldots, \mathrm{j}_n \} \subset \mathbb{Z} \,, \quad n \geq 2 \,, \qquad (3.3)$$

the equation (3.1)-(3.2) *admits families of small-amplitude, analytic, quasi-periodic solutions of the form*

$$y(t,x) = \sum_{j \in \mathcal{I}} \sqrt{\xi_j} \, \cos(\omega_j^\infty(\xi) \, t + \varphi_j + jx) + o(\sqrt{\xi}) \,, \quad \omega_j^\infty(\xi) \overset{\xi \to 0}{\to} \sqrt{j^2 + m} \,,$$

for a Cantor like set of parameters ξ *with asymptotically full measure as* $\xi \to 0$. *Such quasi-periodic solutions have zero Lyapunov exponents and the linearized equation is reducible to constant coefficients.*

Ideas of proof. For proving Theorem 3.1 we write the equation (3.1) as the infinite dimensional Hamiltonian system

$$u_t = -\mathrm{i}\partial_{\bar{u}}H\,, \qquad \bar{u}_t = \mathrm{i}\partial_u H\,,$$

with Hamiltonian

$$H(u,\bar{u}) := \int_{\mathbb{T}} \bar{u}Du + F\Big(\frac{u+\bar{u}}{\sqrt{2}}\Big)\,dx\,, \quad F(s) := \int_0^s f\,, \tag{3.4}$$

in the complex unknown

$$u := \frac{1}{\sqrt{2}}(Dy + \mathrm{i}y_t)\,, \qquad \bar{u} := \frac{1}{\sqrt{2}}(Dy - \mathrm{i}y_t)\,, \qquad \mathrm{i} := \sqrt{-1}\,.$$

Then, setting $u = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$, $\bar{u} = \sum_{j\in\mathbb{Z}} \bar{u}_j e^{-\mathrm{i}jx}$ we obtain the Hamiltonian in infinitely many coordinates

$$H = \sum_{j\in\mathbb{Z}} \lambda_j u_j \bar{u}_j + \int_{\mathbb{T}} F\Big(\frac{1}{\sqrt{2}}\sum_{j\in\mathbb{Z}}(u_j e^{\mathrm{i}jx} + \bar{u}_j e^{-\mathrm{i}jx})\Big)\,dx \tag{3.5}$$

where

$$\lambda_j := \sqrt{j^2 + \mathrm{m}} \tag{3.6}$$

are the eigenvalues of the diagonal operator D.

Remark 3.1. *The nonlinearity in (3.1) is x-independent implying, for (3.4), the conservation of the momentum $-\mathrm{i}\int_{\mathbb{T}} \bar{u}\partial_x u\,dx$. This symmetry simplify somehow the KAM proof (see also Geng-You [21]).*

For every choice of the *tangential sites* $\mathcal{I} := \{\mathrm{j}_1,\ldots,\mathrm{j}_n\} \subset \mathbb{Z}$ as in (3.3) the integrable Hamiltonian $\sum_{j\in\mathbb{Z}} \lambda_j u_j \bar{u}_j$ has the invariant tori

$$\Big\{u_j \bar{u}_j = \xi_j > 0\,, \text{ for } j \in \mathcal{I}\,, \ u_j = \bar{u}_j = 0 \text{ for } j \notin \mathcal{I}\Big\} \tag{3.7}$$

parametrized by the actions $\xi = (\xi_j)_{j\in\mathcal{I}} \in \mathbb{R}_+^n$ (these tori correspond to the solutions of the linear wave equation). The goal of the KAM iteration is prove the existence of nearby invariant tori for the complete Hamiltonian H in (3.5).

After a Birkhoff normal form step (which depends on the term as^3 of the nonlinearity in (3.2)), we introduce action-angle coordinates on the tangential variables:

$$u_j = \sqrt{\xi_j + y_j}e^{\mathrm{i}x_j}\,, \quad \bar{u}_j = \sqrt{\xi_j + y_j}e^{-\mathrm{i}x_j}\,, \ |y_j| < \xi_j\,, \ j \in \mathcal{I}\,, \quad (u_j, \bar{u}_j) = (z_j, \bar{z}_j)\,, \ j \notin \mathcal{I}\,. \tag{3.8}$$

Then we reduce to consider a parameter dependent family of analytic Hamiltonians of the form

$$H = \omega(\xi) \cdot y + \Omega(\xi) \cdot z\bar{z} + P(x, y, z, \bar{z}; \xi) \tag{3.9}$$

where the frequencies $\Omega_j(\xi)$ are close to the unperturbed frequencies λ_j in (3.6).

KAM theorem. The goal of the KAM iteration is to continue the torus $\{x \in \mathbb{T}^n, y = 0, z = \bar{z} = 0\}$ (which is (3.7) in the coordinates (3.8)) when the perturbation $P \neq 0$ is sufficiently small. This is achieved finding a change of variables Φ^∞ close to the identity which transforms the Hamiltonian (3.9) into

$$H^\infty := H \circ \Phi^\infty = \omega^\infty(\xi) \cdot y + \Omega^\infty(\xi) \cdot z\bar{z} + P^\infty(x, y, z, \bar{z}; \xi) \tag{3.10}$$

whose perturbation satisfies

$$\Big(\partial_{y^i z^{j_1} \bar{z}^{j_2}} P^\infty\Big)(x, 0, 0, 0; \xi) = 0\,, \quad 0 \leq 2i + j_1 + j_2 \leq 2\,.$$

As well known, the main difficulty is to fulfill, at each iterative step, the *second order Melnikov non-resonance conditions*. Actually it is sufficient to verify

$$|\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| \geq \frac{\gamma}{1 + |k|^\tau}, \quad \gamma > 0, \tag{3.11}$$

only for the "final" frequencies $\omega^\infty(\xi)$ and $\Omega^\infty(\xi)$ and not along the inductive iteration (see the formulation of the KAM theorem given in [5], [4]).

The application of the usual KAM theory (see e.g. [25], [28]), to the DNLW equation provides only the asymptotic decay estimate

$$\Omega_j^\infty(\xi) = j + O(1) \quad \text{for} \quad j \to +\infty. \tag{3.12}$$

Such a bound is not enough: the set of parameters ξ satisfying (3.11) could be empty. Note that for the semilinear NLW equation (see e.g. [28]) the frequencies decay asymptotically faster, namely like $\Omega_j^\infty(\xi) = j + O(1/j)$.

The key idea for verifying the second order Melnikov non-resonance conditions (3.11) for DNLW is to prove the higher order asymptotic decay estimate

$$\Omega_j^\infty(\xi) = j + a_+(\xi) + \frac{\mathrm{m}}{2j} + O(\frac{\gamma^{2/3}}{j}) \quad \text{for} \quad j \geq O(\gamma^{-1/3}) \tag{3.13}$$

where $a_+(\xi)$ is a constant independent of j (an analogous expansion holds for $j \to -\infty$ with a possibly different limit constant $a_-(\xi)$). In this way infinitely many conditions in (3.11) are verified by imposing only first order Melnikov conditions like $|\omega^\infty(\xi) \cdot k + h| \geq 2\gamma^{2/3}/|k|^\tau$, $h \in \mathbb{Z}$. Indeed, for $i > j > O(|k|^\tau \gamma^{-1/3})$, we get

$$
\begin{aligned}
|\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| &= |\omega^\infty(\xi) \cdot k + i - j + \frac{\mathrm{m}(i-j)}{2ij} + O(\gamma^{2/3}/j)| \\
&\geq 2\gamma^{2/3}|k|^{-\tau} - O(|k|/j^2) - O(\gamma^{2/3}/j) \geq \gamma^{2/3}|k|^{-\tau}
\end{aligned}
$$

noting that $i - j$ is integer and $|i - j| = O(|k|)$ (otherwise no small divisors occur).

Quasi-Töplitz perturbations. The asymptotic decay (3.13) for the perturbed frequencies $\Omega^\infty(\xi)$ is achieved thanks to the "quasi-Töplitz" property of the perturbation. Let us roughly explain this notion. The new normal frequencies after each KAM step are $\Omega_j^+ = \Omega_j + P_j^0$ where the corrections P_j^0 are the coefficients of the quadratic form

$$P^0 z\bar{z} := \sum_j P_{ij}^0 z_i \bar{z}_j, \quad P_j^0 := \int_{\mathbb{T}^n} (\partial_{z_j \bar{z}_j}^2 P)(x, 0, 0, 0; \xi) \, dx.$$

We say that a quadratic form P^0 is quasi-Töplitz if it has the form

$$P^0 = T + R$$

where T is a Töplitz matrix (i.e. constant on the diagonals) and R is a "small" remainder satisfying $R_{jj} = O(1/j)$ Then (3.13) follows with $a := T_{jj}$ which is independent of j.

Since the quadratic perturbation P^0 along the KAM iteration does not depend only on the quadratic perturbation at the previous steps, we need to extend the notion of quasi-Töplitz to general (non-quadratic) analytic functions.

The preservation of the quasi-Töplitz property of the perturbations P at each KAM step (with just slightly modified parameters) holds in view of the following key facts:

1. the Poisson bracket of two quasi-Töplitz functions is quasi-Töplitz (Proposition 3.1 of [5]),

2. the Lie transform of a quasi-Töplitz function is quasi-Töplitz (Proposition 3.2 of [5]),

3. the solution of the homological equation with a quasi-Töplitz perturbation is quasi-Töplitz (Proposition 5.1 of [5]).

We note that, in [20], the analogous properties 1 (and therefore 2) for Töpliz-Lipschitz functions is proved only when one of them is quadratic.

Remark 3.2. *We also mention the recent KAM theorem of Grebért-Thomann [22] for the quantum harmonic oscillator with semilinear nonlinearity. Also here the eigenvalues grow to infinity only linearly.*

Finally, we plan to extend these results also to derivative wave equations

$$y_{tt} - y_{xx} + \mathrm{m}y = g(x, y, y_x, y_t)$$

under the reversibility assumptions

$$g(x, y, y_x, -v) = g(x, y, y_x, v), \quad g(-x, y, -y_x, y_t) = g(x, y, y_x, y_t).$$

The algebraic scheme employs the ideas of reversible KAM theory. The asymptotic analysis of the perturbed frequencies after the KAM iteration is obtained as before.

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