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## ON CONTROLLABILITY OF WAVES AND GEOMETRIC CARLEMAN ESTIMATES

ARICK SHAO

ABSTRACT. The present article is a brief summary of the paper [27], which established new Carleman and observability estimates for a general class of linear wave equations. The main features of these estimates are that (a) they apply to a fully general class of *time-dependent domains*, with *timelike moving boundaries*, (b) they apply to linear wave equations in *any spatial dimension* and with *general time-dependent lower-order coefficients*, and (c) they allow for *smaller time-dependent observation regions* than previously obtained from existing Carleman estimate methods. In particular, the results of [27] imply exact controllability for general linear waves, again in settings of time-dependent domains and regions of control.

### 1. INTRODUCTION

This article provides an introduction to the recent paper [27], and it serves as a companion to the presentation given at the *Séminaire Laurent Schwartz* in February 2019.

The main result of [27] is a novel Carleman estimate for the wave equation, proved using a Lorentzian geometric approach. This estimate was then used to derive novel observability inequalities for linear wave equations, with the following features:

- (I) The estimates apply to a general class of *time-dependent domains* with *moving boundaries*.
- (II) The estimates apply to wave equations in *any spatial dimension*.
- (III) The estimates apply to general linear waves with *time-dependent lower-order coefficients*.
- (IV) The estimates apply for *time-dependent observation regions* that are smaller than those from standard Carleman-based observability inequalities.

An immediate corollary of these observability estimates is a corresponding set of exact controllability results for general linear waves, on the same general class of time-dependent domains.

**1.1. Background.** In evolutionary PDEs, the question of *exact controllability* concerns whether one can drive solutions from any prescribed initial state to any desired final state at a later time, under the constraint that only some limited parameters in the system—the *controls*—can be set. In other words, the above asks whether one can fully govern the system through its controls.

In this article, we restrict our attention to linear wave equations,

$$(1.1) \quad (\square\phi + \nabla_X\phi + V\phi)|_{\mathcal{U}} = 0,$$

defined on a spacetime domain  $\mathcal{U} \subseteq \mathbb{R}^{1+n}$ , where:

- $\square = -\partial_t^2 + \Delta_x$  is the standard wave operator on  $\mathbb{R}^{1+n}$ .
- $X$  is a smooth vector field on  $\bar{\mathcal{U}}$  (and  $\nabla_X$  denotes the derivative along  $X$ ).
- $V$  is a smooth scalar-valued potential on  $\bar{\mathcal{U}}$ .

In particular, both  $X$  and  $V$  are allowed to be *non-analytic* and *time-dependent*.

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In this setting, the initial and final states are each given by a pair of functions— $(\phi_0^-, \phi_1^-)$  and  $(\phi_0^+, \phi_1^+)$ , respectively—representing the values of  $\phi$  and  $\partial_t \phi$  at two fixed times  $T_- < T_+$ . There are several possibilities for what one can take as the control:

- *Interior control*:  $\phi$  is steered by an extra forcing term on the right-hand side of (1.1).
- *Dirichlet boundary control*:  $\phi$  is steered by Dirichlet data on the boundary of  $\mathcal{U}$ .
- *Neumann boundary control*:  $\phi$  is steered by Neumann data on the boundary of  $\mathcal{U}$ .

In this article, we will focus exclusively on Dirichlet boundary control.

For simplicity, and for the sake of exposition, let us first consider the case of a static domain

$$(1.2) \quad \mathcal{U} := \mathbb{R} \times \Omega,$$

where  $\Omega$  is a bounded and open subset of  $\mathbb{R}^n$  that also has a smooth boundary. In this case, our main problem can be more precisely expressed as follows:

**Problem 1.1** (Dirichlet boundary exact controllability). Let  $\mathcal{U}$  be as in (1.2). Fix also initial and final times  $T_\pm$ , as well as a boundary region  $\Gamma \subseteq (T_-, T_+) \times \partial\Omega$ . Given any initial and final data  $(\phi_0^\pm, \phi_1^\pm) \in L^2(\Omega) \times H^{-1}(\Omega)$ , find a control  $\phi_b \in L^2(\mathbb{R} \times \partial\Omega)$ , with  $\phi_b$  supported in  $\Gamma$ , such that the solution  $\phi$  of (1.1), with initial and Dirichlet boundary data<sup>1</sup>

$$(\phi, \partial_t \phi)|_{t=T_-} = (\phi_0^-, \phi_1^-), \quad \phi|_{\mathbb{R} \times \partial\Omega} = \phi_b,$$

also achieves the final state

$$(\phi, \partial_t \phi)|_{t=T_+} = (\phi_0^+, \phi_1^+).$$

Problem 1.1 has been studied extensively over the past fifty years. Modern treatments of controllability are based on the *Hilbert Uniqueness Method (HUM)* of Lions [19, 20], which are closely related to the earlier works of Dolecki and Russell [10]. The key idea is that by duality, controllability is equivalent to quantitative uniqueness properties of the adjoint wave equation. In particular, an affirmative answer to Problem 1.1 is equivalent to solving the following problem:

**Problem 1.2** (Dirichlet boundary observability). Let  $\mathcal{U}$ ,  $T_\pm$ , and  $\Gamma$  be as in Problem 1.1. For any finite-energy solution  $\psi$  to (1.1) with zero Dirichlet boundary data, show that<sup>2</sup>

$$(1.3) \quad \|\psi(T_\pm)\|_{H^1(\Omega)} + \|\partial_t \psi(T_\pm)\|_{L^2(\Omega)} \lesssim \|\partial_\nu \psi\|_{L^2(\Gamma)},$$

where  $\nu$  denotes the outward unit normal on the boundary of  $\mathcal{U}$ .<sup>3</sup>

The estimate (1.3) is commonly known as an *observability inequality*. Since Problems 1.1 and 1.2 are equivalent, we now focus our attention solely on the latter.

Observe that for hyperbolic equations, finite speed of propagation provides an important obstruction to controllability, as some minimum amount of time is required for information on  $\partial\Omega$  to travel to all of  $\Omega$ . This results in a fundamental lower bound on the timespan  $T_+ - T_-$  required for any controllability, and hence observability, result to hold.<sup>4</sup>

Many methods have been developed for proving the crucial observability estimate (1.3). Below, we briefly survey the main classes of techniques that have proved fruitful in this effort:

- (1) *Fourier methods* (see [4] for details), which were some of the earliest employed, have been applied extensively to wave equations in one spatial dimension, with  $X = 0$  and  $V$  constant. Results along this direction revolve around applying Ingham's inequality [15] and its refinements to the Fourier series expansions of  $\psi$ , and are often capable of retrieving the optimal timespan, as dictated by finite speed of propagation.

<sup>1</sup>It is well-known that linear wave equations are well-posed with data in the above classes; see [21].

<sup>2</sup>More accurately,  $\psi$  should solve the equation adjoint to (1.1), but this has the same form as (1.1).

<sup>3</sup>In particular,  $\partial_\nu \psi$  denotes the Neumann trace for  $\psi$  on  $\mathbb{R} \times \partial\Omega$ .

<sup>4</sup>This contrasts with heat and Schrödinger equations, which can generally be controlled in arbitrarily small times.

- (2) *Multiplier methods*, on the other hand, can be applied to waves in arbitrary dimensions, though they are mostly applicable only to free waves (i.e.,  $X = 0$  and  $V = 0$ ). Here, the key step is to integrate by parts an expression of the form

$$\int_{(T_-, T_+) \times \Omega} \square \psi S \psi,$$

where  $S$  represents an appropriately chosen first-order operator. Using this technique, it was shown (see [12, 20]) that given any  $x_0 \in \mathbb{R}^n$ , one could establish (1.3) with timespan

$$(1.4) \quad T_+ - T_- > 2R, \quad R := \max_{y \in \partial\Omega} |y - x_0|,$$

and with observation/control region given by<sup>5</sup>

$$(1.5) \quad \Gamma := (T_-, T_+) \times \{y \in \partial\Omega \mid (y - x_0) \cdot \nu > 0\}.$$

Here, neither the timespan (1.4) nor the observation region (1.5) is necessarily optimal.

- (3) *Microlocal methods* provide some of the most powerful tools in this area, yielding optimal results with regards to the requisite timespan and observation region. Of particular note is the seminal result of Bardos, Lebeau, and Rauch [6], which established that (1.3) holds for  $\Gamma = (T_-, T_+) \times \Lambda$  if and only if the *geometric control condition* (abbreviated *GCC*) is satisfied.<sup>6</sup> Further extensions of this result were given in [8, 24]; in particular, the latter extended the results to *time-dependent* regions  $\Gamma$  satisfying the GCC.

However, one important caveat is that these microlocal methods only apply when the lower-order coefficients  $X$  and  $V$  are *time-independent*, or at most *time-analytic*.<sup>7</sup>

- (4) *Carleman estimates* are weighted (spacetime) integral inequalities of the form

$$(1.6) \quad \|e^{\lambda F} \nabla_{t,x} \psi\|_{L^2}^2 + \|e^{\lambda F} \psi\|_{L^2}^2 \lesssim \lambda^{-2} \|e^{\lambda F} \square \psi\|_{L^2}^2 + \dots,$$

containing an additional free parameter  $\lambda > 0$  and weight  $e^{\lambda F}$ . These inequalities have been applied extensively toward questions of unique continuation; see [13, 29] for general results. Moreover, some global Carleman estimates have been applied toward proving the observability estimate (1.3); see the pioneering work of [30], as well as [7, 11, 16, 33, 34].

With regards to observability, Carleman estimates are advantageous due to their robustness. For instance, they can be used to obtain the same results as from the multiplier methods, but also for wave equations with arbitrary (sufficiently regular)  $X$  and  $V$ . (In particular, by taking  $\lambda$  in (1.6) to be as large as necessary, one can freely “absorb” away potentially dangerous contributions from lower-order terms.)<sup>8</sup>

On one hand, methods based on Carleman estimates lack the precision of microlocal methods and cannot achieve the GCC in general. However, Carleman methods do apply to wave equations with general *time-dependent* lower-order coefficients.

For this article, we are particularly concerned with lower-order coefficients  $X$  and  $V$  that can vary in both space and time, *without any presumption of analyticity*. Since robustness is our primary priority, we resort to exploring Carleman estimates methods here.

<sup>5</sup>Note (1.5) consists of all  $x \in \partial\Omega$  such that the ray emanating from  $x_0$  and passing through  $x$  is leaving  $\Omega$  at  $x$ .

<sup>6</sup>Roughly, the GCC states that every null geodesic in  $(T_-, T_+) \times \Omega$ —with the additional condition that it is reflected via geometric optics at the boundary  $(T_-, T_+) \times \partial\Omega$ —intersects some point of  $\Gamma$ .

<sup>7</sup>This requirement of time-analyticity is a consequence of the unique continuation results of [14, 23, 31].

<sup>8</sup>We remark that for technical reasons, existing Carleman-based results [7, 11, 16, 33, 34] dealt only with the case in which  $x_0 \notin \bar{\Omega}$ . The present work [27] also improves upon these results by removing this restriction.

**1.2. Non-static domains.** While there has been extensive research in the case of static domains (1.2), the literature on more general *time-dependent domains* is far more sparse.

**Assumption 1.3.** We now consider a spacetime domain of the form

$$(1.7) \quad \mathcal{U} := \bigcup_{\tau \in \mathbb{R}} (\{\tau\} \times \Omega_\tau),$$

where the  $\Omega_\tau$ 's are bounded open subsets of  $\mathbb{R}^n$ , again with smooth boundaries, that also vary smoothly with respect to  $\tau$ . In particular, the boundary of  $\mathcal{U}$  is then given by

$$(1.8) \quad \mathcal{U}_b = \bigcup_{\tau \in \mathbb{R}} (\{\tau\} \times \partial\Omega_\tau).$$

In addition, we assume that

- $\mathcal{U}_b$  is *timelike*, that is,  $\mathcal{U}_b$  “moves at less than the wave, or characteristic, speed”.<sup>9</sup>
- $\mathcal{U}$  “can be transformed into a static region (1.2) via an appropriate change of variables”.<sup>10</sup>

(In [27], the above was captured more precisely through the notion of a *generalized timelike cylinder*, or *GTC*. For details, see [27, Definition 2.11] and the discussion thereafter.)

By applying a change of coordinates to transform  $\mathcal{U}$  into a static region of the form (1.2) (and thus transforming (1.1) into a more complicated hyperbolic equation), one can show that the standard well-posedness results for (1.1) on a static domain have direct analogues on time-dependent domains (1.7). As a result, one can pose direct analogues of Problems 1.1 and 1.2 in the current setting:

**Problem 1.4** (Dirichlet boundary exact controllability). Let  $\mathcal{U}$  satisfy Assumption 1.3, fix initial and final times  $T_\pm$ , and fix a boundary region  $\Gamma \subseteq \partial\mathcal{U} \cap \{T_- < t < T_+\}$ . Given any initial and final data  $(\phi_0^\pm, \phi_1^\pm) \in L^2(\Omega_{T_\pm}) \times H^{-1}(\Omega_{T_\pm})$ , find a control  $\phi_b \in L^2(\partial\mathcal{U})$ , with  $\phi_b$  supported in  $\Gamma$ , such that the solution  $\phi$  of (1.1), and with initial and Dirichlet boundary data

$$(\phi, \partial_t \phi)|_{t=T_-} = (\phi_0^-, \phi_1^-), \quad \phi|_{\partial\mathcal{U}} = \phi_b,$$

also achieves the final state

$$(\phi, \partial_t \phi)|_{t=T_+} = (\phi_0^+, \phi_1^+).$$

**Problem 1.5** (Dirichlet boundary observability). Let  $\mathcal{U}$ ,  $T_\pm$ , and  $\Gamma$  be as in Problem 1.4. For any finite-energy solution  $\psi$  to (1.1) with zero Dirichlet boundary data, show that

$$(1.9) \quad \|\psi(T_\pm)\|_{H^1(\Omega_{T_\pm})} + \|\partial_t \psi(T_\pm)\|_{L^2(\Omega_{T_\pm})} \lesssim \|\partial_\nu \psi\|_{L^2(\Gamma)},$$

where  $\partial_\nu \psi$  denotes the Neumann trace for  $\psi$  on  $\partial\mathcal{U}$ .

Using the standard HUM machinery, one can again show that Problems 1.4 and 1.5 are equivalent. Thus, we will focus our attention on proving observability.

To this point, the literature on time-dependent domains have only treated special cases:

- A early result [5] of Bardos and Chen, which predated the HUM, proved interior controllability for free waves on domains  $\mathcal{U}$  that are expanding in time.
- Using the HUM with multiplier methods, Miranda [22] established Dirichlet control for free waves. Moreover, while the domain  $\mathcal{U}$  can be time-dependent and needs not be expanding, the result assumes that  $\mathcal{U}$  is self-similar and becomes “asymptotically static”.<sup>11</sup>

<sup>9</sup>This is needed to ensure that  $\mathcal{U}_b$  is an appropriate place for one to impose boundary data.

<sup>10</sup>This is needed in order to establish basic well-posedness results for (1.1) on  $\mathcal{U}$ .

<sup>11</sup>Roughly, each  $\Omega_\tau$  is of the form  $k(\tau) \cdot \Omega_0$ , and  $k'(\tau)$  decays sufficiently at large times.

- More recently, various authors [9, 26, 25, 28, 32] studied the problem in one spatial dimension using Fourier and multiplier methods. Here,  $\mathcal{U}$  describes the region between two timelike lines or curves; in some cases, the optimal observation time was achieved.

In particular, what was missing—and also what was accomplished by the article [27]—are results for general linear waves on general time-dependent domains, in any dimension.

An informal statement of the main result of [27] can be stated as follows:

**Theorem 1.6.** *Consider a general linear wave equation (1.1) on a domain  $\mathcal{U}$  satisfying Assumption 1.3. In addition, fix initial and final times  $T_{\pm}$ , fix a point  $x_0 \in \mathbb{R}^n$ , and assume*

$$(1.10) \quad T_+ - T_- > R_+ - R_-, \quad R_{\pm} := \sup_{y \in \partial\Omega_{T_{\pm}}} |y - x_0|.$$

*Then, for any finite-energy solution  $\psi$  to (1.1) on  $\mathcal{U}$ , with zero Dirichlet boundary data, we have that the observability estimate (1.9) holds. Moreover, the observation region*

$$\Gamma \subseteq \partial\mathcal{U} \cap \{T_- < t < T_+\}$$

*is time-dependent and “much improved” compared to previous results based on multiplier methods and Carleman estimates (even in the case of static domains).*

In the special case of one spatial dimension, Theorem 1.6 applies to general regions  $\mathcal{U}$  lying between two timelike curves and recovers all the existing results in the literature. Furthermore, Theorem 1.6 achieves the optimal observation time in this setting.

Furthermore, in higher dimensions, Theorem 1.6 provides, to the author’s best knowledge, the first observability result for general time-dependent domains and linear wave operators. In particular, even when  $\mathcal{U}$  is static, the result still achieves an observation region  $\Gamma$  that is a proper subset of (1.5). On the other hand, in contrast to microlocal results (which are not applicable in our present non-analytic setting), Theorem 1.6 does not generally recover the GCC.

More precise versions of Theorem 1.6 will be given in later sections, once we have developed the requisite ideas and terminology. As we hinted at earlier, Theorem 1.6 was proved using novel Carleman estimates, which will be discussed in further detail later in this article.

**1.3. Other directions.** There exist numerous observability and exact controllability results for wave equations on product manifolds  $\mathbb{R} \times M$  (with  $M$  being a Riemannian manifold)—see, for example, [11, 17, 24] among many others. However, as far as the author is aware, there is no literature addressing control for waves on general Lorentzian manifolds with time-dependent geometry. A result of this type would be a natural step beyond [27].

**Remark 1.7.** Using the conformal invariance of the characteristics of the wave equation (which play integral roles in our upcoming Carleman estimates), the results of Theorem 1.6 can already be directly extended to curved backgrounds conformally equivalent to our flat setting.

A less ambitious direction, but one which still contains numerous technical obstacles, is to use the methods of [27] to prove interior observability inequalities. Here, the challenge is to obtain an  $L^2$ -Carleman estimate (as opposed to the  $H^1$ -Carleman estimate described in this article) that can be applied to wave equations (1.1) with general lower-order coefficients.

Another future direction is to turn the techniques developed here toward treating nonlinear wave equations. One advantage of [27] here is that it needs not assume any (time-)analyticity for the coefficients  $X$  and  $V$  in (1.1). (We note that similar Carleman estimates have been applied in [3] toward studying singularity formation for nonlinear waves.)

In the remainder of the article, we give the main ideas behind the proof of Theorem 1.6. To make the exposition more transparent, we present, in the following sections, a succession of partial results, each demonstrating different aspects of the main ideas behind Theorem 1.6.

## 2. A PRELIMINARY MULTIPLIER RESULT

The objective of this section is to present a “warm-up” multiplier-based observability estimate that applies to the free wave equation on general time-dependent domains. While this estimate is strictly weaker than our main results, it allows us to discuss the effects of time-dependent domains on observability apart from the other aspects of our main result.

We mention now that ideas from Lorentzian geometry play a prominent role in all our proofs. In particular, the wave operator  $\square$  is connected to the Minkowski metric,<sup>12</sup>

$$(2.1) \quad g := -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2,$$

in the same way that the Laplacian is connected to the Euclidean metric. Moreover, many analytical aspects of Riemannian geometry, such as the divergence theorem and integrations by parts, also have direct Lorentzian analogues; these tools are used heavily throughout the computations in [27].

**2.1. Multiplier arguments revisited.** In order to better understand the impact of non-static domains, let us first briefly recall the classical multiplier approach to observability estimates on static domains. The starting point of this argument is the identity

$$(2.2) \quad 0 = \int_{(T_-, T_+) \times \Omega} \square \psi S \psi,$$

where  $\psi$  is a solution of (1.1), with  $X = 0$  and  $V = 0$ , and where

$$(2.3) \quad S \psi := (x - x_0) \cdot \nabla_x \psi + \frac{n-1}{2} \cdot \psi, \quad x_0 \in \mathbb{R}^n.$$

Applying various integrations by parts to the right-hand side of (2.2) along with the conservation of energy property of free waves, one arrives at the estimate

$$(2.4) \quad (T_+ - T_-) \cdot \mathcal{E}(T_\pm) \leq 2R \cdot \mathcal{E}(T_\pm) + \frac{1}{2} \int_{(T_-, T_+) \times \partial\Omega} [(x - x_0) \cdot \nu] |\partial_\nu \psi|^2,$$

where  $R$  is defined as in (1.4), and where  $\mathcal{E}$  denotes the standard energy:

$$\mathcal{E}(t) := \frac{1}{2} \int_{\{t\} \times \Omega} [(\partial_t \psi)^2 + |\nabla_x \psi|^2].$$

Observe now that as long as  $T_+ - T_- > 2R$ , the first term on the right-hand side of (2.4) can be absorbed into the left, and we can thus control the energy of  $\psi$  by its Neumann trace. Moreover, note that the boundary integral on the right-hand side of (2.4) needs not include all of  $(T_-, T_+) \times \partial\Omega$ . Indeed, any point for which the coefficient  $(x - x_0) \cdot \nu$  is non-positive can be freely excluded from this integral. Finally, combining the above steps with (2.4) results in the desired observability estimate (1.3), with the observation region  $\Gamma$  given by (1.5).

**2.2. Adaptation to time-dependent domains.** Let us now see how the classical multiplier method can be adapted to deal with a time-dependent domain  $\mathcal{U}$  satisfying Assumption 1.3.

The first point is to notice that the classical multiplier result can be viewed as being “centered about the point  $x_0$ ” in space. To handle non-static domains, however, *we replace this reference point  $x_0$  by a reference event  $(t_0, x_0) \in \mathbb{R}^{1+n}$* . Intuitively speaking, we center our argument about not only a fixed location  $x_0$ , but also a fixed time  $t_0 \in \mathbb{R}$ .<sup>13</sup>

To be more concrete, we now replace the multiplier  $S \psi$  from (2.3) by

$$(2.5) \quad S_* \psi := (t - t_0) \partial_t \psi + (x - x_0) \cdot \nabla_x \psi + \frac{n-1}{2} \cdot \psi.$$

<sup>12</sup>The Minkowski spacetime  $(\mathbb{R}^{1+n}, g)$  is the setting of *special relativity*.

<sup>13</sup>In particular, the contribution from  $t_0$  is not seen whenever  $\mathcal{U}$  is time-independent.

Compared to  $S$ , the operator (2.5) contains an extra term with a time derivative.<sup>14</sup> This term yields the same effect as the applications of energy conservation in the classical argument.

Similar to before, we now begin with the expression

$$0 = \int_{\mathcal{U} \cap \{T_- < t < T_+\}} \square \psi S_* \psi.$$

(Again, we assume  $\psi$  satisfies the free wave equation.) From here, however, we *apply the Lorentzian version of integrations by parts*, with respect to the *Minkowski* metric. In particular, as we are dealing directly with the Minkowski geometry of  $\mathbb{R}^{1+n}$ , rather than the Euclidean geometry of  $\mathbb{R}^n$ , the process can deal directly with objects that are curved in time as well as in space.

**Remark 2.1.** One point to note is that through this Lorentzian integration by parts, one obtains the *Minkowski* unit normal  $\mathcal{N}$  to  $\mathcal{U}$ . More concretely, this Minkowski normal is the same as the Euclidean normal, except that its time component has the opposite sign.

By completing this computation using the above ideas (see [27, Section 2.2.2] for details), we arrive at our multiplier-based observability estimate for time-dependent domains:

**Theorem 2.2** ([27, Theorem 2.23]). *Let  $\mathcal{U}$  be a time-dependent domain satisfying Assumption 1.3, fix initial and final times  $T_\pm$ , and fix a reference point  $x_0 \in \mathbb{R}^n$ . Moreover, assume that*

$$(2.6) \quad T_+ - T_- > R_+ - R_-, \quad R_\pm := \sup_{y \in \partial\Omega_{T_\pm}} |y - x_0|,$$

and fix a reference time  $t_0 \in (T_-, T_+)$  such that<sup>15</sup>

$$(2.7) \quad T_+ - t_0 > R_+, \quad t_0 - T_- > R_-.$$

Then, any finite-energy solution  $\psi$  of the free wave equation  $\square \psi = 0$  on  $\mathcal{U}$ , with zero Dirichlet boundary data, satisfies the observability inequality

$$(2.8) \quad \int_{\mathcal{U} \cap \{t=T_\pm\}} [(\partial_t \psi)^2 + |\nabla_x \psi|^2 + \psi^2] \lesssim \int_{\Gamma_0} |\mathcal{N} \psi|^2,$$

where  $\mathcal{N}$  denotes the Minkowski outer-pointing unit normal to  $\mathcal{U}$ , where

$$(2.9) \quad \Gamma_0 := \mathcal{U}_b \cap \{T_- < t < T_+\} \cap \{\mathcal{N} f_0 > 0\},$$

and where  $f_0$  is the function

$$(2.10) \quad f_0 := \frac{1}{4}[|x - x_0|^2 - (t - t_0)^2].$$

The proof of Theorem 2.2 is a relatively short computation that carries out the above-mentioned argument.<sup>16</sup> We remark that even though Theorem 2.2 only holds for free waves, this result already applies to general time-dependent domains satisfying Assumption 1.3—in particular, Theorem 2.2 already achieves the features (I) and (II) described at the beginning of the article.

Some additional comments on the statement of Theorem 2.2 are in order:

- Theorem 2.2 requires a much smaller observation time than previous results for non-static domains in higher dimensions. In many cases, our  $T_+ - T_-$  is optimal.
- The condition (2.6) is a direct generalization of (1.4) for static domains. One interpretation is that one needs enough time for information to travel from  $\partial\Omega_{T_-}$  to  $x_0$  and then to  $\partial\Omega_{T_+}$ .

<sup>14</sup>In fact, the principal part of  $S_*$  is precisely the Minkowski gradient of the function  $f_0$  from (2.10).

<sup>15</sup>Observe (2.6) implies that a  $t_0$  satisfying (2.7) exists.

<sup>16</sup>In full, it comprises only two pages within [27].



- The function  $f_0$  in (2.10) plays a special role in Minkowski geometry. In particular, its level sets—which are hyperboloids in  $\mathbb{R}^{1+n}$ —are invariant under Lorentz transformations.
- The condition  $\mathcal{N}f_0 > 0$  in (2.9) is a generalization of the standard condition  $(x-x_0)\cdot\nu > 0$  from multiplier methods. In fact, when  $\mathcal{U}$  is static, (2.9) reduces to (1.5).

Finally, a rough qualitative characterization of how the time-dependence of  $\mathcal{U}$  affects the observation region  $\Gamma_0$  in Theorem 2.2 can be described as follows:

- Where  $\mathcal{U}$  is expanding away from time  $t_0$ , the region  $\Gamma_0$  is smaller than that of (1.5).
- Where  $\mathcal{U}$  is shrinking away from time  $t_0$ , the region  $\Gamma_0$  is larger than that of (1.5).

### 3. THE NEW CARLEMAN ESTIMATE

While Theorem 2.2 already captures the essential ideas behind dealing with time-dependent domains, its main drawback is that it only applies to the free wave equation. To extend our results to general linear wave equations (1.1)—without assuming analyticity for  $X$ ,  $V$ , or  $\mathcal{U}_b$ —we will have to establish a Carleman-based analogue of Theorem 2.2. Moreover, we have yet to improve the region of observation compared to existing Carleman-type results.

Recall that Dirichlet observability can be derived from global Carleman estimates of the form

$$(3.1) \quad \|e^{\lambda F} \nabla_{t,x} \psi\|_{L^2(\mathcal{V})}^2 + \|e^{\lambda F} \psi\|_{L^2(\mathcal{V})}^2 \lesssim \lambda^{-2} \|e^{\lambda F} \square \psi\|_{L^2(\mathcal{V})}^2 + \lambda \|e^{\lambda F} \mathcal{N} \psi\|_{L^2(\Gamma)}, \quad \lambda \gg 1,$$

where  $\Gamma \subseteq \mathcal{U}_b$  is the desired observation region, where  $\mathcal{V}$  is an appropriate subset of the domain  $\mathcal{U}$ , and where  $\mathcal{N}$  denotes the Minkowski outer-pointing unit normal to  $\mathcal{U}$ . Indeed, from (3.1), one then usually obtains an observability estimate through fairly standard methods:

- Using energy estimates, the bulk integrals on the left-hand side of (3.1) can be bounded from below by the energy of  $\psi$  at a fixed time.
- $\square \psi$  can be replaced by lower-order terms using (1.1). These terms can then be absorbed into the left-hand side using the smallness of  $\lambda^{-2}$ .

Thus, the critical task here is to derive a *novel global Carleman inequality* of the form (3.1) that successfully leads to Theorem 2.2, as well as accomplishes objectives (III) and (IV) from the beginning of the article. This is the objective of the present section.

**3.1. An almost correct weight.** The proof of (3.1) can be superficially described in terms of multipliers. Indeed, the starting point is to integrate by parts an expression of the form

$$\int (e^{\lambda F} \square \psi) \mathcal{S}(e^{\lambda F} \psi).$$

In other words, the rough idea is to apply a multiplier-type argument for the *conjugated wave operator*  $e^{\lambda F} \square e^{-\lambda F}$  and multiplier  $\mathcal{S}$  to the function  $\psi^* := e^{\lambda F} \psi$ . If the function  $F$  and the multiplier  $\mathcal{S}$  are appropriately chosen, then one can manipulate this computation to ensure that the resulting bulk integrals are positive, as in the left-hand side of (3.1).<sup>17</sup>

Thus, to describe our Carleman estimate, we must first specify the  $F$  and  $\mathcal{S}$  that we use. For the present discussion, let us take a weight  $e^{\lambda F}$  that is *incorrect*—in that *it does not quite suffice for proving Theorem 1.6*—but is particularly useful for explaining some key ideas. (We will proceed to pick the true Carleman weight later in this section.)

We begin by returning to the function  $f_0$  defined in (2.10),

$$f_0 := \frac{1}{4} [|x - x_0|^2 - (t - t_0)^2].$$

Observe that  $f_0$  has the following properties:

- The level sets of  $f_0$  are hyperboloids centered about  $(t_0, x_0)$ .

<sup>17</sup>More specifically, the main condition required for positivity is *pseudoconvexity* for the level sets of  $F$ ; see [13].

- These level sets are invariant under spatial rotations and Lorentz boosts about  $(t_0, x_0)$ .
- The level set  $\{f_0 = 0\}$  is precisely the double null cone with vertex at  $(t_0, x_0)$ ,

$$(3.2) \quad \mathcal{C}_{t_0, x_0} = \{|t - t_0| = |x - x_0|\}.$$

In addition, we denote the exterior of  $\mathcal{C}_{t_0, x_0}$ , where  $f_0$  is positive, by

$$(3.3) \quad \mathcal{D}_{t_0, x_0} := \{f_0 > 0\}.$$

Now, since  $f_0$  played such a central role in Theorem 2.2, it is not surprising that it would be similarly involved in the Carleman estimate. Indeed, we set our Carleman weight as

$$(3.4) \quad e^{\lambda F} = f_0^\lambda e^{2\lambda f_0^{\frac{1}{2}}}, \quad F := \lambda \log f_0 + 2\lambda b f_0^{\frac{1}{2}}.$$

In particular, by being based off of  $f_0$ , our weight has the advantage of being well-adapted to the wave operator and its underlying Minkowski geometry.

A particularly important, and novel, feature of the weight (3.4) is that  $e^{\lambda F}$  *vanishes whenever*  $f_0 = 0$ . The upshot of this is that the cone  $\mathcal{C}_{t_0, x_0}$  can be used as an extra boundary for our Carleman inequality, on which no boundary terms are produced. In other words, by using (3.4), we can *obtain a Carleman estimate that is entirely supported within*  $\mathcal{D}_{t_0, x_0} \cap \mathcal{U}$ .<sup>18</sup>

**Remark 3.1.** The above should be contrasted with more classical applications of Carleman estimates, for which the support of the estimate is constrained using an additional cutoff function.<sup>19</sup> Here, because of the vanishing of our Carleman weight  $e^{\lambda F}$  on  $\mathcal{C}_{t_0, x_0}$ , we can naturally restrict our estimate to  $\mathcal{D}_{t_0, x_0}$ , without employing any cutoff function.

By carrying out the usual computations behind Carleman estimates (from the geometric point of view) using the multiplier  $\mathcal{S} := S_*$  and the weight (3.4), we can obtain the following inequality:

$$(3.5) \quad \begin{aligned} & 0 \cdot \|e^{\lambda F} \nabla_{t,x} \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0, x_0})}^2 + \|e^{\lambda F} \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0, x_0})}^2 \\ & \lesssim \lambda^{-2} \|e^{\lambda F} \square \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0, x_0})}^2 + \|e^{\lambda F} \mathcal{N} \psi\|_{L^2(\Gamma_*)}, \end{aligned}$$

where  $\Gamma_*$  is given by

$$(3.6) \quad \Gamma_* := \mathcal{U}_b \cap \mathcal{D}_{t_0, x_0} \cap \{\mathcal{N} f_0 > 0\}.$$

Let us ignore, for the moment, the factor 0 in blue in (3.5) (which, admittedly, is quite problematic and ultimately causes our weight (3.4) to fail). Observe that *the boundary region*  $\Gamma_*$  *that we achieve in (3.5) is precisely*  $\Gamma_0$  (defined in (2.9)) from our previous multiplier result, *but further restricted to*  $\mathcal{D}_{t_0, x_0}$ .<sup>20</sup> In particular, the restriction to  $\mathcal{D}_{t_0, x_0}$  makes  $\Gamma_*$  inherently time-dependent.

In fact, this additional restriction of the boundary term to  $\mathcal{D}_{t_0, x_0}$  is precisely the source of our improvement of the observation region described in (IV) at the beginning of the article.<sup>21</sup>

**3.2. A relativistic interpretation.** Before discussing how  $F$  is to be modified so that the resulting Carleman estimate suffices to imply Theorem 1.6, let us first gain a better understanding of the region  $\Gamma_*$  from (3.6) by giving an interpretation of it through special relativity.

Consider a point  $(t_b, x_b) \in \Gamma_*$ . Since  $(t_b, x_b)$  lies in the exterior of the light cone  $\mathcal{C}_{t_0, x_0}$ , so that  $(t_b, x_b)$  and  $(t_0, x_0)$  are not causally related, one can apply an appropriate Lorentz boost,

<sup>18</sup>Similar Carleman estimates with vanishing weights have been used in different contexts; see, for instance, [2] for global unique continuation results and [3] for singularity formation results.

<sup>19</sup>In the context of proving observability results, one uses an cutoff function depending only on  $t$ .

<sup>20</sup>The further restriction on  $t$  in (2.9) is already implied here by the restriction to  $\mathcal{D}_{t_0, x_0}$  in (3.6).

<sup>21</sup>Recall that classical Carleman methods would also yield observation regions analogous to  $\Gamma_0$  from (2.9).

centered about  $(t_0, x_0)$ , in order to obtain an inertial coordinate system  $(t', x')$  on  $\mathbb{R}^{1+n}$  such that<sup>22</sup>

$$t'(t_b, x_b) = t'(t_0, x_0) = t(t_0, x_0) = t_0.$$

In other words,  $(t_b, x_b)$  and  $(t_0, x_0)$  are simultaneous in the  $(t', x')$ -coordinates.

Recall that  $f_0$  is invariant under Lorentz boosts. Thus, the defining condition  $\mathcal{N}f_0 > 0$  for  $\Gamma_*$  at  $(t_b, x_b)$ , in terms of the new boosted coordinates, is given by

$$(3.7) \quad 0 < \mathcal{N}f_*|_{(t_b, x_b)} = \frac{1}{4}\mathcal{N}[|x' - x_0|^2]|_{(t_b, x_b)} = \frac{1}{2}[(x' - x_0) \cdot \nu']|_{(t_b, x_b)},$$

where  $\nu'$  denotes the spatial component of  $\mathcal{N}$  in  $(t', x')$ -coordinates. In other words, (3.7) is simply the standard condition  $(x - x_0) \cdot \nu > 0$  from the classical estimates, but now with respect to a boosted inertial coordinate system in which  $(t_0, x_0)$  and  $(t_b, x_b)$  are simultaneous.

Finally, Figure 3.1 below gives some graphical examples of regions  $\Gamma_*$ .

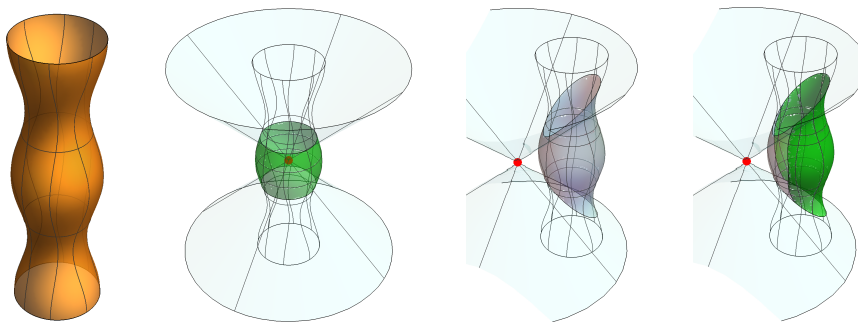


FIGURE 3.1. The first image shows the boundary  $\mathcal{U}_b$  (in orange) of a non-static domain  $\mathcal{U}$ . The second image shows one case in which the point  $(t_0, x_0)$  (in red) lies within  $\mathcal{U}$ ; here,  $\Gamma_*$  (drawn in green) is the full intersection of  $\mathcal{U}_b$  with the null cone exterior  $\mathcal{D}_{t_0, x_0}$ . The last two images demonstrate another case with  $(t_0, x_0)$  outside of  $\mathcal{U}$ : in the third image, the shaded piece (in light purple) is the full intersection of  $\mathcal{U}_b$  with the  $\mathcal{D}_{t_0, x_0}$ , while in the fourth image, the highlighted piece (in green) is the strictly smaller region  $\Gamma_*$ . All of the images were generated using *Mathematica*.

**3.3. The conformal transformation.** The final task of this section is to return to the issue that the Carleman weight  $e^{\lambda F}$  fails to achieve the desired observability estimate in Theorem 1.6, and to discuss how this shortcoming can be remedied.

First, let us briefly recall why  $F$  fails. Recall that the main requirement for a weight to produce a viable Carleman estimate is that its level sets are *pseudoconvex* (with respect to  $\square$ ).<sup>23</sup> Unfortunately, one can see that the level sets of  $f_0$  (and hence the level sets of  $F$ ) barely fail to be pseudoconvex.<sup>24</sup> The upshot of this is that the resulting Carleman estimate can no longer control the derivative of the solution  $\psi$ , so we only obtain an estimate of the form

$$(3.8) \quad \|e^{\lambda F} \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0, x_0})}^2 \lesssim \lambda^{-2} \|e^{\lambda F} \square \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0, x_0})}^2 + \|e^{\lambda F} \mathcal{N} \psi\|_{L^2(\Gamma_*)}.$$

<sup>22</sup>By inertial, we mean that the Minkowski metric in  $(t', x')$ -coordinates is given by  $g = -(dt')^2 + (dx')^2 + \dots (dx'^n)^2$ .

<sup>23</sup>In the context of wave equations, this pseudoconvexity can be roughly described as follows: any null geodesic that hits such a level set tangentially lies entirely on one side of this set nearby; see [13, 18].

<sup>24</sup>These level sets are *zero pseudoconvex*, in that there are null geodesics lying exactly on these hyperboloids.

(This was indicated in (3.4) by the blue factor of zero.) Since (3.8) does not control the full  $H^1$ -norm, we cannot hope to obtain the  $H^1$ -control needed for observability.

To overcome this, the general idea is to perturb  $f_0$  into a slightly different function  $f_\varepsilon$  whose level sets are pseudoconvex. Previous Carleman-based observability results [7, 16, 33, 34] used

$$f_{\varepsilon,c} := \frac{1}{4} [|x - x_0|^2 - (1 - \varepsilon)(t - t_0)^2],$$

whose level sets are hyperboloids associated to waves with a slightly slower speed. However, a significant drawback to this approach is that this  $f_{\varepsilon,c}$  is not well-adapted to the characteristics of the wave equation. In particular, since  $f_{\varepsilon,c} = 0$  is no longer the null cone  $\mathcal{C}_{t_0,x_0}$ , one cannot use weights based on  $f_{\varepsilon,n}$  to obtain Carleman estimates that are naturally supported on  $\mathcal{D}_{t_0,x_0}$ . In other words, we would no longer be able to achieve objective (IV).

Therefore, to preserve our improved observation regions, we must adopt another strategy for perturbing  $f_0$ . For this, we take a very different conformal geometric approach, in which we perturb the spacetime geometry rather than the weight function.

To be more precise, we define the following “warped” Minkowski metric in polar coordinates,

$$(3.9) \quad g_\varepsilon := -dt^2 + dr^2 + (r^2 + 2\varepsilon f_0)\hat{\gamma},$$

where  $\hat{\gamma}$  denotes the round metric for the unit sphere  $\mathbb{S}^{n-1}$ . Although  $f_0$  fails to be pseudoconvex with respect to  $\square$ , one can observe, on the other hand, that  $f_0$  is pseudoconvex with respect to the wave (i.e., Laplace–Beltrami) operator associated with  $g_\varepsilon$ .

The second key observation is that  $(\mathcal{D}_{t_0,x_0}, g)$  is conformally related to  $(\mathcal{D}_{0,0}, g_\varepsilon)$ —via an appropriate change of coordinates, the pullback of  $g_\varepsilon$  is a conformal factor times  $g$ . Now, since pseudoconvexity is a conformally invariant property, one can then obtain a perturbed pseudoconvex weight  $f_\varepsilon$  by pulling back  $f_0$  through the above conformal isometry.<sup>25</sup>

**Remark 3.2.**  $f_\varepsilon$  can be written explicitly as

$$f_\varepsilon := \frac{-uv}{(1 + \varepsilon u)(1 - \varepsilon v)}, \quad v := \frac{1}{2}(t + r), \quad u := \frac{1}{2}(t - r).$$

Thus, we can use  $f_\varepsilon$  in the place of  $f_0$  to obtain a Carleman estimate that controls the full  $H^1$ -norm. Furthermore, as the region  $\{f_\varepsilon > 0\}$  still corresponds to the null cone exterior  $\mathcal{D}_{t_0,x_0}$ , one can ensure that this new Carleman estimate is supported entirely within  $\mathcal{D}_{t_0,x_0}$ .<sup>26</sup> In particular, objective (IV) is still achieved, and we end up with a Carleman estimate of the form

$$(3.10) \quad \begin{aligned} & \|e^{\lambda F_\varepsilon} \nabla_{t,x} \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0,x_0})}^2 + \|e^{\lambda F_\varepsilon} \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0,x_0})}^2 \\ & \lesssim \lambda^{-2} \|e^{\lambda F_\varepsilon} \square \psi\|_{L^2(\mathcal{U} \cap \mathcal{D}_{t_0,x_0})}^2 + \|e^{\lambda F_\varepsilon} \mathcal{N} \psi\|_{L^2(\mathcal{Y}_*)}^2, \end{aligned}$$

where:<sup>27</sup>

- $F_\varepsilon$  is the function obtained by replacing all instances of  $f_0$  in (3.4) by  $f_\varepsilon$ .
- $\mathcal{Y}_*$  can be any open subset of  $\mathcal{U}_b$  that contains the closure of  $\Gamma_*$ .

A precise statement of the Carleman estimate is given in [27, Theorem 3.1].

Note that the boundary region  $\mathcal{Y}_*$  in (3.10) now must be larger than the region  $\Gamma_*$  from our failed estimate. This comes from the fact that the previous condition  $\mathcal{N}f_0 > 0$  from (3.6) is now replaced by  $\mathcal{N}f_\varepsilon > 0$ . On the other hand, since  $f_\varepsilon$  can be made an arbitrarily small perturbation of  $f_0$  by taking  $\varepsilon$  to be small enough,  $\mathcal{Y}_*$  needs differ from  $\Gamma_*$  only by an arbitrarily small amount.

<sup>25</sup>The ideas for warpings and their conformal relations, as well as their applications to Carleman estimates, originated in [1, 2], though these papers treated rather different problems and settings.

<sup>26</sup>Observe that null cones are invariant under Lorentzian conformal transformations.

<sup>27</sup>Here, the associated multiplier  $\mathcal{S}$  is given by the gradient of  $f_\varepsilon$ , rather than  $f_0$  (as was in the case of  $S_*$  from (2.5)).

Finally, we note that we do not actually prove the Carleman estimate (3.10) directly using  $f_\varepsilon$ . Instead, we first prove a preliminary Carleman estimate using  $f_0$ , but within the “warped” spacetime (3.9). We then pull this Carleman estimate back through the above-mentioned conformal relation in order to obtain (3.10). This has the advantage of simplifying many computations, since  $f_0$  is much easier to deal with in the warped spacetime than  $f_\varepsilon$  is in Minkowski spacetime.

#### 4. OBSERVABILITY

Having described our Carleman estimate for time-dependent domains, we can now state our main observability result, that is, the precise version of Theorem 1.6.

**Theorem 4.1** ([27, Theorem 4.1, Theorem 4.5]). *Consider a general linear wave equation (1.1) on a domain  $\mathcal{U}$  satisfying Assumption 1.3. Also, fix  $T_- < T_+$ ,  $x_0 \in \mathbb{R}^n$ , and  $t_0 \in \mathbb{R}$ , and assume:*

- *The condition (1.10) holds for  $T_+ - T_-$ .*
- *The condition (2.7) holds for  $t_0$ .*

*Then, for any finite-energy solution  $\psi$  to (1.1) on  $\mathcal{U}$ , with zero Dirichlet boundary data, we have*

$$(4.1) \quad \int_{\mathcal{U} \cap \{t=T_\pm\}} [(\partial_t \psi)^2 + |\nabla_x \psi|^2 + \psi^2] \lesssim \int_{\mathcal{Y}_*} |\mathcal{N}\psi|^2,$$

where:

- $\mathcal{N}$  denotes the Minkowski outer-pointing unit normal to  $\mathcal{U}$ .
- $\mathcal{Y}_*$  is an open subset of  $\mathcal{U}_b$  that contains the closure of the region  $\Gamma_*$  from (3.6).

Again, note that Theorem 4.1 extends the multiplier-based result of Theorem 2.2 to general wave equations (1.1). In particular, when  $\mathcal{U}$  is static, Theorem 4.1 achieves a smaller observation region than previous results based on multiplier and Carleman estimates (that is, (1.5)).

While the proof of Theorem 4.1 follows the same template as described in the beginning of Section 3, there are a few differences in the argument due to the nature of our Carleman estimate. Furthermore, in contrast to previous Carleman-based observability results, we can now handle the case in which our central event  $(t_0, x_0)$  lies within the domain  $\mathcal{U}$ .

Below, we make a few brief comments regarding the above points of the proof.

**4.1. Exterior observability.** Let us first consider the case  $(t_0, x_0)$  does not lie on  $\mathcal{U}$  nor  $\mathcal{U}_b$ . Starting from the Carleman estimate (3.10), the main idea is to observe that the region  $\mathcal{U} \cap \mathcal{D}_{t_0, x_0}$  contains the entire cross-section  $\mathcal{U} \cap \{t = t_0\}$ , on which the weight  $e^{\lambda F_\varepsilon}$  is uniformly bounded from below. Therefore, using local energy estimates, we can bound the left-hand side of (3.10) from below by the left-hand side of (4.1), completing the proof of Theorem 4.1 in this case.

For details, the reader is referred to [27, Section 4.1].

**Remark 4.2.** In more standard proofs of observability, one must apply a Carleman estimate to  $\xi\psi$ , where  $\xi$  is an appropriate cutoff function depending on  $t$ . However, since our estimate (3.10) is naturally restricted to  $\mathcal{D}_{t_0, x_0}$ , we do not need such a cutoff function here.

**Remark 4.3.** In addition, the borderline case  $(t_0, x_0) \in \mathcal{U}_b$  can be treated by shifting  $(t_0, x_0)$  slightly away from  $\mathcal{U}$  and then applying the preceding argument.

**4.2. Interior observability.** For the case  $(t_0, x_0) \in \mathcal{U}$  (which has not been treated in previous Carleman-based arguments), observe it is no longer the case that  $\mathcal{U} \cap \mathcal{D}_{t_0, x_0}$  contains a cross-section of  $\mathcal{U}$ . As a result, we cannot recover the energy of  $\psi$  directly from (3.10).

To deal with this issue, we choose two distinct points  $x_1, x_2$  very close to  $x_0$ , we apply the (3.10) twice—to  $(t_0, x_1)$  and  $(t_0, x_2)$ , and we sum the results of both Carleman estimates. Then, the combination of both Carleman estimates contains the full cross-section  $\mathcal{U} \cap \{t = t_0\}$ , and the

sum of the two Carleman weights is now uniformly bounded from below on this cross-section. Thus, the observability estimate (4.1) again follows by local energy estimates.<sup>28</sup>

See [27, Section 4.2] for details of this argument.

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<sup>28</sup>The idea for summing two Carleman estimates with nearby centers originated in [3].

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