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CONTROLLABILITY OF A PARABOLIC SYSTEM WITH A DIFFUSIVE INTERFACE

by

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Abstract. — We consider a linear parabolic transmission problem across an interface of codimension one in a bounded domain or on a Riemannian manifold, where the transmission conditions involve an additional parabolic operator on the interface. This system is an idealization of a three-layer model in which the central layer has a small thickness δ. We prove a Carleman estimate in the neighborhood of the interface for an associated elliptic operator by means of partial estimates in several microlocal regions. In turn, from the Carleman estimate, we obtain a spectral inequality that yields the null-controllability of the parabolic system. These results are uniform with respect to the small parameter δ.

1. Introduction

When considering elliptic and parabolic operators in $\mathbb{R}^n$ with a diffusion coefficient that jumps across an interface of codimension one, say $\{x_n = 0\}$, we can interpret the associated equations as two equations with solutions that are coupled at the interface via transmission conditions at $x_n = 0$, viz. in the parabolic case,

$$(1.1) \qquad \partial_t y_1 - \nabla_x c_1 \nabla_x y_1 = f_1 \ \text{ in } \{x_n < 0\}, \quad \partial_t y_2 - \nabla_x c_2 \nabla_x y_2 = f_2 \ \text{ in } \{x_n > 0\},$$

and

$$(1.2) \qquad y_{1|x_n=0^-} = y_{2|x_n=0^+}, \qquad c_1 \partial_{x_n} y_{1|x_n=0^-} = c_2 \partial_{x_n} y_{2|x_n=0^+}.$$

Here, we are interested in parabolic/elliptic models in which part of the diffusion occurs along the interface. Then the transmission conditions are of higher order, involving differentiations in the direction of the interface. Such a model can be viewed as an idealization of two diffusive media separated by a thin membrane. This model can be derived starting from three media and formally letting the thickness of the intermediate layer become very small. A small parameter $\delta > 0$ then measures the thickness of this layer. Questions such as unique continuation, observation and controllability are natural for such a model. This is the main goal of the present article.

Most of the analysis that we shall carry concerns a related elliptic operator, including an additional variable. Our key result is the derivation of a Carleman estimate for this operator (see Theorem 1.2 below). The general form of Carleman estimates for a second-order elliptic operator P is (local form)

$$(1.3) \qquad h\|e^{\varphi/h}w\|_{L^2}^2 + h^3\|e^{\varphi/h}\nabla w\|_{L^2}^2 \le Ch^4\|e^{\varphi/h}Pw\|_{L^2}^2,$$

for h sufficiently small, an appropriately chosen weight function φ, and for smooth compactly supported functions w. We then deduce an interpolation inequality and a spectral

inequality for the original operator in the spirit of the work [**19**]. This spectral inequality then yields the null controllability of the considered parabolic system. A important feature of the results we obtain here is their uniformity in the thickness parameter δ. In particular this allows us to recover the earlier results obtained on (1.1)–(1.2) in [**15**]; this corresponds to the limit $\delta \to 0$ in the model we consider here.

1.1. Setting. — Let (Ω, g) be a smooth compact n-dimensional ($n \geq 2$) connected Riemannian manifold (with or without boundary), with g denoting the metric, and S a $n-1$-dimensional smooth submanifold of Ω (without boundary). We assume that $\Omega \backslash S = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$, so that Ω_1 and Ω_2 are two smooth open subsets of Ω. Endowed with the metric $g_{|T(S)}$, S has a Riemannian structure. We denote by ∂_η a non vanishing vector field defined in a neighborhood of S and normal to S (for the Riemannian metric). We choose the vector field ∂_η outgoing from Ω_1, incoming in Ω_2. In local coordinates, we have

$$\partial_\eta = \sum_j \eta^j \partial_{x_j}, \quad \text{with} \ \ \eta^j = \lambda \sum_k n_k g^{jk}, \quad |\eta|_g = 1,$$

where $g^{ij} g_{jk} = \delta^i_k$, $\lambda^2 = (g^{ij} n_i n_j)^{-1}$, and n is the normal to S for the Euclidean metric in the local coordinates, outgoing from Ω_1, incoming in Ω_2. In fact $\lambda^2_{|S} = \det(g)/\det(g_{|T(S)})$ at S.

The covariant gradient and the divergence operators are given in local coordinates by

$$\nabla_g = \sum_i g^{ij} \partial_{x_i}, \qquad \text{div}_g \, v = \frac{1}{\sqrt{\det(g)}} \sum_i \partial_{x_i}(\sqrt{\det(g)} v_i),$$

with similar definition for the gradient $\nabla^s = \nabla_{g_{|T(S)}}$ and divergence $\text{div}^s = \text{div}_{g_{|T(S)}}$ on the interface S with the metric $g_{|T(S)}$.

We consider a (scalar) diffusion coefficient $c(x)$ with $c_{|\Omega_i} \in \mathscr{C}^\infty(\overline{\Omega_i})$, $i = 1, 2$, yet discontinuous across S and satisfying $c(x) \geq c_{\min} > 0$ uniformly for $x \in \Omega_1 \cup \Omega_2$. We set

$$\Delta_c = \text{div}_g \, c(x) \nabla_g = \frac{1}{\sqrt{\det(g)}} \sum_{i,j} \partial_{x_i}(c g^{ij} \sqrt{\det(g)} \partial_{x_j}), \quad \text{in} \ \Omega_1 \cup \Omega_2,$$

in local coordinates. Let us denote c^s a smooth (scalar) diffusion coefficient on S satisfying $c^s(x) \geq c^s_{\min} > 0$. Similarly we define $\Delta_{c^s} = \text{div}^s \, c^s \nabla^s$ as a second-order elliptic differential operator on S.

In what follows, we shall use the notation $z_{|S_j} = (z_{|\Omega_j})_{|S}$, $j = 1, 2$, for the traces of functions on S.

Given a time $T > 0$, we consider the following parabolic control problem

$$(1.4) \quad \begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z^s - \Delta_{c^s} z^s = \frac{1}{\delta}\big((c\partial_\eta z)_{|S_2} - (c\partial_\eta z)_{|S_1}\big) & \text{in } (0, T) \times S, \\ z_{|S_1} = z^s = z_{|S_2} & \text{in } (0, T) \times S, \\ z_{|\partial\Omega} = 0; \end{cases}$$

with some initial data in $L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$. Here, δ denotes a *bounded* parameter, $0 < \delta \leq \delta_0$, and ω is an open nonempty subset of $\Omega_1 \cup \Omega_2$. Let us suppose for instance that $\omega \subset \Omega_2$. The function u is a control function and the null-controllability problem concerns the ability to drive the solution (z, z^s) to zero at the final time T.

Such a coupling condition at the interface was considered in [**11**] and [**22**] for the associated hyperbolic system. This model corresponds to two diffusive media separated by

a thin layer in which diffusion also occurs. The parameter δ is then a measure of the thickness of this intermediate layer. In the derivation of the model δ is assumed small.

We introduce the Hilbert space $\mathcal{H}_\delta^0 = L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$ with the inner product

$$\big(Z, \tilde{Z}\big)_{\mathcal{H}_\delta^0} = (z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} + \delta\,(z^s, \tilde{z}^s)_{L^2(S)}, \qquad Z = (z, z^s),\ \tilde{Z} = (\tilde{z}, \tilde{z}^s),$$

where

$$(1.5) \qquad (z, \tilde{z})_{L^2(\Omega_1 \cup \Omega_2)} = \underset{\Omega_1 \cup \Omega_2}{\int} z\overline{\tilde{z}}\ d\nu, \qquad (z^s, \tilde{z}^s)_{L^2(S)} = \underset{S}{\int} z^s\overline{\tilde{z}^s}\ d\nu^s,$$

with $d\nu = \sqrt{\det(g)}\ dx$ and $d\nu^s = \sqrt{\det(g_{|T(S)})}\ dy$. We also introduce the following Hilbert space

$$(1.6) \qquad \mathcal{H}_\delta^1 = \{Z = (z, z^s) \in H^1(\Omega_1 \cup \Omega_2) \times H^1(S)\ ;\ z_{|\partial\Omega} = 0\ ;\ z_{|S_1} = z^s = z_{|S_2}\},$$

with the inner product, with $Z = (z, z^s)$, $\tilde{Z} = (\tilde{z}, \tilde{z}^s)$,

$$\big(Z, \tilde{Z}\big)_{\mathcal{H}_\delta^1} = \big(Z, \tilde{Z}\big)_{\mathcal{H}_\delta^0} + \big(c\nabla_g z, \nabla_g \tilde{z}\big)_{L^2(\Omega_1 \cup \Omega_2)} + \delta\big(c^s\nabla^s z, \nabla^s \tilde{z}^s\big)_{L^2(S)}.$$

Problem (1.4) can be written as

$$(1.7) \qquad\qquad\qquad\qquad \partial_t Z + A_\delta Z = Bu,$$

where the state is $Z = (z, z^s) \in \mathcal{H}_\delta^0$ and the operator A_δ reads

$$(1.8) \qquad\qquad A_\delta Z = \begin{pmatrix} -\Delta_c z \\ -\Delta_{c^s} z^s - \frac{1}{\delta}\big((c\partial_\eta z)_{|S_2} - (c\partial_\eta z)_{|S_1}\big) \end{pmatrix},$$

with domain

$$(1.9) \qquad\qquad D(A_\delta) = \{(z, z^s) \in \mathcal{H}_\delta^1;\ A_\delta(z, z^s) \in \mathcal{H}_\delta^0\}.$$

The operator $(A_\delta, D(A_\delta))$ is nonnegative self-adjoint on $\mathcal{H}_\delta^0$. The control operator B is the bounded operator from $L^2(\Omega_1 \cup \Omega_2)$ into $L^2(\Omega_1 \cup \Omega_2) \times L^2(S)$ given by $B : u \mapsto {}^t(\mathbb{1}_\omega u, 0)$. Note that System (1.7), i.e. System (1.4), is well-posed for an initial condition in $\mathcal{H}_\delta^0$.

Remark 1.1. — In the limit $\delta \to 0$, from System (1.4), we obtain the following system (see Section 2 for a proof of convergence)

$$(1.10) \qquad \begin{cases} \partial_t z - \Delta_c z = \mathbb{1}_\omega u & \text{in } (0, T) \times \Omega_1 \cup \Omega_2, \\ (c\partial_\eta z)_{|S_2} = (c\partial_\eta z)_{|S_1} \text{ and } z_{|S_1} = z_{|S_2} & \text{in } (0, T) \times S, \\ z_{|\partial\Omega} = 0; \end{cases}$$

which corresponds to the case studied in [**15**]. We also refer to the recent works [**6, 2, 12, 3, 5, 16, 14, 4**] for the derivation of Carleman estimates for elliptic and parabolic operators with such coefficients with applications to controllability and inverse problems.

1.2. Statement of the main results. —

1.2.1. *Carleman estimate.* — The Carleman estimate we prove concerns an augmented elliptic operator: we introduce an additional coordinate, $x_0 \in (0, X_0) \subset \mathbb{R}$, so that $(x_0, x) \in (0, X_0) \times \Omega$. This variable x_0 was introduced in [**19**]; there it allowed to obtain the null-controllability of the heat equation. This approach was followed in several works [**21, 10, 15**]. It was also used to prove stabilization properties of the wave equation [**18, 20**].

We consider the $n+1$-dimensional partially determined elliptic problem

(1.11)
$$\begin{cases} -\partial_{x_0}^2 w - \Delta_c w + \nabla_a w + bw = f & \text{in } (0, X_0) \times (\Omega_1 \cup \Omega_2), \\ -\partial_{x_0}^2 w^s - \Delta_{c^s} w^s + \nabla_a^s w^s + b^s w^s \\ \quad = \frac{1}{\delta}\big((c\partial_\eta w)_{|(0,X_0)\times S_2} - (c\partial_\eta w)_{|(0,X_0)\times S_1} + \theta^s\big) & \text{in } (0, X_0) \times S, \\ w_{|(0,X_0)\times S_1} = w^s + \theta^1 \quad \text{and} \quad w_{|(0,X_0)\times S_2} = w^s + \theta^2 & \text{in } (0, X_0) \times S. \end{cases}$$

Note that we add lower-order terms to the elliptic operators here: ∇_a (resp. ∇_a^s) denotes any *smooth* vector field on $\Omega_1 \cup \Omega_2$ (resp. S) and b (resp. b^s) are some *bounded* functions on $\Omega_1 \cup \Omega_2$ (resp. S). Moreover, we include source terms θ^j, $j = 1, 2$, θ^s at the interface through the transmission conditions. This system is not fully determined as we do not prescribe any boundary condition on $\{0\} \times \Omega$ and $\{X_0\} \times \Omega$.

In Section 3, we introduce a small neighborhood V_ε of S in Ω, where we can use coordinates of the form (y, x_n) with $y \in S$ and $x_n \in [-2\varepsilon, 2\varepsilon]$. We then set $\mathcal{M} = (0, X_0) \times V_\varepsilon$ and $\mathcal{M}_j = \mathcal{M} \cap \big((0, X_0) \times \Omega_j\big)$, $j = 1, 2$.

For a properly chosen weight function φ (see Section 3.1), for some $0 < \alpha_0 < X_0/2$, and a cut-off function $\zeta = \zeta(x_n) \in \mathscr{C}_c^\infty([0, 2\varepsilon))$, with $\zeta = 1$ on $[0, \varepsilon)$, one can prove the following theorem.

Theorem 1.2. — *For all $\delta_0 > 0$, there exist $C > 0$, and $h_0 > 0$ such that*

$$(1.12) \quad h\|e^{\varphi/h}w\|_0^2 + h^3\|e^{\varphi/h}\nabla_{x_0,x}w\|_0^2 + h\sum_{j=1,2}|e^{\varphi/h}w_{|S_j}|_0^2 + h^3\sum_{j=1,2}|e^{\varphi/h}\nabla_{x_0,x}w_{|S_j}|_0^2$$

$$\leq C\Big(h^4\|e^{\varphi/h}f_{|\mathcal{M}_1}\|_0^2 + h^4\|e^{\varphi/h}f_{|\mathcal{M}_2}\|_0^2 + h^2\delta^2\|\zeta e^{\varphi/h}f_{|\mathcal{M}_2}\|_0^2$$

$$+ h|e^{\varphi/h}\theta^1|_0^2 + \big(h + \frac{\delta^2}{h}\big)|e^{\varphi/h}\theta^2|_0^2 + h^3|e^{\varphi/h}\nabla_{x_0,S}\theta^1|_0^2 + h^3|e^{\varphi/h}\nabla_{x_0,S}\theta^2|_0^2 + h^3|e^{\varphi/h}\theta^s|_0^2\Big),$$

for all $0 < \delta < \delta_0$, $0 < h \leq h_0$, for $(w, \theta^1, \theta^2, \theta^s, f)$ satisfying (1.11), $w_{|\mathcal{M}_j} \in \mathscr{C}^\infty(\overline{\mathcal{M}_j})$, and $w^s \in \mathscr{C}^\infty\big((0, X_0) \times S\big)$ with

$$\operatorname{supp}(w) \subset (\alpha_0, X_0 - \alpha_0) \times S \times (-2\varepsilon, 2\varepsilon), \qquad \operatorname{supp}(w^s) \subset (\alpha_0, X_0 - \alpha_0) \times S.$$

Here $\nabla_{x_0,x} = (\partial_{x_0}, \nabla_g)^t$, $\nabla_{x_0,S} = (\partial_{x_0}, \nabla^s)^t$ and $\|.\|_0$, $|.|_0$ are L^2-norms on $\mathcal{M}$ and $(0, X_0) \times S$ respectively. The weight function φ will be chosen increasing when crossing S from $\mathcal{M}_1$ to $\mathcal{M}_2$, which corresponds to an observation on the side $(0, X_0) \times \Omega_2$. Observe the non symmetric form of the r.h.s. of the estimate above. This originates from our choice of observing the solution w in $(0, X_0) \times \Omega_2$.

This type of Carleman estimate is well known away from the interface S (see [7], and [19] for an estimate at the Dirichlet boundary $\partial\Omega$).

Remark 1.3. — The additional variable x_0 is used here to obtain the spectral inequality of Theorem 1.5 below. The same Carleman inequality holds for the operator A_δ.

Following [15] we shall introduce microlocal regions that are defined on the whole (cotangent bundle of) S. For each region we shall obtain a partial Carleman estimate. The different estimates can then be patched together to yield (1.12).

1.2.2. *Interpolation inequality.* — With the Carleman estimate of Theorem 1.2 we can deduce an interpolation inequality of the form of that introduced in [19]. Let $\alpha_1 \in [0, X_0/2)$, we set $\mathcal{K}_\delta^0(\alpha_1) = L^2\big((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^0\big)$ with also $\mathcal{K}_\delta^0 = \mathcal{K}_\delta^0(0)$, and the following Sobolev

spaces

$$\mathcal{K}_\delta^1(\alpha_1) = L^2\big((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^1\big) \ \cap \ H^1\big((\alpha_1, X_0 - \alpha_1); \mathcal{H}_\delta^0\big), \quad \mathcal{K}_\delta^1 = \mathcal{K}_\delta^1(0),$$

and

$$\mathcal{K}_\delta^2 = L^2\big((0, X_0); D(A_\delta)\big) \ \cap \ H^1\big((0, X_0); \mathcal{H}_\delta^1\big) \ \cap \ H^2\big((0, X_0); \mathcal{H}_\delta^0\big).$$

Theorem 1.4. — *For all $\delta_0 > 0$, there exist $C \geq 0$ and $\nu_0 \in (0, 1)$ such that for all $\delta \in (0, \delta_0)$ we have*

$$(1.13) \qquad \|U\|_{\mathcal{K}_\delta^1(\alpha_1)} \leq C\|U\|_{\mathcal{K}_\delta^1}^{1-\nu_0}\Big(\big\|\big(-\partial_{x_0}^2 + A_\delta\big)U\big\|_{\mathcal{K}_\delta^0} + \|\partial_{x_0}u(0, x)\|_{L^2(\omega)}\Big)^{\nu_0},$$

for all $U = (u, u^s) \in \mathcal{K}_\delta^2$ with $u_{|x_0=0} = 0$ in $\Omega_1 \cup \Omega_2$.

An important consequence of this interpolation inequality is the spectral inequality that we present in the next section.

1.2.3. *Spectral inequality and null-controllability result.* — From the above interpolation inequality we deduce a spectral inequality for the elliptic operator A_δ defined in (1.8). We consider $\mathscr{E}_{\delta,j} = (e_{\delta,j}, e_{\delta,j}^s)$, $j \in \mathbb{N}$, a Hilbert basis of $\mathcal{H}_\delta^0$ composed of eigenfunctions of the operator A_δ associated with the nonnegative eigenvalues $\mu_{\delta,j} \in \mathbb{R}$, $j \in \mathbb{N}$, sorted in an increasing sequence (see Proposition 2.2).

Theorem 1.5. — *For $\delta_0 > 0$, there exists $C > 0$ such that for all $0 < \delta \leq \delta_0$ and $\mu \in \mathbb{R}$, we have*

$$(1.14) \qquad \|Z\|_{\mathcal{H}_\delta^0} \leq Ce^{C\sqrt{\mu}}\|z\|_{L^2(\omega)}, \qquad Z = (z, z^s) \in \text{span}\{\mathscr{E}_{\delta,j}; \mu_{\delta,j} \leq \mu\}.$$

Following [19], this estimation then yields a construction of the control function $u_\delta(t, x)$ in (1.4), by sequentially acting on a finite yet increasing number of eigenspaces, and we hence obtain the following *δ-uniform* controllability theorem. The proof can adapted to those in [19] or [21, Section 5, Proposition 2] and the uniformity w.r.t. the parameter $\delta > 0$ comes naturally. We refer also to [13] for an exposition of the method and to [23, 17, 24, 25] for further developments.

Theorem 1.6. — *Let $\delta_0 > 0$. For an arbitrary time $T > 0$ and an arbitrary nonempty open subset $\omega \subset \Omega$ there exists $C > 0$ such that: for all initial conditions $Z_0 = (z_0, z_0^s) \in \mathcal{H}_\delta^0$ and all $0 < \delta \leq \delta_0$, there exists $u_\delta \in L^2((0, T) \times \omega)$ such that the solution (z, z^s) of (1.4) satisfies $(z(T), z^s(T)) = (0, 0)$ and moreover*

$$\|u_\delta\|_{L^2((0,T)\times\omega)} \leq C\|Z_0\|_{\mathcal{H}_\delta^0}.$$

An important feature of this result is that the control is uniformly bounded as $\delta \to 0$, so that we can extract a subsequence u_δ weakly convergent in $L^2((0, T) \times \omega)$. Below, Corollary 2.4 states that the associated solution of Problem (1.4) converges towards a controlled solution of Problem (1.10). For this last control problem (previously treated in [15]), we hence construct a control function which is robust with respect to small viscous perturbations in the interface.

1.3. Notation: semi-classical operators and geometrical setting. —

1.3.1. *Semi-classical operators on $\mathbb{R}^d$.* — We shall use of the notation $\langle \eta \rangle := (1 + |\eta|^2)^{\frac{1}{2}}$. We denote by $S^m(\mathbb{R}^d \times \mathbb{R}^d)$, S^m for short, the space of smooth functions symbols $a(z, \zeta, h)$ and we define Ψ^m as the space of the associated semi-classical operators $A = \mathrm{Op}(a)$, for $a \in S^m$, formally defined by

$$A\, u(z) = (2\pi h)^{-d} \iint e^{i\langle z-t, \zeta \rangle / h} a(z, \zeta, h)\, u(t)\, dt\, d\zeta, \quad u \in \mathscr{S}'(\mathbb{R}^d).$$

We shall denote the principal symbol a_m by $\sigma(A)$. In the main text the variable z will be $(x_0, x) \in \mathbb{R}^{n+1}$ and $\zeta = (\xi_0, \xi) \in \mathbb{R}^{n+1}$. In particular we set

$$D = \frac{h}{i} \partial, \quad \text{and we have} \quad \sigma(D) = \xi.$$

We introduce Sobolev spaces on $\mathbb{R}^d$ and Sobolev norms which are adapted to the scaling parameter h. The natural norm on $L^2(\mathbb{R}^d)$ is written as $\|u\|_{L^2(\mathbb{R}^d)} = \|u\|_0 := (\int |u(x)|^2 \, dx)^{\frac{1}{2}}$. Let $r \in \mathbb{R}$; we then set

$$\|u\|_r = \|u\|_{\mathscr{H}^r(\mathbb{R}^d)} = \|\Lambda^r u\|_0, \quad \text{with } \Lambda^r := \mathrm{Op}(\langle \xi \rangle^r)$$

and

$$\mathscr{H}^r(\mathbb{R}^d) := \{u \in \mathscr{S}'(\mathbb{R}^d);\ \|u\|_r < \infty\}.$$

The space $\mathscr{H}^r(\mathbb{R}^d)$ is algebraically equal to the classical Sobolev space $H^r(\mathbb{R}^d)$. For a fixed value of h, the norm $\|.\|_r$ is equivalent to the classical Sobolev norm that we write $\|.\|_{H^r(\mathbb{R}^d)}$. However, these norms are not uniformly equivalent as h goes to 0.

1.3.2. *Tangential semi-classical operators on $\mathbb{R}^d$, $d \geq 2$.* — We set $z = (z', z_d)$, $z' = (z_1, \ldots, z_{d-1})$ and $\zeta' = (\zeta_1, \ldots, \zeta_{d-1})$ accordingly. We denote by $S_{\mathcal{T}}^m(\mathbb{R}^d \times \mathbb{R}^{d-1})$, $S_{\mathcal{T}}^m$ for short, the space of smooth functions $b(z, \zeta', h)$, defined for $h \in (0, h_0]$ for some $h_0 > 0$, that satisfy the following property: for all α, β multi-indices, there exists $C_{\alpha,\beta} \geq 0$, such that

$$\left| \partial_z^\alpha \partial_{\zeta'}^\beta b(z, \zeta', h) \right| \leq C_{\alpha,\beta} \langle \zeta' \rangle^{m-|\beta|}, \quad z \in \mathbb{R}^d,\ \zeta' \in \mathbb{R}^{d-1},\ h \in (0, h_0].$$

We define $\Psi_{\mathcal{T}}^m$ as the space of tangential semi-classical operators $B = \mathrm{Op}_{\mathcal{T}}(b)$, for $b \in S_{\mathcal{T}}^m$, formally defined by

$$B\, u(z) = (2\pi h)^{-(d-1)} \iint e^{i\langle z'-t', \zeta' \rangle / h} b(z, \zeta', h)\, u(t', z_d)\, dt'\, d\zeta', \quad u \in \mathscr{S}'(\mathbb{R}^d).$$

In the main text the variable z will be $(x_0, x', x_n) \in \mathbb{R}^{n+1}$ and $\zeta' = (\xi_0, \xi') \in \mathbb{R}^n$. We shall also denote the principal symbol b_m by $\sigma(B)$. We shall denote by $\Lambda_{\mathcal{T}}^s$ the tangential pseudo-differential operator whose symbol is $\langle \zeta' \rangle^s$.

For function defined on $z_d = 0$ or restricted to $z_d = 0$, following [**19**, **20**], we shall denote by $(.,.)_0$ the inner product, i.e., $(f, g)_0 := \iint f(z')\, \overline{g}(z')\, dz'$. The induced norm is denoted by $|.|_0$, i.e., $|f|_0^2 = (f, f)_0$. For $r \in \mathbb{R}$ we introduce

$$(1.15) \qquad |f|_{\mathscr{H}^r(\mathbb{R}^{d-1})} = |f|_r := |\Lambda_{\mathcal{T}}^r f|_0.$$

1.3.3. *Local charts, pullbacks, and Sobolev norms.* — The submanifold S is of dimension $n - 1$ and is furnished with a *finite* atlas (U_j, ϕ_j), $j \in J$. The maps $\phi_j : U_j \to \tilde{U}_j \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism. If $U_j \cap U_k \neq \emptyset$ we also set

$$\phi_{jk}: \quad \phi_j(U_j \cap U_k) \subset \tilde{U}_j \to \phi_k(U_j \cap U_k) \subset \tilde{U}_k,$$
$$y \mapsto \phi_k \circ \phi_j^{-1}(y).$$

We shall use semi-classical Sobolev norms over the manifold S together with a finite atlas $(U_j, \phi_j)_j$, $\phi_j : U_j \to \mathbb{R}^{n-1}$, and a partition of unity $(\psi_j)_j$ subordinated to this covering of S:

$$\psi_j \in \mathscr{C}^{\infty}(S), \quad \operatorname{supp}(\psi_j) \subset U_j, \quad 0 \leq \psi_j \leq 1, \quad \sum_j \psi_j = 1.$$

We then set:

$$(1.16) \qquad |u|_{\mathscr{H}^r(S)} = \sum_j \left| \left(\phi_j^{-1} \right)^* \psi_j u \right|_{\mathscr{H}^r(\mathbb{R}^{n-1})}.$$

1.3.3.1. *Norms in codimension 1.* — For a function u defined on $(0, X_0) \times \mathbb{R}^{n-1}$ we set

$$|u|_0 = |u|_{L^2((0,X_0) \times \mathbb{R}^{n-1})}, \qquad |u|_1^2 = |D_{x_0} u|_0^2 + \int_0^{X_0} |u|_{\mathscr{H}^1(\mathbb{R}^{n-1})}^2 \, dx_0.$$

For a function u defined on $(0, X_0) \times S$, we set

$$(1.17) \qquad |u|_{\ell} = \sum_j \left| \left(\phi_j^{-1} \right)^* \psi_j u \right|_{\ell}, \qquad \ell = 0, 1,$$

where ϕ_j stands for $\operatorname{Id} \otimes \phi_j$.

1.3.3.2. *Norms in all dimensions.* — For a function u defined on $(0, X_0) \times \mathbb{R}^{n-1} \times \mathbb{R}$ we set

$$\|u\|_0 = \|u\|_{L^2((0,X_0) \times \mathbb{R}^{n-1} \times \mathbb{R})}, \qquad \|u\|_1^2 = \|D_{x_0} u\|_0^2 + \int_0^{X_0} \int_{\mathbb{R}} \|u\|_{\mathscr{H}^1(\mathbb{R}^{n-1})}^2 \, dx_0 \, dx_n + \|D_{x_n} u\|_0^2.$$

Note that the latter norm is equivalent to $\|u\|_{\mathscr{H}^1(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})}$ if moreover the function u is compactly supported in the x_0 variable. For a function u defined on $(0, X_0) \times S \times \mathbb{R}$, we set

$$(1.18) \qquad \|u\|_{\ell} = \sum_j \left\| \left(\phi_j^{-1} \right)^* \psi_j u \right\|_{\ell}, \qquad \ell = 0, 1,$$

where ϕ_j stands for $\operatorname{Id} \otimes \phi_j \otimes \operatorname{Id}$.

1.3.3.3. *Tangential semi-classical operators on a manifold.* — We can define tangential semi-classical operators on a manifold by means of local representations. This relies on the change of variables formula for semi-classical operators in $\mathbb{R}^d$. In Section 3.6 below we introduce a particular class of tangential operators that will allow us to separate the analysis into microlocal regions.

2. Well-posedness and asymptotic behavior

We introduce a more general operator

$$\mathcal{A}_{\delta} Z = \begin{pmatrix} -\Delta_c z + \nabla_a z + b z \\ -\Delta_{c^s} z^s + \nabla_a^s z^s + b^s z^s - \frac{1}{\delta} \left((c \partial_{\eta} z)_{|S_2} - (c \partial_{\eta} z)_{|S_1} \right) \end{pmatrix},$$

with domain $D(\mathcal{A}_{\delta}) = D(A_{\delta})$ (see (1.9)), where ∇_a (resp. ∇_a^s) denotes a smooth vector field $a(x) \nabla_g$ (resp. $a^s(x) \nabla^s$), and b (resp. b^s) is a bounded function.

Proposition 2.1. — *Let a, b, a^s, b^s be bounded coefficients. Then, the operator $(-\mathcal{A}_{\delta}, D(\mathcal{A}_{\delta}))$ generates a $\mathscr{C}^0$-semigroup on $\mathcal{H}_{\delta}^0$. If moreover $a = 0$, $a^s = 0$ and $b, b^s \in \mathbb{R}$, then $\mathcal{A}_{\delta}$ is self-adjoint on $\mathcal{H}_{\delta}^0$.*

Proposition 2.2. — *There exists a Hilbert basis of $\mathcal{H}_{\delta}^0$ formed of eigenfunctions $\mathscr{E}_j = (e_{\delta,j}, e_{\delta,j}^s)$, $j \in \mathbb{N}$, of the self-adjoint operator A_{δ} (given in (1.8)) associated with the eigenvalues $0 \leq \mu_{\delta,0} \leq \mu_{\delta,1} \leq \cdots \leq \mu_{\delta,j} \leq \cdots$.*

Note that if Ω is a manifold with no boundary then 0 is an eigenfunction for A_δ. If Ω has a boundary, the Dirichlet boundary condition that we prescribe yield the first eigenvalue to be positive.

Now, we discuss, for some $\lambda > 0$ (one can take $\lambda = 0$ if $\partial\Omega \neq \emptyset$) the convergence properties of the solution $Z_\delta = (z_\delta, z_\delta^s)$ of

$$(2.1) \quad \begin{cases} \partial_t z_\delta - \Delta_c z_\delta + \lambda z_\delta = f_\delta & \text{in } (0,T) \times \Omega_1 \cup \Omega_2, \\ \partial_t z_\delta^s - \Delta_{c^s} z_\delta^s + \lambda z_\delta^s = \frac{1}{\delta}\left((c\partial_\eta z_\delta)_{|S_2} - (c\partial_\eta z_\delta)_{|S_1}\right) + f_\delta^s & \text{in } (0,T) \times S, \\ z_{\delta|S_1} = z_\delta^s = z_{\delta|S_2} & \text{in } (0,T) \times S, \\ z_{\delta|\partial\Omega} = 0 & \text{in } (0,T), \\ z_{\delta|t=0} = z_0 \text{ and } z_{\delta|t=0}^s = z_0^s, \end{cases}$$

towards the solution z of

$$(2.2) \quad \begin{cases} \partial_t \mathsf{z} - \Delta_c \mathsf{z} + \lambda \mathsf{z} = \mathsf{f} & \text{in } (0,T) \times \Omega_1 \cup \Omega_2, \\ \mathsf{z}_{|S_1} = \mathsf{z}_{|S_2} \text{ and } (c\partial_\eta \mathsf{z})_{|S_2} = (c\partial_\eta \mathsf{z})_{|S_1} & \text{in } (0,T) \times S, \\ \mathsf{z}_{|\partial\Omega} = 0 & \text{in } (0,T), \\ \mathsf{z}_{|t=0} = z_0 & \text{in } \Omega. \end{cases}$$

Proposition 2.3. — *Suppose that $\|F_\delta\|_{L^2(0,T;\mathcal{H}_\delta^0)} \leq C$ uniformly in δ, that $f_\delta \rightharpoonup \mathsf{f}$ in $L^2((0,T) \times \Omega_1 \cup \Omega_2)$ as $\delta \to 0$ and that $z_0 \in H_0^1(\Omega)$ and $z_0^s \in H^1(S)$. Then, we have, $z_{\delta|\Omega_j} \rightharpoonup \mathsf{z}_{|\Omega_j}$ in $L^2(0,T;H^2(\Omega_j)) \cap H^1(0,T;L^2(\Omega_j))$ and $*$-weak in $L^\infty(0,T;H^1(\Omega_j))$, and there exists $C' > 0$ such that for all $t \in [0,T]$, $\|z_{\delta|\Omega_j}(t)\|_{H^1(\Omega_j)} \leq C'$ for $j = 1,2$.*

As a consequence, we can obtain a convergence result for the control problem under view. We denote by u_δ the control function given by Theorem 1.6, that satisfies

$$\begin{cases} \partial_t Z_\delta + A_\delta Z_\delta = B u_\delta \\ Z_{\delta|t=0} = Z_0 \\ Z_{\delta|t=T} = 0. \end{cases}$$

According to Theorem 1.6, u_δ is uniformly bounded in $L^2((0,T) \times \omega)$, so that we can extract a subsequence (also denoted by u_δ) weakly converging in this space towards u. We also consider the solution $\tilde{Z}_\delta = (\tilde{z}_\delta, \tilde{z}_\delta^s)$ of

$$(2.3) \quad \begin{cases} \partial_t \tilde{Z}_\delta + A_\delta \tilde{Z}_\delta = B\mathsf{u} \\ \tilde{Z}_{\delta|t=0} = Z_0. \end{cases}$$

The following result is a consequence of Proposition 2.3.

Corollary 2.4. — *As $\delta \to 0$, the limit u is a null-control function for the limit system (1.10). Moreover, $(\tilde{z}_\delta - z_\delta)_{|\Omega_j} \rightharpoonup 0$ in $L^2(0,T;H^2(\Omega_j)) \cap H^1(0,T;L^2(\Omega_j))$ and $*$-weak in $L^\infty(0,T;H^1(\Omega_j))$, and there exists $C > 0$ such that for all $t \in [0,T]$, $\|z_{\delta|\Omega_j}(t) - \tilde{z}_{\delta|\Omega_j}(t)\|_{H^1(\Omega_j)} \leq C$ for $j = 1,2$.*

In particular, we have $\tilde{z}_\delta(T) \rightharpoonup 0$ in $H^1(\Omega)$. This shows that the limit u is a control function for the limit system (1.10) which is robust with respect to small viscous perturbations. Indeed, it realizes an approximate control for System (2.3).

3. Local setting in a neighborhood of the interface

In a sufficiently small neighborhood of S, say V_ε, we place ourselves in normal geodesic coordinates (w.r.t. to the spatial variables x). More precisely (see [**8**, Appendix C.5]) for ε sufficiently small, there exists a diffeomorphism

$$\mathsf{F} : S \times [-2\varepsilon, 2\varepsilon] \to V_\varepsilon$$
$$(y, x_n) \mapsto \mathsf{F}(y, x_n),$$

so that the differential operator $-\partial_{x_0}^2 - \Delta_c + \nabla_a$ takes the form on both sides of the interface:

$$-\partial_{x_0}^2 - c(y, x_n)\left(\partial_{x_n}^2 - R_2(y, x_n)\right) + R_1(y, x_n),$$

and the differential operator $-\partial_{x_0}^2 - \Delta_c^s + \nabla_a^s$ takes the form on the interface

$$-\partial_{x_0}^2 + c^s(y)R_2(y, x_n = 0) + R_1^s(y),$$

where $R_2(y, x_n)$ is a x_n-family of second-order elliptic differential operators on S, i.e., a tangential operator, with principal symbol $r(y, x_n, \eta)$, $\eta \in T_y^*(S)$, that satisfies

(3.1) $$r(y, x_n, \eta) \in \mathbb{R}, \quad \text{and} \quad C_1 |\eta|_g^2 \le r(y, x_n, \eta) \le C_2 |\eta|_g^2,$$

for some $0 < C_1 \le C_2 < \infty$, and $R_1(y, x_n)$ is a first-order operator on $S \times \left([-2\varepsilon, 0) \cup (0, 2\varepsilon]\right)$ (involving partial derivatives in all variables and having a jump across $S \times \{0\}$), $R_1^s(y)$ is a first-order operator on S.

By abuse of notation we shall write V_ε in place of $S \times [-2\varepsilon, 2\varepsilon]$. In this setting, we have

$$V_\varepsilon^- = \mathsf{F}(S \times [-2\varepsilon, 0)) = V_\varepsilon \cap \Omega_1, \quad V_\varepsilon^+ = \mathsf{F}(S \times (0, 2\varepsilon]) = V_\varepsilon \cap \Omega_2,$$

and we recall that the observation region ω is in Ω_2.

In the sequel, we shall often write

$$x := (y, x_n), \quad \text{and} \quad \mathbf{x} := (x_0, x) = (x_0, y, x_n) \in [0, X_0] \times S \times [-2\varepsilon, 2\varepsilon].$$

We set

$$P = -\frac{1}{c}\partial_{x_0}^2 - \left(\partial_{x_n}^2 - R_2(x)\right) + \frac{1}{c}R_1(x), \quad P^s = -\frac{1}{c^s}\partial_{x_0}^2 + R_2(y, x_n = 0) + \frac{1}{c^s}R_1^s(y).$$

In this framework, in the neighborhood V_ε of S, System (1.11) becomes

(3.2)
$$\begin{cases} Pw = F, & \text{in } (0, X_0) \times S \times \left([-2\varepsilon, 0) \cup (0, 2\varepsilon]\right), \\ P^s w^s = \frac{1}{c^s \delta}\left((c\partial_{x_n}w)_{|x_n=0^+} - (c\partial_{x_n}w)_{|x_n=0^-} + \Theta^s\right) & \text{in } (0, X_0) \times S, \\ w_{|x_n=0^-} = w^s + \theta^1 \quad \text{and} \quad w_{|x_n=0^+} = w^s + \theta^2, & \text{in } (0, X_0) \times S, \end{cases}$$

with

(3.3) $$F = \frac{1}{c}f + R_0 w, \quad \Theta^s = \theta^s + \delta R_0^s w^s,$$

where R_0 and R_0^s are zero-order operators with *bounded* coefficients on $S \times \left([-2\varepsilon, 0) \cup (0, 2\varepsilon]\right)$ and S respectively.

3.1. Properties of the weight functions. — We denote by $\tilde{r}(x, \eta, \eta')$ the symmetric bilinear form associated with the quadratic principal symbol $r(x, \eta)$. We introduce the following symmetric bilinear form

$$(3.4) \qquad \tilde{\beta}(x; \xi_0, \eta; \xi_0', \eta') = \frac{1}{c(x)} \xi_0 \xi_0' + \tilde{r}(x, \eta, \eta').$$

and the associated positive definite quadratic form $\beta(x; \xi_0, \eta)$. We choose a positive bounded continuous function $\gamma(x)$ in $\overline{V_\varepsilon^+}$ such that

$$(3.5) \quad \beta(y, -x_n; \xi_0, \eta) - \gamma(y, x_n)\beta(y, x_n; \xi_0, \eta) \geq C|(\xi_0, \eta)|^2 > 0, \quad (\xi_0, \eta) \in \mathbb{R} \times T_y^*(S),$$

for $x = (y, x_n) \in \overline{V_\varepsilon^+}$.

We then choose a function $\varphi = \varphi(\mathbf{x})$ on $[0, X_0] \times V_\varepsilon$ that is smooth on both sides of the interface and simply continuous across the interface, that moreover satisfies the following properties.

1. For a function γ' such that $0 < \gamma'(x) \leq \gamma(x) - \epsilon$ in $\overline{V_\varepsilon^+}$, for some $\epsilon > 0$, we have

$$(3.6) \qquad \gamma'(y, x_n)(\partial_{x_n}\varphi)^2(x_0, y, x_n) - (\partial_{x_n}\varphi)^2(x_0, y, -x_n) \geq C > 0,$$

   for $x_0 \in [0, X_0]$, and $x = (y, x_n) \in V_\varepsilon^+$.
2. For a given value of $\nu > 0$ sufficiently small we have

$$(3.7) \qquad |\partial_{x_0}\varphi(\mathbf{x})| + |\nabla^s\varphi(\mathbf{x})|_g \leq \nu \inf_{\overline{V_\varepsilon}} |\partial_{x_n}\varphi|, \quad \mathbf{x} = (x_0, x) \in [0, X_0] \times V_\varepsilon.$$

3. We have

$$(3.8) \qquad |\partial_{x_0}\varphi| + |\nabla^s\varphi|_g + |\partial_{x_n}\varphi| > 0$$

   in $[0, X_0] \times V_\varepsilon$ and Hörmander's sub-ellipticity condition is satisfied on both sides of the interface. This condition will be precisely stated below after the introduction of the conjugated operator (see (3.13)).

Note that we have $\inf_{V_\varepsilon^+} |\partial_{x_n}\varphi| \geq C > 0$.

The first condition states the increase in the normal slope of the weight function when crossing the interface. We thus ask the weight function to be relatively flat in the tangent directions to the interface as compared to its variations in the normal direction. We explain below how a weight function satisfying the sub-ellipticity condition can be built through a convexification procedure (see Remark 3.3).

Remark 3.1. — Property (3.6) and $|\partial_{x_0}\varphi| + |\nabla^s\varphi|_g + |\partial_{x_n}\varphi| > 0$ can be obtained by choosing φ such that $(\partial_{x_n}\varphi)_{|[0,X_0]\times S} \geq C > 0$ and assuming that (3.6) only holds on $[0, X_0] \times S$ and then shrinking the neighborhood V_ε by choosing ε sufficiently small.

Remark 3.2. — Note that the conditions we impose on the weight function are proven sharp in [**14**] in the limiting case $\delta \to 0$. If (3.6) is not satisfied, i.e., the increase in the normal slope of the weight function is chosen too small, one can then build a quasi-mode that concentrates at the interface and shows that the Carleman estimate cannot hold.

3.2. A system formulation. — Following [**1, 15**], we shall consider (3.2) as a system of two equations coupled at the boundary $x_n = 0^+$. Here, the coupling involves a tangential second-order elliptic operator. In $[0, X_0] \times S \times [-2\varepsilon, 0)$, we make the change of variables x_n to $-x_n$. For a function ψ defined in V_ε, we set

$$\psi^r(y, x_n) = \psi(y, x_n) \quad \text{and} \quad \psi^l(y, x_n) := \psi(y, -x_n), \quad \text{for } x_n \geq 0,$$

and similarly for symbols and operators, e.g.,

$$r^r(y, x_n, \eta) = r(y, x_n, \eta) \quad \text{and} \quad r^l(y, x_n, \eta) = r(y, -x_n, \eta), \quad \text{for } x_n \geq 0.$$

We set $V_\varepsilon^+ = S \times (0, 2\varepsilon]$. System (3.2) then takes the form

$$(3.9) \quad \begin{cases} P^{r/l} w^{r/l} = F^{r/l}, & \text{in } (0, X_0) \times V_\varepsilon^+, \\ P^s w^s = \frac{1}{c^s \delta} \big((c^r \partial_{x_n} w^r)_{|x_n=0^+} + (c^l \partial_{x_n} w^l)_{|x_n=0^+} + \Theta^s \big) & \text{in } (0, X_0) \times S, \\ w^{r/l}_{|x_n=0^+} = w^s + \theta^{r/l} & \text{in } (0, X_0) \times S. \end{cases}$$

3.3. Conjugation by the weight function. — We now consider the weight functions $\varphi^{r/l}$ built up as above from the *continuous* function φ defined on V_ε. We introduce the following conjugated differential operators

$$P_\varphi^{r/l} = h^2 e^{\varphi^{r/l}/h} P^{r/l} e^{-\varphi^{r/l}/h}, \quad P_\varphi^s = h^2 e^{\varphi_{|S}/h} P^s e^{-\varphi_{|S}/h}.$$

With the functions

$$v^{r/l} = e^{\varphi^{r/l}/h} w^{r/l}, \quad v^s = e^{\varphi_{|S}/h} w^s,$$

$$F_\varphi^{r/l} = h^2 e^{\varphi^{r/l}/h} F^{r/l}, \quad \Theta_\varphi^s = -ihe^{\varphi_{|S}/h} \Theta^s, \quad \theta_\varphi^{r/l} = e^{\varphi_{|S}/h} \theta^{r/l},$$

with $0 < h < h_0$, System (3.9) can be rewritten as

$$(3.10) \quad \begin{cases} P_\varphi^{r/l} v^{r/l} = F_\varphi^{r/l} & \text{in } (0, X_0) \times V_\varepsilon^+, \\ P_\varphi^s v^s = \frac{hi}{c^s \delta} \big(c^r (D_{x_n} + i\partial_{x_n} \varphi^r) v^r_{|x_n=0^+} \\ \qquad + c^l (D_{x_n} + i\partial_{x_n} \varphi^l) v^l_{|x_n=0^+} + \Theta_\varphi^s \big) & \text{in } (0, X_0) \times S, \\ v^{r/l}_{|x_n=0^+} = v^s + \theta_\varphi^{r/l} & \text{in } (0, X_0) \times S. \end{cases}$$

Recall that $D = h\partial/i$ here. We shall consider the operators $P_\varphi^{r/l}$ and P_φ^s as semi-classical differential operators.

We separate the self- and anti-adjoint parts of the operators $P_\varphi^{r/l}$, viz.,

$$\tilde{Q}_2^{r/l} = \frac{1}{2} \big(P_\varphi^{r/l} + (P_\varphi^{r/l})^* \big), \quad \tilde{Q}_1^{r/l} = \frac{1}{2i} \big(P_\varphi^{r/l} - (P_\varphi^{r/l})^* \big).$$

The (semi-classical) principal symbols $\tilde{q}_j$ of $\tilde{Q}_j$, $j = 1, 2$ are then

$$\tilde{q}_2^{r/l}(\mathbf{x}, \xi_0, \eta, \xi_n) = \xi_n^2 + q_2^{r/l}(\mathbf{x}, \xi_0, \eta),$$

$$\tilde{q}_1^{r/l}(\mathbf{x}, \xi_0, \eta, \xi_n) = 2\xi_n \partial_{x_n} \varphi^{r/l} + 2q_1^{r/l}(\mathbf{x}, \xi_0, \eta),$$

for $(y, \eta) \in T^*(S)$, with

$$q_2^{r/l}(\mathbf{x}, \xi_0, \eta) = \frac{\xi_0^2}{c^{r/l}} + r^{r/l}(x, \eta) - \Big(\frac{(\partial_{x_0} \varphi^{r/l})^2}{c^{r/l}} + r^{r/l}(x, d_y \varphi^{r/l}) + (\partial_{x_n} \varphi^{r/l})^2 \Big)$$

$$q_1^{r/l}(\mathbf{x}, \xi_0, \eta) = \frac{\xi_0 \partial_{x_0} \varphi^{r/l}}{c^{r/l}} + \tilde{r}^{r/l}(x; \eta, d_y \varphi^{r/l}).$$

Recall that $\tilde{r}^{r/l}(x, \eta, \eta')$ stands for the symmetric bilinear form associated with the quadratic principal symbol $r^{r/l}(x, \eta)$. The principal symbol of $P_\varphi^{r/l}$ is naturally

$$(3.11) \quad p_\varphi^{r/l} = \tilde{q}_2^{r/l} + i\tilde{q}_1^{r/l} = \xi_n^2 + 2i\xi_n \partial_{x_n} \varphi^{r/l} + q_2^{r/l} + 2iq_1^{r/l}.$$

For the sake of concision we have at places omitted some of the variable dependencies, e.g. writing $\varphi^{r/l}$ in place of $\varphi^{r/l}(\mathbf{x})$.

Note also that the symbol of P_φ^s is given by

$$p_\varphi^s = \frac{\xi_0^2}{c^s} + r(x,\eta) - \Big(\frac{(\partial_{x_0}\varphi)^2}{c^s} + r(x, d_y\varphi_{|x_n=0}) \Big)\Big|_{x_n=0}$$

(3.12)
$$+ 2i\Big(\frac{\xi_0 \partial_{x_0}\varphi}{c^s} + \tilde{r}(x;\eta, d_y\varphi_{|x_n=0}) \Big)\Big|_{x_n=0}.$$

(Recall that r^l and r^r (resp. φ^l and φ^r) coincide at $x_n = 0^+$.)

After the introduction of the conjugated operator we can introduce the sub-ellipticity property satisfied by the weight function:

$$(3.13) \quad \forall \mathbf{x} \in [0, X_0] \times \overline{V_\varepsilon^+}, \ (\xi_0, \eta, \xi_n) \in \mathbb{R} \times T_y^*(S) \times \mathbb{R},$$

$$p_\varphi^{r/l}(\mathbf{x}, \xi_0, \eta, \xi_n) = 0 \ \Rightarrow \ \{\tilde{q}_2^{r/l}, \tilde{q}_1^{r/l}\}(\mathbf{x}, \xi_0, \eta, \xi_n) > 0.$$

The sub-ellipticity property (3.13) is necessary for the derivation of the Carleman estimate and is geometrically invariant (see e.g. [**7**, Section 8.1, page 186], see also [**13**]).

Remark 3.3. — A weight function φ that satisfies the properties of Section 3.1 can be obtained in the following classical way. Choose a continuous function ψ, smooth on both sides of S, such that $\psi^{r/l}$ satisfies conditions (3.6)–(3.8). These conditions are then also satisfied by $\varphi = e^{\lambda\psi}$, $\lambda \geq 1$. For the parameter λ sufficiently large φ will also fulfill the sub-ellipticity condition (see e.g. Lemma 3 in [**19**, Section 3.B], Theorem 8.6.3 in [**7**, Chapter 8], or Proposition 28.3.3 in [**9**, Chapter 28]).

3.4. Phase-space regions. — Following [**20, 15**] we introduce the following quantity

$$(3.14) \qquad \mu^{r/l}(\mathbf{x}, \xi_0, \eta) = q_2^{r/l}(\mathbf{x}, \xi_0, \eta) + \frac{\big(q_1^{r/l}(\mathbf{x}, \xi_0, \eta)\big)^2}{\big(\partial_{x_n}\varphi^{r/l}\big)^2},$$

and the following sets in the (tangential) phase space:

(3.15)
$$E^{r/l, \pm} = \big\{ (x_0, y, x_n; \xi_0, \eta) \in [0, X_0] \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S); \mu^{r/l}(x_0, y, x_n; \xi_0, \eta) \gtrless 0 \big\},$$
(3.16)
$$Z^{r/l} = \big\{ (x_0, y, x_n; \xi_0, \eta) \in [0, X_0] \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S); \mu^{r/l}(x_0, y, x_n; \xi_0, \eta) = 0 \big\}.$$

The analysis we carry on will make precise the behavior of the roots of $p_\varphi^{r/l}$ (viewing $p_\varphi^{r/l}$ as a second-order polynomial in the variable ξ_n, see (3.11)) as $(\mathbf{x}, \xi_0, \eta)$ varies.

The assumption we have formulated yields the following *key* property.

Proposition 3.4. — *There exists $C_0 > 0$ such that in the neighborhood V_ε we have*

$$\big(\mu^l - \gamma(x)\mu^r\big)(\mathbf{x}, \xi_0, \eta) \geq C_0 \langle(\xi_0, \eta)\rangle^2 > 0, \quad \mathbf{x} = (x_0, x) = (x_0, y, x_n), \ (\xi_0, \eta) \in \mathbb{R} \times T_y^*(S).$$

In particular, $E^{r,+} \cup Z^r \subset E^{l,+}$.

Proposition 3.5. — *With the properties of the weight function of Section 3.1 we have*

$$\text{Char}(p_\varphi^s) \subset \text{Char}(\text{Re}\, p_\varphi^s) \subset (E^{l,-} \cap \{x_n = 0\}).$$
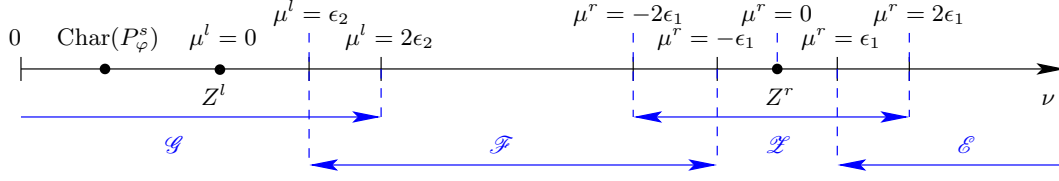
FIGURE 1. Sketch of the relative localization of the different phase-space regions. Here ν represents the norm of the tangential frequencies.

3.5. Root properties. — The following lemma describes the position of the roots of $p_\varphi^{r/l}$ of (3.11) viewed as a second-order polynomial in ξ_n.

Lemma 3.6. — *We have the following root properties.*

1. *In the region $E^{r/l,+}$, the polynomial $p_\varphi^{r/l}$ defined in (3.11) has two distinct roots that satisfy $\operatorname{Im} \rho^{r/l,+} > 0$ and $\operatorname{Im} \rho^{r/l,-} < 0$. Moreover we have*

$$\mu^{r/l} \geq C > 0 \quad \Leftrightarrow \quad \operatorname{Im} \rho^{r/l,+} \geq C' > 0 \text{ and } \operatorname{Im} \rho^{r/l,-} \leq -C' < 0,$$

2. *In the region $E^{r/l,-}$, the imaginary parts of the two roots have the same sign as that of $-\partial_{x_n}\varphi^{r/l}$.*

3. *In the region $Z^{r/l}$, one of the roots is real.*

Moreover, there exist $C > 0$ and $H > 0$ such that $|\rho^{r/l,+} - \rho^{r/l,-}| \geq |\operatorname{Im} \rho^{r/l,+} - \operatorname{Im} \rho^{r/l,-}| \geq C > 0$ in the region $\{\mu^{r/l} \geq -H\}$.

Remark 3.7. — Note that $(\mathbf{x}, \xi_0, \eta) \in E^{r/l,+}$ for $|\xi_0| + |\eta|_g$ sufficiently large, say $|\xi_0| + |\eta|_g \geq R$, uniformly in $\mathbf{x} \in [0, X_0] \times \overline{V_\varepsilon^+}$ and for h bounded. Note also that in the region $\{\mu^{r/l} \geq -H\}$, the roots $\rho^{r/l,\pm}$ are smooth since they do not cross.

3.6. Microlocalisation operators. — We call

$$\mathcal{M}_+ = (0, X_0) \times S \times [0, 2\varepsilon].$$

We also set

$$\mathcal{M}_+^* := \left\{ (x_0, y, x_n, \xi_0, \eta) \in (0, X_0) \times S \times [0, 2\varepsilon] \times \mathbb{R} \times T_y^*(S) \right\} \simeq T^* \left((0, X_0) \times S \right) \times [0, 2\varepsilon].$$

We define the following open sets in (tangential) phase-space:

(3.17)
$$\begin{aligned}
\mathscr{E} &= \left\{ (\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \ \epsilon_1 < \mu^r(\mathbf{x}, \xi_0, \eta) \right\}, \\
\mathscr{L} &= \left\{ (\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \ -2\epsilon_1 < \mu^r(\mathbf{x}, \xi_0, \eta) < 2\epsilon_1 \right\}, \\
\mathscr{F} &= \left\{ (\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \ \epsilon_2 < \mu^l(\mathbf{x}, \xi_0, \eta), \text{ and } \mu^r(\mathbf{x}, \xi_0, \eta) < -\epsilon_1 \right\}, \\
\mathscr{G} &= \left\{ (\mathbf{x}, \xi_0, \eta) \in \mathcal{M}_+^*; \ \mu^l(\mathbf{x}, \xi_0, \eta) < 2\epsilon_2 \right\}.
\end{aligned}$$

The constants ϵ_1 and ϵ_2 are taken such that $\sup(\gamma)\epsilon_1 + \epsilon_2 < C_0/2$, with C_0 as in Proposition 3.4. Our analysis in the region $\mathscr{L}$ will require ϵ_1 to be small (see Section 4.3 below). Recall that γ is defined in Section 3.1. This yields $\overline{\mathscr{G}} \cap \overline{\mathscr{L}} = \emptyset$. As a consequence of Propositions 3.4 and 3.5, the localization of the different microlocal zones can be represented as in Figure 1. In particular, we have $\operatorname{Char}(p_\varphi^s) \subset (\mathscr{G} \setminus \mathscr{F}) \cap \{x_n = 0\}$.

With the open covering of $\mathcal{M}_+^*$ by $\mathscr{E}$, $\mathscr{L}$, $\mathscr{F}$ and $\mathscr{G}$ we introduce a $\mathscr{C}^\infty$ partition of unity,

$$\chi_{\mathscr{E}} + \chi_{\mathscr{L}} + \chi_{\mathscr{F}} + \chi_{\mathscr{G}} = 1, \quad 0 \leq \chi_\bullet \leq 1, \ \operatorname{supp}(\chi_\bullet) \subset \bullet, \ \bullet = \mathscr{E}, \mathscr{L}, \mathscr{F}, \mathscr{G}.$$

The sets $\mathscr{L}$, $\mathscr{F}$ and $\mathscr{G}$ are relatively compact which gives $\chi_{\mathscr{L}}, \chi_{\mathscr{F}}, \chi_{\mathscr{G}} \in S_{\mathcal{T}}^{-\infty}(\mathcal{M}_+^*) = \bigcap_{m>0} S^{-m}(\mathcal{M}_+^*)$ and consequently $\chi_{\mathscr{E}} \in S_{\mathcal{T}}^0(\mathcal{M}_+^*)$. Associated with these symbols we now define tangential pseudo-differential operators on $\mathcal{M}_+$.

Given $0 < \alpha_0 < X_0/2$, we choose a function $\zeta^1 \in \mathscr{C}_c^\infty(0, X_0)$ that satisfies $\zeta^1 = 1$ on a neighborhood of $(\alpha_0, X_0 - \alpha_0)$ and $0 \leq \zeta^1 \leq 1$. Setting

$$(3.18) \qquad \zeta_j(x_0, y, x_n) = \zeta^1(x_0)\psi_j(y)$$

gives a partition of unity on $(\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon]$. Recall that $(\psi_j)_{j \in J}$ is a partition of unity on S (see Section 1.3.3).

We define the following operators on $\mathcal{M}_+$:

$$(3.19) \quad \Xi_\bullet = \sum_{j \in J} \Xi_{\bullet, j}, \quad \text{with} \quad \Xi_{\bullet, j} = \phi_j^* \operatorname{Op}_{\mathcal{T}}(\chi_{\bullet, j})(\phi_j^{-1})^* \zeta_j, \quad j \in J, \quad \bullet = \mathscr{E}, \mathscr{Z}, \mathscr{F}, \mathscr{G},$$

where ϕ_j^* denotes the pullback by the function ϕ_j and

$$(3.20) \qquad \chi_{\bullet, j} = \tilde{\zeta}_j (\phi_j^{-1})^* \chi_\bullet,$$

and $\tilde{\zeta}_j$ denotes a function in $\mathscr{C}_c^\infty((0, X_0) \times \tilde{U}_j)$ with $\tilde{\zeta}_j = 1$ in a neighborhood of $\operatorname{supp}((\phi_j^{-1})^* \zeta_j)$.

The operators $\Xi_\bullet$ are zero-order tangential semi-classical operators on $\mathcal{M}_+$, with principal symbol $\zeta^1(x_0)\chi_\bullet(\mathbf{x}, \xi_0, \eta)$.

Remark 3.8. — The role of the parameter α_0 introduced here is to avoid considering boundary problems on $(\{0\} \cup \{X_0\}) \times S \times [0, 2\varepsilon]$.

4. Proof of the Carleman estimate in a neighborhood of the interface

In this section, we prove Carleman estimates in the four microlocal regions described above, that is, for functions $\Xi_\bullet v^{\tau/h}$, with $v^{\tau/h} \in \mathscr{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon))$ and $\bullet = \mathscr{E}, \mathscr{Z}, \mathscr{F}, \mathscr{G}$. Two main technics can be used to obtained these microlocal estimates: Calderón projectors and the standard Carleman method. The first one exploits ellipticity; one has to be away from the characterisitic set of the conjugated operator; there is no loss of derivative in such estimate which can be observed in the powers of the semi-classical parameter h. The second one is based on the computation of an L^2 norm and uses a sub-ellipticity argument; it can be used in the neighborhood of the characterisitic set of the conjugated operator; there is a loss of derivative there which shows in the the powers of the semi-classical parameter h.

4.1. Estimate in the region $\mathscr{G}$. — We introduce a microlocal cut-off function $\chi_{\mathscr{G}\mathscr{F}} \in \mathscr{C}_c^\infty(\mathcal{M}_+^*)$, $0 \leq \chi_{\mathscr{G}\mathscr{F}} \leq 1$, satisfying

$$(4.1) \qquad \begin{aligned} &\chi_{\mathscr{G}\mathscr{F}} = 1 \text{ on a neighborhood of } \operatorname{supp}(\chi_{\mathscr{G}}), \\ &\chi_{\mathscr{G}} + \chi_{\mathscr{F}} = 1 \text{ on a neighborhood of } \operatorname{supp}(\chi_{\mathscr{G}\mathscr{F}}). \end{aligned}$$

We choose $\zeta^2 \in \mathscr{C}_c^\infty(0, X_0)$ such that $0 \leq \zeta^2 \leq 1$, $\zeta^2 = 1$ on a neighborhood of $\operatorname{supp}(\zeta^1)$ (with ζ^1 defined in (3.18)), and such that $\tilde{\zeta}_j = 1$ on $\operatorname{supp}((\phi_j^{-1})^* \zeta_j^2)$ where $\zeta_j^2(x_0, y) = \zeta^2(x_0)\psi_j(y)$. As in (3.20) we set

$$\chi_{\mathscr{G}\mathscr{F}, j} = \tilde{\zeta}_j (\phi_j^{-1})^* \chi_{\mathscr{G}\mathscr{F}},$$

and we define the associated tangential pseudo-differential operator $\Xi_{\mathscr{G}\mathscr{F}}$ by

$$\Xi_{\mathscr{G}\mathscr{F}} = \sum_{j \in J} \Xi_{\mathscr{G}\mathscr{F}, j}, \quad \text{with} \quad \Xi_{\mathscr{G}\mathscr{F}, j} = \phi_j^* \operatorname{Op}_{\mathcal{T}}(\chi_{\mathscr{G}\mathscr{F}, j})(\phi_j^{-1})^* \zeta_j^2, \quad j \in J,$$

Note that the local symbol of $\Xi_{\mathscr{G}\mathscr{F}}$ in each chart is equal to one in the support of that of $\Xi_{\mathscr{G}}$.

We recall that the function $\zeta = \zeta(x_n) \in \mathscr{C}_c^\infty([0, 2\varepsilon))$ satisfies $\zeta(0) = 1$ on $[0, \varepsilon]$.

Making use of the Calderón projector technique for $P^r_{\varphi,j}$ and of the standard Carleman techniques for $P^l_{\varphi,j}$, we obtain the following partial estimate.

Proposition 4.1. — *Suppose that the weight function φ satisfies the properties listed in Section 3.1. Then, for all $\delta_0 > 0$, there exist $C > 0$ and $h_0 > 0$ such that, for all $0 < \delta \le \delta_0$ and $0 < h \le h_0$, $v^{r/l} \in \mathscr{C}^\infty_c((0, X_0) \times S \times [0, 2\varepsilon))$ and $v^s \in \mathscr{C}^\infty_c((0, X_0) \times S)$ satisfying (3.10), we have*

(4.2)

$$\|\Xi_{\mathscr{G}} v^r\|^2_1 + h|\Xi_{\mathscr{G}} v^r_{|x_n=0^+}|^2_1 + h|D_{x_n}\Xi_{\mathscr{G}} v^r_{|x_n=0^+}|^2_0$$
$$\le C\Big(\|P^r_\varphi v^r\|^2_0 + h^2\|v^r\|^2_1 + h^4|D_{x_n} v^r_{|x_n=0^+}|^2_0\Big),$$

and

(4.3)
$$h\|\Xi_{\mathscr{G}} v^l\|^2_1 + h|\Xi_{\mathscr{G}} v^l_{|x_n=0^+}|^2_1 + h|D_{x_n}\Xi_{\mathscr{G}} v^l_{|x_n=0^+}|^2_0$$
$$\le C\Big(1 + \frac{\delta^2}{h^2}\Big)\Big(\|\zeta P^r_\varphi v^r\|^2_0 + h^2\|\Xi_{\mathscr{G}\mathscr{F}} v^r\|^2_1 + h^4|D_{x_n} v^r_{|x_n=0^+}|^2_0 + h^4\|v^r\|^2_1 + h^3|v^s|^2_1\Big)$$
$$+ C\Big(\|P^l_\varphi v^l\|^2_0 + h^2\|v^l\|^2_1 + h|\theta^l_\varphi|^2_1 + \frac{\delta^2}{h}|\theta^r_\varphi|^2_0 + h|\theta^r_\varphi|^2_1 + h|\Theta^s_\varphi|^2_0\Big).$$

Note the difference in the powers of the semi-classical parameter h in the term $\|\Xi_{\mathscr{G}} v^r\|_1$ and $\|\Xi_{\mathscr{G}} v^l\|_1$ in the l.h.s. of these estimates: there is no loss of derivative in (4.2) and a half derivative is lost in (4.3). The factor $(1 + \delta/h)$ in the r.h.s. of (4.3) is important: it originates from the non-ellipticity of the operator P^s_φ on the interface in the region $\mathscr{G}$ (see figure 1).

4.2. Estimate in the region $\mathscr{F}$. — Making use of the Calderón projector technique for both $P^r_{\varphi,j}$ and $P^l_{\varphi,j}$, we obtain the following partial estimate.

Proposition 4.2. — *Suppose that the weight function φ satisfies the properties listed in Section 3.1. Then, for all $\delta_0 > 0$, there exist $C > 0$ and $h_0 > 0$ such that, for all $0 < \delta \le \delta_0$ and $0 < h \le h_0$, $v^{r/l} \in \mathscr{C}^\infty_c((0, X_0) \times S \times [0, 2\varepsilon))$ and $v^s \in \mathscr{C}^\infty_c((0, X_0) \times S)$ satisfying (3.10), we have*

(4.4) $\quad \|\Xi_{\mathscr{F}} v^r\|^2_1 + h|\Xi_{\mathscr{F}} v^r_{|x_n=0^+}|^2_1 + h|D_{x_n}\Xi_{\mathscr{F}} v^r_{|x_n=0^+}|^2_0$
$$\le C\Big(\|P^r_\varphi v^r\|^2_0 + h^2\|v^r\|^2_1 + h^4|D_{x_n} v^r_{|x_n=0^+}|^2_0\Big),$$

and

(4.5) $\quad \|\Xi_{\mathscr{F}} v^l\|^2_1 + h|\Xi_{\mathscr{F}} v^l_{|x_n=0^+}|^2_1 + h|D_{x_n}\Xi_{\mathscr{F}} v^l_{|x_n=0^+}|^2_0$
$$\le C\Big(\|P^l_\varphi v^l\|^2_0 + h^2\|v^l\|^2_1 + h^4|D_{x_n} v^l_{|x_n=0^+}|^2_0 + \|P^r_\varphi v^r\|^2_0$$
$$+ h^2\|v^r\|^2_1 + h^4|D_{x_n} v^r_{|x_n=0^+}|^2_0 + h|\theta^l_\varphi|^2_1 + h|\theta^r_\varphi|^2_1\Big).$$

4.3. Estimate in the region $\mathscr{Z}$. — As a consequence of property (3.6) of the weight function and the compactness of $[0, X_0] \times S \times [0, 2\varepsilon]$, we remark that in the region $\mathscr{Z}$, there exists $K_1 > 0$ such that

(4.6) $$\big(\partial_{x_n}\varphi^r\big)^2 - \mu^r \ge \min\big(\partial_{x_n}\varphi^r\big)^2 - 2\epsilon_1 \ge K_1 > 0$$

for ϵ_1 sufficiently small (the constant ϵ_1 is used in the definition of the microlocal regions in (3.17)).

Making use of the Calderón projector technique for $P_{\varphi,j}^l$, and standard techniques to prove Carleman estimates for $P_{\varphi,j}^r$, we obtain the following partial estimate.

Proposition 4.3. — *Suppose that the weight function φ satisfies the properties listed in Section 3.1. Then, for all $\delta_0 > 0$, there exist $C > 0$ and $h_0 > 0$ such that, for all $0 < \delta \leq \delta_0$ and $0 < h \leq h_0$, $v^{r/l} \in \mathscr{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon))$ and $v^s \in \mathscr{C}_c^\infty((0, X_0) \times S)$ satisfying (3.10), we have*

(4.7)

$$
h\|\Xi_{\mathscr{X}}v^r\|_1^2 + h\Big(1 + \frac{\delta^2}{h^2}\Big)|\Xi_{\mathscr{X}}v^r_{|x_n=0^+}|_1^2 + h|D_{x_n}\Xi_{\mathscr{X}}v^r_{|x_n=0^+}|_0^2
$$
$$
\leq C\Big(\|P_\varphi^r v^r\|_0^2 + h^2\|v^r\|_1^2 + h(\delta^2 + h^2)|v^s|_1^2 + \|P_\varphi^l v^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n}v^l_{|x_n=0^+}|_0^2
$$
$$
+ \frac{\delta^2}{h}|\theta_\varphi^r|_0^2 + h|\theta_\varphi^l|_1^2 + h|\theta_\varphi^r|_1^2 + h|\Theta_\varphi^s|_0^2\Big),
$$

and

(4.8)

$$
\|\Xi_{\mathscr{X}}v^l\|_1^2 + h|\Xi_{\mathscr{X}}v^l_{|x_n=0^+}|_1^2 + h|D_{x_n}\Xi_{\mathscr{X}}v^l_{|x_n=0^+}|_0^2
$$
$$
\leq C\Big(\|P_\varphi^l v^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n}v^l_{|x_n=0^+}|_0^2 + h^3|v^s|_1^2 + \frac{h^2}{\delta^2 + h^2}\big(\|P_\varphi^r v^r\|_0^2 + h^2\|v^r\|_1^2\big)
$$
$$
+ h|\theta_\varphi^l|_1^2 + h|\theta_\varphi^r|_1^2 + \frac{h^3}{\delta^2 + h^2}|\Theta_\varphi^s|_0^2\Big).
$$

Transmission conditions are important in the derivation of this microlocal estimate; information is transported from the 'l' side to the 'r' side through such conditions; in the proof, a quadratic form is estimated at the interface, positivity is obtained thanks to the tranmission conditions; this explains the factor $(1 + \delta^2/h^2)$ in (4.7).

4.4. Estimate in the region $\mathscr{E}$. — Using in this region the ellipticity of $P_{\varphi,j}^s$ and the Calderón projector technique for both $P_{\varphi,j}^r$ and $P_{\varphi,j}^l$, we obtain the following partial estimate.

Proposition 4.4. — *Suppose that the weight function φ satisfies the properties listed in Section 3.1. Then, for all $\delta_0 > 0$, there exist $C > 0$ and $h_0 > 0$ such that, for all $0 < \delta \leq \delta_0$ and $0 < h \leq h_0$, $v^{r/l} \in \mathscr{C}_c^\infty((0, X_0) \times S \times [0, 2\varepsilon))$ and $v^s \in \mathscr{C}_c^\infty((0, X_0) \times S)$ satisfying (3.10), we have*

(4.9)

$$
\|\Xi_{\mathscr{E}}v^{r/l}\|_1^2 + h|\Xi_{\mathscr{E}}v^{r/l}_{|x_n=0^+}|_1^2 + h|D_{x_n}\Xi_{\mathscr{E}}v^{r/l}_{|x_n=0^+}|_0^2
$$
$$
\leq C\Big(\|P_\varphi^r v^r\|_0^2 + h^2\|v^r\|_1^2 + h^4|D_{x_n}v^r_{|x_n=0^+}|_0^2 + \|P_\varphi^l v^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n}v^l_{|x_n=0^+}|_0^2
$$
$$
+ h^3|v^s|_1^2 + h|\Theta_\varphi^s|_0^2 + h|\theta_\varphi^r|_1^2 + h|\theta_\varphi^l|_1^2\Big).
$$

4.5. A semi-global Carleman estimate: proof of Theorem 1.2. — In this section, we explain how we can patch together the four microlocal estimate of Propositions 4.1, 4.2, 4.3 and 4.4, to obtain a global Carleman estimate in a neighborhood of S, and prove Theorem 1.2.

First, let us introduce some notation. We set

$$\mathrm{BT}(w) := h|w_{|x_n=0^+}|_1^2 + h|D_{x_n}w_{|x_n=0^+}|_0^2,$$

$$\mathrm{RHS}^{r/l}(w) := \|P_\varphi^{r/l}w\|_0^2 + h^2\|w\|_1^2 + h^4|D_{x_n}w_{|x_n=0^+}|_0^2,$$

$$R_\theta := h|\Theta_\varphi^s|_0^2 + h|\theta_\varphi^r|_1^2 + h|\theta_\varphi^l|_1^2.$$

This allows us to formulate concisely the four microlocal estimates of Propositions 4.1, 4.2, 4.3 and 4.4.

$$(4.10) \qquad \|\Xi_{\mathscr{G}}v^r\|_1^2 + \mathrm{BT}(\Xi_{\mathscr{G}}v^r) \lesssim \mathrm{RHS}^r(v^r),$$

$$(4.11) \quad \varepsilon h\|\Xi_{\mathscr{G}}v^l\|_1^2 + \varepsilon\,\mathrm{BT}(\Xi_{\mathscr{G}}v^l) \lesssim \Big(1 + \frac{\delta^2}{h^2}\Big)\Big(\varepsilon\|\zeta P_\varphi^r v^r\|_0^2 + \varepsilon h^4|D_{x_n}v_{|x_n=0^+}^r|_0^2 + \varepsilon h^4\|v^r\|_1^2\Big)$$

$$+ \varepsilon\,\mathrm{RHS}^l(v^l) + \boxed{\varepsilon(h^2 + \delta^2)\|\Xi_{\mathscr{G}\mathscr{F}}v^r\|_1^2} + \boxed{\varepsilon h(h^2 + \delta^2)|v^s|_1^2} + \varepsilon R_\theta + \varepsilon\frac{\delta^2}{h}|\theta_\varphi^r|_0^2,$$

$$(4.12) \qquad \|\Xi_{\mathscr{F}}v^r\|_1^2 + \mathrm{BT}(\Xi_{\mathscr{F}}v^r) \lesssim \mathrm{RHS}^r(v^r),$$

$$(4.13) \qquad \|\Xi_{\mathscr{F}}v^l\|_1^2 + \mathrm{BT}(\Xi_{\mathscr{F}}v^l) \lesssim \mathrm{RHS}^l(v^l) + \mathrm{RHS}^r(v^r) + R_\theta,$$

$$(4.14)$$

$$\varepsilon h\|\Xi_{\mathscr{L}}v^r\|_1^2 + \varepsilon\,\mathrm{BT}(\Xi_{\mathscr{L}}v^r) \lesssim \varepsilon\,\mathrm{RHS}^r(v^r) + \varepsilon\,\mathrm{RHS}^l(v^l) + \boxed{\varepsilon h(h^2 + \delta^2)|v^s|_1^2} + \varepsilon\frac{\delta^2}{h}|\theta_\varphi^r|_0^2 + \varepsilon R_\theta,$$

$$(4.15) \qquad \|\Xi_{\mathscr{L}}v^l\|_1^2 + \mathrm{BT}(\Xi_{\mathscr{L}}v^l) \lesssim \mathrm{RHS}^l(v^l) + h^3|v^s|_1^2 + \frac{h^2}{\delta^2 + h^2}\mathrm{RHS}^r(v^r) + R_\theta,$$

$$(4.16) \qquad \|\Xi_{\mathscr{E}}v^{r/l}\|_1^2 + \mathrm{BT}(\Xi_{\mathscr{E}}v^{r/l}) \lesssim \mathrm{RHS}^l(v^l) + \mathrm{RHS}^r(v^r) + h^3|v^s|_1^2 + R_\theta.$$

To derive the final Carleman estimate we need to sum together these microlocal estimate and many terms in the r.h.s. need to be "absorbed" by those in the l.h.s.. This is a standard procedure usually making use of the powers of the parameter h in front of these terms and by choosing h sufficiently small. Note, however, that some powers of h are critical here so that the related terms (in frames) in the right hand-sides cannot be "absorbed" directly. To overcome this problem, we have multiplied the two concerned equations by a small parameter $\varepsilon > 0$ whose value is independent of h and δ.

Note that these three atypical terms are the reason for the introduction of the microlocal region $\mathscr{F}$ (compare with the microlocal regions used in [**15**]). In fact, the microlocal region $\mathscr{F}$ acts as a buffer: as $\mathscr{F}$ is an elliptic region for both the operators $P_\varphi^{r/l}$, it provides terms in the l.h.s. of the associated microlocal estimates of better quality than those obtained in the regions $\mathscr{G}$ and $\mathscr{L}$ (compare the powers of h in the l.h.s. terms of these estimates).

Observe that the property $\chi_\mathscr{E} + \chi_\mathscr{L} + \chi_\mathscr{F} + \chi_\mathscr{G} = 1$ implies, see Section 3.6,

$$\Xi_{\mathscr{G},j} + \Xi_{\mathscr{F},j} + \Xi_{\mathscr{L},j} + \Xi_{\mathscr{E},j} = \zeta_j(x_0, y).$$

As a consequence of the definition of the operators $\Xi_\bullet$, $\bullet = \mathscr{E}, \mathscr{L}, \mathscr{F}, \mathscr{G}$, given in (3.19)-(3.20), this yields

$$(4.17) \qquad \Xi_\mathscr{E} + \Xi_\mathscr{L} + \Xi_\mathscr{F} + \Xi_\mathscr{G} = \zeta^1(x_0).$$

We now treat the three atypical terms and use the small parameter ε.

As $\text{supp}(v^s) \subset (\alpha_0, X_0 - \alpha_0) \times S$ (see the statement of Theorem 1.2 and Section 3.3), with (4.17), and using the transmission conditions (3.10), we have

$$v^s = \zeta^1 v^s = \Xi_{\mathscr{E}} v^s + \Xi_{\mathscr{X}} v^s + \Xi_{\mathscr{F}} v^s + \Xi_{\mathscr{G}} v^s$$
$$= \Xi_{\mathscr{E}} v^l - \Xi_{\mathscr{E}} \theta_\varphi^l + \Xi_{\mathscr{X}} v^l - \Xi_{\mathscr{X}} \theta_\varphi^l + \Xi_{\mathscr{F}} v^l - \Xi_{\mathscr{F}} \theta_\varphi^l + \Xi_{\mathscr{G}} v^r - \Xi_{\mathscr{G}} \theta_\varphi^r, \quad \text{at } x_n = 0^+.$$

Hence, for $\delta \leq \delta_0$ and $h \leq h_0$ we can estimate the two atypical terms concerning v^s in (4.11) and (4.14) as

$$\varepsilon h \delta^2 |v^s|_1^2 \lesssim \varepsilon h |\Xi_{\mathscr{E}} v^l|_1^2 + \varepsilon h |\Xi_{\mathscr{X}} v^l|_1^2 + \varepsilon h |\Xi_{\mathscr{F}} v^l|_1^2 + \varepsilon h |\Xi_{\mathscr{G}} v^r|_1^2 + \varepsilon R_\theta.$$

When summing all the estimates (4.10)-(4.16) together and taking ε sufficiently small, the four terms $\varepsilon h |\Xi_{\mathscr{E}} v^l|_1^2$, $\varepsilon h |\Xi_{\mathscr{X}} v^l|_1^2$, $\varepsilon h |\Xi_{\mathscr{F}} v^l|_1^2$, $\varepsilon h |\Xi_{\mathscr{G}} v^r|_1^2$ can be "absorbed" by the l.h.s. of (4.16), (4.15), (4.13), and (4.10) respectively.

The remaining atypical term is in (4.11):

$$\varepsilon \big(h^2 + \delta^2 \big) \| \Xi_{\mathscr{G}\mathscr{F}} v^r \|_1^2 \lesssim \varepsilon \| \Xi_{\mathscr{G}\mathscr{F}} v^r \|_1^2.$$

We choose a function $\zeta^4 \in \mathscr{C}_c^\infty(0, X_0)$ such that $\zeta^4 = 1$ on a neighborhood of $(\alpha_0, X_0 - \alpha_0)$, $\zeta^1 = 1$ on a neighborhood of $\text{supp}(\zeta^4)$ and $0 \leq \zeta^4 \leq 1$. Since $\text{supp}(v^r) \subset (\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon)$, we have

$$(4.18) \qquad \Xi_{\mathscr{G}\mathscr{F}} v^r = \Xi_{\mathscr{G}\mathscr{F}} (\Xi_{\mathscr{G}} + \Xi_{\mathscr{F}}) v^r + \Xi_{\mathscr{G}\mathscr{F}} (1 - \Xi_{\mathscr{G}} - \Xi_{\mathscr{F}}) \zeta^4 v^r.$$

The principal symbol of the operator $\Xi_{\mathscr{G}\mathscr{F}} (1 - \Xi_{\mathscr{G}} - \Xi_{\mathscr{F}}) \zeta^4$ is

$$\zeta^2 \chi_{\mathscr{G}\mathscr{F}} \big(1 - \zeta^1 (\chi_{\mathscr{G}} + \chi_{\mathscr{F}}) \big) \zeta^4 = \zeta^2 \chi_{\mathscr{G}\mathscr{F}} \big(1 - (\chi_{\mathscr{G}} + \chi_{\mathscr{F}}) \big) \zeta^4 = 0$$

since $\chi_{\mathscr{G}} + \chi_{\mathscr{F}} = 1$ on $\text{supp}(\chi_{\mathscr{G}\mathscr{F}})$ by (4.1). We thus have $\Xi_{\mathscr{G}\mathscr{F}} (1 - \Xi_{\mathscr{G}} - \Xi_{\mathscr{F}}) \zeta^4 \in h \Psi_{\mathcal{T}}^{-1}(\mathcal{M}_+)$, so that (4.18) gives

$$\varepsilon (h^2 + \delta^2) \| \Xi_{\mathscr{G}\mathscr{F}} v^r \|_1^2 \lesssim \varepsilon \| \Xi_{\mathscr{G}} v^r \|_1^2 + \varepsilon \| \Xi_{\mathscr{F}} v^r \|_1^2 + \varepsilon h^2 \| v^r \|_1^2.$$

When summing all the estimates (4.10)-(4.16) together and taking ε sufficiently small, the two terms $\varepsilon \| \Xi_{\mathscr{G}} v^r \|_1^2$, $\varepsilon \| \Xi_{\mathscr{F}} v^r \|_1^2$ in this expression can be absorbed by the l.h.s. of (4.10) and (4.12), respectively. This is possible since these two estimates are obtained in elliptic regions yielding better powers in h.

Now, if we sum all the partial estimates (4.10)-(4.16), and handle the atypical terms as explained above, we obtain

$$(4.19) \qquad \| \Xi_{\mathscr{G}} v^r \|_1^2 + \text{BT}(\Xi_{\mathscr{G}} v^r) + h \| \Xi_{\mathscr{G}} v^l \|_1^2 + \text{BT}(\Xi_{\mathscr{G}} v^l)$$
$$+ \| \Xi_{\mathscr{F}} v^r \|_1^2 + \text{BT}(\Xi_{\mathscr{F}} v^r) + \| \Xi_{\mathscr{F}} v^l \|_1^2$$
$$+ \text{BT}(\Xi_{\mathscr{F}} v^l) + h \| \Xi_{\mathscr{X}} v^r \|_1^2 + \text{BT}(\Xi_{\mathscr{X}} v^r)$$
$$+ \| \Xi_{\mathscr{X}} v^l \|_1^2 + \text{BT}(\Xi_{\mathscr{X}} v^l) + \| \Xi_{\mathscr{E}} v^{\tau/l} \|_1^2 + \text{BT}(\Xi_{\mathscr{E}} v^{\tau/l})$$
$$\lesssim \text{RHS}^r(v^r) + \text{RHS}^l(v^l) + \big(1 + \frac{\delta^2}{h^2} \big) \| \zeta P_\varphi^r v^r \|_0^2 + h^3 |v^s|_1^2$$
$$+ h^2 |D_{x_n} v^r_{|x_n=0^+}|_0^2 + R_\theta + \frac{\delta^2}{h} |\theta_\varphi^r|_0^2.$$

Using $\text{supp}(v^{\tau/l}) \subset (\alpha_0, X_0 - \alpha_0) \times S \times [0, 2\varepsilon)$ and (4.17), we can write

$$\| v^{\tau/l} \|_1 \leq \| \Xi_{\mathscr{G}} v^{\tau/l} \|_1 + \| \Xi_{\mathscr{F}} v^{\tau/l} \|_1 + \| \Xi_{\mathscr{X}} v^{\tau/l} \|_1 + \| \Xi_{\mathscr{E}} v^{\tau/l} \|_1,$$

together with

$$|v^{\tau/l}_{|x_n=0^+}|_1 \leq |\Xi_{\mathscr{G}} v^{\tau/l}_{|x_n=0^+}|_1 + |\Xi_{\mathscr{F}} v^{\tau/l}_{|x_n=0^+}|_1 + |\Xi_{\mathscr{X}} v^{\tau/l}_{|x_n=0^+}|_1 + |\Xi_{\mathscr{E}} v^{\tau/l}_{|x_n=0^+}|_1,$$

and
$$|D_{x_n} v^{r/l}_{|x_n=0^+}|_0 \le |D_{x_n} \Xi_{\mathscr{G}} v^{r/l}_{|x_n=0^+}|_0 + |D_{x_n} \Xi_{\mathscr{F}} v^{r/l}_{|x_n=0^+}|_0 + |D_{x_n} \Xi_{\mathscr{X}} v^{r/l}_{|x_n=0^+}|_0 + |D_{x_n} \Xi_{\mathscr{E}} v^{r/l}_{|x_n=0^+}|_0.$$

These three inequalities together with (4.19) give

$$h\|v^{r/l}\|_1^2 + h|v^{r/l}_{|x_n=0^+}|_1^2 + h|D_{x_n} v^{r/l}_{|x_n=0^+}|_0^2$$

$$\lesssim \|P_\varphi^l v^l\|_0^2 + h^2\|v^l\|_1^2 + h^4|D_{x_n} v^l_{|x_n=0^+}|_0^2 + \big(1 + \frac{\delta^2}{h^2}\big)\|\zeta P_\varphi^r v^r\|_0^2 + \|P_\varphi^r v^r\|_0^2$$

$$+ h^2\|v^r\|_1^2 + h^2|D_{x_n} v^r_{|x_n=0^+}|_0^2 + h^3|v^s|_1^2 + R_\theta + \frac{\delta^2}{h}|\theta_\varphi^r|_0^2,$$

Taking $0 < h \le h_0$ with h_0 sufficiently small in this expression gives

$$h\|v^{r/l}\|_1^2 + h|v^{r/l}_{|x_n=0^+}|_1^2 + h|D_{x_n} v^{r/l}_{|x_n=0^+}|_0^2$$

$$\lesssim \|P_\varphi^l v^l\|_0^2 + \|P_\varphi^r v^r\|_0^2 + \big(1 + \frac{\delta^2}{h^2}\big)\|\zeta P_\varphi^r v^r\|_0^2 + R_\theta + \frac{\delta^2}{h}|\theta_\varphi^r|_0^2.$$

Recalling the definitions of $v^{r/l} = e^{\varphi^{r/l}/h} w^{r/l}$, $F_\varphi^{r/l}$, $\theta_\varphi^{r/l}$, Θ_φ^s (see Section 3.3 and Equation (3.3)), and observing that we have

$$\|e^{\varphi^{r/l}/h} D_{x_k} w^{r/l}\|_0 \le \|D_{x_k}\big(e^{\varphi^{r/l}/h} w^{r/l}\big)\|_0 + \|(\partial_{x_k}\varphi^{r/l})e^{\varphi^{r/l}/h} w^{r/l}\|_0,$$

and similar inequalities for the norms at the interface $\{x_n = 0^+\}$, we can "absorb" the zero-order terms in (3.3), which concludes the proof of Theorem 1.2. □

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