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# On asymptotic stability of nonlinear waves\*

Michał Kowalczyk<sup>†</sup>, Yvan Martel<sup>‡</sup>, Claudio Muñoz<sup>§</sup>

## Abstract

We review some results on asymptotic stability of nonlinear waves for a few dispersive or wave models, like the nonlinear Schrödinger equation, the generalized Korteweg-de Vries equation, and the nonlinear wave and Klein-Gordon equations. Then, we focus on recent results of the authors concerning the asymptotic stability of the kink for the  $\phi^4$  equation under odd perturbations. We also present two results (one of which seems previously unknown) of non-existence of small breathers for some nonlinear Klein-Gordon equations.

## 1 Introduction

In these notes, we review some results and techniques concerning the *asymptotic stability* of solitary waves or kinks for a few typical dispersive equations, like the nonlinear Schrödinger equation, the Korteweg-de Vries in generalized form, and for wave problems, like the nonlinear wave equations, the nonlinear Klein-Gordon equations, the sine-Gordon and  $\phi^4$  equations.

*The literature on the subject is huge. We present only a few selected results on asymptotic stability related to the authors' interests. Because of possible oversimplifications or missing references, we systematically refer to the original papers and to the references therein.*

An intuition of what we mean by asymptotic stability for solitary waves or kinks may be developed using the analogy with finite dimensional problems, but dealing with partial differential equations not only highly complicates the analysis, but also forces us to adapt the notion of convergence to each given situation.

Basically, one can identify two approaches to asymptotic stability.

- (1) The first method is to use integrability techniques. When the equation at hand is completely integrable and inverse scattering is available and effective, a global reduction of the nonlinear problem to a linear problem is possible. One may obtain the generic

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behavior of any solution (at the cost of technical assumptions), typically, under the form of the *soliton resolution*, see §2.1. In some sense, this gives much more information than asymptotic stability in a neighbourhood of the waves. Note that such *global results* have recently started to appear also for non-integrable models (see §8.2).

When the inverse scattering does not provide all the setting for such global results a remnant of the integrability structure may give some important spectral information on the linearized operator around solitary waves which can be useful in applying the second approach (see §2.2, §2.3). In numerous situations the inverse scattering method also provides us with explicit solutions giving valuable information, like for example possible obstructions to general asymptotic stability results (see §2.2 and §9.1).

- (2) The second approach is *perturbative* which means that one studies the flow only in a neighbourhood of some family of solitary waves. This can be done either by using suitable assumptions on the initial data and stability theory (in stable cases, see §3), or by imposing a global in time assumption on the solution, or by working on a restricted set of initial data (see §5), typically a manifold of finite co-dimension (center-stable manifold). Then, one decomposes the solution into a wave, known up to a finite number of geometrical parameters (mainly corresponding to the invariances of the equation and satisfying a system of equations), plus a small residue. The asymptotic stability problem then reduces to the understanding of the long time behavior of the small residue and of the evolution of the parameters. Once the residue is controlled, the behavior of the parameters is generally not difficult to handle by standard ODE theory. On the residue, ideally, one would like to obtain *scattering*, which means that the residue behaves like the solution of a linear dispersive equation. In practice, it is not always true (modified scattering is proved in some situations) or provable by current technology and convergence to zero in large time in some weaker sense is already quite interesting. The perturbative approach in principle applies to any model (no special structure is needed, in contrast with the rigidity of integrability techniques), but the linearized PDE satisfied by the residue (in general coupled with the equations of the geometrical parameters) may be quite hard to study. Here, we identify two main strategies to handle the residue.
- (2a) Dispersive estimates: the idea is essentially to extend the small data global Cauchy theory to a linearized equation with potential. In general, this requires rather strong spectral information on the corresponding Schrödinger operator, rarely fully proved (see §5). This method has the advantage to provide detailed information on the behavior of the residue, but often requires high power nonlinearities and/or large space dimension. Some other assumptions, like the *Fermi golden rule* may be needed (see §5 and §9).
- (2b) Liapunov functionals: virial type arguments can provide convergence to zero in a weaker sense, under different and in some sense weaker spectral information (sometimes still difficult to prove analytically). This method is especially useful in low dimensional problems with low power nonlinearities, since dispersive estimates, even for the free flow, are not needed (see §6, §7, §8, §9).

Generally speaking, it seems that proving asymptotic stability for nonlinear waves is a case-by-case problem, escaping any kind of universal criterium or approach, as developed for example for the stability theory of ground states (see §3).

## 2 Some results from inverse scattering

### 2.1 The KdV equation

The Korteweg-de Vries equation

$$\partial_t u + \partial_x(\partial_x^2 u + u^2) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1)$$

is a typical model where the inverse scattering method has been extremely successful. We refer to the following books and papers [1, 49, 50, 63, 66, 68, 106, 107, 112, 129, 151, 169] and to the references therein. In particular, for this equation, a satisfactory version of the *decomposition into solitons* is available.

Here, we set  $Q(x) = \frac{3}{2} \cosh^{-2}(x/2)$  and for  $c > 0$ ,  $Q_c(x) = cQ(c^{1/2}x)$ . Since  $Q_c$  is solution of  $Q_c'' + Q_c^2 = cQ_c$ , the function defined by  $q_c(t, x) = Q_c(x - ct - \sigma)$  (where  $\sigma \in \mathbb{R}$ ) is solution of equation (1) (called a *soliton*).

**Theorem 1** (Decomposition into solitons [63, 151]). *Let  $u_0$  be a  $C^4$  function such that for any  $j \in \{0, \dots, 4\}$ , for all  $x \in \mathbb{R}$ ,  $|(\partial^j u / \partial x^j)(0, x)| \lesssim \langle x \rangle^{-10}$ . Let  $u$  be the solution of (1) with initial data  $u_0$ . Then, there exist  $K \in \mathbb{N}$ ,  $\sigma_1, \dots, \sigma_K$  and  $c_1 > \dots > c_K > 0$  such that,*

$$u(t, x) - \sum_{k=1}^K Q_{c_k}(x - c_k t - \sigma_k) \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \text{ for all } x > 0.$$

The inverse scattering method used to treat integrable problems decouples the localized part (solitary waves) and the *dispersive part*. It requires some regularity and decay on the solutions. From [30], the assumptions on the initial data in the above result are sufficient to apply the results in [63, 151], but they are certainly not optimal. It is not clear from the literature what are the optimal conditions. Note also that the asymptotic behavior of the solution is described only for  $x > 0$  (see results in [151] for slight improvement). The question of the exact asymptotic behavior of the residue for  $x < 0$  is delicate; see *e.g.* [51, 63, 151] and references therein.

Recall that the inverse scattering method also provides exceptional solutions (called multi-solitons) for which the residue strongly converges to zero for all  $x \in \mathbb{R}$ , and describing for all time the elastic collisions of any number of solitons; see *e.g.* [169, 114, 120, 129] (by elastic we mean that the interacting solitons recover their exact sizes and speeds after a collision). Recall also that [71] proves the existence of solutions with an infinite number of solitons.

The modified Korteweg-de Vries equation, *i.e.*,  $\partial_t u + \partial_x(\partial_x^2 u \pm u^3) = 0$  is also a completely integrable model, related to (1) by the Miura transform. We refer to [129, 151]. For related results based on PDE techniques for these integrable models, see for example [21, 70, 126].

### 2.2 The cubic 1D NLS

Recall that the 1D cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (2)$$

is also an integrable equation, widely studied as such. See *e.g.* [35, 46, 49, 50, 65, 88, 137, 176, 167, 177, 178]. Here, we only reproduce the recent asymptotic stability result from [46].

Let  $Q(x) = \sqrt{2} \cosh^{-1}(x)$  denote the unique (up to translation) positive solution of the equation  $Q'' + Q^3 = Q$ , and let  $Q_c(x) = c^{1/2}Q(c^{1/2}x)$  for  $c > 0$ .

**Theorem 2** (Asymptotic stability for 1D cubic NLS, [46]). *Let  $s > 1$ . There exists  $\delta > 0$  such that for any  $u_0 \in L^2$  with  $\langle x \rangle^s u_0 \in L^2$  and  $\|\langle x \rangle^s (u_0 - Q)\|_{L^2} \leq \delta$ , the solution  $u$  of (2) with initial data  $u_0$  satisfies, for  $t \gg 1$ ,*

$$\|u(t) - e^{i\Gamma(t, \cdot)} Q_{c^+}(\cdot - \sigma(t))\|_{L^\infty} \lesssim t^{-1/2},$$

for some  $c^+$  close to 1, some translation and phase parameters  $\sigma$  and  $\Gamma$ .

The proof of this result relies on integrability techniques (see references to works by Deift-Park and Deift-Zhou in [46]), and it is especially striking. Indeed, in strong contrast with KdV, *there is an explicit obstruction to asymptotic stability for (2)*, which is the existence of two-solitons (solutions behaving like the sum of two solitons in some sense, see [178]) for which the solitons have parallel trajectories and arbitrary different sizes. In particular, there are two-solitons of (2) containing a soliton of size 1 and a parallel arbitrarily small soliton, for all time (such solutions are certainly unstable by nature). Since small solitons are small in  $H^1$ , these solutions are explicit obstructions to asymptotic stability in the energy space for single solitons. However, small solitons are not small in the norm of  $L^2(\langle x \rangle^s dx)$ , for  $s \geq 1$ .

Concerning the soliton decomposition for (2), we refer to the recent work [20].

### 2.3 The 2D KP-II equation

Recall that the integrable Kadomtsev-Petviashvili-II equation

$$\partial_x(\partial_t u + \partial_x(\partial_x^2 u + u^2)) + \partial_y^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad (3)$$

is a 2D extension of the classical KdV equation which does not admit localized solitons. However, the solitons  $q_c(t, x) = Q_c(x - ct - \sigma)$  of KdV, defined above ( $Q_c'' + Q_c^2 = cQ_c$ ) provide traveling waves solutions of KP-II independent of  $y$ , called *line-solitons*. It is a classical question to study the stability and asymptotic stability of these line-solitons with respect to localized (or periodic) perturbations. The problem was recently settled in [132, 133]. For  $a > 0$ , define the weighted norm  $\|\cdot\|_{L_a}$  by

$$\|f\|_{L_a}^2 = \|e^{ax} f\|_{L_{xy}^2}^2 + \|e^{ax} f\|_{L_y^1 L_x^2}^2 + \|f\|_{L_{xy}^2}^2.$$

**Theorem 3** (Asymptotic stability of line-solitons for KP-II, [132]). *For  $a > 0$  small, there exists  $\delta > 0$  such that if  $u_0$  satisfies  $\|u_0 - Q\|_{L_a} \leq \delta$ , then the solution  $u$  of (3) with initial data  $u_0$  satisfies, for all  $t \gg 1$ ,*

$$\|e^{ax} (u(t, x + \sigma(t, y), y) - Q_{c(t, y)}(x))\|_{L_{xy}^2} \lesssim |t|^{-3/4},$$

where  $\sup_y |c(t, y) - 1| + |\partial_y \sigma(t, y)| \lesssim |t|^{-1/2}$ .

Note that the argument in [132] uses a combination of algebraic properties coming from integrability theory (linearized Miura transformation) and of refined PDE tools. The use of exponentially weighted spaces in this context is also related to [143]. Interestingly, it is necessary to consider parameters  $\sigma$  and  $c$  depending on  $t$  and  $y$ . We refer to [132] for more precise statements. Note that the scaling parameter  $c(t, y)$  converges to 1; indeed, a localized perturbation cannot provoke a change in the scaling of the infinite energy line-soliton. Another recent result [133] weakens the assumption on the initial data.

For the integrable KP-I equation (the term  $\partial_y^2 u$  in (3) has the opposite sign), we refer to the result of instability of line-solitons by transverse perturbations proved earlier in [146].

## 2.4 Other models

Numerous other integrable models, like the Toda Lattice [10, 134], have been considered. The case of the sine-Gordon equation is briefly discussed in §9.1. We also mention recent progress on the stability of breathers for the integrable case in [2, 3], related to techniques used for KdV multi-solitons in [112].

For nearly integrable models, we refer to the review paper [92]. For models close to integrable KdV, we also refer to [83, 84, 85] on the FPU model and to [39, 140, 141, 147] on the water-wave problem. See [69] and references therein on the Szëgo and half wave equations. Finally, [4] and [135] describe other applications of the Virial technique to fluids and quasilinear wave equations.

## 3 Solitary waves

### 3.1 Existence and uniqueness results

In the 80s, the important development of the elliptic theory provided many results on the existence, uniqueness and further properties of (mainly radial) solutions of nonlinear elliptic equations on  $\mathbb{R}^d$ , notably equations of the form

$$\Delta u + f(u) = cu, \quad x \in \mathbb{R},$$

for  $c > 0$  and for general nonlinearities, typically  $f(u) = |u|^{p-1}u$ , for  $p > 1$ . From variational methods, or relatively soft spectral methods, stability issues were reduced to simple criteria. We refer to [11, 12, 13, 14, 15, 18, 19, 25, 27, 28, 29, 72, 76, 77]. In one dimension or in any dimension in the radial case the ground state solutions are well-understood.

In these notes, we do not discuss the question of local or global well-posedness of the Cauchy problem for the various nonlinear PDE considered. We mostly consider energy solutions and we refer to [25, 26, 73, 91] for the nonlinear Schrödinger equation, the gKdV equations and the wave equation. For the critical wave equation, we refer to the references given in [90]. We usually denote by  $u_0$  the initial data of a solution  $u(t)$  at  $t = 0$ .

### 3.2 Stability result for sub-critical gKdV

Consider the generalized KdV equations with pure power nonlinearity

$$u_t + (u_{xx} + u^p)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (4)$$

For  $p > 1$ , extending the previous formulas, we define  $Q$  to be the unique positive solution (up to translation) of  $Q'' + Q^p = Q$  in  $H^1$ . For  $c > 0$ , let  $Q_c(x) = c^{1/(p-1)}Q(c^{1/2}x)$ . It has been proved by variational arguments and the conservation of mass and energy

$$M(u) = \int u^2, \quad E(u) = \int \left( \frac{1}{2}u_x^2 - \frac{1}{p+1}u^{p+1} \right)$$

that the solitons are (orbitally) stable in  $H^1$  in the following sense.

**Theorem 4** (Stability of the soliton for sub-critical gKdV [11, 18, 27, 174]). *Let  $1 < p < 5$ . For all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that if  $\|u_0 - Q\|_{H^1} \leq \delta$ , then the solution  $u$  of (4) with initial data  $u_0$  satisfies, for all  $t \in \mathbb{R}$ ,  $\|u(t, \cdot + \sigma(t)) - Q\|_{H^1} \leq \epsilon$  for some translation parameter  $\sigma$ .*

*Remark 1.* By time reversibility, it follows from the stability statement that if an  $H^1$  solution  $u(t)$  of (4) satisfies  $\lim_{t \rightarrow +\infty} \|u(t, \cdot + \sigma(t)) - Q\|_{H^1} = 0$ , for some translation parameter  $\sigma(t)$ , then, for all time  $t \in \mathbb{R}$ ,  $u(t, x) = Q(x - t - \sigma_0)$ , for some  $\sigma_0$ , *i.e.* the solution  $u$  is a soliton.

Note that the translation parameter satisfies from the proof  $|\sigma'(t) - 1| \lesssim \epsilon$ , but we cannot expect  $|\sigma(t) - t|$  to be uniformly controled (take as  $u$  a soliton  $Q_c(x - ct)$  where  $c \sim 1$ ,  $c \neq 1$ ).

For the 1D sub-critical nonlinear Schrödinger equation, taking into account additional symmetries of this equation, a completely analogous result exists. In higher dimension, the same result was proved in sub-critical cases, for the *ground states solitary waves*, see [25, 174].

### 3.3 Instability

For critical and super-critical nonlinearity  $p \geq 5$ , the solitons are unstable by the gKdV flow. The instability phenomenon is quite different in the critical case (linear stability holds and the nonlinear instability is related to the scaling parameter) and in the super-critical case (linear exponential instability). Similar results exist for NLS. See [19, 25, 77, 115, 142, 173].

### 3.4 Recent developments and perspectives

There are still many open problems on existence, uniqueness, non-degeneracy and orbital stability of waves, mainly on truly nonradial situations and for excited states. Recent results have been obtained on several delicate cases: see the general result of uniqueness and non-degeneracy of the soliton for Benjamin-Ono type equations obtained in [67], and [108] on the non-degeneracy of the lump solution of the KP-I equation. The construction of non-radial solutions is also a quite active field.

## 4 General issues for proving asymptotic stability

From the integrable cases discussed in §2 and the stability statement in §3, we can identify the following issues when trying to prove asymptotic stability.

- (i) *Time reversibility.* It is well-known that time reversibility of a model is a strong restriction to obtain smoothing effects in the case where some persistence of regularity also holds. A similar problem, illustrated in Remark 1 above, exists for asymptotic stability. If stability holds in some topology, then asymptotic stability cannot be true in exactly the same topology, since by reversing time, one would obtain the contradiction that any solution close to a soliton is exactly a soliton. This means that the topology has to be adapted to the problem, and that the irreversibility mechanism underlying a statement of asymptotic stability has to be understood in some way.
- (ii) *Difficult spectral problems.* Once the nonlinear model has been linearized around a nonlinear wave proving convergence to 0 for the residue is similar to proving scattering of small solutions of nonlinear models (or some weaker statement), but as we had mentioned before, that problem in turn has a non-zero potential coming from linearization around the nonlinear wave. This potential is usually not explicit (except in some specific 1D situations) and dispersive estimates for linear operators with general potential are hard to prove as they require detailed spectral information on the corresponding linear Schrödinger operator. Such estimates are known only in few situations and require a

case-by-case study, sometimes accessible only by computer-assisted proofs (see §5). Two common ways to handle the situation is to add a further potential to the equation (with convenient assumptions) and consider waves close to ground states (see for example Theorem 5), or to assume the desired properties on the potential (without much hope to actually verify these assumptions).

- (iii) *Scattering to 0.* In many nonlinear problems already the asymptotic stability of the zero solution (no unknown potential appears in the equation) can be hard to prove, especially for low power focusing nonlinearities in low dimensions. Some notion of modified scattering may be needed, or some weaker notion of convergence to 0. Many interesting physical situations correspond to quadratic or cubic nonlinearities. Working in space dimensions higher than or equal to 3 make things easier from the point of view of the dispersion, but problems in 1D and 2D are also quite important and numerous.
- (iv) *Instability directions.* When considering large dimension or strong nonlinearity, the solitary waves may become unstable, see §3.3. It may seem an important obstacle, but actually, the strategy of replacing the general statement of asymptotic stability by the construction of a finite co-dimensional manifold of initial data leading to asymptotic stability has been especially successful recently, see §5.3. If the instability mechanism is well-known, the problem is not much harder than the stable case, and only requires a special treatment of the instability directions. In some *critical* cases, the instability mechanism may become degenerate (no exponential instability) and related to the scaling instability. It does not prevent from proving some sort of asymptotic stability, even in focusing contexts, see §7 and §8.

A different but related problem in the instable case is the existence of a finite dimensional stable manifold, see [32, 33, 34, 57, 76, 87, 142].

- (v) *Direct obstructions to asymptotic stability.* The notion of asymptotic stability has to take into account obvious obstructions from the symmetries of the flow, for example by letting free some geometrical parameters. Other deeper obstructions may come from arbitrarily small waves. It is not always the case that small solitons are geographically decoupled from the large ones. A typical example are the multi-solitons for the 1D cubic NLS, as explained in §2.2. There are several ways to get round such problems like changing the topology (then small traveling waves becomes large, see §2.2) or removing the small traveling waves of the model by perturbing the nonlinearity close to 0 (see §5 and Theorem 15). For the sine-Gordon model, the existence of small breathers implies that the kink cannot be asymptotically stable. See §9.2.
- (vi) *Less obvious obstructions at the linear level.* For some models, there are less obvious obstructions, like the existence of internal modes. These internal modes are obstructions at the linear level and may or may not correspond to real nonlinear obstructions. The comprehension of such phenomenon seems to be widely open. See Section §9.5 for an example of such difficulty at the linear level, handled at the nonlinear level by considering quadratic terms and using a variant of the Fermi golden rule.
- (vii) *Choice of the functional space.* As a consequence of (i), (iii) and (v), one has to choose suitable norms or notion of asymptotic stability.



## 5 Spectral methods for NLS

In the context of the nonlinear Schrödinger equation, the first pioneering results are [22, 23, 156, 157]. These papers initiated the method of separating modes and using dispersive estimates with potential (usually in weighted spaces), under assumptions on the spectrum of the linearized operator.

### 5.1 NLS with potential

The presence of a well-behaved potential  $V$  in the equation facilitates the understanding of the behavior of small solutions, since for such solutions, the potential of the linearized operator is  $V$  at the main order. We present from [79] a typical result with potential. For a complete list of references concerning such questions, we refer to the introduction of [79]. Consider the following model

$$i\partial_t u + \Delta u - Vu + g(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (5)$$

We assume that

- (1) the potential  $V$  is real-valued and satisfies  $\sup_{\mathbb{R}^3} \langle x \rangle^{3+\epsilon} |V(x)| < +\infty$ ;
- (2) the operator  $-\Delta + V$  has only one eigenvalue  $e_0 < 0$ , associated to a normalized positive eigenfunction  $\phi_0$ ;
- (3) the nonlinearity  $g$  satisfies  $|g(v)| \lesssim |v|^{2/3}$  for  $|v| < 1$  and  $|g(v)| \lesssim |v|^2$  for  $|v| > 1$ .

Now, we consider the family of small nonlinear bound states  $Q_z$ , parametrized by small  $z \in \mathbb{C}$ , and satisfying the linearization  $Q_z - z\phi_0 = o(z) \perp \phi_0$ ,

$$(-\Delta + V)Q_z + g(|Q_z|^2)Q_z = \lambda_z Q, \quad \lambda_z = e_0 + o(z) \in \mathbb{R}.$$

**Theorem 5** (Scattering around solitons for NLS with potential, [79]). *Assume (1), (2), (3). Any solution  $u$  of (5) with initial data sufficiently small in  $H^1$  decomposes for all  $t \geq 0$ , as*

$$u(t) = e^{i\Gamma(t)} Q_{z(t)} + w(t),$$

for parameters  $\Gamma$  and  $z$ , with  $|z(t)| \rightarrow m_\infty \geq 0$  as  $t \rightarrow +\infty$ ,  $|m_\infty - |z(0)|| \lesssim \|u(0)\|_{H^1}^2$ , and where the remainder  $w$  scatters in  $H^1$ .

In [165], in the case of several bound states (and among them, a ground state), another important question whether solutions initially close to excited bound states will approach the ground state in large time is studied.

In [22, 23, 24, 79] the nonlinearity is supposed to be flat enough at 0 so that dispersive estimates are enough to control nonlinear terms. In one dimension, this means at least a quintic power near zero. A recent work of Delort [54] addresses the case of a Schrödinger equation in one dimension with cubic power and variable coefficients. Under natural assumptions on the potential and using oddness assumption, the asymptotic behavior of the solution in large time is described, in particular, the solution is proved not to scatter. We also refer to the discussion in the introduction of [54] for more references on *modified scattering*.

## 5.2 Stable NLS solitons without potential

The problem of the asymptotic stability of stable solitary waves for the nonlinear Schrödinger equations has been extensively studied, see *e.g.* [22, 23, 24, 40, 41, 43, 44, 46, 74, 144, 145, 148, 149, 150, 151] and references therein. (In this list, we especially highlight [24, 44, 144] as typical works.) See also the related works [45, 78]. The situation is hard to summarize since all these works (except in the integrable case [46], presented in §2.2) are subject to specific assumptions, like spectral assumptions or suitable dispersive estimates for equations with unknown potential, a suitable Fermi Golden Rule, specific flatness conditions on the nonlinearities at 0, etc. To the authors' knowledge, no result of asymptotic stability is fully proved for any pure power NLS equation with stable solitons, except for the integrable cubic 1D NLS discussed in §2.2. The explanation for such a situation is that this problem gathers most of the difficulties pointed out in §3. Indeed, for pure power NLS equations, stability of the solitons means low dimension or low nonlinearity. Moreover, except in dimension one, the solitons are not explicit. Note that the existence of internal modes is studied in [28, 31].

## 5.3 Unstable NLS solitons

Part of the difficulty for the NLS equation comes from low dispersion and low nonlinearity. It is thus natural to consider larger dimensions, or higher order nonlinearities. As discussed above, in the focusing pure power case, this leads to unstable solitons. This section concerns the notion of *conditional asymptotic stability* (asymptotic stability in unstable cases) or the construction of *center-stable manifolds* with exact behavior on this manifold.

We focus on the 3D cubic NLS equation ( $\dot{H}^{1/2}$  critical)

$$i\partial_t u + \Delta u + |u|^2 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (6)$$

where the theory has been especially well-developed and successful, at least in the radial case, in a series of works [8, 9, 36, 136, 148, 151]. The necessary spectral assumptions to obtain dispersive estimates for the linearized equation around the soliton have been checked, first numerically and then rigorously by computer assisted proof (see [36] and references therein).

In the following theorem  $Q$  is the unique positive radial solution of  $\Delta Q + Q^3 = Q$  in  $H^1$  and  $Q_\lambda = \lambda^{-1}Q(\lambda^{-1}x)$ .

**Theorem 6** (Center stable manifold around soliton for 3D cubic NLS, [9, 36, 136, 150]). *There exists  $\delta > 0$  and a smooth manifold  $\mathcal{M} \subset H_{\text{rad}}^1$  included in a ball  $\mathcal{B}$  of  $H^1$  of center  $Q$  and radius  $\delta$  which divides  $\mathcal{B}$  into two connected components. For any  $u_0 \in \mathcal{M}$ , the solution  $u$  of (6) with initial data  $u_0$  decomposes as*

$$u(t) = e^{i\Gamma(t)}Q_{\lambda(t)} + v(t),$$

where  $v$  is small  $\sup_{t \geq 0} \|v\|_{H^1} \lesssim \delta$  and scatters, i.e.  $v(t) = e^{-it\Delta}v_\infty + o_{H^1}(1)$  as  $t \rightarrow +\infty$ , for some small function  $v_\infty$ . Moreover, a solution  $u$  of (6) with initial data  $u_0 \in \mathcal{B}$  stays close to the family of solitons for all time if and only if  $u_0 \in \mathcal{M}$ .

Later, Nakanishi and Schlag classified all possible behaviors in the same framework, complementing the above result by determining the behavior of the solutions for initial data on each of the two regions separated by the manifold. See §6.3 of [136].

We also refer to [103] for previous related results in the 1D super-critical case.

## 6 Asymptotic stability of solitons for gKdV

For the generalized KdV equation (4), Pego and Weinstein [143] have first considered the question of the asymptotic stability of the family of solitons. They proved asymptotic stability for initial data close to solitons in some exponential weighted norms. Their proof essentially relies on linear theory and a spectral property of the linearized operator around  $Q$ . The interest of considering weighted spaces is to remove the possibility of a soliton of size 1 perturbed by an arbitrarily small soliton  $Q_c$  (the construction of such solutions is explicit in the integrable case  $p = 2$ , see section 4). This result has been improved with the same kind of technique in [130], using polynomial weighted spaces.

From [116, 117, 119], we state the following result in the energy space.

**Theorem 7** (Asymptotic stability of the gKdV soliton in  $H^1$ , [117]). *Let  $p = 2, 3, 4$ . For any  $\beta > 0$ , there exists  $\delta = \delta(\beta) > 0$  such that the following is true. Let  $u_0 \in H^1$  be such that  $\|u_0 - Q\|_{H^1} \leq \delta$ . Then, the global solution  $u$  of (4) with initial data  $u_0$  satisfies*

$$\lim_{t \rightarrow +\infty} \|u(t) - Q_{c^+}(\cdot - \sigma(t))\|_{H^1(x > \beta t)} = 0,$$

for some  $c^+ > 0$  with  $|c^+ - 1| \lesssim \delta$  and some  $C^1$  function  $\sigma$  with  $\sigma'(t) \rightarrow c^+$  as  $t \rightarrow +\infty$ .

Theorem 1 claims strong convergence in  $H^1$  in the region  $x > \beta t$ . Strong convergence in  $H^1(\mathbb{R})$  is never true since it would imply that  $u(t)$  is a soliton, see Remark 1. The region where the convergence is obtained in Theorem 1 is sharp since one can construct a solution which behaves asymptotically as  $t \rightarrow +\infty$  as  $Q(x - t) + Q_c(x - ct)$ , where  $c > 0$  arbitrary (see [114, 129, 169]). In particular, choosing  $c > 0$  small, the  $H^1$  norm of  $Q_c(x - ct)$  is small, and this soliton travels on the line  $x = ct$ . This explains the necessity of a positive  $\beta$  in the convergence result. The papers [116, 117, 119] present different variants of proofs. We also refer to the survey [163].

For the case  $p = 4$ , i.e.

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (7)$$

this theorem has been much improved in [95, 162] where it is shown that the residue scatters in the Besov space  $\dot{B}_\infty^{-1/6,2}$  close to the critical Sobolev space  $\dot{H}^{-1/6}$ .

**Theorem 8** (Scattering to a soliton for quartic gKdV, [95, 162]). *Let  $c_0 > 0$ . There exists  $\delta > 0$  such that for any  $u_0 \in \dot{B}_\infty^{-1/6,2}$  satisfying  $\|u(0) - Q\|_{\dot{B}_\infty^{-1/6,2}} \leq \delta$ , the solution  $u$  of (7) with initial data  $u_0$  satisfies*

$$\lim_{t \rightarrow +\infty} \|u(t) - Q_{c^+}(\cdot - \sigma(t)) - v_L(t)\|_{\dot{B}_\infty^{-1/6,2}} = 0,$$

where  $c^+ > 0$  and  $v_L$  is a solution of the linear Airy equation.

A remarkable feature of this result is to obtain asymptotic stability with assumptions purely in the (negative regularity) scaling space; see the discussion in [95].

The strategy of the proof of Theorem 7 have been extended to the Gross-Pitaevski equation, the Benjamin-Bona-Mahony equation, the Benjamin-Ono equation (and other KdV-type models) see e.g. [6, 16, 38, 64, 75, 95, 131]. For the peakons of the Camassa-Holm equation, Molinet proved recently strong related results (private communication).

## 7 Solitons as blow up profiles

In this section, we recall a typical result of universality of *blow up profile* for the 1D mass critical NLS equation

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (8)$$

Recall the three conservation laws  $\int |u|^2$ ,  $E(u) = \int (\frac{1}{2}u_x^2 - \frac{1}{6}|u|^6)$ ,  $\Im \int (u_x \bar{u})$ . In the next result,  $Q$  denotes the unique positive even solution of  $Q'' + Q^5 = Q$ .

**Theorem 9** (Blow up profile for mass critical NLS, [124, 125]). *There exists  $\delta > 0$  such that if  $u_0 \in H^1$  satisfies  $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \delta$  and  $E(u_0) < 0$ , then the solution  $u$  of (8) with initial data  $u_0$  blows up in finite time  $T > 0$ , and*

$$\left\| e^{i\gamma(t)} \lambda^{1/2}(t) u(t, \lambda(t) \cdot + \sigma(t)) - Q \right\|_{\dot{H}^1} \rightarrow 0 \quad \text{as } t \uparrow T,$$

for some parameters  $\lambda$ ,  $\gamma$  and  $\sigma$ , where  $\lambda(t) \sim c_0 ((T-t)/\log|\log(T-t)|)^{1/2}$  for  $t \sim T$ .

See Theorem 2 in [124] and Theorem 1 in [125], where the result is proved also in higher space dimensions. Note that the soliton was previously known to be deeply related to the blow up threshold for critical NLS, [172]. The above result is an analogue to asymptotic stability in the neighbourhood of  $Q$ , since up to the translation and scaling parameters, solutions converge to  $Q$  in a strong topology. Note also that much more information is known on such blow up solutions, especially in the case of the above *log-log* blow up rate (see [123, 125]).

The analogy is also clear for gKdV, where techniques used to study blow up results close to solitons in the mass critical case were adapted to the sub-critical case to prove asymptotic stability around the solitons, see [116, 117, 118, 119]. Indeed, in both situations, the main ingredient is a virial identity, serving as a Liapunov functional: once all the free parameters have been adjusted (in particular the scaling in blow up problem), the residue satisfies orthogonality conditions and it is forced to converge to zero, at least in some local norms.

For blow up results related to solitons for the semilinear wave equation, see [48, 127].

## 8 Energy critical wave equation

In this section, we briefly recall some recent results related to blow up profile and the soliton resolution conjecture for the 3D energy critical focusing wave equation

$$\partial_t^2 u - \Delta u = u^5, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (9)$$

Denoting  $E(u, v) = \int (\frac{1}{2}|\nabla u|^2 + \frac{1}{2}v^2 - \frac{1}{6}|u|^6)$ , the energy of a solution  $(u, \partial_t u)$  of (9), defined by  $E(u, \partial_t u)$ , is conserved by the flow.

The function defined by  $W(x) = (1 + \frac{|x|^2}{3})^{-1/2}$  is the unique (up to translation) positive  $\dot{H}^1$  solution of  $\Delta W = W^5$ . Recall that  $W$  has a variational characterization, related to the best constant in the inequality  $\|v\|_{L^6} \lesssim \|\nabla v\|_{L^2}$ , see [5, 90, 161]. However, it has one direction of exponential instability.

For  $\ell \in \mathbb{R}^3$ ,  $|\ell| < 1$ , we denote

$$W_\ell(x) = W \left( \left( \frac{1}{\sqrt{1-|\ell|^2}} - 1 \right) \frac{\ell(\ell \cdot x)}{|\ell|^2} + x \right),$$

so that  $w_\ell(t, x) = W_\ell(x - \ell t)$  is solution of (9) (Lorentz transformation).

### 8.1 Blow up profile for critical NLW

In [90], Kenig and Merle gave a classification of all possible behaviors (blow up or scattering) of solutions whose initial data  $(u_0, u_1)$  satisfies  $E(u_0, u_1) < E(W, 0)$ . Next, in [57], Duyckaerts and Merle studied the *threshold case*  $E(u_0, u_1) = E(W, 0)$  and constructed the stable manifold around  $W$ . Then, Duyckaerts, Kenig and Merle [58, 59] proved the following result for solutions slightly above the threshold.

**Theorem 10** (Blow up profile for 3D critical NLW, [58, 59]). *There exists  $\delta > 0$  such that if  $u$  is a solution of (9) that blows up in finite time  $T > 0$  and*

$$\sup_{[0, T)} \left( \|\nabla u(t)\|_{L^2} + \frac{1}{2} \|\partial_t u(t)\|_{L^2} \right) \leq \|\nabla W\|_{L^2} + \delta,$$

then, as  $t \uparrow T$ ,

$$\left\| (u(t), \partial_t u(t)) - (v_0, v_1) \mp \left( \frac{1}{\lambda^{1/2}(t)} W_\ell \left( \frac{\cdot - \sigma(t)}{\lambda(t)} \right), -\frac{1}{\lambda^{3/2}(t)} (\ell \cdot \nabla W_\ell) \left( \frac{\cdot - \sigma(t)}{\lambda(t)} \right) \right) \right\|_{L^2} \rightarrow 0,$$

for some parameters  $\sigma, \lambda$  and  $\ell \in \mathbb{R}^3$ ,  $|\ell| < 1$  and functions  $(v_0, v_1) \in \dot{H}^1 \times L^2$ .

We see that in the radial and in the nonradial cases, the family  $\{\pm W_\ell\}$  is the universal blow up profile. We refer to the original paper for more results and details.

We refer to [101, 102] for classification results for solutions with energy at most slightly above the one of the ground state, and to [104, 86] for constructions of solutions with prescribed blow up rates (called *type II blow up*).

### 8.2 The soliton resolution conjecture for critical NLW

In [60], Duyckaerts, Kenig and Merle proved a complete result of decomposition into solitons for equation (9) in the radial case.

**Theorem 11** (Soliton resolution for the 3D radial critical wave equation, [60]). *Let  $u$  be a global radial solution of (9). Then, there exist a solution  $v_L$  of the linear wave equation,  $K \in \mathbb{N}$ , and for  $k \in \{1, \dots, K\}$ ,  $\epsilon_k \in \{-1, 1\}$ , and  $\lambda_k > 0$ , such that, as  $t \rightarrow +\infty$ ,*

$$\lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_K(t) \ll t,$$

$$\left\| (u(t), \partial_t u(t)) - \left( v_L(t) + \sum_{k=1}^K \frac{\epsilon_k}{\lambda_k^{1/2}(t)} W \left( \frac{\cdot}{\lambda_k(t)} \right), \partial_t v_L(t) \right) \right\|_{\dot{H}^1 \times L^2} \rightarrow 0.$$

Note that the above result is even more complete than in the integrable gKdV case (§2.1), since the residue is proved to scatter. A similar result holds for blow up solutions, provided they exhibit type II blow up. The soliton resolution conjecture was later proved in non-radial case for a subsequence of time for the 3, 4, 5D energy critical wave equation in [61, 62]. See other similar results for the wave map problem in [37, 89].

These results are the first examples of soliton decomposition in non-integrable cases. Such results go much beyond asymptotic stability since they apply to any initial data.

## 9 One dimensional wave problems

### 9.1 Dynamics of NLKG around solitons

We recall from [100] (see also [136], §6.2) a result of classification around solitons (including conditional asymptotic stability) for the 1D nonlinear Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (10)$$

Denote  $E(u, v) = \int \left( \frac{1}{2}u_x^2 + \frac{1}{2}v^2 - \frac{1}{p+1}|u|^{p+1} \right)$ , so that the energy  $E(u, \partial_t u)$  of a solution is conserved by the flow. As before, the soliton is the unique (up to translation) positive solution  $Q$  of  $Q'' + Q^p = Q$ . Recall that for any  $p > 1$ , it has one direction of exponential instability.

**Theorem 12** (Classification around the soliton for NLKG, [100, 136]). *Let  $p > 5$ . There exists  $\delta > 0$  such that for any even  $(u_0, u_1) \in H^1 \times L^2$ , with energy  $E(u_0, u_1) < E(Q, 0) + \delta$ , the solution  $u$  of (10) with initial data  $(u_0, u_1)$  satisfies one of the following*

- (1) Blow up: *the solution  $u$  blows up in finite time.*
- (2) Scattering: *the solution  $u$  is global and scatters to 0 as  $t \rightarrow +\infty$ .*
- (3) Scattering to  $Q$ :  *$u$  exists globally for  $t \geq 0$  and scatters to  $Q$ .*

*In addition, the set of even initial data such that  $E(u_0, u_1) < E(Q, 0) + \delta$  splits into nine non-empty disjoint sets corresponding to all possible combinations of this trichotomy as  $t \rightarrow \pm\infty$ .*

Note that case (2) corresponds to a local manifold of co-dimension one separating the other two cases. A main point in the proof of Theorem 12 is to obtain dispersive estimates in local norms using a suitable *distorted Fourier transform* (see Lemma 6.9 in [136]). Note also that Hamza and Zaag gave the unique blow up rate in case (1) (see reference in [136]).

Similar results for  $1 < p \leq 5$  seem to be open. Numerical and analytical arguments in [17] suggest that the rate of convergence of solutions in case (2) to solitons should depend strongly on the nonlinearity: for  $p = 3$ , the expected rate is  $t^{-1/2}$  and slower for  $1 < p < 3$ .

### 9.2 The sine-Gordon equation

The one dimensional sine-Gordon equation  $\square u + \sin u = 0$  has been widely studied as a physically relevant completely integrable model (see *e.g.* [49, 106]). This equation has an explicit kink solution  $S(x) = 4 \arctan(e^x)$  and besides it has other exceptional solutions, among them a one parameter family of explicit, odd, time-periodic solutions called *wobbling kinks* (see [47, 152]). Such wobbling kinks exist arbitrarily close to the kink. Since these solutions are periodic in time, *the sine-Gordon kink is not asymptotically stable in the energy space*. Actually, from these solutions, asymptotic stability does not hold for the sine-Gordon equation even in a stronger topology, in contrast with the one dimensional cubic Schrödinger equation for which changing the topology is enough to remove obstruction to asymptotic stability (see §2.2).

Recall that this equation also has arbitrarily small breathers (see [106]), which means that small solutions do not all converge to zero, even a weak sense (see §9.5).

### 9.3 Asymptotic stability of the $\phi^4$ kink under odd perturbations

We consider in this section the  $\phi^4$  model (see *e.g.* [49, 113, 166, 175]) in one dimension

$$\partial_t^2 \phi - \partial_x^2 \phi = \phi - \phi^3, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (11)$$

Recall that the energy  $E(\phi, \partial_t \phi) = \int \frac{1}{2} |\partial_t \phi|^2 + \frac{1}{2} |\partial_x \phi|^2 + \frac{1}{4} (1 - |\phi|^2)^2$  is formally conserved along the flow. The *kink*, defined by  $H(x) = \tanh(x/\sqrt{2})$  is the unique (up to sign change), bounded, odd solution of the equation  $-H'' = H - H^3$ . We recall that the orbital stability of the kink with respect to small perturbations in the energy space has been proved in [82] using the energy conservation. Here, we consider only odd perturbations in the energy space. Note that for odd initial data, the corresponding solution of (11) is odd. Rewrite

$$\phi = H + \varphi_1, \quad \partial_t \phi = \varphi_2, \quad \varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}.$$

Then,  $\varphi$  satisfies

$$\begin{cases} \partial_t \varphi_1 = \varphi_2 \\ \partial_t \varphi_2 = -L\varphi_1 - (3H\varphi_1^2 + \varphi_1^3) \end{cases} \quad (12)$$

where  $L = -\partial_x^2 - 1 + 3H^2 = -\partial_x^2 + 2 - 3\operatorname{sech}^2(x/\sqrt{2})$  is the linearized operator around  $H$ .

**Theorem 13** (Asymptotic stability of the kink by odd perturbations, [98]). *There exists  $\delta > 0$  such that for any odd  $\varphi_0 \in H^1 \times L^2$  with  $\|\varphi_0\|_{H^1 \times L^2} < \delta$ , the solution  $\varphi$  of (12) with initial data  $\varphi_0$  satisfies, for any bounded interval  $I \subset \mathbb{R}$ ,*

$$\lim_{t \rightarrow \pm\infty} \|\varphi(t)\|_{H^1(I) \times L^2(I)} = 0.$$

The notion of asymptotic stability introduced in Theorem 13 is related to difficulty (i). Indeed, as before for the (gKdV) equation, we observe that if a solution  $\varphi$  of (12) satisfies  $\lim_{t \rightarrow +\infty} \|\varphi(t)\|_{H^1 \times L^2} = 0$ , then by the orbital stability result [82],  $\varphi(t) \equiv 0$  for all  $t \in \mathbb{R}$ .

### 9.4 Convergence to zero of small solutions to NLKG

We see from (12) that the problem of asymptotic stability of the kink for  $\phi^4$  reduces to the *convergence to zero of any small solution of a nonlinear Klein-Gordon equation with variable coefficients*. This 1D Klein-Gordon equation contains quadratic and cubic terms. More generally, and independently, the long time behaviour for small solutions to NLKG equations has been widely studied starting with the pioneering papers [93, 94, 154]. A main delicate point is to deal with the quadratic nonlinearity, which complicates the analysis even in dimension 3 and higher due to slow rate of decay of linear Klein-Gordon waves. Of course, the situation is even more delicate in 1D and 2D. We refer to [52, 53, 55, 7, 109, 110, 160] for results about small solutions for semilinear and quasilinear Klein-Gordon equations. Finally, we mention the works [80, 81] on the modified scattering procedure for cubic and quadratic constant coefficients NLKG equations in one dimension. Note that none of these works is directly applicable to (12).

In addition to the aforementioned difficulties of the NLKG equations in 1D with quadratic and cubic nonlinearities, what makes problem (12) challenging is the existence of an internal

mode of oscillation (see §9.5). The mechanism of exchange of energy between internal oscillations and the radiation part was first studied by Soffer and Weinstein [159] for a class of nonlinear Klein-Gordon equations with potential (see also [153, 158]). More precisely, they study the question of asymptotic stability of the vacuum state (the zero solution) for the following Klein-Gordon equation in  $\mathbb{R}^3$ :

$$\partial_t^2 u = (\Delta - V(x) - m^2)u + \lambda u^3, \quad \lambda \in \mathbb{R}, \lambda \neq 0. \quad (13)$$

In addition to some natural hypothesis on the decay of the potential  $V$ , it is assumed that

- (1) the operator  $L_V = -\Delta + V + m^2$  has a continuous spectrum  $\sigma_{\text{cont}} = [m, \infty)$ , a single eigenvalue  $\Omega^2 < m^2$ , and the bottom of its continuous spectrum is not a resonance;
- (2) the Fermi Golden Rule holds (see precise statement in [159]).

Under such assumptions, they show that  $u = 0$  is asymptotically stable, and give detailed information on the rate of convergence of small solutions: the internal oscillation mode decays as  $t^{-1/4}$ , while the radiation decays as  $t^{-3/4}$  in the space  $L^8$ . We observe both an anomalously slow time decay rate and the existence of different decay rates for each component of the solution. This discordance seems to hold in general, and it is also a characteristic of the  $\phi^4$  problem. Note that dispersive estimates are used ultimately on the radiation part and the fact to work in 3D (or higher) is decisive to estimate the nonlinear terms.

Returning to the  $\phi^4$  problem, we mention two results more closely related to Theorem 13. In [96, 97], the asymptotic stability of the kink is studied for the 1D equation  $\partial_t^2 u = \partial_x^2 u + F(u)$ , where the nonlinearity writes  $F = -W'$ , for a smooth double well potential  $W$  such that  $W(x) = \frac{m^2}{2}(x \pm a)^2 + O(|x \pm a|^{14})$  as  $x \rightarrow \pm a$ . This guarantees existence of a kink  $U$  but it excludes the  $\phi^4$  model. Under further hypothesis similar to (1) and (2) above, [96] shows asymptotic stability of the kink  $U$  with respect to odd perturbations, with explicit decay rates:  $t^{-1/2}$  decay for the internal oscillations, and for the radiation,  $t^{-1}$  decay in weighted Sobolev spaces. The method, inspired by [24, 22, 23, 165, 144], is based on Poincaré normal forms and dispersive estimates.

In [42], the stability and asymptotic stability of the *one dimensional kink* for the  $\phi^4$  model, subject to *localized three dimensional* perturbations, is studied. The method used in this paper combines dispersive estimates from [170, 171] (see also [74]), together with Klainerman vectors fields and normal forms. The fact that the space dimension is three, with better decay estimates for free solutions, is essential in order to close the nonlinear estimates.

## 9.5 Sketch of the proof of Theorem 13

Here, we reproduce the sketch of the proof from [98], adding explanations related to the general difficulties identified in §4. The operator  $L$  appearing in (12) is classical and it is well-known (see *e.g.* [138]) that  $\text{spec } L = \{0, \frac{3}{2}\} \cup [2, +\infty)$ . The discrete spectrum consists of two simple eigenvalues  $\lambda_0 = 0$  and  $\lambda_1 = \frac{3}{2}$ , with  $L^2$  normalized eigenfunctions, respectively given by

$$Y_0(x) = c_0 \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) \quad \text{and} \quad Y_1(x) = c_1 \tanh\left(\frac{x}{\sqrt{2}}\right) \operatorname{sech}\left(\frac{x}{\sqrt{2}}\right).$$

Note that  $Y_0(x) = cH'(x)$  is related to the invariance of equation (11) by space translation. Since we restrict ourselves to odd perturbations of the stationary kink, this direction is not



relevant here. In contrast, the eigenfunction  $Y_1$ , usually referred as the *internal mode* of oscillation of the kink, is not related to any invariance and plays a key role in the analysis of the long time dynamics of  $\varphi$ . Indeed, *at the linear level*, the internal mode is a typical obstacle to asymptotic stability, since the time-periodic function defined by

$$\varphi_1^L(t, x) = \cos(\mu t)Y_1(x), \quad \varphi_2^L(t, x) = -\mu \sin(\mu t)Y_1(x), \quad \mu = \sqrt{\frac{3}{2}},$$

satisfies the linear system

$$\begin{cases} \partial_t \varphi_1^L = \varphi_2^L \\ \partial_t \varphi_2^L = -L\varphi_1^L. \end{cases}$$

However, in contrast with the sine-Gordon equation, no obstruction to asymptotic stability is known at the nonlinear level so that one can hope that because of nonlinear dumping this mode goes to zero for large time (see [105]). This observation leads to separate the mode  $Y_1$  at the linear level, and to study carefully quadratic terms for this mode. It also suggest that the rate of convergence to zero should be rather weak. We proceed in four steps.

1. *Spectral decomposition and coupling.* Define

$$\begin{aligned} z_1 &= \langle \varphi_1, Y_1 \rangle, \quad z_2 = \frac{1}{\mu} \langle \varphi_2, Y_1 \rangle, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ u_1 &= \varphi_1 - z_1 Y_1, \quad u_2 = \varphi_2 - \mu z_2 Y_1, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

so that  $\langle u_1, Y_1 \rangle = \langle u_2, Y_1 \rangle = 0$ . Since  $LY_1 = \mu^2 Y_1$ , we obtain

$$\begin{cases} \dot{z}_1 = \mu z_2, \\ \dot{z}_2 = -\mu z_1 - \frac{3}{\mu} \langle HY_1^2, Y_1 \rangle z_1^2 + \mathcal{O}_3, \end{cases}$$

where we use the notation  $\mathcal{O}_3 = \mathcal{O}(|z|^3, |z| \cdot \|u\|, \|u\|^2)$ . Recall that  $z^2$  terms are important and not considered as perturbation. In contrast, quadratic terms in  $u$  are discarded in this discussion. The norm  $\|\cdot\|$  used for  $u$  is an important issue, but we will not discuss it here. For the rigorous treatment of the nonlinear terms (quadratic in  $u$  and cubic in  $z$ ), we refer to [98]. Note that the above system in  $z$  is not enough to understand the exact behavior of  $(z_1, z_2)$ . Indeed, the quadratic term in  $z$  can be removed by changing variable, and the terms in  $\mathcal{O}_3$  can drastically change the long time behavior of  $z$ . At this point, we can only deduce that

$$\frac{d}{dt}(|z|^2) = \mathcal{O}_3 \quad \text{where} \quad |z|^2 = z_1^2 + z_2^2.$$

For the other component  $(u_1, u_2)$ , one checks that

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = -Lu_1 - 2z_1^2 f + \mathcal{O}_3, \end{cases}$$

where  $f = \frac{3}{2} (HY_1^2 - \langle HY_1^2, Y_1 \rangle Y_1)$  is an odd Schwartz function satisfying  $\langle f, Y_1 \rangle = 0$ .

The equation of  $z_1^2$  is not very handy and thus, we prefer to deal with the following auxiliary functions

$$\alpha = z_1^2 - z_2^2, \quad \beta = 2z_1z_2, \quad \text{satisfying} \quad \begin{cases} \dot{\alpha} = 2\mu\beta + \mathcal{O}_3 \\ \dot{\beta} = -2\mu\alpha + \mathcal{O}_3 \end{cases}$$

Indeed, there is a simple way to replace the term  $z_1^2 f$  in the equation of  $u_2$  by a term involving only  $\alpha$  by introducing the unique odd solution  $q \in H^1(\mathbb{R})$  of  $Lq = f$ . Indeed, we check that

$$v_1 = u_1 + |z|^2 q, \quad v_2 = u_2, \quad v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \quad \text{satisfies} \quad \begin{cases} \dot{v}_1 = v_2 + \mathcal{O}_3, \\ \dot{v}_2 = -Lv_1 - \alpha f + \mathcal{O}_3, \end{cases}$$

and  $\langle v_1, Y_1 \rangle = \langle v_2, Y_1 \rangle = 0$ . At this point, we have obtained a *coupled system* in  $(\alpha, \beta, v_1, v_2)$ , with nonlinear perturbations terms in  $\mathcal{O}_3$ , which takes into account the *nonlinear coupling* between the mode  $Y_1$  and the infinite dimensional part,  $\alpha$  and  $\beta$  being quadratic in  $z$ .

To study the above system, it is natural to try to remove the term  $-\alpha f$  in the equation of  $v_2$  to decouple at the linear level the equations of  $(v_1, v_2)$  and  $(\alpha, \beta)$ . However, this does not seem possible. Indeed, setting, for some function  $g$  to be determined,

$$v_1 = w_1 + \alpha g, \quad v_2 = w_2 + \dot{\alpha} g$$

we see that removing the term in  $\alpha$  in the equation of  $w_2$  requires  $\ddot{\alpha} g + \alpha Lg + \alpha f = 0$ . Thus, from the equation of  $\alpha$ , we impose that  $g$  satisfies  $-(L - 4\mu^2)g = f$ . One checks that setting

$$k(x) = e^{i2x} \left( 1 + \frac{1}{2} \operatorname{sech}^2 \left( \frac{x}{\sqrt{2}} \right) + i\sqrt{2} \tanh \left( \frac{x}{\sqrt{2}} \right) \right),$$

the equation  $-(L - 4\mu^2)g = f$  has a Schwartz solution if and only if  $\langle \Im k, f \rangle = 0$  (see [155]). In [98], we check (numerically) that  $\langle \Im k, f \rangle \neq 0$ . This is a variant of the Fermi golden rule, in the following sense: since no special vanishing occurs, it is not possible to decouple  $(v_1, v_2)$  and  $(\alpha, \beta)$ , which means that the mode  $Y_1$  has strong chance to scatter at the quadratic order in any direction through the infinite dimensional unknown  $(v_1, v_2)$ .

2. *Orbital stability.* Using the stability result in [82], we recall that if  $\varphi_0$  is small enough then  $\varphi(t)$  is global in time and uniformly small in  $H^1 \times L^2$ . It follows that  $(u_1, u_2)$ ,  $z$  and thus  $(v_1, v_2)$  and  $\alpha, \beta$  are also small. More precisely, for all  $t \in \mathbb{R}$ ,

$$\|u(t)\|_{H^1 \times L^2} + \|v(t)\|_{H^1 \times L^2} + |z(t)| \lesssim \|\varphi_0\|_{H^1 \times L^2}.$$

3. *Virial type arguments.* The large time behavior of solutions of the system in  $(v_1, v_2, \alpha, \beta)$  is now studied by a virial argument, inspired by [116, 117, 119]. The objective of virial argument is to prove the following estimate:

$$\int_{-\infty}^{\infty} (|z(t)|^4 + \|v(t)\|_{\omega}^2) dt \lesssim \|\varphi_0\|_{H^1 \times L^2}. \quad (14)$$

Here and below  $\|v\|_{\omega}$  means the  $\dot{H}^1 \times L^2$  norm of  $v$  with a suitable exponential weight,  $\|v_1\|_{\omega}$  and  $\|v_2\|_{\omega}$  the  $\dot{H}^1$  norm of  $v_1$  and the  $L^2$  norm of  $v_2$ , respectively, also with a suitable exponential weight (as discussed above, one cannot expect such information for the global energy norm).

The proof of (14) is based on several *ad hoc* functionals. For  $\lambda = 8$ , let

$$\psi(x) = \lambda\sqrt{2}H\left(\frac{x}{\lambda}\right) = \lambda\sqrt{2}\tanh\left(\frac{x}{\lambda\sqrt{2}}\right), \quad \zeta(x) = \sqrt{\psi'(x)} = \operatorname{sech}\left(\frac{x}{\lambda\sqrt{2}}\right)$$

and, for a function  $g$  to be chosen, let

$$\mathcal{I} = \int \left( \psi \partial_x v_1 + \frac{1}{2} \psi' v_1 \right) v_2, \quad \mathcal{J} = \alpha \langle v_2, g \rangle - 2\mu\beta \langle v_1, g \rangle.$$

Using the systems for  $(\alpha, \beta)$  and  $(v_1, v_2)$ , we find

$$-\frac{d}{dt}(\mathcal{I} + \mathcal{J}) = \mathcal{B}(v_1) + \alpha \langle v_1, h \rangle + \alpha^2 \langle f, g \rangle + o(|z|^4, \|v\|_\omega^2),$$

where  $\mathcal{B}(v_1) = \int \left\{ \psi' (\partial_x v_1)^2 - \frac{1}{4} \psi''' v_1^2 - 3\psi H H' v_1^2 \right\}$  and  $h = -(\psi f' + \frac{1}{2} \psi' f - Lg + 4\mu^2 g)$ . As above, one checks that it is not possible to find  $g$  so that  $h = 0$ . Rewrite  $\mathcal{B}$  using the auxiliary function  $w = \zeta v_1$

$$\mathcal{B}(v_1) = \mathcal{B}^\sharp(w) = \int \left\{ w_x^2 + \left[ \frac{1}{2} \left( \frac{\zeta''}{\zeta} - \frac{(\zeta')^2}{\zeta^2} \right) - 3 \frac{\psi}{\psi'} H H' \right] w^2 \right\}.$$

By direct computations, one has  $\mathcal{B}^\sharp(w) = \int (w_x^2 - V w^2)$ , where the potential  $V$  is defined by

$$V(x) = \frac{1}{4\lambda^2} \operatorname{sech}^2\left(\frac{x}{\lambda\sqrt{2}}\right) + 3\lambda \tanh\left(\frac{x}{\lambda\sqrt{2}}\right) \cosh^2\left(\frac{x}{\lambda\sqrt{2}}\right) \tanh\left(\frac{x}{\sqrt{2}}\right) \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right).$$

This potential is simple, explicit and exponentially decay in space. Its spectral properties can be studied by standard methods. First, we prove that the orthogonality condition  $\langle v_1, Y_1 \rangle = 0$  and the oddness of  $v_1$  imply  $\mathcal{B}(v_1) = \mathcal{B}^\sharp(w) \gtrsim \|w_x\|_{L^2}^2$ . Second, we find a special choice of  $g$  such that

$$\mathcal{B}(v_1) + \alpha \langle v_1, h \rangle + \alpha^2 \langle f, g \rangle \gtrsim \|v_1\|_\omega^2 + \alpha^2. \quad (15)$$

The coercivity property (15) is the key estimate. Note that, as in [116, 117], we rely on the numerical computations of some integrals to prove it. In practice, the function  $g$  is chosen as the unique Schwartz solution of the equation (other choices are certainly possible)

$$Lg - 4\mu^2 g = \psi f' + (a + \frac{1}{2}) \psi' f,$$

where  $a$  is adjusted so that the right hand side is orthogonal to  $\mathfrak{S}(k)$  ensuring that  $g$  is a Schwartz function. It is at this point that we make use of  $\langle \mathfrak{S}k, f \rangle \neq 0$  which is the Fermi Golden Rule.

From the previous observations, we obtain the following estimate

$$-\frac{d}{dt}(\mathcal{I} + \mathcal{J}) \gtrsim \alpha^2 + \|v_1\|_\omega^2 + o(|z|^4, \|v\|_\omega^2).$$

Formally, integrating in time and using the global bounds due to stability, this estimate says that  $\int_0^{+\infty} (\alpha^2 + \|v_1\|_\omega^2) dt \lesssim \|\varphi_0\|_{H^1 \times L^2}^2$ , provided one can control the higher order error terms. For this, we just need to control  $\beta$  (or  $|z|^2$ ) and  $v_2$ .

Now, we set  $\gamma = \alpha\beta$ . Then, by similar computations,  $\dot{\gamma} = 2\mu(\beta^2 - \alpha^2) + o(|z|^4, \|v\|_\omega^2)$ . This means that a bound on  $\alpha^2$  gives a bound  $\beta^2$  and thus on  $|z|^4$ . Moreover, by direct computations, it can be proven that

$$\|v_2\|_\omega^2 \lesssim \frac{d}{dt} \int \operatorname{sech}\left(\frac{x}{2\sqrt{2}}\right) v_1 v_2 + C(|z|^4 + \|v_1\|_\omega^2).$$

We see that for small perturbations, integrating in time a suitable linear combination of the above estimates gives (14).

4. *Convergence to the zero state for a weighted norm.* From (14), it follows that there exists a sequence  $t_n \rightarrow +\infty$  such that  $\lim_{n \rightarrow +\infty} \|v(t_n)\|_\omega + |z(t_n)| = 0$ . For  $z(t)$ , from the estimate  $\frac{d}{dt}|z|^2 = \mathcal{O}_3$ , we obtain

$$\left| \frac{d}{dt}|z|^4 \right| \lesssim |z|^3(|z|^2 + \|v\|_\omega^2).$$

Integrating on  $[t, t_n]$ , taking the limit  $n \rightarrow +\infty$  and using (14), we see that  $\lim_{t \rightarrow +\infty} |z(t)| = 0$ . For  $v(t)$ , we consider the functional

$$\mathcal{H} = \int (|\partial_x v_1|^2 + 2|v_1|^2 + |v_2|^2) \operatorname{sech}(cx),$$

and we check by direct computation that for  $c > 0$  large enough,  $|\dot{\mathcal{H}}| \lesssim |z|^4 + \|v\|_\omega^2$ . Integrating on  $[t, t_n]$ , and using (14), we deduce that  $\lim_{t \rightarrow +\infty} \mathcal{H}(t) = 0$ , which proves the result.

## 9.6 Other applications of the virial argument to NLKG

As observed recently in [99], virial arguments in simpler situations also give some new results with elementary proofs. Consider the general 1D nonlinear Klein-Gordon equation

$$\partial_t^2 u - \partial_x^2 u + u = f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (16)$$

where the nonlinearity  $f$  satisfies  $f(0) = 0$  and for some  $p > 1$ ,  $|f(u)| \lesssim |u|^p$  for all  $|u| < 1$ . Note that using the energy conservation, small solutions of this equation are global and uniformly bounded in time (see *e.g.* Chapter 6 of [26]).

**Theorem 14** (Small odd solutions of NLKG, [99]). *Assume that  $f$  is odd. There exists  $\delta > 0$  such that for any odd  $(u_0, u_1) \in H^1 \times L^2$  such that  $\|(u_0, u_1)\|_{H^1 \times L^2} < \delta$ , the global odd solution  $u$  of (16) with initial data  $(u_0, u_1)$  satisfies, for any bounded interval  $I \subset \mathbb{R}$ ,*

$$\lim_{t \rightarrow +\infty} \|(u(t), \partial_t u(t))\|_{H^1(I) \times L^2(I)} = 0, \quad \int_0^{+\infty} \|(u(t), \partial_t u(t))\|_{H^1(I) \times L^2(I)}^2 dt \lesssim 1.$$

**Theorem 15** (Small solutions to NLKG with supercritical nonlinearity). *Assume that for some  $C > 0$ ,  $f$  satisfies, for all  $|u| < 1$ ,  $\frac{1}{2}uf(u) - F(u) \leq C|u|^6$ , where  $F(u) = \int_0^u f$ . Then, there exists  $\delta > 0$  such that, for any  $(u_0, u_1) \in H^1 \times L^2$  such that  $\|(u_0, u_1)\|_{H^1 \times L^2} < \delta$ , the global solution  $u$  of (16) with initial data  $(u_0, u_1)$  satisfies, for any bounded interval  $I \subset \mathbb{R}$ ,*

$$\int_0^{+\infty} (1+t)^{-4/5} \|(u(t), \partial_t u(t))\|_{H^1(I) \times L^2(I)}^2 dt \lesssim 1.$$

Both results imply the non-existence of small breathers under the specified assumptions. The assumptions of Theorem 15 are not satisfied by the sine-Gordon equation, which is consistent with the existence of arbitrarily small breathers for this equation. We refer to [105], for a discussion on *approximate breathers slowly radiating their energy*, and to [56, 168] for other closely related results of non-existence of breathers.

Note that because of the term  $(1+t)^{-4/5}$ , the information on the solution in Theorem 15 is weaker than the one obtained in Theorem 14 (the exponent  $4/5$  is not optimal). For a sequence  $t_n \rightarrow +\infty$ , it holds  $\lim_{n \rightarrow +\infty} \|(u(t_n), \partial_t u(t_n))\|_{H^1(I) \times L^2(I)} = 0$ , but we do not obtain convergence for the whole sequence of time.

For the sake of completeness, we briefly prove Theorem 15. For  $\lambda(t) = (1+t)^{1/2}$ , let

$$\psi(t, x) = \tanh\left(\frac{x}{\lambda(t)}\right) \quad \text{and} \quad \mathcal{I} = \int \left( \psi \partial_x u + \frac{1}{2} (\partial_x \psi) u \right) \dot{u}.$$

Using the equation satisfied by  $u$ , we find

$$\frac{d}{dt} \mathcal{I} = \int \left( \dot{\psi} \partial_x u + \frac{1}{2} (\partial_x \dot{\psi}) u \right) \dot{u} - \int \left\{ (\partial_x u)^2 \partial_x \psi - \frac{1}{4} u^2 \partial_x^3 \psi + \left( F(u) - \frac{1}{2} u f(u) \right) \partial_x \psi \right\}.$$

From the definition of  $\psi$ , we have

$$\begin{aligned} \dot{\psi}(t, x) &= -\frac{\dot{\lambda} x}{\lambda^2} \cosh^{-2}\left(\frac{x}{\lambda}\right) \quad \text{and thus} \quad |\dot{\psi}| \lesssim \frac{1}{1+t} \cosh^{-1}\left(\frac{x}{\lambda}\right), \\ \partial_x \psi(t, x) &= \frac{1}{\lambda} \cosh^{-2}\left(\frac{x}{\lambda}\right), \quad \partial_x^3 \psi(t, x) = \frac{1}{\lambda^3} \left( 4 \cosh^{-2}\left(\frac{x}{\lambda}\right) - 6 \cosh^{-4}\left(\frac{x}{\lambda}\right) \right). \end{aligned}$$

Therefore, we obtain the following estimates

$$\begin{aligned} \left| \int \dot{\psi} (\partial_x u) \dot{u} \right| &\lesssim \frac{1}{1+t} \int \cosh^{-1}\left(\frac{x}{\lambda}\right) |\partial_x u| |\dot{u}| \lesssim \delta \int (\partial_x u)^2 \partial_x \psi + C_\delta (1+t)^{-3/2} \int |\dot{u}|^2, \\ \left| \int (\partial_x \dot{\psi}) u \dot{u} \right| + \left| \int u^2 \partial_x^3 \psi \right| &\lesssim (1+t)^{-3/2} \int (|u|^2 + |\dot{u}|^2). \end{aligned}$$

Last, by the assumption of  $f$ ,

$$\int \left( F(u) - \frac{1}{2} u f(u) \right) \partial_x \psi \gtrsim - \int |u|^6 \partial_x \psi.$$

Moreover, from Lemma 3.3 in [121] (see also Lemma 6 in [122] and Lemma 2.1 in [139]), we have

$$\int |u|^6 \partial_x \psi \lesssim \left( \int u^2 \right)^2 \left\{ \int |u_x|^2 \partial_x \psi + \int |u|^2 \frac{(\partial_x^2 \psi)^2}{\partial_x \psi} \right\} \lesssim \delta^4 \int |u_x|^2 \partial_x \psi + (1+t)^{-3/2}.$$

Thus, we obtain (for  $\delta > 0$  small enough),

$$-\frac{d}{dt} \mathcal{I} \gtrsim \int (\partial_x u)^2 \partial_x \psi - C(1+t)^{-3/2}.$$

Integrating on  $(0, +\infty)$ , using the bound  $\sup_t |\mathcal{I}| \lesssim 1$ , we obtain

$$\int_0^{+\infty} (1+t)^{-1/2} \int (\partial_x u)^2 \cosh^{-2}\left(\frac{x}{\lambda}\right) dt \lesssim 1.$$

Let  $I$  be a bounded interval of  $\mathbb{R}$ . From the uniform bound  $\int u^2 \lesssim 1$ , for any  $t \in \mathbb{R}$ , there exists  $|\sigma(t)| \lesssim (1+t)^{1/4}$  such that  $u^2(t, \sigma(t)) \lesssim (1+t)^{-1/4}$ . Therefore, for any  $x \in I$ ,

$$\begin{aligned} u^2(t, x) &\leq u^2(t, \sigma(t)) + 2 \int_{|y| \lesssim (1+t)^{1/4}} |\partial_x u(t, y)| |u(t, y)| dy \\ &\lesssim (1+t)^{-1/4} + (1+t)^{1/4} \int_{|y| \lesssim (1+t)^{1/4}} |\partial_x u(t, y)|^2 dy. \end{aligned}$$

This implies that

$$\int_0^{+\infty} (1+t)^{-4/5} \int_I (u^2 + (\partial_x u)^2)(t) dt \lesssim 1 + \int_0^{+\infty} (1+t)^{-11/20} \int_{|y| \lesssim (1+t)^{1/4}} |\partial_x u(t, y)|^2 dy \lesssim 1,$$

and the result is proved.

*Remark 2.* For NLKG with negative mass  $\partial_t^2 u - \partial_x^2 u - u = f(u)$ , small solutions are not necessarily global. However, considering small global solutions, it is easy to prove that they converge to zero as in Theorem 14, without any oddness assumption or any assumption of  $g$ . The idea is to consider  $\psi(x) = \tanh(x)$  and  $\mathcal{I} = \int \psi(\partial_x u) \dot{u}$ . Then,

$$\frac{d}{dt} \mathcal{I} = -\frac{1}{2} \int ((\partial_t u)^2 + (\partial_x u)^2 + u^2 - F(u)) \psi'.$$

For small  $u$ , we have  $|F(u)| \lesssim \delta |u|^2$ , and thus,  $-4 \frac{d}{dt} \mathcal{I} \geq \int ((\partial_t u)^2 + (\partial_x u)^2 + u^2) \psi'$ . The rest of the proof is identical to Theorem 14 in [99]. Other results can also be proved similarly for the zero mass case (the wave equation).

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