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Discretization of Euler's equations using optimal transport: Cauchy and boundary value problems

Quentin Mérigot*

Abstract

This note presents a numerical method based on optimal transport to construct minimal geodesics along the group of volume preserving maps, equipped with the L^2 metric. As observed by Arnold, such geodesics solve the Euler equations of inviscid incompressible fluids. The method relies on the generalized polar decomposition of Brenier, numerically implemented through semi-discrete optimal transport and it is robust enough to extract non-classical, multi-valued solutions of Euler's equations predicted by Brenier and Schnirelman [Mérigot and Mirebeau, SIAM J. Num. Anal., 54**(6)**, 2016]. In a second part, we explain how this approach also leads to a numerical scheme able to approximate regular solutions to the Cauchy problem for Euler's equations [Gallouët and Mérigot, J. Found Comput Math, 2017].

1 Introduction

The motion of an inviscid incompressible fluid, moving in a compact domain $\Omega \subset \mathbb{R}^d$, is described by the Euler equations [Eul65]

$$\begin{cases} \partial_t v + (v \cdot \nabla) v = -\nabla p \\ \mathrm{div} v = 0, \end{cases} \tag{1}$$

coupled with the impervious boundary condition $v \cdot n = 0$ on $\partial\Omega$. As noticed by Arnold [Arn66], when expressed in Lagrangian coordinates, Euler's equations can be interpreted as the equation of geodesics in the infinite-dimensional group of measure-preserving diffeomorphisms of Ω. To see this, consider the flow map $s : [0, T] \times \Omega \to \Omega$ induced by the vector field v, that is $\frac{d}{dt} s(t, x) = v\,(t, s(t, x))$ and $s(0) = \mathrm{id}$. Euler's equations (1) can therefore be reformulated as

$$\frac{d^2}{dt^2} s(t) = -\nabla p(t, s(t, x)). \tag{2}$$

Using $\frac{d}{dt} \det \mathrm{D}s(t, x) = \mathrm{div}(v(t, x)) \det \mathrm{D}s(t, x)$, the incompressibility constraint $\mathrm{div}(v(t, x)) = 0$ and the initial condition $s(0) = \mathrm{id}$, one can check that $s(t, \cdot)$

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belongs to the set SDiff of diffeomorphisms of Ω with unit Jacobian. The pressure term in the evolution equation in (2) expresses that the acceleration of s should be orthogonal to the tangent plane to SDiff at s, where SDiff is regarded as a submanifold of $\mathrm{L}^2(\Omega, \mathbb{R}^d)$. Indeed, the tangent plane to SDiff at $s(t)$ is the set $\{v \circ s(t) \mid \mathrm{div}(v) = 0\}$, whose orthogonal is the space of gradients of functions. The evolution equation in (2) expresses that $\frac{d^2}{dt^2} s(t) \perp T_{s(t)}\mathrm{SDiff}$, in other words that $t \mapsto s(t, \cdot)$ is a geodesic of SDiff. This formalism leads to two natural problems:

- The standard Cauchy problem, forward in time: given the initial position and velocity of the fluid particles, find their subsequent positions at all positive times. This amounts to computing the exponential map on the Lie group SDiff.
- The boundary value problem: given some observed initial and final positions of the fluid particles, find their intermediate states. This amounts to computing a minimizing geodesic on SDiff.

Our purpose here is to show how Arnold's geometric interpretation of Euler's equations, refined by Brenier with the help of optimal transport, naturally lead to numerical schemes for both problems [MM16, GM17], improving on early numerical experiments by Brenier [Bre89a].

2 Polar factorization and semi-discrete optimal transport

In order to perform computations, the diffeomorphism $s(t) \in \mathrm{SDiff}$ will be approximated by a piecewise constant map $m(t)$. A piecewise constant map is never a diffeomorphism and cannot be measure-preserving. One way to express the incompressibility constraint is to use the geometry of the ambient space $\mathrm{L}^2(\Omega, \mathbb{R}^d)$ in which both SDiff and piecewise-constant maps embed. Our scheme will make sure that the L^2 distance between $m(t)$ and SDiff remain small. Since SDiff is not closed with respect to the L^2 metric, we will use instead the larger subset of *measure preserving maps* $\mathbb{S}$.

Definition 2.1 (Measure-preserving maps). *Let $\mathbb{S}$ be the set of measurable maps $\sigma : \Omega \to \mathbb{R}^d$ sending the restriction of the Lebesgue measure on Ω (henceforth denoted* Leb*) to itself. The measure-preserving condition can be written as* $\sigma_\#\mathrm{Leb} = \mathrm{Leb}$*, which reads* $\mathrm{Leb}(\sigma^{-1}(A)) = \mathrm{Leb}(A)$ *for all $A \subseteq \mathbb{R}^d$ measurable. We also denote $\mathrm{d}_\mathbb{S} : m \in \mathrm{L}^2(\Omega, \mathbb{R}^d) \to \mathbb{R}$ the distance function to $\mathbb{S}$:*

$$\mathrm{d}_\mathbb{S}(m) := \inf_{\sigma \in \mathbb{S}} \|m - \sigma\|_{\mathrm{L}^2(\Omega, \mathbb{R}^d)}.$$

The polar decomposition theorem of Brenier [Bre91] allows to compute the distance to the set of measure-preserving maps explicitely.

Theorem 2.2 (Brenier). *Any $m \in \mathrm{L}^2(\Omega, \mathbb{R}^d)$ admits a projection on $\mathbb{S}$. Given any such projection σ, there exists $\phi : \Omega \to \mathbb{R}^d$ convex such that $m = \nabla\phi \circ \sigma$.*

Moreover,

$$\mathrm{d}_{\mathbb{S}}^2(m) = \|\nabla\phi\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^d)}$$

This theorem can be reformulated in terms of optimal transport. The *Wasserstein distance* between two probability measures μ and ν on $\mathbb{R}^d$, where μ is absolutely continuous with respect to Lebesgue, can be defined as

$$\mathrm{W}_2^2(\mu,\nu) = \inf_{T_\#\mu=\nu} \int \|T(x)-x\|^2 \mathrm{d}\mu(x). \tag{3}$$

The polar factorization theorem implies that $\mathrm{d}_{\mathbb{S}}^2(m) = \mathrm{W}_2^2(m_\#\mathrm{Leb}, \mathrm{Leb})$: computing the distance to $\mathbb{S}$ amounts to solving an optimal transport problem. Before explaining how to use the squared distance $\mathrm{d}_{\mathbb{S}}^2$ to approximate geodesics in SDiff, we will first explain how to discretize the space $\mathbb{M}$.

Finite-dimensional approximation In order to perform numerics, we replace the infinite-dimensional ambient space $\mathbb{M} = \mathrm{L}^2(\Omega,\mathbb{R}^d)$ by a finite-dimensional space $\mathbb{M}_N$, where $N \geq 1$. We fix a partition of Ω into N subdomains $(\omega_j)_{1\leq j\leq N}$ with equal area and with diameter $\leq C(1/N)^{1/d}$. We then denote $\mathbb{M}_N$ the space of piecewise-constants functions over this partition:

$$\mathbb{M}_N = \{m \in \mathrm{L}^2(\Omega,\mathbb{R}^d) \mid m|_{\omega_j} = \mathrm{cst}\}.$$

Computing $\mathrm{d}_{\mathbb{S}}^2(m) = \mathrm{W}_2(m_\#\mathrm{Leb},\mathrm{Leb})^2$ therefore amounts to the computation of the Wasserstein distance between the Lebesgue measure and the finitely supported measure $m_\#\mathrm{Leb}$. Such optimal transport problems are often called semi-discrete, because only one measure is discretized, and they can be solved efficiently relying on Kantorovich duality and tools from computational geometry [AHA98, Mer11, dGBOD12, Lév14]. Let $\nu = m_\#\mathrm{Leb} = \frac{1}{N}\sum_{y\in Y}\delta_y$ for some set Y with cardinal N. Kantorovich duality asserts that

$$W_2^2(\mathrm{Leb},\nu) = \sup_{\substack{\phi:\Omega\to\mathbb{R},\psi:Y\to\mathbb{R}\\ \phi(x)-\psi(y)\leq\|x-y\|^2}} \int_\Omega \phi(x)\, d\mathrm{Leb}(x) - \int \psi(y)\, d\nu(y).$$

For any fixed $\psi : Y \to \mathbb{R}$ the largest function $\phi : \Omega \to \mathbb{R}$ obeying the constraint is given by $\phi(x) := \min_{y\in Y} \|x-y\|^2 - \psi(y)$. This function is piecewise quadratic over a partition $(\mathrm{Lag}_y(g))_{y\in Y}$ of Ω into convex polyhedra

$$\mathrm{Lag}_y(\psi) := \left\{x \in \Omega \mid \forall z \in Y,\ \|x-y\|^2 + \psi(y) \leq \|x-z\|^2 + \psi(z)\right\}.$$

Eliminating the optimization variable ϕ, Kantorovich duality reads

$$W_2^2(\mathrm{Leb},\nu) = \sup_{\psi:Y\to\mathbb{R}} \sum_{y\in Y} \int_{\mathrm{Lag}_y(\psi)} (\|x-y\|^2 + \psi(y))\, dx - \frac{1}{N}\sum_{y\in Y}\psi(y). \tag{4}$$

This partition of Ω induced by $(\mathrm{Lag}_y(\psi))_{y\in Y}$ is called the Laguerre diagram of (Y,ψ). It can be be computed in near-linear time in $\mathbb{R}^2$ using existing software [cga]. The maximization problem (4) is an unconstrained, concave and twice continuously differentiable maximization problem, which is efficiently solved via Newton or quasi-Newton methods. Semi-discrete optimal transport has become a reliable and efficient building block for PDE discretizations [BCMO14].

Gradient of $\mathrm{d}_{\mathbb{S}}^2$ The numerical schemes will use the gradient of $\mathrm{d}_{\mathbb{S}}^2$, so we will say a few word about it. As the square of a distance, the function $\mathrm{d}_{\mathbb{S}}^2$ is semi-concave, and therefore its restriction to the finite-dimensional space $\mathbb{M}_N$ is differentiable almost everywhere. One also expects that $\nabla \mathrm{d}_{\mathbb{S}}^2(m) = 2(m - P_{\mathbb{S}}(m))$ where $P_{\mathbb{S}}(m)$ is the closest point to m on $\mathbb{S}$. However, $\mathbb{S}$ is not convex and this projection map is not always uniquely defined, and is actually *never* uniquely defined on $\mathbb{M}_N$. Nonetheless, the following proposition can be proven using elementary tools from convex analysis. The diagonal $\mathbb{D}_N$ denotes the set of functions m in $\mathbb{M}_N$ such that $m(\omega_j) = m(\omega_k)$ for some $j \neq k$.

Proposition 2.3. *The functional $\mathrm{d}_{\mathbb{S}}^2$ is continuously differentiable on $\mathbb{M}_N \setminus \mathbb{D}_N$. If T is the optimal transport map between* Leb *and* $m_\#\mathrm{Leb}$*, then,*

$$\frac{1}{2}\nabla \mathrm{d}_{\mathbb{S}}^2|_{\mathbb{M}_N}(m) = m - P_{\mathbb{M}_N} P_{\mathbb{S}}(m), \tag{5}$$

where $P_{\mathbb{M}_N}$ is the orthogonal projection on the space of piecewise-constant maps.

More precisely, let $Y = \{m(\omega_j) \mid 1 \leq j \leq N\}$ and consider $\psi : Y \to \mathbb{R}$ solving the optimal transport problem (4). Define $b_j(m)$ as the isobarycenter of the jth Laguerre cell. Then, letting $\mathbf{1}_{\omega_j}$ be the indicator function of ω_j,

$$\frac{1}{2}\nabla \mathrm{d}_{\mathbb{S}}^2|_{\mathbb{M}_N}(m) = m - \sum_{1 \leq j \leq N} b_j(m) \mathbf{1}_{\omega_i}. \tag{6}$$

The first column of Figure 1 display two point sets Y^1 and Y^2 of the unit square with $N = 400$ points. The second column displays the Laguerre cells associated to the computation of $\mathrm{d}_{\mathbb{S}}^2(m^i)$ with $m^i = \frac{1}{N}\sum_{y \in Y^j} \delta_{y_j}$. The last two rows display minus the gradient of the squared distance to the set of measure preserving maps and $m^i - \nabla \mathrm{d}_{\mathbb{S}}^2|_{\mathbb{M}_N}(m^i) = P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(m^i)$.

3 Minimizing geodesics in SDiff

In this first section, we present the strategy introduced in [MM16] to construct approximate minimizing geodesics in SDiff. Let $s_* = \mathrm{id}$ and let $s^* \in \mathrm{SDiff}$ be a map which gives the final position $s^*(x)$ of each fluid particle initially at position $x \in \Omega$. The minimizing geodesics problem is

$$\text{minimize} \int_0^1 \|\dot{s}(t)\|^2_{\mathrm{L}^2(\Omega,\mathbb{R}^d)} dt, \quad \text{subject to} \begin{cases} \forall t \in [0,1],\ s(t) \in \mathbb{S}, \\ s(0) = s_*,\ s(1) = s^*. \end{cases} \tag{7}$$

The set of unit Jacobian diffeomorphism SDiff is not closed for the L^2 metric, and we have therefore replaced it with the space of measure-preserving maps $\mathbb{S}$, which in dimension $d \geq 3$ is the closure of SDiff. Despite this relaxation, the optimization problem (7) needs not have a minimizer in $s \in H^1([0,1],\mathbb{S})$ in dimension $d \geq 3$ [Shn94], and minimizing sequences $(s_n)_{n\geq 1}$ may instead display oscillations reminiscent of an homogeneization phenomenon. A second relaxation is therefore necessary, we refer the reader to [FD12] for a review.

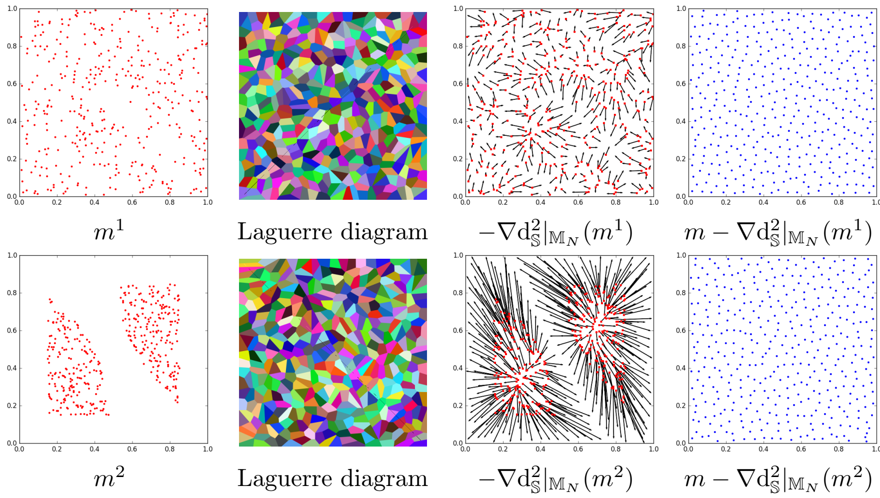


Figure 1: Projection of a piecewise constant map on the set of measure preserving maps.

3.1 Generalized flows

Brenier introduced in [Bre89b] a second relaxation of the minimizing geodesic problem (7) based on generalized flows, which allow particles to split and their paths to cross. This surprising behavior seems to be an unavoidable consequence of the lack of viscosity in Euler's equations. Generalized flows are also relevant in dimension $d \in \{1,2\}$ if the underlying physical model actually involves a three dimensional domain $\Omega \times [0,\varepsilon]^{3-d}$ in which one neglects the fluid acceleration in the extra dimensions [Bre08].

Definition 3.1 (Generalized flow)**.** *Consider the space of continuous paths (of fluid particles)* $\Gamma := C^0([0,1],\Omega)$*. A* generalized flow*, in Brenier's sense [Bre89b], is a probability measure* μ *over* Γ*. We denote by* $\mathrm{e}_t(\gamma) := \gamma(t)$ *the evaluation map at time* $t \in [0,1]$*, so that the pushforward measure* $\mathrm{e}_t\#\mu$ *can be understood as the distribution of particles at time* t *under the flow.*

Any classical flow $s \in H^1([0,1],\mathbb{S})$ can be represented by a generalized flow μ_s, supported on the paths $\gamma_x : t \mapsto s(t,x)$, weighted by the Lebesgue measure on $x \in \Omega$. The use of generalized flows turns the non-linear incompressibility constraint $s(t) \in \mathbb{S}$ into the linear constraint $\mathrm{e}_t\#\mu = \mathrm{Leb}$. This similar to Kantorovich's linearization of the non-linear mass preservation constraint in Monge's optimal transport problem. This idea leads to a convex relaxation of the minimizing geodesic distance problem (7), for which the existence of a minimizer is guaranteed:

$$d^2(s_*, s^*) := \min_{\mu\in\mathcal{P}(\Gamma)} \int_\Gamma \mathcal{A}(\gamma) d\mu(\gamma), \qquad \text{where } \mathcal{A}(\gamma) = \int_0^1 |\dot\gamma(t)|^2 dt$$
$$\text{subject to} \begin{cases} \forall t\in[0,1],\ \mathrm{e}_t\#\mu = \mathrm{Leb}, \\ (\mathrm{e}_0,\mathrm{e}_1)\#\mu = (s_*, s^*)\#\mathrm{Leb}. \end{cases} \tag{8}$$

The second constraint $(\mathrm{e}_0,\mathrm{e}_1)\#\mu = (s_*, s^*)\#\mathrm{Leb}$ rephrases the prescription of the endpoints ($s(0) = s_*$ and $s(1) = s^*$) in the minimizing geodesic problem (7) by imposing a coupling between particle positions at initial and final times.

Pressure. The incompressibility constraint in (8) gives rise to a Lagrange multiplier, called the pressure and which generalises the field p in (1). Surprisingly, the pressure turns out to be the *unique* maximizer to a concave optimization problem dual to (8), up to trivial invariance (that is, the addition of a function depending only on time) [Bre93]. Moreover, the pressure is a classical function $p \in L^2_{\mathrm{loc}}(]0,1[, \mathrm{BV}(\Omega))$, at least when the domain Ω is a d-dimensional torus [AF07]. This regularity is sufficient to show that any solution s to (7) (resp. μ-almost any path γ, for any solution μ to (8)) satisfies

$$\partial_{tt} s(t,x) = -\nabla p(t, s(t,x)), \qquad \text{resp.} \quad \ddot\gamma(t) = -\nabla p(t,\gamma(t)). \tag{9}$$

This implies that the support of μ is contained in the space of solutions to a second-order ODE, a fact that is used in our convergence estimates.

3.2 Discretization and convergence

Given two measure-preserving maps $s_*, s^* \in \mathbb{S}$, discretization parameters $T, N \geq 1$, and a penalization factor $\lambda \gg 1$, we introduce the functional which to $m \in \mathbb{M}_N^{T+1}$ associates

$$\mathcal{E}_{T,N,\lambda}(m) := T \sum_{0\leq i<T} \|m_{i+1} - m_i\|^2 + \lambda\left(\|m_0 - s_*\|^2 + \|m_T - s^*\|^2 + \sum_{1\leq i<T} d^2_{\mathbb{S}}(m_i)\right). \tag{10}$$

Comparing this with (7), we recognize the standard discretization of the length of the discrete path $(m_0, \cdots, m_T)$, as well as an implementation by penalization of the boundary value constraints. The last term corresponds to a penalization of the incompressibility constraints, or more precisely a penalization of the squared distance between the maps m_i and the set $\mathbb{S}$ of measure preserving maps. The discrete optimization problem we consider is then

$$\min_{m\in\mathbb{M}_N^{T+1}} \mathcal{E}_{T,N,\lambda}(m). \tag{11}$$

The problem (11), considered as a function of $m \in \mathbb{M}_N^{T+1}$, is an $N(T+1)d$-dimensional unconstrained optimization problem. Note that the functional $\mathcal{E}$ is non-convex and $\mathcal{C}^2$ smooth only on a dense open set. However, the use of a quasi-Newton method (L-BFGS) gives satisfactory results.

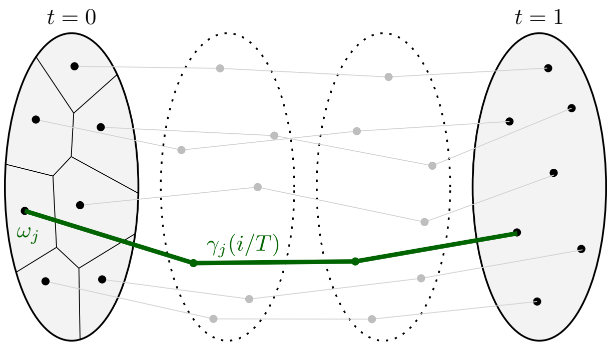


Figure 2: From a chain of maps to a probability measure over paths.

Convergence towards generalized geodesics In order to prove convergence of discrete solutions towards generalized geodesics, we need to embed $\mathbb{M}_N^{T+1}$ in the space $\mathcal{P}(\Gamma)$. To do so, note that $m \in \mathbb{M}_N^{T+1}$ is equivalently described by a family of points $(M_{i,j})_{1\leq i\leq T, 1\leq j\leq N}$ in $\mathbb{R}^d$, where $M_{i,j}$ is the position of the point j at time i. The chain $M_{1,j}, \dots, M_{T,j}$ describes the successive positions of the jth particle, which can be interpolated into a piecewise-affine path $\gamma_j : [0,1] \to \mathbb{R}^d$ satisfying $\gamma_j(\frac{i}{T}) = M_{ij}$. One can then associate to $m \in \mathbb{M}_N^T$ a generalized flow $\mu[m] \in \mathcal{P}(\Gamma)$,

$$\mu[m] := \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\gamma_j}.$$

This construction is illustrated in Figure 2.

Definition 3.2 (Regular flow)**.** *We call regular flow between s_* and $s^* \in$ SDiff a probability measure $\mu \in \mathcal{P}(\Gamma)$ satisfying*

(i) the incompressibility constraint $e_{t\#}\mu = \mathrm{Leb}$
(ii) the boundary conditions $(e_0, e_1)_\#\mathrm{Leb} = (s_*, s^*)_\#\mathrm{Leb}$
(iii) μ-almost every $\gamma \in \Gamma$ satisfies $\ddot{\gamma} = -\nabla p \circ \gamma$, where $p : \Omega \times [0,1]$ has Lipschitz gradient.

Note that if the Hessian of the pressure p is sufficiently small, namely that

$$\forall t \in [0,1],\ \forall x \in \Omega,\ \nabla^2 p(t,x) \prec \pi^2 \mathrm{id} \tag{12}$$

in the sense of symmetric matrices, then a regular flow is actually a minimizing geodesic. This fact has been used by Brenier to prove that classical solutions to Euler's equations whose pressure satisfy this condition are in fact minimizing the relaxed energy [Bre89b]. The following theorem proves the convergence of solutions of the discrete problems (11) towards regular minimizing geodesics [MM16].

Theorem 3.3. *Let $s_*, s^* \in$ SDiff, and assume that there exists a generalized minimizing geodesic, solving* (8)*, which is also a regular flow. Let $m_N \in$*

$\arg\min \mathcal{E}_{N,T_N,\lambda_N}$, *where* $\lambda_N = N^{2d}$ *and* $\lim_{N\to\infty} T_N\lambda_N = 0$. *Then, up to subsequences,* $\mu[m_N] \in \mathcal{P}(\Gamma)$ *converges weakly to a probability measure* $\mu \in \mathcal{P}(\Gamma)$ *which is a generalized minimizing geodesic between* s_* *and* s^*.

This theorem is proven in the spirit of Γ-convergence. The main difficulty is to establish that $\limsup_{N\to\infty} \mathcal{E}_{N,T_N,\lambda_N} \leq \mathcal{E}(\mu)$, where $\mu \in \mathcal{P}(\Gamma)$ is the regular generalized minimizing geodesic and where $\mathcal{E}(\mu) = \int \mathcal{A}(\gamma)\mathrm{d}\mu(\gamma)$. To do that, it suffices to construct sequences $\tilde{m}_N \in \mathbb{M}_N^{T_N}$ from μ such that $\mathcal{E}_{N,T_N,\lambda_N}(\tilde{m}_N) \leq (1+o(1))\mathcal{E}(\mu)$. The main idea is to use optimal quantization [Gru04] to approximate μ by an empirical measure of the form

$$\mu_N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\gamma_i},$$

and to control the rate of convergence of μ_N to μ. This is where the regularity hypothesis plays a role, as it implies that μ is not supported on the whole infinite-dimensional space $\Gamma = \mathcal{C}^0([0,1],\Omega)$ but on the subset

$$\Gamma_p = \{\gamma \in \mathcal{C}^2([0,1],\Omega) \mid \ddot{\gamma} = -\nabla p \circ \gamma\},$$

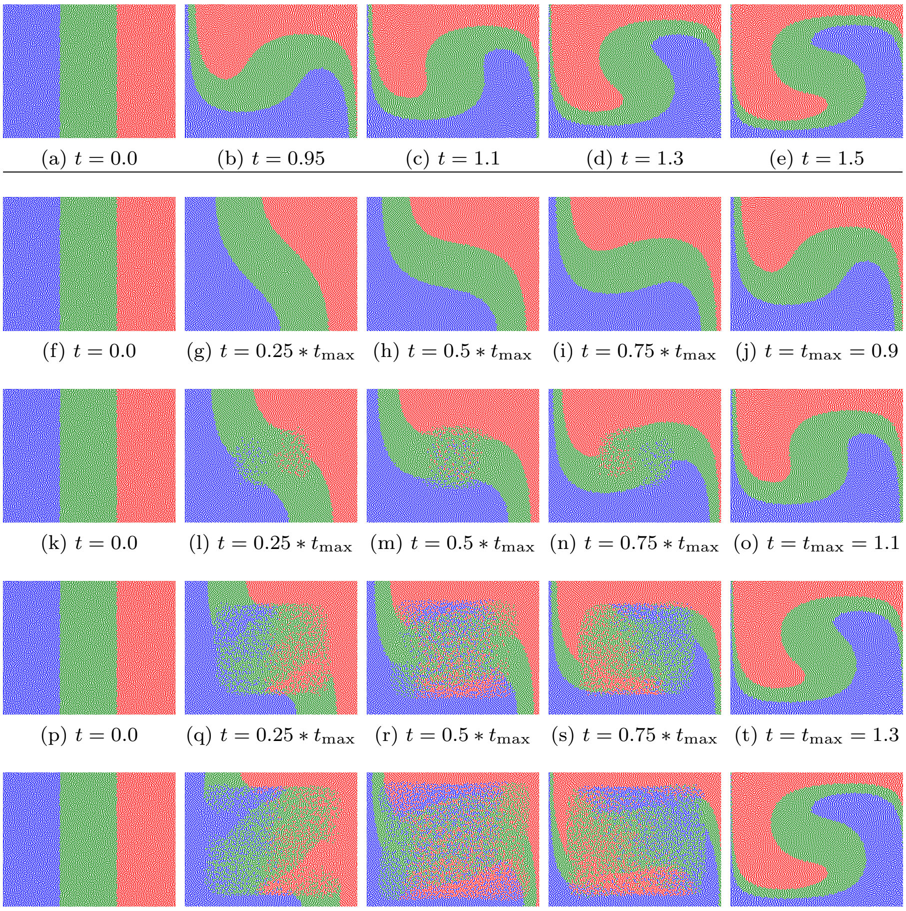
which has dimension $2d$ by the Cauchy-Lipschitz theorem. Thanks to this upper bound on the dimension of $\mathrm{spt}(\mu)$, one can control the rate of convergence of μ_N towards μ, and finally the energy of the discrete chain constructed from μ_N to the energy of μ. The condition $\lambda_N = N^{2d}$ comes from $\dim(\Gamma_p) = 2d$. More generally, if $\dim(\Gamma_p) = D$ one can choose $\lambda_N = N^D$ in order to get convergence.

A numerical result Figures 3 and 4 from [MM16] illustrate, for the first time, a sharp result of Brenier [Bre89b]: for the Beltrami flow on the square, the classical solution to Euler's equation is also a minimizing geodesic as long as $t_{\max} < 1$. If this threshold is exceeded, then several minimizers of (8) may exist, and some or all of them may be non-classical generalized flows $\mu \in \mathcal{P}(\Gamma)$. In that case the support of the flow μ has dimension $> d$, so that some fluid particles follow non-deterministic paths.

4 Cauchy problem for Euler's equations

In this section, we present the construction of a numerical scheme solutions to the Cauchy problem for Euler's equation (1). This scheme has been introduced in [GM17], and is strongly inspired by a particle scheme of Brenier [Bre00]. The solution to Euler's equations (1) is considered as a geodesic in SDiff, which is then approximated by the solution to a differential equation in the finite-dimensional space $\mathbb{M}_N$:

$$\begin{cases} \ddot{m}(t) + \dfrac{\nabla \mathrm{d}^2_{\mathbb{S}}(m(t))}{2\varepsilon^2} = 0, \quad \text{for } t \in [0,T], \\ (m(0), \dot{m}(0)) \in \mathbb{M}_N^2 \end{cases} \tag{13}$$



(a) $t = 0.0$ (b) $t = 0.95$ (c) $t = 1.1$ (d) $t = 1.3$ (e) $t = 1.5$

(f) $t = 0.0$ (g) $t = 0.25 * t_{\max}$ (h) $t = 0.5 * t_{\max}$ (i) $t = 0.75 * t_{\max}$ (j) $t = t_{\max} = 0.9$

(k) $t = 0.0$ (l) $t = 0.25 * t_{\max}$ (m) $t = 0.5 * t_{\max}$ (n) $t = 0.75 * t_{\max}$ (o) $t = t_{\max} = 1.1$

(p) $t = 0.0$ (q) $t = 0.25 * t_{\max}$ (r) $t = 0.5 * t_{\max}$ (s) $t = 0.75 * t_{\max}$ (t) $t = t_{\max} = 1.3$

(u) $t = 0.0$ (v) $t = 0.25 * t_{\max}$ (w) $t = 0.5 * t_{\max}$ (x) $t = 0.75 * t_{\max}$ (y) $t = t_{\max} = 1.5$

Figure 3: *(First row)* The Beltrami flow in the unit square, shown at various timesteps, is a classical solution to Euler equations. The particles color depends on their initial position. *(Second to fifth row)* Generalized fluid flows reconstructed by our algorithm, using the boundary conditions displayed in the first and last column. When $t_{\max} < 1$ we recover the classical flow, while for $t_{\max} \geq 1$ the solution is not classical anymore and includes some mixing.

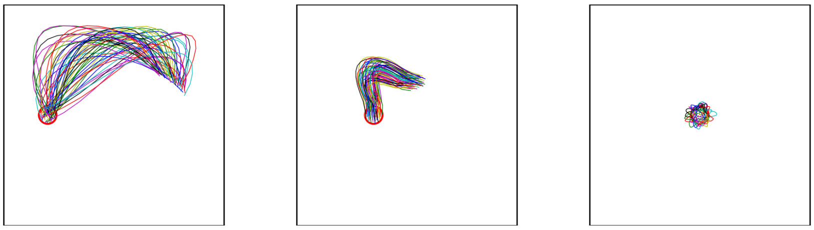


Figure 4: Particles trajectories in the case $t_{\max} = 1.5$, see Figure 3. Trajectories originating from inside the red circle on the left seem strongly non-deterministic.

The idea is that the point $m(t)$ want to move in straight line, but is forced by the term $\frac{1}{2\varepsilon^2}\nabla \mathrm{d}^2_{\mathbb{S}}(m(t))$ to remain close to the space $\mathbb{S}$. One difficulty, as before, is that we want to approximate a geodesic in $\mathbb{S}$ by a curve in $\mathbb{M}_N$ while the intersection $\mathbb{S} \cap \mathbb{M}_N$ is empty (and actually, these two sets are at positive distance from each other). This differential system is induced by the Hamiltonian $H(m, \dot{m}) = \frac{1}{2}\|\dot{m}\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}}(m)}{2\varepsilon^2}$. Equation (13) can be rewritten as a system of N particles in interaction, whose positions are denoted $M_1(t), \dots, M_N(t) \in \mathbb{R}^d$. Given $m(t) = \sum_j M_j(t)\mathbf{1}_{\omega_j(t)}$ and using (6), it is possible to rewrite this differential system in terms of barycenters of Laguerre cells, which depend on the solution to the optimal transport problem between Leb and the empirical measure $\frac{1}{N}\sum_{1\leq i\leq N}\delta_{M_i}$. With these notations, (13) is then equivalent to

$$\begin{cases} \ddot{M}_j(t) + \frac{1}{\varepsilon^2}\left(M_j(t) - b_j\left(\sum_i M_j(t)\mathbf{1}_{\omega_j(t)}\right)\right) = 0 \\ (M(0), \dot{M}(0)) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \end{cases} \tag{14}$$

Loosely speaking, equation (14) describe a physical system where each particle $M_j(t)$ is subject to the force of a spring with stiffness $\frac{1}{\varepsilon}$ attached to the point $b_j(M_1(t), \dots, M_N(t))$ which varies in time and depends on the position of all the particles.

The main result of [GM17] is that the system of equations (13) can be used to approximate regular solutions to Euler's equations (1). It relies on a modulated energy technique which is similar to that used in [Bre00] and requires $\mathcal{C}^{1,1}$ regularity assumptions on the solution to Euler's equations. See also [BL04, CGP07] for related works.

Theorem 4.1 ([GM17])**.** *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary. Let v, p be a strong solution of Euler's equations* (1)*, let s be the flow map induced by v (see* (2)*) and assume that $v, p, \partial_t v, \partial_t p, \nabla v$ and ∇p are Lipschitz on Ω, uniformly on $[0, T]$. Suppose in addition that there exists a $\mathcal{C}^1$ curve $m : [0, T] \to \mathbb{M}_N$ satisfying the initial conditions*

$$m(0) = P_{\mathbb{M}_N}(\mathrm{id}), \quad \dot{m}(0) = P_{\mathbb{M}_N}(v(0, \cdot)),$$

which is twice differentiable and satisfies the second-order equation (13) *for all* *times in* $[0, T]$*, possibly up to a countable number of exceptions. Then,*

$$\max_{t\in[0,T]} \|\dot{m} - v(t, \phi(t, \cdot))\|^2_{\mathbb{M}} \leq C \left(\frac{h_N^2}{\varepsilon^2} + \varepsilon^2 + h_N \right). \tag{15}$$

Numerical simulations show that this scheme is also able to recover the qualitative behavior of fluids, such as the Kelvin-Helmoltz and Rayleigh-Taylor instability, even if these situations are not encompassed in the current convergence theory. We refer to [GM17] for details on the time-discretization and for 2D numerical results. We also refer to [dGWH+15] for impressive 3D simulations using a similar discretization.

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