Séminaire Laurent Schwartz

EDP et applications

Année 2011-2012

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Séminaire Laurent Schwartz — EDP et applications (2011-2012), Exposé n° XXIII, 10 p.

<http://slsedp.cedram.org/item?id=SLSEDP_2011-2012____A23_0>

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2011-2012
Exposé n° XXIII, 1-10

HERMITE BASIS DIAGONALIZATION FOR THE NON-CUTOFF RADIALLY SYMMETRIC LINEARIZED BOLTZMANN OPERATOR

N. LERNER, Y. MORIMOTO, K. PRAVDA-STAROV & C.-J. XU

Abstract. We provide some new explicit expressions for the linearized non-cutoff radially symmetric Boltzmann operator with Maxwellian molecules, proving that this operator is a simple function of the standard harmonic oscillator. A detailed article is available on arXiv [15].

1. Introduction

1.1. **The Boltzmann equation.** It describes the behaviour of a dilute gas when the only interactions taken into account are binary collisions. It reads as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{1.1}$$

for the density distribution of the particles $f = f(t,x,v) \geq 0$ at time t, having position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. The term appearing in the right-hand-side of this equation $Q(f,f)$ is the so-called quadratic Boltzmann collision operator associated to the Boltzmann bilinear operator,

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \quad f|_{t=0} = f_0,$$

$$Q(g,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(v-v_*, \sigma)\big(g'_* f' - g_* f\big) d\sigma dv_*, \tag{1.2}$$

with $d \geq 2$, where $f'_* = f(t,x,v'_*)$, $f' = f(t,x,v')$, $f_* = f(t,x,v_*)$, $f = f(t,x,v)$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

for $\sigma \in \mathbb{S}^{d-1}$. Those relations between pre and post collisional velocities follow from the conservations of momentum and kinetic energy in the binary collisions:

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \quad f|_{t=0} = f_0,$$

$$Q(g,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(v-v_*, \sigma)\big(g'_* f' - g_* f\big) d\sigma dv_*, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 35Q20, 35S05, 76P05, 82B40, 35R11.

Key words and phrases. Non-cutoff Boltzmann equation, Non-cutoff Kac equation, Spectral analysis, Microlocal analysis, Harmonic oscillator.

The term $Q(f,f)$ is expected to provide some smoothing and decay effect, e.g. to behave as a negative globally elliptic operator. We consider cross sections of the type

$$B(v-v_*,\sigma)=\Phi(|v-v_*|)b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big),\quad \cos\theta=\frac{v-v_*}{|v-v_*|}\cdot\sigma, \tag{1.4}$$

$|\theta|\leq\frac{\pi}{2}$, with a kinetic factor

$$\Phi(|v-v_*|)=|v-v_*|^{\gamma},\quad \gamma\in]-d,+\infty), \tag{1.5}$$

and a factor related to the collision angle with a singularity

$$(\sin\theta)^{d-2}b(\cos\theta)\underset{\theta\to 0}{\approx}|\theta|^{-1-2s}, \tag{1.6}$$

for some $0<s<1$. Notice that this singularity is not integrable, but a finite part argument gives a meaning to the integrals involved: for $\varphi\in C^2$,

$$\int_{|\theta|\leq\pi/2}|\theta|^{-1-2s}\big(\varphi(\theta)+\varphi(-\theta)-2\varphi(0)\big)d\theta\quad\text{makes sense,}$$

$$\text{as well as}\quad\int_{|\theta|\leq\pi/2}|\theta|^{-1-2s}\psi(\theta)d\theta\quad\text{for }\psi\text{ even, }C^2,\ \psi(0)=0.$$

This non-integrability property plays a major rôle regarding the qualitative behaviour of the solutions of the Boltzmann equation and for the smoothing effect to be present, that non-integrability feature is essential. Indeed, as first observed by Desvillettes for the Kac equation in [7], grazing collisions (that account for the non-integrability of the angular factor near $\theta=0$) do induce smoothing effects for the solutions of the non-cutoff Kac equation, or more generally for the solutions of the non-cutoff Boltzmann equation.

On the other hand, these solutions are at most as regular as the initial data (see [24]), when the collision cross section is assumed to be integrable, or after removing the singularity by using a cutoff function (Grad's angular cutoff assumption).

1.2. **The linearized Boltzmann collision operator.** We are concerned with a close-to-equilibrium framework, so we consider the fluctuation around μ given by the Maxwellian

$$\mu(v)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v|^2}{2}}, \tag{1.7}$$

setting $f=\mu+\sqrt{\mu}g$. Since $Q(\mu,\mu)=0$ by the conservation of the kinetic energy, the Boltzmann collision operator can be split into three terms,

$$Q(\mu+\sqrt{\mu}g,\mu+\sqrt{\mu}g)=Q(\mu,\sqrt{\mu}g)+Q(\sqrt{\mu}g,\mu)+Q(\sqrt{\mu}g,\sqrt{\mu}g),$$

whose linearized part is $Q(\mu,\sqrt{\mu}g)+Q(\sqrt{\mu}g,\mu)$. Setting

$$\mathscr{L}g=\mathscr{L}_1g+\mathscr{L}_2g,$$

$$\text{with}\quad \mathscr{L}_1g=-\mu^{-1/2}Q(\mu,\mu^{1/2}g),\quad \mathscr{L}_2g=-\mu^{-1/2}Q(\mu^{1/2}g,\mu),$$

the original Boltzmann equation (1.1) is reduced to the Cauchy problem for the fluctuation g,

$$\begin{cases}\partial_tg+v\cdot\nabla_xg+\mathscr{L}g=\mu^{-1/2}Q(\sqrt{\mu}g,\sqrt{\mu}g),\\ g|_{t=0}=g_0,\end{cases} \tag{1.8}$$

with

$$\mathscr{L}g = \mathscr{L}_1 g + \mathscr{L}_2 g,$$

$$\text{and} \quad \mathscr{L}_1 g = -\mu^{-1/2} Q(\mu, \mu^{1/2} g), \quad \mathscr{L}_2 g = -\mu^{-1/2} Q(\mu^{1/2} g, \mu).$$

This linearized Boltzmann operator $\mathscr{L}$ is known to be an unbounded symmetric operator on $L^2(\mathbb{R}^d_v)$ (acting in the velocity variable) such that its Dirichlet form satisfies $(\mathscr{L}g, g)_{L^2(\mathbb{R}^d_v)} \geq 0$. Alexandre, Desvillettes, Villani and Wennberg have highlighted in [2] that the non-cutoff Boltzmann operator enjoys remarkable coercive properties. The unraveling of these special features of the non-cutoff Boltzmann operator have led them to conjecture that this collision operator behaves and induces smoothing effects as a fractional Laplacian. The following coercive estimate was later proven in [5] (see also [4, 9, 18, 19])

$$\|(\mathrm{Id} - \mathbf{P})g\|^2_{H^s_{\frac{\gamma}{2}}} + \|(\mathrm{Id} - \mathbf{P})g\|^2_{L^2_{s+\frac{\gamma}{2}}} \lesssim (\mathscr{L}g,\, g)_{L^2(\mathbb{R}^d)} \lesssim \|(\mathrm{Id} - \mathbf{P})g\|^2_{H^s_{s+\frac{\gamma}{2}}}, \tag{1.9}$$

where the weighted Sobolev space is defined as

$$H^k_\ell = H^k_\ell(\mathbb{R}^d) = \big\{ f \in \mathscr{S}'(\mathbb{R}^d) : \ (1 + |v|^2)^{\ell/2} f \in H^k(\mathbb{R}^d) \big\},$$

and $\mathbf{P}$ is the L^2 orthogonal projection onto the space of collisional invariants

$$\mathrm{Span}\{\mu^{1/2}, v_j \mu^{1/2}, |v|^2 \mu^{1/2}\}.$$

1.3. **The present work.** We consider the case of the non-cutoff Boltzmann operator with Maxwellian molecules acting on radially symmetric functions with respect to the velocity variable and the case of the non-cutoff Kac operator.

We aim at studying the spectral properties and the structure of these collision operators linearized around a normalized Maxwellian distribution. We shall display some explicit expressions for these operators, using essentially two major tools: functional calculus of operators and pseudodifferential calculus with a key rôle for Mehler's formula. More specifically, these linearized operators are shown to be explicit functions of the contraction semigroup and the spectral projections of the harmonic oscillator

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}. \tag{1.10}$$

The linearized Kac operator is shown to be diagonal in the Hermite basis and to behave essentially as $\mathcal{H}^s$ where $s \in (0, 1)$ is the singularity exponent appearing in the expression of the cross-section (1.6).

2. Main results

2.1. **Radially symmetric Boltzmann operator.** We consider the case of the non-cutoff Boltzmann operator with Maxwellian molecules acting on the radially symmetric Schwartz space on $\mathbb{R}^d$

$$\mathscr{S}_r(\mathbb{R}^d) = \{f(|v|)\}_{f \text{ even } \in \mathscr{S}(\mathbb{R})}. \tag{2.1}$$

The case of Maxwellian molecules corresponds to the case when the parameter $\gamma = 0$ in the kinetic factor (1.5),

$$Q(g, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b\Big(\frac{v - v_*}{|v - v_*|} \cdot \sigma\Big) \big(g'_* f' - g_* f\big) d\sigma dv_*, \tag{2.2}$$

where $g'_* = g(v'_*)$, $f' = f(v')$, $g_* = g(v_*)$, $f = f(v)$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and $\sigma \in \mathbb{S}^{d-1}$. The non-negative cross section

$$b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big) = b(\cos\theta), \quad \text{with} \quad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma, \tag{2.3}$$

is assumed to be supported where $\cos\theta \geq 0$ and to satisfy the singularity assumption (1.6). We consider the linearized Boltzmann operator $\mathscr{L}u = \mathscr{L}_1 u + \mathscr{L}_2 u$, where

$$\mathscr{L}_1 u = -\mu^{-1/2} Q(\mu, \mu^{1/2} u), \quad \mathscr{L}_2 u = -\mu^{-1/2} Q(\mu^{1/2} u, \mu) \tag{2.4}$$

where μ is the Maxwellian given in (1.7). We set

$$\beta(\theta) = |\mathbb{S}^{d-2}| |\sin 2\theta|^{d-2} b(\cos 2\theta) \underset{\theta\to 0}{\approx} |\theta|^{-1-2s}, \tag{2.5}$$

for some $0 < s < 1$.

Theorem 2.1. *When it acts on $\mathscr{S}_r(\mathbb{R}^d)$, the first part of the linearized Boltzmann operator defined by $\mathscr{L}_1 f = -\mu^{-1/2} Q(\mu, \mu^{1/2} f)$, is equal to*

$$\mathcal{L}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta) \Big[\mathrm{Id} - (\sec\theta)^{\frac{d}{2}} \exp\big(-\mathcal{H}\ln(\sec\theta)\big)\Big] d\theta, \tag{2.6}$$

where $\mathcal{H} = -\Delta + \frac{|v|^2}{4}$ is the harmonic oscillator. Also

$$\mathcal{L}_1 = \sum_{k\geq 1} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\big(1 - (\cos\theta)^k\big) d\theta \ \mathbb{P}_k. \tag{2.7}$$

See a reminder on the spectral decomposition of the harmonic oscillator in Section 4: here we have used

$$\mathrm{Id} = \sum_{k\geq 0} \mathbb{P}_k, \ \mathbb{P}_k^2 = \mathbb{P}_k = \mathbb{P}_k^*, \ \mathbb{P}_k\mathbb{P}_l = \delta_{k,l}\mathbb{P}_k, \quad \mathcal{H} = \sum_{k\geq 0} (\frac{d}{2} + k)\mathbb{P}_k.$$

We note that

$$\mathcal{L}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \underbrace{\beta(\theta)}_{\text{even, } \approx |\theta|^{-1-2s}} \underbrace{\Big[\mathrm{Id} - (\sec\theta)^{\frac{d}{2}} \exp\big(-\mathcal{H}\ln(\sec\theta)\big)\Big]}_{\text{even, vanishing at } 0} d\theta. \tag{2.8}$$

and

$$\mathcal{L}_1 = \sum_{k\geq 1} \underbrace{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\big(1 - (\cos\theta)^k\big) d\theta}_{\sim c_s k^s \quad \text{for } k\to+\infty} \ \mathbb{P}_k. \tag{2.9}$$

The domain of $\mathcal{L}_1$ can be taken as

$$\mathcal{D} = \{u \in L^2(\mathbb{R}^d), \ \sum_{k\geq 1} k^{2s} \|\mathbb{P}_k u\|_{L^2}^2 < +\infty\} = \{u \in L^2(\mathbb{R}^d), \mathcal{H}^s u \in L^2(\mathbb{R}^d)\}. \tag{2.10}$$

Theorem 2.2. *When it acts on $\mathscr{S}_r(\mathbb{R}^d)$, the second part of the linearized Boltzmann operator defined by $\mathscr{L}_2 f = -\mu^{-1/2}Q(\mu^{1/2}f, \mu)$, is equal to*

$$\mathcal{L}_2 = -\sum_{l\geq 1}\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2l} d\theta\right)\mathbb{P}_{2l}. \tag{2.11}$$

For all $s \in (0,1)$, there exist positive constants $C(s,d), c(d)$ such that

$$0 \leq -\mathcal{L}_2 \leq C(s,d)\exp -c(d)\mathcal{H}. \tag{2.12}$$

N.B. $\mathcal{L}_2$ is a trace class operator on $L^2(\mathbb{R}^d)$ (even $\mathcal{H}^N\mathcal{L}_2$ is trace-class for all $N \in \mathbb{N}$), which is diagonal in the Hermite basis. Nonetheless $\mathcal{L}_2$ is smoothing (induces regularity), but also induces exponential decay.

Corollary 2.3. *When it acts on $\mathscr{S}_r(\mathbb{R}^d)$, the linearized Boltzmann operator $\mathscr{L}$ is equal to $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 = \sum_{k\geq 1}\lambda_k\mathbb{P}_k$ with*

$$\lambda_k \approx k^s \quad \text{when } k \to +\infty, \tag{2.13}$$

$$\lambda_{2l+1} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\big(1-(\cos\theta)^{2l+1}\big)d\theta, \quad l \geq 0, \tag{2.14}$$

$$\lambda_{2l} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\big(1-(\sin\theta)^{2l}-(\cos\theta)^{2l}\big)d\theta, \quad l \geq 1, \tag{2.15}$$

$\mathcal{L}$ is a nonnegative unbounded operator which is diagonal in the Hermite basis. $\mathcal{L}$ is essentially equal to $\mathcal{H}^s$.

2.2. **On the non-cutoff Kac operator.** Here the velocity variable $v \in \mathbb{R}$ is one-dimensional. The non-cutoff Kac collision operator is defined as

$$K(g,f)(v) = \int_{|\theta|\leq\pi/4}\beta(\theta)\left(\int_{\mathbb{R}}(g'_*f' - g_*f)dv_*\right)d\theta \tag{2.16}$$

where $f'_* = f(v'_*)$, $f' = f(v')$, $f_* = f(v_*)$, $f = f(v)$, with

$$v' = v\cos\theta - v_*\sin\theta, \quad v'_* = v\sin\theta + v_*\cos\theta, \quad v, v_* \in \mathbb{R}. \tag{2.17}$$

As previously, the main assumption concerning the non-negative cross-section is the presence of a non-integrable singularity for grazing collisions

$$\beta(\theta) \underset{\theta\to 0}{\approx} |\theta|^{-1-2s}, \quad \beta(-\theta) = \beta(\theta), \tag{2.18}$$

for some $0 < s < 1$ (with $\beta \in L^1_{loc}(0,1)$). The relations between the pre and post collisional velocities follow from the conservation of kinetic energy

$$v^2 + v_*^2 = v'^2 + v'^2_*.$$

As before for the general Boltzmann equation, we consider a fluctuation around the normalized Maxwellian distribution (1.7) (with $d = 1$) by setting $f = \mu + \sqrt{\mu}h$. Since $K(\mu,\mu) = 0$ by conservation of the kinetic energy, we may write

$$K(\mu + \sqrt{\mu}h, \mu + \sqrt{\mu}h) = K(\mu, \sqrt{\mu}h) + K(\sqrt{\mu}h, \mu) + K(\sqrt{\mu}h, \sqrt{\mu}h)$$

and consider the linearized Kac operator $\mathcal{K}h = \mathcal{K}_1 h + \mathcal{K}_2 h$, with

$$\mathcal{K}_1 h = -\mu^{-1/2}K(\mu, \mu^{1/2}h), \quad \mathcal{K}_2 h = -\mu^{-1/2}K(\mu^{1/2}h, \mu). \tag{2.19}$$

Theorem 2.4. *Defining the first part of the linearized Kac operator as* $f \mapsto \mathcal{K}_1 f = -\mu^{-1/2} K(\mu, \mu^{1/2} f)$*, we have*

$$\mathcal{K}_1 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\Big[\mathrm{Id} - (\sec\theta)^{1/2} \exp\big(-\mathcal{H}\ln(\sec\theta)\big)\Big] d\theta.$$

Theorem 2.5. *Defining the second part of the linearized Kac operator as* $f \mapsto \mathcal{K}_2 f = -\mu^{-1/2} K(\mu^{1/2} f, \mu)$*, we have*

$$\mathcal{K}_2 = -\sum_{l=1}^{+\infty} \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)(\sin\theta)^{2l} d\theta \right) \mathbb{P}_{2l}.$$

Corollary 2.6. *The linearized Kac operator is a non-negative unbounded operator, diagonal in the Hermite basis:*

$$\begin{aligned}
\mathcal{K} &= \sum_{k\geq 1} \lambda_k \mathbb{P}_k, \\
\lambda_{2k+1} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\left[1 - (\cos\theta)^{2k+1}\right] d\theta \geq 0, \quad k \geq 0 \\
\lambda_{2k} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \beta(\theta)\left[1 - (\cos\theta)^{2k} - (\sin\theta)^{2k}\right] d\theta \geq 0, \quad k \geq 1, \\
\lambda_k &\approx k^s \quad \text{when } k \to +\infty.
\end{aligned}$$

$\mathcal{K}$ *is essentially equal to* $\mathcal{H}^s$*.*

2.3. **Pseudodifferential framework.** The previous diagonalization in the Hermite basis is satisfactory and it is much simpler to deal with infinite diagonal matrices than with pseudodifferential operators. However the following result is interesting.

Theorem 2.7. *The linearized Kac operator* $\mathcal{K}$ *is a pseudodifferential operator whose Weyl symbol* $l(v,\xi)$ *is real-valued, belongs to the symbol class* $\mathbf{S}^s(\mathbb{R}^2)$ *(see the definition below) and admits the following asymptotic expansion:*

$$l(v,\xi) \sim c_0\Big(1 + \xi^2 + \frac{v^2}{4}\Big)^s - d_0 + \sum_{k=1}^{+\infty} c_k \Big(1 + \xi^2 + \frac{v^2}{4}\Big)^{s-k}.$$

The symbol $l(v,\xi)$ is smooth on $\mathbb{R}^2$ and satisfies

$$|(\partial_v^\alpha \partial_\xi^\beta l)(v,\xi)| \leq C_{\alpha\beta}(1 + |v|^2 + |\xi|^2)^{s - \frac{|\alpha|+|\beta|}{2}},$$

so it belongs to $\mathbf{S}^s(\mathbb{R}^2)$ (this is a definition). One may object that this makes the harmonic oscillator (symbol $|\xi|^2 + |v|^2/4$) of order 1, i.e. in $\mathbf{S}^1$, but it is precisely the correct scaling since taking for instance $p_j(v,\xi)$, $j = 1,2$, polynomials of v,ξ with degree $2m_j$, thus in $\mathbf{S}^{m_j}$ their Poisson bracket

$$\{p_1, p_2\} = \partial_\xi p_1 \cdot \partial_x p_2 - \partial_x p_1 \cdot \partial_\xi p_2$$

is a polynomial of degree $2m_1 + 2m_2 - 2$ thus in $\mathbf{S}^{m_1+m_2-1}$ as expected in a standard symbolic calculus.

3. Proofs

1. We compute the distribution-kernels of the operators.
2. We use a formula to get the Weyl symbols from the kernels.
3. We get plenty of exponential terms in the symbols.
4. We identify these terms via Mehler's formula.

3.1. **Mehler's formula.** Let $z \in \mathbb{C}$ with $|z| < 1, \operatorname{Re} z \geq 0$. Then,

$$\left[\exp -\Big(2z\big(|\xi|^2 + \frac{|v|^2}{4}\big)\Big)\right]^{Weyl} = \frac{1}{(1-z^2)^{d/2}} \exp\left(-\mathcal{H} \ln \frac{1+z}{1-z}\right). \tag{3.1}$$

In other words, an operator with Weyl symbol $\exp -\Big(2z\big(|\xi|^2 + \frac{|v|^2}{4}\big)\Big)$ is (up to a scalar factor) the exponential, in the operator-theoretic sense of $-\alpha(z)\mathcal{H}$, where $\mathcal{H}$ is the harmonic oscillator and $\operatorname{Re} \alpha(z) \geq 0$.

3.2. **From the kernel to the symbol.** Let us simply outline the computation for the linearized Kac operator $\mathcal{K}_1 u = -\mu^{-1/2} K(\mu, \mu^{1/2} u)$. It follows from Bobylev formula and Fourier inversion formula that

$$-\mu^{-1/2} K(\mu, \mu^{1/2} u)(v) = \frac{e^{\frac{v^2}{4}}}{(2\pi)^{\frac{3}{4}}} \iint_{\mathbb{R}\times(-\frac{\pi}{4},\frac{\pi}{4})} \beta(\theta) \left[\widehat{\mu}(0)\widehat{\mu^{1/2}u}(\eta) - \widehat{\mu}(\eta\sin\theta)\widehat{\mu^{1/2}u}(\eta\cos\theta)\right] e^{iv\eta} d\eta d\theta.$$

Easy (but tedious) to compute the distribution-kernel, then the Weyl symbol. It follows that

$$\begin{aligned} &-\mu^{-1/2} K(\mu, \mu^{1/2} u)(v) \\ &= \frac{1}{2\pi} \iint_{\mathbb{R}\times(-\frac{\pi}{4},\frac{\pi}{4})} \beta(\theta) \left(\int_{\mathbb{R}} e^{\frac{v^2-y^2}{4}} \left[e^{-iy\eta} - e^{-\frac{\eta^2\sin^2\theta}{2}} e^{-iy\eta\cos\theta}\right] e^{iv\eta} u(y) dy\right) d\eta d\theta \\ &= \int_{|\theta|\leq\pi/4} \beta(\theta)(\mathcal{K}_{1,\theta} u)(v) d\theta, \end{aligned}$$

where the distribution-kernel of the operator $\mathcal{K}_{1,\theta}$ is given by the oscillatory integral

$$\begin{aligned} \mathfrak{K}_{1,\theta}(v,y) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{v^2-y^2}{4}} \left[e^{-iy\eta} - e^{-\frac{\eta^2\sin^2\theta}{2}} e^{-iy\eta\cos\theta}\right] e^{iv\eta} d\eta \\ &= \delta_0(v-y) - \frac{1}{2\pi} e^{\frac{v^2-y^2}{4}} \int_{\mathbb{R}} e^{-\frac{\eta^2\sin^2\theta}{2}} e^{-iy\eta\cos\theta} e^{iv\eta} d\eta \\ &= \delta_0(v-y) - \frac{e^{\frac{v^2-y^2}{4}}}{(2\pi)^{1/2}|\sin\theta|} \exp -\frac{(v-y\cos\theta)^2}{2\sin^2\theta}. \end{aligned}$$

Since we have from the computation above $\mathfrak{K}_{1,\theta}(v - \frac{y}{2}, v + \frac{y}{2}) =$

$$\delta_0(y) - \frac{e^{-\frac{vy}{2}}}{(2\pi)^{1/2}|\sin\theta|} \exp -\left\{\frac{\big(v - \frac{y}{2} - (v+\frac{y}{2})\cos\theta\big)^2}{2\sin^2\theta}\right\},$$

we obtain from that the Weyl symbol $l_{1,\theta}$ of $\mathcal{K}_{1,\theta}$ is $l_{1,\theta}(v,\xi)=1-\ell_{1,\theta}(v,\xi)$, with

$$\ell_{1,\theta}(v,\xi)=\int e^{iy\xi}\frac{1}{(2\pi)^{1/2}|\sin\theta|}\exp-\left\{\frac{\left(v-\frac{y}{2}-(v+\frac{y}{2})\cos\theta\right)^2+vy\sin^2\theta}{2\sin^2\theta}\right\}dy.$$

implying that

Lemma 3.1. *The Weyl symbol l_1 of the operator $\mathcal{K}_1$ is equal to*

$$l_1(v,\xi)=\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\left[1-\sec^2(\frac{\theta}{2})\exp\left\{-2\tan^2(\frac{\theta}{2})(\xi^2+\frac{v^2}{4})\right\}\right]d\theta. \tag{3.2}$$

N.B. The functions of θ inside the integrals factoring β are even, vanish at 0 and are smooth on the compact interval of integration: l_1 is indeed given by a Lebesgue integral.

Without the Weyl quantization, it would be pretty hard to sort out the selfadjoint and skew-adjoint (which is zero here) parts and essentially impossible to recognize Mehler's formula.

A nice feature of the Weyl quantization is $(\bar a)^w=(a^w)^*$.

$$(a^wu)(x)=\iint e^{i\langle x-y,\xi\rangle}a\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi(2\pi)^{-d}.$$

$$a(x,\xi)=\int k(x-\frac{t}{2},x+\frac{t}{2})e^{it\xi}dt,$$

$$k(x,y)=\int e^{i\langle x-y,\xi\rangle}a\Big(\frac{x+y}{2},\xi\Big)d\xi(2\pi)^{-d}.$$

$$l_1(v,\xi)=\int_{|\theta|\le\frac{\pi}{4}}\beta(\theta)\left[1-\sec^2(\frac{\theta}{2})\exp\left\{-2\tan^2(\frac{\theta}{2})(\xi^2+\frac{v^2}{4})\right\}\right]d\theta. \tag{3.3}$$

Looking at the previous formula, we see that we are in the range of application of Mehler's formula and we obtain indeed

Theorem 3.2. *Defining the first part of the linearized Kac operator as $f\mapsto\mathcal{K}_1f=-\mu^{-1/2}K(\mu,\mu^{1/2}f)$, we have*

$$\mathcal{K}_1=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\beta(\theta)\Big[\mathrm{Id}-(\sec\theta)^{1/2}\exp\big(-\mathcal{H}\ln(\sec\theta)\big)\Big]d\theta.$$

Asymptotic equivalent: a typical computation. We consider

$$\mu_k=\int_{0\le\theta\le\pi/4}\underbrace{\theta^{-1-2s}}_{u'(\theta)}\underbrace{\big(1-e^{-k\theta^2}\big)}_{v(\theta)}d\theta,\quad k\in\mathbb{N}. \tag{3.4}$$

We want to find an equivalent when $k\to+\infty$.

$$\mu_k=\left[\frac{\theta^{-2s}}{-2s}\big(1-e^{-k\theta^2}\big)\right]_0^{\pi/4}+\int_0^{\pi/4}\frac{\theta^{-2s}}{2s}2k\theta e^{-k\theta^2}d\theta$$

$$\mu_k=C+O(e^{-ck})+\frac{k}{s}\int_0^{\pi/4}\theta^{1-2s}e^{-k\theta^2}d\theta,\quad\theta=k^{-\frac12}\tau,$$

$$\mu_k\sim\frac{k}{s}k^{-\frac12+s}\int_0^{+\infty}\tau^{1-2s}e^{-\tau^2}d\tau k^{-\frac12}=k^s\frac{\Gamma(1-s)}{2s}.$$

3.3. **Perspectives.** Without the radially symmetric assumption, the computations get wilder. However it seems quite likely that the Harmonic Oscillator should be replaced by the Landau operator

$$\mathcal{L} = -\Delta + \frac{|v|^2}{4} - \frac{d}{2} + \big(\|v \wedge \xi\|^2\big)^{Weyl} - \frac{d(d-1)}{4}$$

and that the smoothing effect is due to a diffusive term of type $\mathcal{L}^s$.

4. Appendix

The standard Hermite functions $\{\phi_n\}_{n\in\mathbb{N}}$ are defined on $\mathbb{R}$ by

$$\phi_n(x) = (2^n n!)^{-1/2}\pi^{-1/4}\big(x - \frac{d}{dx}\big)^n(e^{-x^2/2}) = (n!)^{-1/2}a_+^n\phi_0,$$

where a_+ is the creation operator $2^{-1/2}(x - d/dx)$. The $(\phi_n)_{n\in\mathbb{N}}$ make an orthonormal basis of $L^2(\mathbb{R})$. We define for $n \in \mathbb{N}, \alpha = (\alpha_j)_{1\le j\le d} \in \mathbb{N}^d, x \in \mathbb{R}, v \in \mathbb{R}^d$,

$$\psi_n(x) = 2^{-1/4}\phi_n(2^{-1/2}x), \quad \psi_n = (n!)^{-1/2}\big(\frac{x}{2} - \frac{d}{dx}\big)^n\psi_0,$$

$$\Psi_\alpha(v) = \prod_{j=1}^{d}\psi_{\alpha_j}(v_j), \quad \mathcal{E}_k = \mathrm{Span}\{\Psi_\alpha\}_{\alpha\in\mathbb{N}^d,|\alpha|=k},$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The $(\Psi_\alpha)_{\alpha\in\mathbb{N}^d}$ make an orthonormal basis of $L^2(\mathbb{R}^d)$ composed by the eigenfunctions of the d-dimensional harmonic oscillator:

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4} = \sum_{k\ge 0}\big(\frac{d}{2} + k\big)\mathbb{P}_k, \quad \mathrm{Id} = \sum_{k\ge 0}\mathbb{P}_k, \tag{4.1}$$

where $\mathbb{P}_k$ is the orthogonal projection onto $\mathcal{E}_k$,

$$\text{whose dimension is } \binom{k+d-1}{d-1} \underset{k\to+\infty}{\sim} \frac{k^{d-1}}{(d-1)!}.$$

The eigenvalue $d/2$ is simple in all dimensions and $\mathcal{E}_0$ is generated by

$$\Psi_0(v) = (2\pi)^{-d/4}e^{-|v|^2/4} = \mu^{1/2}(v).$$

References

[1] Alexandre, R.: Remarks on 3D Boltzmann linear operator without cutoff. Transport Theory Statist. Phys. 28-5, 433473 (1999).

[2] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg, *Entropy dissipation and long-range interactions*, Arch. Rational Mech. Anal. 152 (2000) 327-355.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T.Yang, *Uncertainty principle and kinetic equations*, J. Funct. Anal. 255 (2008) 2013-2066.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, *Global existence and full regularity of the Boltzmann equation without angular cutoff*, Comm. Math. Phys. 304 (2011), 513-581.

[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, *Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential*, J. Funct. Anal., Vol. 262 (2012), 915-1010.

[6] C. Cercignani, *The Boltzmann Equation and its Applications*, Applied Mathematical Sciences, vol. 67 (1988), Springer-Verlag, New York

[7] L. Desvillettes, *About the regularization properties of the non cut-off Kac equation*, Comm. Math. Phys. 168 (1995) 417-440.
[8] L. Desvillettes, G. Furioli, E. Terraneo, *Propagation of Gevrey regularity for solutions of Boltzmann equation for Maxwellian molecules*, Trans. Amer. Math. Soc. 361 (2009) 1731–1747.
[9] P.-T. Gressman, R.-M. Strain, *Global classical solutions of the Boltzmann equation without angular cut-off*, J. Amer. Math. Soc. 24 (2011), 771-847.
[10] L. Hörmander, *The analysis of linear partial differential operators*, vol. I–IV, (1985) Springer Verlag.
[11] L. Hörmander, *Symplectic classification of quadratic forms and general Mehler formulas*, Math. Z. 219, 3 (1995) 413–449.
[12] N. Lekrine, C.-J. Xu, *Gevrey regularizing effect of the Cauchy problem for non-cutoff homogeneous Kac's equation,* Kinetic and Related Models, 2 (2009), 647-666.
[13] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudodifferential operators*, (2010) Pseudo-Differential Operators. Theory and Applications, Birkhäuser Verlag, Basel.
[14] N. Lerner, Y. Morimoto, K. Pravda-Starov, *Hypoelliptic estimates for a linear model of the Boltzmann equation without angular cutoff*, (2011) to appear in Comm. Partial Differential Equations, arXiv:1012.4915.
[15] N. Lerner, Y. Morimoto, K. Pravda-Starov, C.-J. Xu, *Diagonalization of the Linearized Non-Cutoff Radially Symmetric Boltzmann Operator*, preprint, arXiv:1111.0423v1.
[16] Y. Morimoto, C.-J. Xu, *Hypoellipticity for a class of kinetic equations*, J. Math. Kyoto Univ. 47, no. 1 (2007), 129-152.
[17] Y. Morimoto, C.-J. Xu, *Ultra-analytic effect of Cauchy problem for a class of kinetic equations,* J. Differential Equations, 247 (2009) 596-617.
[18] C. Mouhot, *Explicit coercivity estimates for the Boltzmann and Landau operators*, Comm. Partial Differential Equations, 31 (2006) 1321-1348.
[19] C. Mouhot, R.M. Strain, *Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff*, J. Math. Pures Appl. (9) 87 (2007), no. 5, 515-535.
[20] Y.P. Pao, *Boltzmann collision operator with inverse-power intermolecular potentials. I*, Comm. Pure Appl. Math. 27 (1974) 407-428.
[21] Y.P. Pao, *Boltzmann collision operator with inverse-power intermolecular potentials. II*, Comm. Pure Appl. Math. 27 (1974) 559-581.
[22] A. Unterberger, *Oscillateur harmonique et opérateurs pseudodifférentiels*, Ann. Institut Fourier, 29 (3) (1979), 201-221.
[23] C. Villani, *A review of mathematical topics in collisional kinetic theory*, Handbook of Mathematical Fluid Dynamics, vol. I, North-Holland, Amsterdam (2002) 71-305.
[24] B. Wennberg, *Regularity in the Boltzmann equation and the Radon transform*, Comm. Partial Differential Equations, 19 (1994) 2057-2074.

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