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Chenmin Sun, Nikolay Tzvetkov, and Weijun Xu

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

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UNIVERSALITY RESULTS FOR A CLASS OF NONLINEAR WAVE EQUATIONS AND THEIR GIBBS MEASURES

CHENMIN SUN, NIKOLAY TZVETKOV, AND WEIJUN XU

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1. The microscopic model

For $N \geq 1$, consider as a microscopic model the weakly interacting waves of the type

$$\begin{cases} \partial_t^2 \tilde{u} + |\nabla|^{2\alpha} \tilde{u} + N^{-\theta} \Pi_N V'(\tilde{u}) = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{T}_N^2, \\ \tilde{u}(0,\cdot) = \tilde{\phi}, \quad (\partial_t \tilde{u})(0,\cdot) = \tilde{\psi}, \end{cases} \tag{1.1}$$

where $\mathbf{T}_N^2 = (\mathbf{R}/2\pi N\mathbf{Z})^2$ is the two dimensional torus of side length $2\pi N$, V is the even polynomial

$$V(u) = \sum_{j=0}^{m} a_j u^{2j}, \quad m \geq 2,\, a_m > 0,$$

satisfying certain structural conditions specified below, and Π_N is the projection operator such that

$$\widehat{\Pi_N f}(k) = \mathbf{1}_{|k| \leq N} \widehat{f}(k).$$

The differential operator $|\nabla|^\gamma$ acts on functions on torus of side length L as

$$\widehat{|\nabla|^\gamma f}(k) := \Big|\frac{2\pi k}{L}\Big|^\gamma \widehat{f}(k).$$

Here in the microscopic model, we take $\gamma = 2\alpha$ and $L = 2\pi N$. Without the term $N^{-\theta}\Pi_N V'(\tilde{u})$ in

$$\partial_t^2 \tilde{u} + |\nabla|^{2\alpha} \tilde{u} + N^{-\theta} \Pi_N V'(\tilde{u}) = 0$$

we have N linear waves (each Fourier coefficient) oscillating independently. Our goal is to try to understand how the weak nonlinear interaction $N^{-\theta}\Pi_N V'(\tilde{u})$ modifies the free evolution for $N \gg 1$.

2. On the nature of the nonlinear interaction

The presence of Π_N in

$$\partial_t^2 u + |\nabla|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0,\ (t,x) \in \mathbf{R} \times \mathbf{T}_N^2,$$

is essential for the existence of the dynamics. Indeed, consider

$$\partial_t^2 u + |\nabla|^{2\alpha} u + N^{-\theta} u^{2k+1} = 0,\ (t,x) \in \mathbf{R} \times \mathbf{T}_N^2. \tag{2.1}$$

Then for $k > \alpha/(1-\alpha)$, (2.1) is an energy supercritical problem and it is not clear at all that there is a well-defined flow, even for smooth data (see for example the recent work [11]). More precisely, the energy controls the H^α norm while the scaling invariant norm is $H^{1-\alpha/k}$. Then for $\alpha < 1$,

$$1 - \frac{\alpha}{k} > \alpha \quad \Longleftrightarrow \quad k > \frac{\alpha}{1-\alpha}.$$

3. The initial data

The initial data $\widetilde{\phi}$ and $\widetilde{\psi}$ in (1.1) are two random functions given by

$$\widetilde{\phi}(x) = \frac{1}{2\pi N^{1-\alpha}} \sum_{|k|\leq N} \frac{g_k}{\sqrt{1+|k|^{2\alpha}}}\, e^{ik\cdot x/N}, \qquad \widetilde{\psi}(x) = \frac{1}{2\pi N} \sum_{|k|\leq N} h_k\, e^{ik\cdot x/N},$$

where $\{g_k\}$ and $\{h_k\}$ are standard complex Gaussians with $g_{-k} = \overline{g_k}$, $h_{-k} = \overline{h_k}$ and otherwise independent. We omit their dependence on N to keep notations simple and consistent (omitting the N in notation for microscopic quantities). This type of initial condition is natural since the Gaussian measure it induces is invariant under the perturbed linear evolution above (with the differential operator $|\nabla|^{2\alpha}$ replaced by $\frac{1}{N^{2\alpha}} + |\nabla|^{2\alpha}$ and without nonlinear interaction).

The initial data is, very roughly speaking, of the type

$$\frac{1}{2\pi N} \sum_{|k|\leq N} \rho(k/N) g_k(\omega) e^{ik\cdot x/N}$$

for suitable function $\rho : \mathbf{R}^2 \to \mathbf{R}$. In our case, $\rho(x) = 1/\langle x\rangle^\alpha$ for the initial position, and $\rho(x) \equiv 1$ for the initial velocity. Although natural from the invariance of the perturbed linear dynamics, we should also note that our choice is also very restrictive relating to the support of the corresponding Gibbs measure.

4. Assumptions on the potential

Note that $\widetilde{\phi}$ has a stationary Gaussian distribution with $\widetilde{\phi}(x) \sim \mathcal{N}(0, \sigma_N^2)$, where

$$\sigma_N^2 = \frac{1}{4\pi^2 N^{2(1-\alpha)}} \sum_{|k|\leq N} \frac{1}{1+|k|^{2\alpha}} = \underbrace{\frac{1}{4\pi^2} \int_{|\xi|\leq 1} \frac{1}{|\xi|^{2\alpha}} \mathrm{d}\xi}_{\sigma^2} + \mathcal{O}(N^{-2(1-\alpha)}). \tag{4.1}$$

Let σ^2 be defined as above, $\tilde{\mu}$ be the law of $\mathcal{N}(0, \sigma^2)$, and

$$\langle V\rangle(z) := \int_{\mathbf{R}} V(z+y)\tilde{\mu}(\mathrm{d}y)$$

be the average of V under $\tilde{\mu}$. Our main assumption on V is the criticality and positivity of its averaged version $\langle V\rangle$.

Assumption 4.1. *V is an even polynomial of degree $2m \geq 4$ with the form*

$$V(z) = \sum_{j=0}^{2m} a_j z^{2j}.$$

Furthermore, we assume

(1) *$z = 0$ is a bifurcation point of $\langle V \rangle$ in the sense that $\langle V \rangle''(0) = 0$.*

(2) *$\langle V \rangle(z) - \langle V \rangle(0) > 0$ for all $z \neq 0$.*

The averaged version $\langle V \rangle$ has the expression

$$\langle V \rangle(z) = \sum_{j=0}^{m} \overline{a}_j z^{2j}$$

with

$$\overline{a}_j = \frac{1}{(2j)!} \mathbf{E}\big[V^{(2j)}\big(\mathcal{N}(0,\sigma^2)\big)\big] = \frac{1}{(2j)!} \sum_{k=j}^{m} \frac{(2k)!}{(2k-2j)!!} \cdot a_k \cdot \sigma^{2(k-j)}. \tag{4.2}$$

Hence, Condition (1) above is equivalent to say that $\overline{a}_1 = 0$. Since the renormalisation term in the wave dynamics and the measures are constant multiples of $\overline{a}_1 N^{2(1-\alpha)} u_N$ and $\overline{a}_1 N^{2(1-\alpha)} \phi_N^{\diamond 2}$ respectively, Condition (1) guarantees that the divergent parts in various terms are canceled out automatically, and there is no need to subtract the renormalisation by hand. With Condition (1), Condition (2) is then equivalent to the following positivity condition:

$$\sum_{j=2}^{m} \overline{a}_j z^{2(j-2)} > 0, \qquad \forall z \in \mathbf{R}. \tag{4.3}$$

Exemple 4.2. *If we fix $a_2 > 0, \dots, a_m > 0$, we can find $a_1 < 0$ such that our assumptions on V are satisfied. For example*

$$V(z) = z^6 - 45\sigma^2 z^2$$

satisfies the assumptions. We can also find $V \geq 0$ such that our assumptions are satisfied.

5. The macroscopic model

Our aim is to investigate the influence of the microscopic weak non-linear interaction to the macroscopic behaviour of $\tilde{u}$ under the above assumption on V. For $\mathbf{T}^2 = (\mathbf{R}/2\pi\mathbf{Z})^2$, define the macroscopic process u_N on $\mathbf{R} \times \mathbf{T}^2$ by

$$u_N(t,x) := N^{1-\alpha} \tilde{u}(N^\alpha t, Nx), \quad (t,x) \in \mathbf{R} \times \mathbf{T}^2.$$

It satisfies the equation

$$\partial_t^2 u_N + |\nabla|^{2\alpha} u_N + N^{1+\alpha-\theta} \Pi_N V'(u_N/N^{1-\alpha}) = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{T}^2 \tag{5.1}$$

with initial data

$$(u_N, \partial_t u_N)(0,x) = (\phi_N(x), \psi_N(x)) = \frac{1}{2\pi}\Big(\sum_{|k|\le N} \frac{g_k}{\sqrt{1+|k|^{2\alpha}}} e^{ik\cdot x}, \sum_{|k|\le N} h_k e^{ik\cdot x}\Big). \tag{5.2}$$

In order for u_N to converge to a cubic equation, one necessarily sets $\theta = 4\alpha - 2$ and hence $1+\alpha-\theta = 3(1-\alpha)$. Therefore we expect that under such a scaling at macroscopic level the dynamics is governed by a "cubic equation" (even if there is no cubic term in the polynomial V' !).

For $\alpha \in (3/4, 1)$, let μ be the Gaussian measure on $\mathcal{D}'(\mathbf{T}^2)$ (the space of distributions on $\mathbf{T}^2$) with covariance operator $(1+|\nabla|^{2\alpha})^{-1}$, and μ' be the white noise measure on $\mathbf{T}^2$. Equivalently, the Gaussian measures μ and μ' are induced by the random fields

$$\phi = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}^2} \frac{g_k}{\sqrt{1+|k|^{2\alpha}}} \mathrm{e}^{ik\cdot x}, \quad \psi = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}^2} h_k e^{ik\cdot x}.$$

Let $\mu_N := \mu\circ\Pi_N^{-1}$ and $\mu'_N = \mu'\circ\Pi_N^{-1}$ be the marginals of μ and μ' on frequencies up to N. Hence, the initial data of the macroscopic wave dynamics (5.1) are distributed according to $\mu_N \otimes \mu'_N$. Let $\widetilde{\sigma}_N^2$ be the variance of ϕ under μ_N, which is invariant under translations and hence $\widetilde{\sigma}_N^2$ does not depend on the spatial variable x. In fact, a direct computation shows

$$\widetilde{\sigma}_N^2 := \mathbf{E}^\mu |\Pi_N \phi|^2 = \frac{1}{4\pi^2} \sum_{k\in\mathbf{Z}^2, |k|\le N} \frac{1}{1+|k|^{2\alpha}} = \underbrace{(\sigma^2 + \mathrm{err}_N)}_{=:\sigma_N^2} \cdot N^{2(1-\alpha)}, \tag{5.3}$$

where σ_N^2 and

$$\sigma^2 = \frac{1}{4\pi^2}\int_{|\xi|\le 1} \frac{1}{|\xi|^{2\alpha}} \mathrm{d}\xi \tag{5.4}$$

are as defined in (4.1), and $\mathrm{err}_N = \mathcal{O}(N^{-2(1-\alpha)})$ as $N\to+\infty$.

Now, let V be an even polynomial satisfying Assumption 4.1. For every $N\in\mathbf{N}$, let

$$V_N(\varphi) = N^{4(1-\alpha)} V(\varphi/N^{1-\alpha}), \tag{5.5}$$

and we have

$$V_N(\varphi) = \sum_{j=1}^m \overline{a}_{j,N} N^{-(2j-4)(1-\alpha)} H_{2j}(\varphi; \widetilde{\sigma}_N^2), \tag{5.6}$$

where $H_k(\cdot, \sigma^2)$ is the k-th Hermite polynomial with leading coefficient 1 and variance σ^2. Recall that the Hermite polynomials are defined by

$$e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^\infty \frac{t^k}{k!} H_k(x;\sigma).$$

In particular

$$H_1(x;\sigma) = x, \ H_2(x;\sigma) = x^2 - \sigma, \ H_3(x;\sigma) = x^3 - 3\sigma x.$$

The coefficients $\overline{a}_{j,N}$ can be explicitly computed as

$$\overline{a}_{j,N} = \frac{1}{(2j)!}\mathbf{E}\big[V^{(2j)}\big(\mathcal{N}(0,\sigma_N^2)\big)\big]. \tag{5.7}$$

For every j, we have $\overline{a}_{j,N} \to \overline{a}_j$ as $N \to +\infty$, where $\overline{a}_j$ are as given in (4.2). Furthermore, the following slightly more delicate relation holds.

Proposition 5.1. *Assume that $\alpha \in \big(1/2, 1\big)$. There exists an absolute constant $\lambda_0 \in \mathbf{R}$, such that as $N \to \infty$,*

$$\overline{a}_{1,N} = \overline{a}_1 + \lambda_0 N^{-2(1-\alpha)} + O(N^{-1}) + O(N^{-4(1-\alpha)}).$$

6. Wave dynamics

Our main result concerns the behavior of the macroscopic wave-dynamics as $N \to \infty$.

Theorem 6.1. *Suppose that $\alpha \in (8/9, 1)$. Let $\sigma < \alpha - 1$ and suppose that V satisfies Assumption 4.1 with $\lambda := 4\overline{a}_2 > 0$. Let u_N be the solution of*

$$\partial_t^2 u_N + |\nabla|^{2\alpha} u_N + \Pi_N V_N'(u_N) = 0,$$

with initial data

$$(u_N, \partial_t u_N)|_{t=0} = \frac{1}{2\pi} \sum_{|k| \le N} \Big(\frac{g_k(\omega)}{\sqrt{1+|k|^{2\alpha}}}\ \mathrm{e}^{ik\cdot x},\ h_k(\omega)\ \mathrm{e}^{ik\cdot x} \Big). \tag{6.1}$$

Then solutions of (with $\lambda_0 \in \mathbf{R}$ given in Proposition 5.1)

$$\partial_t^2 v_N + |\nabla|^{2\alpha} v_N + 2\lambda_0 v_N + \lambda \Pi_N\big((v_N)^3 - 3\widetilde{\sigma}_N^2 v_N\big) = 0$$

with initial data (6.1) *converge almost surely in the sense of distribution on $\mathbf{R} \times \mathbf{T}^2$, as $N \to \infty$ and satisfy*

$$\lim_{N\to\infty} \|u_N - v_N\|_{\mathcal{C}([-T,T],H^\sigma(\mathbf{T}^2))} = 0, \quad \forall T > 0.$$

In [13], we have a more detailed convergence statement by decomposing u_N (and also v_N) into a random term with low regularity and a smoother contribution. The latter converges in positive Sobolev norms.

We emphasize that our range of α is *independent* of the degree $2m$ of the potential V. Indeed, the Cauchy problem (7.3) below without the negative powers of N in higher nonlinearities in V is highly supercritical. For large m, this is even supercritical with respect to the probabilistic scaling, a notion introduced in [3, 4]. What saves us here is the truncation Π_N in frequency space and the negative power of N in front of the high-power nonlinearity. The same situation appears in Hairer-Quastel [8] for the KPZ equation (though in a different setup where the problem is the singularity of the driving noise instead of the initial data).

The theorem still holds true if the sharp cutoff in the truncation is replaced by a smoother cutoff with a sufficiently fast decay smooth function. The constant λ in the final statement then will depend on the actual cutoff function.

7. The Gibbs measures

In order to prove Theorem 6.1, we re-write the macroscopic model (5.1) as

$$\partial_t^2 u_N + (1+|\nabla|^{2\alpha})u_N + \Pi_N\big(V_N'(u_N) - u_N\big) = 0, \tag{7.1}$$

still with initial data (5.2). We add a mass term in the linear part in order to control the free evolution of the zero-th Fourier mode, and modified the nonlinear term to compensate the change. In fact, without the mass term, the zero-th mode will grow in time under the linear evolution. Let

$$\widetilde{V_N}(\varphi) := V_N(\varphi) - \frac{1}{2}\big(\varphi^2 - \widetilde{\sigma}_N^2\big),$$

and let ν_N be the probability measure given by

$$\nu_N(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_N} e^{-\int_{\mathbf{T}^2} \widetilde{V_N}(\phi)\mathrm{d}x} \mu_N(\mathrm{d}\phi). \tag{7.2}$$

The measure ν_N is well defined as long as $a_m > 0$, and $\nu_N \otimes \mu_N'$ is invariant under the dynamics (7.1). If $\lambda := \overline{a}_2 > 0$, then the measure

$$\nu(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}} e^{-\lambda \int_{\mathbf{T}^2} \phi^{\diamond 4}\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}^2} \phi^{\diamond 2}\mathrm{d}x} \mu(\mathrm{d}\phi)$$

is also well-defined, where $\phi^{\diamond k}$ denotes the k-th Wick power of ϕ with respect to the Gaussian structure induced by μ. The measure ν is known as the fractional ϕ_2^4 with exponent α.

Note that the measure ν has an additional quadratic term on the exponential with the opposite sign compared to the usual fractional ϕ_2^4. This is because we define the Gaussian measure μ to have covariance $(1+|\nabla|^{2\alpha})^{-1}$. Indeed, the measure ν is the same with the quadratic term removed if the reference Gaussian measure has covariance $|\nabla|^{-2\alpha}$ and 0-mode being a $\mathcal{N}(0,1)$ random variable independent of all other modes.

Recall that μ' is the white noise measure on $\mathbf{T}^2$. We define the measures $\vec{\mu}$, $\vec{\nu}_N$ and $\vec{\nu}$ by

$$\vec{\mu} := \mu \otimes \mu', \qquad \vec{\nu}_N := \nu_N \otimes \mu_N', \qquad \vec{\nu} := \nu \otimes \mu'.$$

More precisely, writing $\vec{\phi} = (\phi, \phi')$, we have

$$\vec{\nu}_N(\mathrm{d}\vec{\phi}) = \nu_N(\mathrm{d}\phi)\mu_N'(\mathrm{d}\phi') = \mathcal{Z}_N^{-1} e^{-\int_{\mathbf{T}^2} \widetilde{V_N}(\phi)\mathrm{d}x} \underbrace{\mu_N(\mathrm{d}\phi)\mu_N'(\mathrm{d}\phi')}_{\vec{\mu}_N(\mathrm{d}\vec{\phi})},$$

and

$$\vec{\nu}(\mathrm{d}\vec{\phi}) = \nu(\mathrm{d}\phi)\mu'(\mathrm{d}\phi') = \mathcal{Z}^{-1} e^{-\lambda\int_{\mathbf{T}^2} \phi^{\diamond 4}\mathrm{d}x} \underbrace{\mu(\mathrm{d}\phi)\mu'(\mathrm{d}\phi')}_{\vec{\mu}(\mathrm{d}\vec{\phi})},$$

where the values of $\mathcal{Z}_N$ and $\mathcal{Z}$ are the same as before. The equation (7.1) can be written as a Hamiltonian system for $\vec{u}_N := (u_N, \partial_t u_N)$ as

$$\partial_t \begin{pmatrix} u_N \\ \partial_t u_N \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial \mathcal{E}_N}{\partial(u_N, \partial_t u_N)}, \tag{7.3}$$

where the Hamiltonian is given by

$$\mathcal{E}_N(f,g) = \frac{1}{2}\Big(\langle |\nabla|^{2\alpha} f, f\rangle_{L^2} + \langle g, g\rangle_{L^2}\Big) + \int_{\mathbf{T}^2} V_N(\Pi_N f)\mathrm{d}x.$$

For every N, the probability measure $\vec{\nu}_N$ is invariant under the above Hamiltonian dynamics. Theorem 7.1 below implies that $\vec{\nu}_N \otimes \mu_N^{\perp} \otimes (\mu_N')^{\perp}$ converges to $\vec{\nu}$ in the sense that the density with respect to $\vec{\mu}$ converges in $L^p(\vec{\mu})$ for every $p \geq 1$. The measures $\vec{\mu}$ and $\vec{\nu}$ are supported on

$$\mathcal{H}^{-(1-\alpha)\text{-}}(\mathbf{T}^2) := H^{-(1-\alpha)\text{-}}(\mathbf{T}^2) \times H^{-1\text{-}}(\mathbf{T}^2),$$

where

$$H^{\gamma\text{-}} := \bigcap_{\varepsilon>0} H^{\gamma-\varepsilon}.$$

The invariance of $\nu_N \otimes \mu_N'$ under the dynamics (7.1) is an essential ingredient in the proof of Theorem 6.1. In addition, convergence of the measures itself may be of independent interest.

7.1. **Convergence of the measures.** We now state our result on the convergence of the Gibbs measures. For convenience, we introduce another measure $\overline{\nu}_N$ by

$$\overline{\nu}_N(\mathrm{d}\phi) := \nu_N \otimes \mu_N^{\perp} = \frac{1}{\mathcal{Z}_N} e^{-\int_{\mathbf{T}^2} \widetilde{V_N}(\Pi_N\phi)\mathrm{d}x} \mu(\mathrm{d}\phi),$$

where the normalization constant $\mathcal{Z}_N$ is the same as the one in (7.2). For every $p \geq 1$, define

$$\mathcal{Z}_N^{(p)} := \mathbf{E}^{\mu}\Big[e^{-p\int_{\mathbf{T}^2} \widetilde{V_N}(\Pi_N\phi)\mathrm{d}x}\Big].$$

Then $\mathcal{Z}_N = \mathcal{Z}_N^{(1)}$. We have the following statement.

Theorem 7.1. *Let $\alpha \in (\frac{3}{4}, 1)$. Suppose that V verifies Assumption 4.1. Then for every $p \geq 1$, we have*

$$\sup_N |\log \mathcal{Z}_N^{(p)}| < +\infty$$

Furthermore, $\lambda := \overline{a}_2 > 0$, and

$$\mathbf{E}^{\mu}\Big| e^{-\int_{\mathbf{T}^2} \widetilde{V_N}(\Pi_N\phi)\mathrm{d}x} - e^{-\lambda\int_{\mathbf{T}^2} \phi^{\diamond 4}\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}^2}\phi^{\diamond 2}\mathrm{d}x}\Big|^p \to 0$$

for every $p \geq 1$. Hence, $\overline{\nu}_N$ converges to the fractional ϕ_2^4 measure ν in the sense that the densities with respect to μ converge in $L^p(\mu)$.

The restriction $\alpha > 3/4$ is natural in the sense that in this range, one can define the ϕ^4 measure ν by an absolutely continuous density with respect to the Gaussian measure μ. The fourth Wick power $\phi^{\diamond 4}$ fails to exist under μ when $\alpha = 3/4$, in which case one expects to end up with a measure (after further renormalization) that is mutually singular with respect to μ.

The next proposition says that (4.3) is actually almost necessary for the main theorem.

Proposition 7.2. *If there exists $\theta \in \mathbf{R}$ such that $\sum_{j=1}^{m} \overline{a}_2 \theta^{2(j-2)} < 0$, then there exists $c > 0$ such that $\log \mathcal{Z}_N > cN^{4(1-\alpha)}$ for all $N \in \mathbf{N}$. As a consequence, the densities $\mathrm{d}\bar{\nu}_N/\mathrm{d}\mu$ do not converge in $L^1(\mu)$.*

8. Comparison with parabolic equations and other dispersive models

This type of weak universality was first studied by Hairer-Quastel ([8]) in deriving the KPZ equation from a large class of microscopic growth models. It has later been extended in various directions in the setting of parabolic singular stochastic PDEs ([10, 9, 12, 6, 5]). A key feature in this type of problem is that every term in the expansion of the nonlinearity has the same size — and hence the constant λ of this limiting equation depends on the whole nonlinearity rather than the naive guess of the corresponding power only. As far as we know, our Theorem 6.1 is the first one for dispersive models fitting in this situation.

Technically, one difference between dispersive and parabolic equations is the lack of L^∞ based estimates in the dispersive setting. Hence, the heuristic reasoning that negative powers of N balance out high powers of singular objects needs more involved justification with the help of dispersive tools. A second technical difference lies in the globalization argument. In the parabolic setting, the global-in-time convergence follows from the global well-posedness of the limiting equation and stability. However in the current dispersive setting, even though the limiting equation is globally well-posed, the stability properties are not good enough here, and we need to make an essential use of invariant measure to get global convergence.

Note that our techniques can be used to extend the weak universality result of Gubinelli-Koch-Oh [7] for the 2D stochastic nonlinear wave equation to the stochastic nonlinear fractional wave equation with space-time white noise, formally written as

$$\partial_t^2 u + |\nabla|^{2\alpha} u + \partial_t u + \lambda u^{\diamond 3} = \xi, \quad (t,x) \in \mathbf{R}^+ \times \mathbf{T}^2$$

when $\alpha > 8/9$. The weak universality result of Gubinelli-Koch-Oh is a consequence of the almost sure global well-posedness for the two-dimensional nonlinear wave equation ($\alpha = 1$) with *any order nonlinearity*, while for the fractional wave equation with $\alpha < 1$, the situation is radically different.

9. Some ideas behind the proof

The invariance of $\nu_N \otimes \mu'_N$ under the dynamics (7.1) is used in two different ways. The first one is that it gives key a priori bounds for truncated dynamics (for fixed N). Second, the convergence of the invariant measures to a limiting measure (as stated in Theorem 7.1) and the invariance of the limiting measure under the limiting dynamics allows us to pass from local to global in time convergence. Another key component of our argument is the use of dispersive estimates giving $L^2_t L^\infty_x$ local bounds.

Let us be slightly more precise. We have that for any $\delta > 0$

$$\Big\| \sum_{|k|\leq N} \frac{g_k(\omega)}{\langle k\rangle^\alpha}\, e^{ik\cdot x} \Big\|_{L^\infty_x} \leq C_\delta N^{1-\alpha+\delta}$$

in a set of residual probability $\lesssim \exp(-N^\theta)$ for some $\theta > 0$. As in the work by Bourgain-Bulut [1] thanks to invariant measure considerations, we can propagate this information to the full solution u_N.

This is unfortunately not sufficient to pass into the limit in terms like

$$u_N^3 \left(N^{-(1-\alpha)} u_N\right)^{2k+1}$$

for $k \gg 1$ because of small losses of power of N in $N^{-(1-\alpha)} u_N$.

We can overcome this difficulty by using dispersive estimates. More precisely, we can write

$$u_N^3 \left(N^{-(1-\alpha)} u_N\right)^{2k+1} = N^{-(1-\alpha)} u_N^4 \left(N^{-(1-\alpha)} u_N\right)^{2k}$$

and exploit the $L^2_t L^\infty_x$ control coming from Strichartz estimates. This leads to local in time convergence.

The global in time convergence crucially relies on the a priori bounds on the global cubic dynamics. These bounds are again relying on invariant measure considerations but this time for the limit dynamics. This essentially explains the basic idea behind the proof.

Let is also mention that as in [1] and [2] in the convergence proof we define suitable quantities $x(t)$ satisfying inequalities of type

$$\dot{x}_N(t) \leq C_\delta (\log(N))^\delta x_N(t),$$

i.e. we allow a slow semi-group growth of order $\exp\left((\log(N))^\delta\right)$, $\delta < 1$. This is compensated by the convergence of $x_N(0)$ which is of order $N^{-\theta}$ for some $\theta > 0$.

10. Perspectives

The method we use is quite general and we expect that it can be extended to other dispersive models. In this context, the Benjamin-Ono equation seems a challenging case.

It would be interesting to obtain triviality results when the assumptions on V are not satisfied. The critical case seems particularly challenging.

It would be interesting to extend our results to more general initial data.

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Université de Cergy-Pontoise, Cergy-Pontoise, F-95000, UMR 8088 du CNRS
Email address: `chenmin.sun@u-cergy.fr`

Université de Cergy-Pontoise, Cergy-Pontoise, F-95000, UMR 8088 du CNRS
Email address: `nikolay.tzvetkov@u-cergy.fr`

Peking University
Email address: `weijunx@gmail.com`