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THE SUB-LEADING COEFFICIENT OF THE L -FUNCTION OF AN ELLIPTIC CURVE

by

Christian Wuthrich

Abstract. — We show that there is a relation between the leading term at $s = 1$ of an L -function of an elliptic curve defined over an number field and the term that follows.

Résumé. — On montre une relation entre le terme dominant de la série L en $s = 1$ d'une courbe elliptique définie sur un corps de nombre et le terme suivant.

Let E be an elliptic curve defined over a number field K . We will assume that the L -function $L(E, s)$ admits an analytic continuation to $s = 1$ and that it satisfies the functional equation. By modularity [1], we know that this holds when $K = \mathbb{Q}$. The conjecture of Birch and Swinnerton-Dyer predicts that the behaviour at $s = 1$ is linked to arithmetic information. More precisely, if

$$L(E, s) = a_r (s - 1)^r + a_{r+1} (s - 1)^{r+1} + \dots$$

is the Taylor expansion at $s = 1$ with $a_r \neq 0$, then r should be the rank of the Mordell-Weil group $E(K)$ and the leading term a_r is equal to a precise formula involving the Tate-Shafarevich group of E . It seems to have passed unnoticed that the sub-leading coefficient a_{r+1} is also determined by the following formula.

Theorem 1. — *With the above assumption, we have the equality*

$$(1) \quad a_{r+1} = \left([K : \mathbb{Q}] \cdot (\gamma + \log(2\pi)) - \frac{1}{2} \log(N) - \log |\Delta_K| \right) \cdot a_r$$

where $\gamma = 0.577216\dots$ is Euler's constant, N is the absolute norm of the conductor ideal of E/K and Δ_K is the absolute discriminant of K/\mathbb{Q} .

In particular, the conjecture of Birch and Swinnerton-Dyer also predicts completely what the sub-leading coefficient a_{r+1} should be. One consequence for $K = \mathbb{Q}$ is that for all curves with conductor $N > 125$, and this is all but 404 isomorphism classes of curves, the sign of a_{r+1} is the opposite of a_r . Of course, it is believed that a_r is positive for all E/\mathbb{Q} .

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Proof. — Set $f(s) = B^s \cdot \Gamma(s)^n$ with $n = [K : \mathbb{Q}]$ and $B = \sqrt{N} \cdot |\Delta_K| / (2\pi)^n$. Then $\Lambda(s) = f(s) \cdot L(E, s)$ is the completed L -function, which satisfies the functional equation $\Lambda(s) = (-1)^r \cdot \Lambda(2 - s)$, see [3]. For $i \equiv r + 1 \pmod{2}$ it follows that $\frac{d^i}{ds^i} \Lambda(s)|_{s=1} = 0$. Hence for $i = r + 1$, we obtain that

$$(r + 1) \cdot f'(s) \cdot \frac{d^r}{ds^r} L(E, s) + f(s) \cdot \frac{d^{r+1}}{ds^{r+1}} L(E, s)$$

is zero at $s = 1$. Therefore $(r + 1) f'(1) r! a_r + f(1) (r + 1)! a_{r+1} = 0$. It remains to note that $f(1) = B$ and $f'(1) = B \cdot (\log(B) + n \cdot \Gamma'(1))$ together with $\Gamma'(1) = -\gamma$. \square

Obviously a similar formula holds for the L -function of a modular form of weight 2 for $\Gamma_0(N)$. More generally, for any L -function with a functional equation there is a relation between the leading and the sub-leading coefficient of the Taylor expansion of the L -function at the central point.

Sub-leading coefficients of Dirichlet L -functions have been investigated; for instance Colmez [2] makes a conjecture, which is partially known. However these concern the much harder case when s is not at the centre but the boundary of the critical strip of the L -function.

References

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