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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
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Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
F-91128 PALAISEAU CEDEX
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THE CUBIC SZEGŐ FLOW AT LOW REGULARITY

PATRICK GÉRARD AND HERBERT KOCH

Abstract. We prove that the cubic Szegő equation is well posed on the space BMO_+ of functions of bounded mean oscillation in the Hardy class of the disc, and we establish the Hölder regularity of this flow in the L^2 distance. We also show that the Cauchy problem is illposed on the corresponding L^∞ space.

1. Introduction

This paper is devoted to low regularity solutions of the cubic Szegő equation on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$,

$$i\partial_t u = \Pi(|u|^2 u) \tag{1}$$

where $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ denotes the orthogonal projector onto the closed subspace $L^2_+(\mathbb{T})$ of $L^2(\mathbb{T})$ defined by the cancellation of all negative Fourier modes,

$$\forall k < 0 \ , \ \widehat{u}(k) = 0 \ .$$

Recall that $L^2_+(\mathbb{T})$ can be identified to the Hardy space $\mathbb{H}^2(\mathbb{D})$ consisting of holomorphic functions u on the unit disc such that

$$\sup_{r<1} \int_0^{2\pi} \left| u(re^{ix}) \right|^2 \, dx < \infty \ .$$

In the sequel, we shall make use of this identification freely.

Equation (1) was introduced by S. Grellier and the first author in [5], where a flow on $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$, $s \geq 1/2$, was defined, and where a Lax pair structure was discovered. In [8], this equation was identified as the time averaged effective system to the half wave equation on $\mathbb{T}$. In [6], more precise integrability properties were established, while in [7] an explicit formula for H^s solutions was derived. Finally, a general nonlinear Fourier transform was constructed in [9], where almost periodicity of solutions in $H^{1/2}_+$ and growth of higher Sobolev

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norms were proved. Furthermore, analyticity of solutions was studied in [10].

Since Π is a pseudodifferential operator of order 0, it is natural to ask about solving Equation (1) for initial data with low regularity. For instance, the ordinary differential equation

$$i\partial_t u = |u|^2 u \tag{2}$$

is wellposed on $L^\infty(\mathbb{T})$, with the explicit formula

$$u(t,x) = \mathrm{e}^{-it|u(0,x)|^2} u(0,x)\ .$$

The purpose of this paper is to investigate how this property is modified by the action of the pseudodifferential operator Π. It is well known that Π is not bounded on $L^\infty(\mathbb{T})$. The space

$$\mathrm{BMO}_+(\mathbb{T}) = \{\Pi(b), b \in L^\infty(\mathbb{T})\}$$

was identified by Fefferman [3] as the intersection of $L^2_+(\mathbb{T})$ with the space $\mathrm{BMO}(\mathbb{T})$ of functions of bounded mean oscillation introduced by John and Nirenberg, see [13], [4], as the space of functions $f \in L^1(\mathbb{T})$ such that

$$\sup_I \frac{1}{|I|}\int_I |f(x) - \langle f\rangle_I|\, dx \ <\ +\infty\ ,\ \langle f\rangle_I := \frac{1}{|I|}\int_I f(x)\, dx\ , \tag{3}$$

where the supremum above is taken on all intervals $I \subset \mathbb{T}$. The space BMO_+ is also the dual of

$$L^1_+(\mathbb{T}) = \{h \in L^1(\mathbb{T}) : \forall k < 0\ ,\ \widehat{h}(k) = 0\}\ .$$

For every $u \in \mathrm{BMO}_+(\mathbb{T})$, we set

$$\|u\|_{\mathrm{BMO}} = \inf\{\|b\|_{L^\infty}, b \in L^\infty(\mathbb{T}), \Pi(b) = u\} = \|u\|_{(L^1_+)'}\ .$$

Our main result is the following.

Theorem 1. *For every $u_0 \in \mathrm{BMO}_+(\mathbb{T})$, there exists a unique function $u \in C^1(\mathbb{R}, L^2_+(\mathbb{T})) \cap C_{w*}(\mathbb{R}, \mathrm{BMO}_+(\mathbb{T}))$, solution of the initial value problem*

$$i\partial_t u = \Pi(|u|^2 u)\ ,\ u(0) = u_0\ . \tag{4}$$

Furthermore, $\|u(t)\|_{\mathrm{BMO}} = \|u_0\|_{\mathrm{BMO}}$. Moreover, if u, v are two BMO *solutions of* (1) *satisfying*

$$\|u(0)\|_{\mathrm{BMO}} + \|v(0)\|_{\mathrm{BMO}} \le M\ ,$$

there exists a constant K, depending only on M, such that, for every $t \in \mathbb{R}$,

$$\|u(t) - v(t)\|_{L^2} \le K\, \|u(0) - v(0)\|_{L^2}^{\alpha(t)}\ ,\ \alpha(t) := \mathrm{e}^{-K|t|}\ . \tag{5}$$

Next we come to propagation of Sobolev regularity. In the low regularity case, it is only partially obtained, as a consequence of the stability estimate (5).

Corollary 1. *Let u be a* BMO *solution of the cubic Szegő equation, as given by Theorem 1. Assume $u(0) = u_0 \in H^s$ for some $s > 0$. Then, if $s \geq 1/2$, $u(t) \in H^s(\mathbb{T})$ for every $t \in \mathbb{R}$. In the case $0 < s < 1/2$, there exists $K > 0$, depending only on a bound of $\|u_0\|_{\mathrm{BMO}}$, such that*

$$\forall t \in \mathbb{R}, u(t) \in H^{s(t)}(\mathbb{T}) \ , \ s(t) := \frac{s\mathrm{e}^{-K|t|}}{1 - 2s + 2s\mathrm{e}^{-K|t|}} \ .$$

Remark 1.
• We do not know whether the above exponent $s(t)$ is optimal or not. If it is optimal, such a loss of regularity could be compared to the one established by Bahouri and Chemin in Theorem 1.3 of [1] for the bidimensional incompressible Euler flow with bounded vorticity.
• The above corollary has a local version, which will be established in the forthcoming paper [11].

In the beginning of this note, we have seen that the ordinary differential equation (2) is well posed on $L^\infty(\mathbb{T})$. In contrast, using the John–Nirenberg definition (3), it is easy to prove that this equation is not wellposed on $\mathrm{BMO}(\mathbb{T})$. Indeed, though $u_0(x) = \log|\sin x|$ belongs to $\mathrm{BMO}(\mathbb{T})$, one can check that, for every $t \neq 0$, the function

$$u(t,x) = (\log|\sin x|)\mathrm{e}^{-it(\log|\sin x|)^2}$$

does not belong to $\mathrm{BMO}(\mathbb{T})$, because its average on $[\varepsilon, 2\varepsilon]$ is bounded as ε tends to 0. Somewhat symmetrically, the next result shows that the Szegő equation is illposed on L^∞. We denote by $C_+(\mathbb{T}) = C(\mathbb{T}) \cap L^2_+(\mathbb{T})$ the Banach space of continuous functions on the circle with nonnegative Fourier modes.

Theorem 2. *There exists a dense G_δ subset $\mathcal{G}$ of $C_+(\mathbb{T})$ such that, for every $u_0 \in \mathcal{G}$, the solution u of (4) satisfies*

$$\forall T > 0, u \notin L^\infty([0,T] \times \mathbb{T}) \ .$$

The present note will give a sketch of the proof of Theorem 1, Corollary 1 and Theorem 2. An extended version with more detailed proofs and additional results is in preparation [11].

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2. Proof of Theorem 1

The proof of Theorem 1 is based on two arguments. The first one is a characterization of $\mathrm{BMO}_+(\mathbb{T})$ which was established by Nehari [16] before the John–Nirenberg paper. Nehari's result — see also Theorem 1.1 of Peller [17] — claims that, given $u \in L^2_+(\mathbb{T})$, the Hankel operator $\Gamma_{\widehat{u}}$ defined on finitely supported sequence $\mathbf{x} := (x_n)_{n\geq 0}$ by

$$[\Gamma_{\widehat{u}}(\mathbf{x})]_p = \sum_{n=0}^{\infty} \widehat{u}(p+n)x_n$$

extends as a bounded operator on $\ell^2(\mathbb{N})$ if and only if $u \in \mathrm{BMO}_+(\mathbb{T})$, and that

$$\|\Gamma_{\widehat{u}}\|_{\ell^2\to\ell^2} = \|u\|_{\mathrm{BMO}} \ .$$

As we will recall below, it turns out that the Lax pair discovered in [5] allows to prove that, if a u is a smooth solution of (1), the operator norm $\|\Gamma_{\widehat{u}(t)}\|_{\ell^2\to\ell^2}$ is independent of t. This provides a BMO bound for the sequence (u_n) of smooth solutions of (1) which approximates u_0 at $t=0$ in $\mathcal{B}_{\mathrm{BMO}}(\|u_0\|_{\mathrm{BMO}})$.

The second argument relies on the John–Nirenberg inequality [13], [4], which claims that $\mathrm{BMO}_+(\mathbb{T}) \subset L^p(\mathbb{T})$ for every $p<\infty$, and that there exists a universal constant $C>0$ such that, for every $v \in \mathrm{BMO}_+(\mathbb{T})$, for every $p\in[1,\infty)$,

$$\|u\|_{L^p} \leq C\,p\,\|v\|_{\mathrm{BMO}} \ .$$

This inequality will allow us to prove that the sequence (u_n) is a Cauchy sequence in $C([-T,T],L^2_+(\mathbb{T}))$ for every $T<\infty$, leading to existence of solution u.

Let us come to the detailed proof of Theorem 1. We first recall the Lax pair structure of the cubic Szegő equation, as established in [5] and revisited in [7]. For every $u \in \mathrm{BMO}_+(\mathbb{T})$, define the antilinear Hankel operator

$$H_u : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$$

by the formula

$$H_u(h) = \Pi(u\overline{h}) \ , \ h\in L^2_+(\mathbb{T}) \ .$$

It is easy to check that H_u is bounded on $L^2 +_(\mathbb{T})$, and that

$$\widehat{H_u(h)} = \Gamma_{\widehat{u}}\left(\overline{\widehat{h}}\right) \ , \ \langle H_u(h_1),h_2\rangle = \langle H_u(h_2),h_1\rangle,$$

where $\langle f,g\rangle$ denotes the usual L^2 inner product. In particular,

$$H_u^2 \simeq \Gamma_{\widehat{u}}\Gamma_{\widehat{u}}^*$$

is a linear positive selfadjoint operator. From Nehari's theorem, we have

$$\|H_u\|_{L^2_+\to L^2_+} = \|u\|_{\mathrm{BMO}} \ . \tag{6}$$

Next we claim that, for every $a,b,c\in L^\infty_+(\mathbb{T})$,

$$H_{\Pi(a\bar{b}c)} = T_{a\bar{b}}H_c + H_aT_{b\bar{c}} - H_aH_bH_c \ , \tag{7}$$

where, for every $m\in L^\infty(\mathbb{T})$, the Toeplitz operator T_m is defined by

$$T_m(h) = \Pi(mh) \ , \ h\in L^2_+(\mathbb{T}) \ .$$

Indeed, given $h \in L^2_+(\mathbb{T})$, we have

$$\begin{aligned} H_{\Pi(a\bar{b}c}(h) &= \Pi(\Pi(a\bar{b}c)\overline{h}) = \Pi(a\bar{b}c\overline{h}) \\ &= \Pi(a\bar{b}\Pi(c\overline{h})) + \Pi(a\bar{b}(I-\Pi)(c\overline{h})) \\ &= T_{a\bar{b}}H_c(h) + H_a\left(b\,\overline{(I-\Pi)(c\overline{h})}\right) \ . \end{aligned}$$

The proof of (7) is completed by observing that

$$b\,\overline{(I-\Pi)(c\overline{h})} = \Pi\left(b\overline{(I-\Pi)(c\overline{h})}\right) = T_{b\bar{c}}(h) - H_bH_c(h) \ .$$

Now assume that u is a smooth solution to (1). Combining the equation and identity (7), we have

$$\frac{d}{dt}H_u = -iH_{\Pi(|u|^2u)} = -i(H_uT_{|u|^2} + T_{|u|^2}H_u - H_u^3) = [B_u, H_u] \ ,$$

where $[B, C]$ denotes the commutator of the operators B, C,

$$B_u := -iT_{|u|^2} + \frac{i}{2}H_u^2 \ ,$$

and where we have used the antilinearity of H_u in writing

$$i(H_uA + AH_u) = [iA, H_u]$$

for every linear operator A. Notice that B_u is an antiselfadjoint linear operator on $L^2_+(\mathbb{T})$. Solving the linear ODE

$$\frac{dU}{dt} = B_uU \ , \ U(0) = I \ . \tag{8}$$

in the space of bounded operators on L^2_+, we get a one parameter family $U(t)$ of unitary operators, which satisfies

$$\forall t \in \mathbb{R} \ , \ H_{u(t)} = U(t)H_{u(0)}U(t)^* \ . \tag{9}$$

From (9) and (6), we conclude

$$\forall t \in \mathbb{R} \ , \ \|u(t)\|_{\mathrm{BMO}} = \|u_0\|_{\mathrm{BMO}} \ . \tag{10}$$

We now come to the second step of the proof, for which the main point is the following stability lemma.

Lemma 1. *Let u, v be two smooth solutions of* (1)*, satisfying*

$$\|u_0\|_{\mathrm{BMO}} + \|v_0\|_{\mathrm{BMO}} \leq M \ .$$

There exists a constant K, depending only on M, such that, for every $t \in \mathbb{R}$,

$$\|u(t) - v(t)\|_{L^2} \leq K\,\|u_0 - v_0\|_{L^2}^{\mathrm{e}^{-K|t|}} \ .$$

Proof. Recall that we denote by

$$\langle f, g\rangle = \int_0^{2\pi} f(\mathrm{e}^{ix})\overline{g\,(\mathrm{e}^{ix})}\,\frac{dx}{2\pi}$$

the inner product on $L^2(\mathbb{T})$. Set $N(t) := \|u(t)-v(t)\|^2_{L^2}$. Assume $t\ge 0$ for simplicity, and compute

$$\frac{dN}{dt} = 2\mathrm{Im}\left\langle \Pi(|u|^2u) - \Pi(|v|^2v), u-v\right\rangle \ .$$

Applying the Taylor formula, we have, introducing $w_\theta := \theta u + (1-\theta)v$ for $\theta \in [0,1]$,

$$\Pi(|u|^2u) - \Pi(|v|^2v) = \int_0^1 (2T_{|w_\theta|^2} + H_{w_\theta^2})(u-v)\, d\theta \ .$$

Since $T_{|w_\theta|^2}$ is selfadjoint, its contribution to the imaginary part of the inner product cancels, and we are left with

$$\frac{dN}{dt} = 2\int_0^1 \mathrm{Im}\left\langle H_{w_\theta^2}(u-v), u-v\right\rangle\, d\theta \ .$$

Using identity (7) with $a=c=w_\theta$ and $b=1$, we obtain

$$\begin{aligned}\frac{dN}{dt} &= 2\int_0^1 \mathrm{Im}\left\langle \left(T_{w_\theta}H_{w_\theta} + H_{w_\theta}T_{\overline{w}_\theta} - H_{w_\theta}H_1H_{w_\theta}\right)(u-v), u-v\right\rangle\, d\theta \\ &= 4\int_0^1 \mathrm{Im}\left\langle H_{w_\theta}(u-v), \overline{w}_\theta(u-v)\right\rangle\, d\theta + 2\int_0^1 \mathrm{Im}\left(\langle w_\theta, u-v\rangle^2\right)\, d\theta \ .\end{aligned}$$

From the conservation of the BMO norm (10), we already know that $\|w_\theta\|_{\mathrm{BMO}} \le M$, and thus

$$\|H_{w_\theta}\|_{L^2_+\to L^2_+} \le M \ , \ \|w_\theta\|_{L^p} \le C\,M\,p \ .$$

Using Hölder's inequality, we infer, for large p and for every time $t\ge 0$,

$$\begin{aligned}|\langle H_{w_\theta}(u-v), \overline{w}_\theta(u-v)\rangle| &\le M\|u-v\|_{L^2}\|w_\theta|u-v|^{2/p}|u-v|^{1-2/p}\|_{L^2} \\ &\le M\|u-v\|_{L^2}\|w_\theta|u-v|^{2/p}\|_{L^p}\||u-v|^{1-2/p}\|_{L^{2p/p-2}} \\ &\le M(CMp)^{1+2/p}\|u-v\|^{2-2/p}_{L^2} \\ &\le \widetilde{C}(M)\,p\,N^{1-1/p} \ .\end{aligned}$$

We now choose, at a given time $t\ge 0$,

$$p = p(t) = 2 + \log(M^2/N(t)) \ge 2 \ ,$$

since, by the conservation of L^2 norms of u and v,

$$N(t) \le (\|u_0\|_{L^2} + \|v_0\|_{L^2})^2 \le M^2 \ .$$

We infer

$$\left|\frac{dN}{dt}\right| \le K(M)N\left(2+\log(M^2/N)\right) \ .$$

Solving this differential inequality, we obtain the lemma. ☐

Let us complete the proof of Theorem 1. Let $u_0 \in \mathrm{BMO}_+(\mathbb{T})$. Select a sequence (u_0^n) of smooth functions in L^2_+ such that

$$\|u_0^n - u_0\|_{L^2} \to 0 \ , \ \limsup \|u_0^n\|_{\mathrm{BMO}} \le \|u_0\|_{\mathrm{BMO}} \ .$$

For instance, one can choose

$$u_0^n(\mathrm{e}^{ix}) = u_0(r_n \mathrm{e}^{ix}) \ ,$$

where r_n is any sequence of positive numbers smaller than 1 converging to 1. Denote by u^n the solution of (1) with initial datum u_0^n. Then Lemma 1 implies that (u^n) is a Cauchy sequence in $C([-T,T], L^2_+)$ for every $T > 0$, hence it converges to $u \in C(\mathbb{R}, L^2_+)$. Furthermore,

$$\|u^n(t)\|_{\mathrm{BMO}} = \|u_0^n\|_{\mathrm{BMO}},$$

hence $u_n(t) \to u(t)$ in L^p for every $p < \infty$, locally uniformly in time. This allows to pass to the limit in Equation (1), so that u is a solution of (4), and moreover

$$\|u(t)\|_{\mathrm{BMO}} \le \limsup \|u_0^n\|_{\mathrm{BMO}} \le \|u_0\|_{\mathrm{BMO}} \ .$$

It remains to prove uniqueness of such solutions, and the conservation of the BMO norm. For uniqueness, we observe that the proof of Lemma 1 can be easily extended to solutions $u, v \in C(\mathbb{R}, L^2_+) \cap C_{w*}(\mathbb{R}, \mathrm{BMO}_+(\mathbb{T})$. Indeed, the only technical point is to extend the identity

$$\Pi(w^2\overline{h}) = wH_w(h) + H_w(\overline{w}h) - H_wH_1H_w(h)$$

to the case $w, h \in \mathrm{BMO}_+$. This can be easily achieved by approximation of w. This leads to estimate (5). Applying this estimate to $u_0 = v_0$, we conclude that there exists only one solution $u \in C(\mathbb{R}, L^2_+) \cap C_{w*}(\mathbb{R}, \mathrm{BMO}_+(\mathbb{T}))$ of (4).

As for the conservation of the BMO norm, it is enough to observe that, given $T \in \mathbb{R}$, that we already have an inequality,

$$\|u(T)\|_{\mathrm{BMO}} \le \|u_0\|_{\mathrm{BMO}} \ .$$

Now, precisely from what we did, the problem

$$i\partial_t v = \Pi(|v|^2 v) \ , \ v(0) = u(T)$$

has only one solution $v \in C(\mathbb{R}, L^2_+)$ and locally bounded in BMO, and $\|v(t)\|_{\mathrm{BMO}} \le \|v(0)\|_{\mathrm{BMO}}$. Therefore $v(t) = u(t+T)$, and applying the above inequality at $t = -T$ yields $\|u_0\|_{\mathrm{BMO}} \le \|u(T)\|_{\mathrm{BMO}}$, whence the desired equality.

3. Proof of Corollary 1

In the case $s \ge 1/2$, Corollary 1 is just a consequence of the uniqueness of the Cauchy problem in Theorem 1 and of the wellposedness theory in H^s [5].

In the case $0 < s < 1/2$, a first idea is to combine the stability estimate (5), the invariance of the flow by translation on $\mathbb{T}$, and the following representation of the H^s norm,

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \int_{-1}^{1}\int_{\mathbb{T}} \frac{|u(x+h)-u(x)|^2}{|h|^{1+2s}}\,dx\,dh\ .$$

However, this provides a result which does not take into account the conservation of the $H^{1/2}$ norm. Therefore we prefer to use the following interpolation argument, which was suggested to us by D. Tataru. Given $\lambda > 1$, one can decompose $u_0 \in H^s$ as

$$u_0 = u_0^{<\lambda} + u_0^{>\lambda},$$

with $\|u_0^{<\lambda}\|_{\mathrm{BMO}} \lesssim 1$,

$$\|u_0^{<\lambda}\|_{H^{1/2}} \lesssim \lambda^{\frac12 - s}\ ,\ \|u_0^{>\lambda}\|_{L^2} \lesssim \lambda^{-s}\ .$$

Then, by the conservation of the $H^{1/2}$ norm, $u^{<\lambda} := Z(u_0^{<\lambda})$ satisfies

$$\|u^{<\lambda}(t)\|_{H^{1/2}} \lesssim \lambda^{\frac12 - s}\ ,$$

while the stability estimate (5) yields, with $\alpha(t) = \mathrm{e}^{-K|t|}$ and $K = K(\|u_0\|_{\mathrm{BMO}})$,

$$\|u(t) - u^{<\lambda}(t)\|_{L^2} \lesssim \|u_0 - u_0^{<\lambda}\|^{\alpha(t)} \lesssim \lambda^{-s\alpha(t)}\ .$$

Therefore the the dyadic component $\Delta_k u(t)$ of $u(t)$ can be estimated, for every $\lambda > 0$, as

$$\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-k/2}\lambda^{\frac12 - s} + \lambda^{-s\alpha(t)}\ .$$

Choosing $\lambda = \lambda(k,t)$ optimally, we obtain

$$\|\Delta_k u(t)\|_{L^2} \lesssim 2^{-ks\alpha(t)/(1-2s+2s\alpha(t))}\ ,$$

and therefore $u(t) \in H^{s(t)}$ with

$$s(t) = \frac{s\mathrm{e}^{-\widetilde{K}|t|}}{1 - 2s + 2s\mathrm{e}^{-\widetilde{K}|t|}}$$

for every $\widetilde{K} > K$. This completes the proof.

4. Proof of Theorem 2

The arguments for Theorem 2 are an adaptation of a method developed by Elgindi and Masmoudi in [2], which leads to ill–posedness for the incompressible Euler equation at the C^1 regularity. The crucial step is the following lemma.

Lemma 2. *Let $u_0 \in C_+(\mathbb{T})$. There exists a sequence (u^n) of smooth solutions to the* (1) *such that*

$$\|u^n(0) - u_0\|_{L^\infty} \to 0\ ,$$

and a sequence of times $T_n > 0$ tending to 0 such that

$$\sup_{t\in[0,T_n]} \|u^n(t)\|_{L^\infty} \to \infty \ .$$

Let us show how Lemma 2 implies Theorem 2. For every $u_0 \in \mathrm{BMO}_+(\mathbb{T})$ and every $t \in \mathbb{R}$, we denote by $Z(t)(u_0)$ the value $u(t)$ at time t of the solution $u := Z(u_0)$ of (4) provided by Theorem 1. For every integer $p \geq 1$, denote by Ω_p the subset of those $u_0 \in C_+(\mathbb{T}))$ such that, for some $r_p \in]0,1[$, we have

$$\sup_{t\in[0,1/p]} \sup_{x\in\mathbb{T}} \left|Z(t)(u_0)\left(r_p \mathrm{e}^{ix}\right)\right| > p \ .$$

We claim that Ω_p is an open subset of $C_+(\mathbb{T})$. Indeed, for every $r < 1$, the map

$$u \in L^2_+(\mathbb{T}) \to u_r \in L^\infty_+(\mathbb{T}) \ , \ u_r(\mathrm{e}^{ix}) := u(r\mathrm{e}^{ix})$$

is continuous in view of the Cauchy integral formula, and the mapping

$$u_0 \in C_+(\mathbb{T}) \mapsto Z(u_0) \in C([0,1], L^2_+(\mathbb{T}))$$

is continuous in view of Theorem 1.

Next we claim that Ω_p is dense in $C_+(\mathbb{T})$. Given $u_0 \in C_+(\mathbb{T})$, we apply Lemma 2. The sequence provided by this lemma converges to u_0 in $C_+(\mathbb{T})$. Furthermore, for n big enough, $T_n < 1/p$ and

$$\sup_{t\in[0,T_n]} \|u^n(t)\|_{L^\infty} > p \ .$$

Since, for every $f \in L^\infty_+(\mathbb{T})$,

$$\|f\|_{L^\infty} = \sup_{r<1} \sup_{x\in\mathbb{T}} \left|f(r\mathrm{e}^{ix})\right| \ ,$$

we conclude that u^n belongs to Ω_p.

Introduce

$$\mathcal{G} = \bigcap_{p\geq 1} \Omega_p \ .$$

Since $C_+(\mathbb{T})$ is a Banach space, the Baire theorem shows that $\mathcal{G}$ is a dense G_δ subset of $C_+(\mathbb{T})$. Furthermore, if $u_0 \in \mathcal{G}$, we have, for every $T > 0$ and every $p \geq T^{-1}$,

$$\sup_{t\in[0,T]} \sup_{r\in[0,1[} \sup_{x\in\mathbb{T}} \left|Z(t)u_0(r\mathrm{e}^{ix})\right| > p \ ,$$

hence $Z(u_0) \notin L^\infty([0,T]\times\mathbb{T})$.

4.1. **Proof of Lemma 2.** We shall make use of a Banach algebra B of functions on the torus, invariant by Π, included into L^∞, such that

$$\|uv\|_B \le C(\|u\|_{L^\infty}\|v\|_B + \|u\|_B\|v\|_{L^\infty}) \ , \tag{11}$$

and which, roughly speaking, has the same scaling properties as L^∞. An example is provided by the Besov space

$$B = B^{1/2}_{2,1} = \{u \in L^2(\mathbb{T}) : \|u\|_B = |\widehat{u}(0)| + \sum_{k=0}^{\infty} 2^{k/2}\|\Delta_k u\|_{L^2} < \infty\} \ ,$$

where $\Delta_k u$ denotes the usual dyadic component of u. Indeed, $\Pi(B) \subset B$ from the definition, the inclusion $B \subset L^\infty$ is a consequence of the standard inequality

$$\|\Delta_k u\|_{L^\infty} \lesssim 2^{k/2}\|\Delta_k u\|_{L^2} \ ,$$

and the tame estimate (11) follows from paralinearising the product uv. The subspace $B_+ = \Pi(B)$ of L^∞_+ can also be characterised by the condition

$$[u]_{B_+} := \int_0^1 \frac{1}{\sqrt{1-r}} \left(\int_0^{2\pi} \left|u'\left(r\mathrm{e}^{ix}\right)\right|^2 \, dx \right)^{1/2} dr < \infty \ , \tag{12}$$

where u' is the holomorphic derivative of u, the norm $|\widehat{u}(0)| + [u]_{B_+}$ being equivalent to $\|u\|_B$ on B_+.

We now fix $\alpha \in]0,\infty[$ and introduce, for every $\rho \in]0,1[$,

$$f_\rho(z) = (1-\rho z)^{i\alpha} = \mathrm{e}^{i\alpha \log(1-\rho z)}, |z| \le 1 \ ,$$

with $\log(1-\rho z) \in \mathbb{R} + i[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Lemma 3. *The following estimates hold as ρ tends to 1,*

$$\|f_\rho\|_{L^\infty} \le C \ , \ \|f_\rho\|_B \le C \log \frac{1}{1-\rho} \ ,$$

and for every trigonometric polynomial $g = g(z) \in L^2_+$ with $g(1) \neq 0$,

$$\|\Pi(|f_\rho|^2 g)\|_{L^\infty} \ge c(g) \log \frac{1}{1-\rho} \ ,$$

for some $c(g) > 0$.

Proof. Notice that, for $x \in \mathbb{T}$,

$$f_\rho(\mathrm{e}^{ix}) = \mathrm{e}^{i\frac{\alpha}{2}\log(1+\rho^2-2\rho\cos x)}\mathrm{e}^{-\alpha A_\rho(x)} \ , \ A_\rho(x) = \arctan\left(\frac{\rho \sin x}{1-\rho\cos x}\right) \ .$$

In particular,

$$\|f_\rho\|_{L^\infty} \le \mathrm{e}^{\alpha\pi/2} \ .$$

On the other hand,

$$f'_\rho(z) = -i\alpha\rho(1-\rho z)^{i\alpha-1} \ ,$$

so that

$$\int_0^{2\pi} \left|f_\rho'\left(r\mathrm{e}^{ix}\right)\right|^2 \, dx \lesssim \frac{1}{1-\rho r} \ ,$$

and

$$[f_\rho]_{B_+} \lesssim \int_0^1 \frac{dr}{\sqrt{(1-r)(1-\rho r)}} \lesssim \log \frac{1}{1-\rho} \ .$$

It remains to prove the last statement. Let $g = g(z) \in L^2_+$ be a trigonometric polynomial. We compute

$$\begin{aligned}\Pi(|f_\rho|^2 g)(\rho) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f_\rho(\mathrm{e}^{ix})|^2 g(\mathrm{e}^{ix})}{1-\rho\mathrm{e}^{-ix}}\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{-2\alpha A_\rho(x)} g(\mathrm{e}^{ix})}{1-\rho\mathrm{e}^{-ix}}\, dx \ .\end{aligned}$$

The above integral is uniformly bounded as ρ tends to 1, except for the contribution of a neighborhood of $x=0$. Symmetrizing the integration domain, we get

$$\Pi(|f_\rho|^2 g)(\rho) = \int_0^{\pi} \frac{h_\rho(x)+h_\rho(-x)}{(1-\rho)^2+2\rho(1-\cos x)}\frac{dx}{2\pi} \ ,$$

with

$$h(x) := (1-\rho\mathrm{e}^{ix})\mathrm{e}^{-2\alpha A_\rho(x)} g(\mathrm{e}^{ix}) \ .$$

Expanding e^{ix} near $x=0$, we obtain,

$$\Pi(|f_\rho|^2 g)(\rho) == O(1) + \frac{i\rho}{2\pi} g(1) \int_0^{\pi} \frac{x\left(\mathrm{e}^{2\alpha A_\rho(x)} - \mathrm{e}^{-2\alpha A_\rho(x)}\right)}{(1-\rho)^2+2\rho(1-\cos x)}\, dx \ .$$

Notice that function A_ρ is nonnegative on $[0,\pi]$ and increasing from $x=0$ to $x=\arccos\rho \sim \sqrt{2(1-\rho)}$. In particular, the integrand of the above integral is nonnegative, and we may restrict x to the domain of integration $[1-\rho, \sqrt{1-\rho}]$, on which $A_\rho(x) \gtrsim \frac{\pi}{4}$, so that

$$\begin{aligned}\int_0^{\pi} \frac{x\left(\mathrm{e}^{2\alpha A_\rho(x)} - \mathrm{e}^{-2\alpha A_\rho(x)}\right)}{(1-\rho)^2+2\rho(1-\cos x)}\, dx &\geq c_\alpha \int_{1-\rho}^{\sqrt{1-\rho}} \frac{x}{(1-\rho)^2+x^2}\, dx \\ &\geq \widetilde{c}_\alpha \log\frac{1}{1-\rho} \ .\end{aligned}$$

This completes the proof of Lemma 3.

Next, we consider, for a given trigonometric polynomial $g = g(z) \in L^2_+$ such that $g(1) \neq 0$, the family of data

$$u_0^{\rho,\varepsilon} = g + \varepsilon f_\rho \ .$$

Applying Lemma 3, we observe that

$$\|u_0^{\rho,\varepsilon} - g\|_{L^\infty} = O(\varepsilon) \ , \ \|u_0^{\rho,\varepsilon}\|_B \lesssim O(1) + \varepsilon \log \frac{1}{1-\rho} \ .$$

Furthermore,

$$\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon}) = \Pi(|g|^2 g) + \varepsilon[2\Pi(|g|^2 f_\rho) + \Pi(g^2 \overline{f}_\rho)] + \\ + \varepsilon^2[2\Pi(|f_\rho|^2 g) + \Pi(f_\rho^2 \overline{g})] + \varepsilon^3 \Pi(|f_\rho|^2 f_\rho) \ .$$

Notice that, if $h \in L^\infty_+$,

$$\Pi(\mathrm{e}^{-ix} h) = \mathrm{e}^{-ix}(h - h(0))$$

belongs to L^∞_+ with $\|\Pi(\mathrm{e}^{-ix}h)\|_{L^\infty} \leq 2\|h\|_{L^\infty}$. Since $\Pi(|g|^2 f_\rho)$ is a finite linear combination of terms of the form $\mathrm{e}^{inx} f_\rho$ and $\Pi(\mathrm{e}^{-inx} f_\rho)$ with $|n|$ not greater than the degree of g, we conclude that $\Pi(|g|^2 f_\rho)$ is bounded in L^∞_+. Similarly, $\Pi(f_\rho^2 \overline{g})$ is bounded in L^∞_+, and so is $\Pi(g^2 \overline{f}_\rho)$, since it is a finite trigonometric polynomial of degree not greater than twice the degree of g, with coefficients estimated by the supremum of Fourier coefficients of f_ρ. Finally, applying (11) and Lemma 3,

$$\|\Pi(|f_\rho|^2 f_\rho)\|_{L^\infty} \lesssim \|\Pi(|f_\rho|^2 f_\rho)\|_B \lesssim \|f_\rho\|_B \lesssim \log \frac{1}{1-\rho} \ .$$

This leads to

$$\|\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon})\|_{L^\infty} \geq \varepsilon^2 (c(g) - \varepsilon C(g)) \log \frac{1}{1-\rho} - O(1) \ .$$

Choosing ε small enough, we infer

$$\|\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon})\|_{L^\infty} \geq \varepsilon^2 \widetilde{c}(g) \log \frac{1}{1-\rho} - O(1) \ , \ \widetilde{c}(g) > 0 \ . \tag{13}$$

Next we consider $u^{\rho,\varepsilon} = Z(u_0^{\rho,\varepsilon})$. We claim that, for every positive time $T \ll 1$, there exists $\rho = \rho(\varepsilon, T)$ such that, for $\varepsilon \ll 1$,

$$\limsup_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u^{\rho(\varepsilon,T),\varepsilon}\|_{L^\infty} = +\infty \ .$$

Indeed, assume by contradiction that, for some $T > 0$ and for some M, we have, for some $\varepsilon_0 > 0$,

$$\sup_{\varepsilon < \varepsilon_0} \sup_{\rho < 1} \sup_{t \in [0,T]} \|u^{\rho,\varepsilon}\|_{L^\infty} \leq M \ .$$

Then, from the equation

$$u^{\rho,\varepsilon}(t) = u_0^{\rho,\varepsilon} - i \int_0^t \Pi(|u^{\rho,\varepsilon}(s)|^2 u^{\rho,\varepsilon}(s))\, ds$$

and using (11), we have, if $t \in [0,T]$,

$$\sup_{s\in[0,t]} \|u^{\rho,\varepsilon}(s)\|_B \le \|u_0^{\rho,\varepsilon}\|_B + CM^2 t \sup_{s\in[0,t]} \|u^{\rho,\varepsilon}(s)\|_B,$$

so that, if $t \le \widetilde{T}^* := \min(T,(2CM^2)^{-1})$,

$$\sup_{s\in[0,t]} \|u^{\rho,\varepsilon}(s)\|_B \le 2\|u_0^{\rho,\varepsilon}\|_B \lesssim O(1) + \varepsilon \log \frac{1}{1-\rho} \ . \tag{14}$$

Then we write the Taylor formula at second order in t,

$$u^{\rho,\varepsilon}(t) = u_0^{\rho,\varepsilon} - it\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon}) + \int_0^t (t-s)\left[-2(T_{|u^{\rho,\varepsilon}(s)|^2})^2 + H_{u^{\rho,\varepsilon}(s)^2} T_{|u^{\rho,\varepsilon}(s)|^2}\right] u^{\rho,\varepsilon}(s)\, ds \ ,$$

so that, using again (11) and (14), for every $t \in [0,T^*]$,

$$\|u^{\rho,\varepsilon}(t) - u_0^{\rho,\varepsilon} + it\Pi(|u_0^{\rho,\varepsilon}|^2 u_0^{\rho,\varepsilon})\|_B \le K(M)\varepsilon t^2 \log \frac{1}{1-\rho} + O(1) \ .$$

Using (13), we infer

$$\forall t \in [0,T^*] \ , \ \|u^{\rho,\varepsilon}(t)\|_{L^\infty} \ge t\varepsilon \log \frac{1}{1-\rho}(\widetilde{c}(g)\varepsilon - tK(M)) - O(1) \ .$$

Choosing $t = T^{**} := \min(T^*, \varepsilon\widetilde{c}(g)/2K(M))$ and $\rho = \rho(\varepsilon,T)$ close enough to 1, we obtain a contradiction.

Summing up, we have proved that, for every trigonometric polynomial $g = g(z) \in L^2_+$ such that $g(1) \neq 0$, there exists a sequence of data u_0^n converging to g in $C_+(\mathbb{T})$, and a sequence of positive times T_n converging to 0, such that

$$\sup_{t\in[0,T_n]} \|Z(t)u_0^n\|_{L^\infty} \to \infty \ .$$

Since any $u_0 \in C_+(\mathbb{T})$ can be approximated by a sequence of trigonometrical polynomials $g \in L^2_+$ with $g(1) \neq 0$, this completes the proof of Lemma 2. □

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(P. Gérard) Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris–Saclay, 91405 Orsay, France
E-mail address: `Patrick.Gerard@math.u-psud.fr`

(H. Koch) Mathematisches Institut, Universität Bonn, D-53115 Bonn, Germany
E-mail address: `koch@math.uni-bonn.de`