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ON SOME NEW CONGRUENCES BETWEEN GENERALIZED BERNOULLI NUMBERS, I

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On some new congruences between generalized Bernoulli numbers, I

by

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Abstract. In the paper some new congruences modulo 64 for generalized Bernoulli numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ belonging to quadratic characters $\left(\frac{d}{\cdot}\right)$, $d < 0$ are proved and for each $-1 \leq \nu \leq 5$ all negative d and odd k satisfying $\mathrm{ord}_2\, B_{k,\left(\frac{d}{\cdot}\right)} = \nu$ are found. In the second part of the paper we shall deal with the case of positive d.

All results are consequences of [1] and [2].

Key words: Bernoulli numbers, Kummer congruences, class numbers.

1. Introduction.

For the discriminant d of a quadratic field, let $\left(\frac{d}{\cdot}\right)$ denote the Kronecker symbol. Denote by $B_{k,\chi}$ the kth generalized Bernoulli number belonging to the Dirichlet character χ.

For $x \geq 0$ put

$$t_k(x) := \sum_{0 \leq a \leq x} \left(\frac{d}{a}\right) a^k,$$

and for $X \subset \mathbf{N} \cup \{0\}$ denote by $t_k(x, a \in X)$, $t_k(x, b \mid a)$, $t_k(x, b \nmid a)$, $t_k(x, b \parallel a)$ or $t_k(x, a \equiv r \pmod b)$ the above sum with the appropriate additional condition: $a \in X$, $b \mid a$, $b \nmid a$, $b \parallel a$ or $a \equiv r \pmod b$. Set $t_k := t_k(\delta)$, where $\delta := |d|$. If $d = -4d^*$ or $\pm 8d^*$, where d^* is the discriminant of a quadratic field then $\delta^* := |d^*|$ and we continue to write t_k^* for the above sums defined for the discriminant d^* in contrast with t_k given for d.

For $d < 0$ we have $t_1 = -dB_{1,\left(\frac{d}{\cdot}\right)}$, and for $d > 0$ we have $t_2 = dB_{2,\left(\frac{d}{\cdot}\right)}$. Put $h(d) := -B_{1,\left(\frac{d}{\cdot}\right)}$, if $d < -4$ and $h(-3) = h(-4) := 1$. Put $k_2(d) := B_{2,\left(\frac{d}{\cdot}\right)}$, if $d > 8$ and $k_2(5) = k_2(8) := 4$. It is known that $h(d)$ equals the class number and $k_2(d)$ probably (certainly up to 2-torsion and in many cases) equals the order of the group K_2 of integers of a quadratic field with the discriminant d.

It is known that $B_{0,\left(\frac{d}{\cdot}\right)} = 0$ and for $k \geq 1$

$$B_{k,\left(\frac{d}{\cdot}\right)} = 0 \tag{1.1}$$

if and only if $d > 0$ and k is odd or $d < 0$ and k is even. Write $\alpha(-3) := \frac{1}{3}$, $\gamma(-4) := \frac{1}{2}$, $\beta(5) := \frac{1}{5}$, $\rho(8) := \frac{1}{2}$, and $\alpha(d)$, $\gamma(d)$, $\beta(d)$, $\rho(d) = 1$, otherwise. Put $\xi(-3) := \alpha(-3)$, $\xi(-4) := \gamma(-4)$, $\eta(5) := \beta(5)$, $\eta(8) := \rho(8)$, and $\xi(d)$, $\eta(d) := 1$, otherwise.

Our purpose is for each $-1 \leq \nu \leq 5$, if $d < 0$ (in the first part of the paper), and for $0 \leq \nu \leq 5$, if $d > 0$ (in the second one) to find all d and k such that $\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = \operatorname{ord}_2 k + \nu$. In order to do it we need some new congruences of the generalized Kummer congruences type modulo 64 but without any assumptions on k.

Set

$$b_k(d) := \frac{B_{k,\left(\frac{d}{\cdot}\right)}}{k}.$$

Let us recall the generalized Kummer congruences (see [3]) imply the following:

$$b_k(d) \equiv -\left(1 - \left(\frac{d}{2}\right)\right) h(d)\xi(d) \pmod{2^{a+1}}, \tag{1.2}$$

if $d < 0$, $d \neq -4, -8$, $k \equiv 1 \pmod{2^a}$, $k \geq a+2$, $a \geq 1$, and

$$b_k(d) \equiv \frac{1}{2}\left(1 - 2\left(\frac{d}{2}\right)\right) k_2(d)\eta(d) \pmod{2^{a+1}}, \tag{1.3}$$

if $d > 0$, $d \neq 8$, $k \equiv 2 \pmod{2^a}$, $k \geq a+2$, $a \geq 1$.

Also (1.2) implies

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} \geq a + 1 \tag{1.4}$$

if $\left(\frac{d}{2}\right) = 1$ or $\left(\frac{d}{2}\right) \neq 1$ and $\operatorname{ord}_2 h(d) \geq a + 1 + \left(\frac{d}{2}\right)$, and

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = \operatorname{ord}_2 h(d) - \left(\frac{d}{2}\right) \tag{1.5}$$

if $\left(\frac{d}{2}\right) \neq 1$ and $\operatorname{ord}_2 h(d) \leq a + \left(\frac{d}{2}\right)$. Similarly (1.3) implies

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} \geq \operatorname{ord}_2 k + a + 1, \tag{1.6}$$

if $\operatorname{ord}_2 k_2(d) \geq a + 2$, and

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = \operatorname{ord}_2 k + \operatorname{ord}_2 k_2(d) - 1, \tag{1.7}$$

otherwise.

In both the papers we discuss the above formulas for any k and $a \leq 5$.

In this part we prove the following:

THEOREM 1. *Let d, $2 \nmid d$ and $k \geq 3$ be the discriminant of an imaginary quadratic field and an odd natural number respectively. With the above notation, the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral and the following congruence holds:*

$$b_k(d) \equiv -k\mu\left(1 - \left(\tfrac{d}{2}\right)\right) h(d)\alpha(d) - \vartheta \tfrac{k-1}{2} k_2(-4d) \pmod{64},$$

where $\mu := \mu_k(d)$, $\vartheta := \vartheta_k(d)$, and $\vartheta_3 d = 5$, $\mu_3 d = -d - 2$, $\mu_5 = -15$, and μ_k, $\vartheta_k = 1$, otherwise.

If $2 \mid d$ we get more complicated congruences. We prove the following:

THEOREM 2. *Let $d = -4d^*$, where d^* is the discriminant of a real quadratic field (i.e. $d \neq -4$), and let $k \geq 3$ be an odd natural number. Then the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral and we have:*

$$b_k(d) \equiv \vartheta_1 k_2(d^*)\beta(d^*) + \vartheta_2 h(d) + \vartheta_3 h(2d) \pmod{64},$$

where $\vartheta_i := \vartheta_i(d, k) \in \mathbb{Z}$ $(i = 1, 2, 3)$ are of the form

$$\vartheta_i = p_i k + q_i,$$

and

$$\begin{aligned}
p_1 &= \tfrac{1}{2}\left(1 - \left(\tfrac{-1}{k}\right)\right), & q_1 &= -2\left(1 - \left(\tfrac{2}{k}\right)\right) + \left(1 - \left(\tfrac{-1}{k}\right)\right)\left(3 - \left(\tfrac{d^*}{2}\right)\right), \\
p_2 &= 1 + \left(\tfrac{-1}{k}\right), & q_2 &= -2\left(3 + 2\left(\tfrac{2}{k}\right)\right)\left(2 - \left(\tfrac{-1}{k}\right)\right) + 7, \\
p_3 &= 2\left(\tfrac{-1}{k}\right), & q_3 &= 2\left(3 - 4\left(\tfrac{-1}{k}\right)\left(\tfrac{2}{k}\right)\right).
\end{aligned}$$

THEOREM 3. *Let $d = \pm 8d^*$, $d < 0$, where d^* is the discriminant of a quadratic field (i.e. $d \neq -8$), and let $k \geq 3$ be an odd natural number. Then the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral and we have:*

$$b_k(d) \equiv \begin{cases} \vartheta_1 k_2(-4d^*) + \vartheta_2 h(d) + \vartheta_3 h(d^*)\alpha(d^*) \pmod{64}, & \text{if } d^* < 0, \\ \mu_1 k_2(d^*)\beta(d^*) + \mu_2 h(d) + \mu_3 h(-4d^*) \pmod{64}, & \text{if } d^* > 0, \end{cases}$$

where $\vartheta_i := \vartheta_i(d, k)$, $\mu_i := \mu_i(d, k) \in \mathbb{Z}$ $(i = 1, 2, 3)$ are of the form $\vartheta, \mu = pk + q$, and

$$\begin{aligned}
\vartheta_1 &= -\tfrac{1}{2}(k-1), & \vartheta_2 &= 4k - 5, \quad \vartheta_3 = \left(7 - 15\left(\tfrac{d^*}{2}\right)\right)(k-1), \\
\mu_1 &= -\tfrac{3}{2}\left(1 - 2\left(\tfrac{d^*}{2}\right)\right)(k-1), & \mu_2 &= 8k - 9, \quad \mu_3 = 5(k-1).
\end{aligned}$$

Combining Thm. 2 and 3 with Cor. 1 to Thm. 1, 2 [2] we can get many new congruences for generalized Bernoulli numbers modulo 64 (or 32).

Lemma 4 and the congruence (5.1) imply a weaker version of Thm. 1 and 2:

THEOREM 4. *Let $d<0$, $2\,|\,d$, $d\neq -4$ be the discriminant of a quadratic field, and let $k\geq 3$ be an odd natural number. Then the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral and we have:*

$$b_k(d)\equiv -h(d)\pmod{2^{6-\operatorname{ord}_2 d}}.$$

∎

2. Lemmas

We have divided the proof of the theorems into a sequence of lemmas.

LEMMA 1. *Let d, $2\nmid d$ be the discriminant of a quadratic field. Then we have:*

(i) *If $d>0$ (i.e. $\delta\equiv 1\pmod 4$) then*

$$\begin{aligned}
t_0(\delta,a\equiv\delta\pmod 8)&=\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+2\pmod 8)&=-\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/4)+\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+4\pmod 8)&=t_0(\delta/4)-\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+6\pmod 8)&=\left(\tfrac{d}{2}\right)t_0(\delta/4)-\left(\tfrac{d}{2}\right)t_0(\delta/8).
\end{aligned}$$

(ii) *If $d<0$ (i.e. $\delta\equiv 3\pmod 4$)) then*

$$\begin{aligned}
t_0(\delta,a\equiv\delta\pmod 8)&=-\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+2\pmod 8)&=\left(\tfrac{d}{2}\right)t_0(\delta/4)-\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+4\pmod 8)&=-t_0(\delta/4)+\left(\tfrac{d}{2}\right)t_0(\delta/8),\\
t_0(\delta,a\equiv\delta+6\pmod 8)&=t_0(\delta/2)-\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/4)+\left(\tfrac{d}{2}\right)t_0(\delta/8).
\end{aligned}$$

PROOF. Let us note that

$$\begin{aligned}
t_0(\delta/2,a\equiv\delta\pmod 4)&=\left(\tfrac{d}{-1}\right)\sum_{\substack{0\leq a\leq\delta/2,\\ a\equiv\delta\ (\mathrm{mod}\ 4)}}\left(\tfrac{d}{\delta-a}\right)=\left(\tfrac{d}{-1}\right)\sum_{\substack{\delta/2\leq a\leq\delta,\\ 4|a}}\left(\tfrac{d}{a}\right)\\
&=\left(\tfrac{d}{-1}\right)\left[t_0(\delta,4\,|\,a)-t_0(\delta/2,4\,|\,a)\right]\\
&=\left(\tfrac{d}{-1}\right)\left[t_0(\delta/4)-t_0(\delta/8)\right],
\end{aligned}\tag{2.1a}$$

and

$$\begin{aligned}
t_0(\delta/2,a\equiv\delta+2\pmod 4)&=\left(\tfrac{d}{-1}\right)\sum_{\substack{0\leq a\leq\delta/2,\\ a\equiv\delta+2\ (\mathrm{mod}\ 4)}}\left(\tfrac{d}{\delta-a}\right)=\left(\tfrac{d}{-1}\right)\sum_{\substack{\delta/2\leq a\leq\delta,\\ a\equiv 2\ (\mathrm{mod}\ 4)}}\left(\tfrac{d}{a}\right)\\
&=\left(\tfrac{d}{-1}\right)\Big[\sum_{\substack{\delta/2\leq a\leq\delta,\\ 2|a}}\left(\tfrac{d}{a}\right)-\sum_{\substack{\delta/2\leq a\leq\delta,\\ 4|a}}\left(\tfrac{d}{a}\right)\Big]\\
&=\left(\tfrac{d}{-1}\right)\left[t_0(\delta,2\,|\,a)-t_0(\delta/2,2\,|\,a)-t_0(\delta,4\,|\,a)+t_0(\delta/2,4\,|\,a)\right]\\
&=\left(\tfrac{d}{-1}\right)\left[\left(\tfrac{d}{2}\right)t_0(\delta/2)-\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/4)+t_0(\delta/8)\right].
\end{aligned}\tag{2.1b}$$

From this we conclude the lemma because

$$
\begin{aligned}
t_0(\delta, a \equiv \delta \pmod 8) &= \left(\tfrac{d}{-1}\right) t_0(\delta, 8 \,|\, a) = \left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_0(\delta/8), \\
t_0(\delta, a \equiv \delta + 2 \pmod 8) &= \left(\tfrac{d}{-1}\right) t_0(\delta, a \equiv -2 \pmod 8) \\
&= \left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_0(\delta/2, a \equiv 3 \pmod 4), \\
t_0(\delta, a \equiv \delta + 4 \pmod 8) &= \left(\tfrac{d}{-1}\right) [t_0(\delta, 4 \,|\, a) - t_0(\delta, 8 \,|\, a)] \\
&= \left(\tfrac{d}{-1}\right) \left[t_0(\delta/4) - \left(\tfrac{d}{2}\right) t_0(\delta/8)\right], \\
t_0(\delta, a \equiv \delta + 6 \pmod 8) &= \left(\tfrac{d}{-1}\right) t_0(\delta, a \equiv 2 \pmod 8) \\
&= \left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_0(\delta/2, a \equiv 1 \pmod 4). \qquad \blacksquare
\end{aligned}
$$

For $i \geq 0$ set $t'_i := t_i(\delta, 2 \nmid a)$. To prove the theorems we apply the following lemmas:

LEMMA 2. *Let d be the discriminant of an imaginary quadratic field. Then we have:*

(i)
$$t'_0 = -\left(\tfrac{d}{2}\right)\left(2 - \left(\tfrac{d}{2}\right)\right) h(d)\alpha(d).$$

(ii)
$$t'_1 = \begin{cases} \left(\tfrac{d}{2}\right) d h(d)\alpha(d), & \text{if } 2 \nmid d, \\ d h(d)\gamma(d), & \text{if } 2 \,|\, d. \end{cases}$$

(iii)
$$t'_2 = -d^2 h(d)\gamma(d), \text{ if } 2 \,|\, d.$$

(iv)
$$t'_2 \equiv C_1(d)h(d)\alpha(d) + k_2(-4d) \pmod{64} \text{ if } 2 \nmid d,$$

where
$$C_1(d) := -2\left(\tfrac{d}{2}\right) d + 2\left(\tfrac{d}{2}\right) - 1.$$

(v)
$$t'_3 \equiv \begin{cases} C_2(d)h(d)\alpha(d) \pmod{64}, & \text{if } 2 \nmid d, \\ 4(15\gamma(d) - 14)h(d) \pmod{64}, & \text{if } 4 \,\|\, d, \\ 8 \operatorname{sgn} d^* h(d) \pmod{64} & \text{if } 8 \,|\, d,\ d = \pm 8d^*, \end{cases}$$

where
$$C_2(d) := 3\left(\tfrac{d}{2}\right) d + 2\left(\tfrac{d}{2}\right) - 4.$$

More generally, for $d \neq -4, -8$ we have

$$t'_3 \equiv \begin{cases} \left(3 - 2\left(\tfrac{d^*}{2}\right)\right) \delta k_2(d^*)\beta(d^*) - 13\delta h(d) + \\ \qquad +8\delta h(2d) \pmod{64\delta}, & \text{if } d = -4d^*, 2 \nmid d^*, \\ -3\delta k_2(-4d^*) - 11\delta h(d) + \\ \qquad +2\left(5 + 3\left(\tfrac{d^*}{2}\right)\right) \delta h(d^*)\alpha(d^*) \pmod{64\delta}, & \text{if } d = 8d^*,\ d^* < 0, \\ \left(3 - 2\left(\tfrac{d^*}{2}\right)\right) \delta k_2(d^*)\beta(d^*) + 13\delta h(d) - \\ \qquad -2\delta h(-4d^*) \pmod{64\delta}, & \text{if } d = -8d^*,\ d^* > 0. \end{cases}$$

Here for $d = -4d^*$ or $\pm 8d^*$, d^* is the discriminant of a quadratic field or $d^* = 1$, if $d = -4, -8$.

PROOF. Since for $d < 0$ (see [3])

$$t_0(\delta/2) = \left(2 - \left(\tfrac{d}{2}\right)\right) h(d)\xi(d), \tag{2.1}$$

(i) of the lemma follows immediately.

On the other hand for any d we have

$$t_1' = t_1 - 2\left(\tfrac{d}{2}\right) t_1(\delta/2). \tag{2.2}$$

Also for $d < 0$ we have (see [3])

$$t_1 = dh(d)\xi(d) \tag{2.3}$$

and

$$t_1(\delta/2) = -\tfrac{1}{2}d\left(1 - \left(\tfrac{d}{2}\right)\right) h(d)\xi(d) \tag{2.4}$$

(cf. p. 255 [2]). Therefore (ii) of the lemma follows from (2.2).

In order to prove (iii) of the lemma, we apply (1.1). Then for $d < 0$ we have

$$B_{2,\left(\frac{d}{\cdot}\right)} = 0.$$

Consequently from (3.1) (with $F = \delta$) we get

$$t_2 = -d^2 h(d)\xi(d). \tag{2.5}$$

To prove (iv) let us note that for any d we have

$$t_2' \equiv 2\sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} rt_1(\delta, a \equiv r \pmod 8) - \sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} r^2 t_0(\delta, a \equiv r \pmod 8) \pmod{64}.$$

Hence for $d < 0$, $2\nmid d$ we obtain

$$\begin{aligned}
t_2' &\equiv 2t_1' - t_0' + 2\sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r-1)t_1(\delta, a \equiv r \pmod 8) - \sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r^2-1)t_0(\delta, a \equiv r \pmod 8)\\
&\equiv 2t_1' - t_0' + 8\sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} rt_0(\delta, a \equiv r \pmod 8) - 4t_1(\delta, a \equiv 3 \pmod 8) +\\
&\quad + 4t_1(\delta, a \equiv 7 \pmod 8) - \sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r^2-1)t_0(\delta, a \equiv r \pmod 8)\\
&\equiv 2t_1' - t_0' + \sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (1 + 8r - r^2)t_0(\delta, a \equiv r \pmod 8) -\\
&\quad - 24t_0(\delta, a \equiv 3 \pmod 8) + 4t_1(\delta, a \equiv 3 \pmod 4)\\
&\equiv 2t_1' - t_0' - 8t_0(\delta, a \equiv 3 \pmod 8) + 16t_0(\delta, a \equiv 5 \pmod 8) +\\
&\quad + 8t_0(\delta, a \equiv 7 \pmod 8) + 4t_1(\delta, a \equiv 3 \pmod 4)\\
&\equiv 2t_1' - t_0' - 8t_0(\delta, a \equiv 3 \pmod 4) + 16t_0(\delta, a \equiv 5, 7 \pmod 8) +\\
&\quad + 4t_1(\delta, a \equiv 3 \pmod 4) \pmod{64}.
\end{aligned}$$

Consequently in view of Lemma 1 (ii) we deduce that

$$\begin{aligned} t_2' &\equiv 2t_1' - t_0' + 8t_0(\delta/4) + 16\left[\tfrac{1}{2}\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/2) - 2t_0(\delta/4)\right] + 16t_1(\delta/4) - 4\delta t_0(\delta/4) \\ &\equiv 2t_1' - t_0' + 4(10-\delta)t_0(\delta/4) + 16t_1(\delta/4) + 8\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/2) \pmod{64}. \end{aligned}$$

Thus by Lemma 2(i), (ii) and (2.1), (iv) of the lemma follows because for $d < 0$, $2 \nmid d$ we have

$$t_0(\delta/4) = \tfrac{1}{2}\left(1+\left(\tfrac{d}{2}\right)\right)h(d)\alpha(d), \tag{2.6}$$

(see Thm. 7.1 [1]), and

$$t_1(\delta/4) = \tfrac{1}{16}k_2(-4d) + \tfrac{1}{8}d\left(1-\left(\tfrac{d}{2}\right)\right)h(d)\alpha(d) \tag{2.7}$$

(see Thm. 2(i) [2]).

In order to prove (v) let us notice that for any d we have

$$\begin{aligned} t_3' \equiv 3\sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} rt_2(\delta, a\equiv r \pmod 8) - 3\sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} r^2t_1(\delta, a\equiv r \pmod 8)+ \\ + \sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} r^3t_0(\delta, a\equiv r \pmod 8) \pmod{2^9}. \end{aligned}$$

Therefore in view of the congruence

$$t_2(\delta, a\equiv r \pmod 8) \equiv 2rt_1(\delta, a\equiv r \pmod 8) - r^2t_0(\delta, a\equiv r \pmod 8) \pmod{64},$$

we see that

$$t_3' \equiv 3\sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} r^2t_1(\delta, a\equiv r \pmod 8) - 2\sum_{\substack{1\le r\le 7,\\ r\text{ odd}}} r^3t_0(\delta, a\equiv r \pmod 8) \pmod{64}.$$

From this it may be concluded that

$$\begin{aligned} t_3' &\equiv 3t_1' + 3\sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r^2-1)rt_0(\delta, a\equiv r \pmod 8) - 2t_0' - \\ &\quad - 2\sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r^3-1)t_0(\delta, a\equiv r \pmod 8) \\ &\equiv 3t_1' - 2t_0' + \sum_{\substack{3\le r\le 7,\\ r\text{ odd}}} (r^3-3r+2)t_0(\delta, a\equiv r \pmod 8) \\ &\equiv 3t_1' - 2t_0' + 20t_0(\delta, a\equiv 3 \pmod 8) - 16t_0(\delta, a\equiv 5 \pmod 8) \\ &\quad + 4t_0(\delta, a\equiv 7 \pmod 8) \\ &\equiv 3t_1' - 2t_0' + 20t_0(\delta, a\equiv 3 \pmod 4) - 16t_0(\delta, a\equiv 5,7 \pmod 8) \pmod{64}. \end{aligned}$$

Consequently in virtue of Lemma 1(ii) we find that

$$\begin{aligned} t_3' &\equiv 3t_1' - 2t_0' - 20t_0(\delta/4) - 16\left[\tfrac{1}{2}\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/2) - 2t_0(\delta/4)\right] \\ &\equiv 3t_1' - 2t_0' + 12t_0(\delta/4) - 8\left(1+\left(\tfrac{d}{2}\right)\right)t_0(\delta/2) \pmod{64}. \end{aligned}$$

Thus by Lemma 2(i), (ii), (2.1) and (2.6) we obtain (v) of the lemma for $2\nmid d$.

Now we are going to consider the case $2\,|\,d$. Then for any $2\,|\,d$, $d \neq -4$ and k we have

$$t_k = t_k(\delta/2) + \left(\tfrac{d}{-1}\right)\sum_{a=0}^{\delta/2}\left(\tfrac{d}{a}\right)(\delta-a)^k = t_k(\delta/4) - \left(\tfrac{d}{-1}\right)\sum_{a=0}^{\delta/4}\left(\tfrac{d}{a}\right)\left(\tfrac{\delta}{2}-a\right)^k +$$
$$+\left(\tfrac{d}{-1}\right)\sum_{a=0}^{\delta/4}\left(\tfrac{d}{a}\right)(\delta-a)^k - \sum_{a=0}^{\delta/4}\left(\tfrac{d}{a}\right)\left(\tfrac{\delta}{2}+a\right)^k, \quad (2.7a)$$

because for $2\,|\,d$ we deduce that

$$\left(\tfrac{d}{\delta/2-a}\right) = \left(\tfrac{d}{-1}\right)\left(\tfrac{d}{-\delta/2+a}\right) = \left(\tfrac{d}{-1}\right)\left(\tfrac{d}{\delta/2+a}\right) = -\left(\tfrac{d}{-1}\right)\left(\tfrac{d}{a}\right).$$

Therefore putting $\tau := \min(k,7)$ for $k \geq 3$ (i.e. $\tau \geq 3$) we have

$$t_k \equiv t_k(\delta/4) - \sum_{i=0}^{\tau}\binom{k}{i}\left(\tfrac{\delta}{2}\right)^i\left[(-1)^{k-i}\left(\tfrac{d}{-1}\right)+1\right]t_{k-i}(\delta/4) +$$
$$+\left(\tfrac{d}{-1}\right)\sum_{i=0}^{3}\binom{k}{i}\delta^i(-1)^{k-i}t_{k-i}(\delta/4) \pmod{64\delta}.$$

By the above, under the condition that $\left(\frac{d}{-1}\right) = (-1)^k$ we conclude that

$$t_k \equiv \varepsilon'_k + \varepsilon''_k \pmod{64\delta}, \quad (2.8)$$

where

$$\begin{aligned}\varepsilon'_k := \varepsilon'_k(d) &= -\sum_{i=1}^{3}\binom{k}{i}(\delta/2)^i\left[(-1)^i(1-2^i)+1\right]t_{k-i}(\delta/4)\\ &= -k\delta t_{k-1}(\delta/4) + \tfrac{1}{2}\delta^2\binom{k}{2}t_{k-2}(\delta/4) - \binom{k}{3}\delta^3 t_{k-3}(\delta/4),\end{aligned}$$

and

$$\begin{aligned}\varepsilon''_k := \varepsilon''_k(d) &= -\sum_{i=4}^{\tau}\binom{k}{i}(\delta/2)^i\left[(-1)^i+1\right]t_{k-i}(\delta/4)\\ &= -\tfrac{1}{8}\binom{k}{4}\delta^4 t_{k-4}(\delta/4) - \tfrac{\lambda_k}{32}\binom{k}{6}\delta^6 t_{k-6}(\delta/4),\end{aligned}$$

if $k \geq 4$, and $\varepsilon''_3 = 0$, where $\lambda_k = 0$, if $k \leq 5$, and $\lambda_k = 1$, otherwise.

Consequently from (2.8) for $k = 3$ and $d < 0$ we obtain

$$t_3 \equiv -3\delta t_2(\delta/4) + \tfrac{3}{2}\delta^2 t_1(\delta/4) - \delta^3 t_0(\delta/4) \pmod{64\delta}. \quad (2.9)$$

We will apply (2.8) in the proof of Lemma 5. For $k = 3$ this congruence is an equality.

We need consider three cases: $d = -4d^*$ or $d = \pm 8d^*$, where d^* is the discriminant of a quadratic field or $d^* = 1$, if $d = -4$, -8. We follow the notation of Introduction.

First, let $d = -4d^*$, $d^* > 0$. Since for $d > 0$, $2\nmid d$ we have

$$t_0(\delta/4) = \tfrac{1}{2}h(-4d) \quad (2.10)$$

(see Thm 7.1 [1]) in our case we observe that

$$\begin{aligned} t_0(\delta/4) &= t_0^*(\delta^*, a \equiv 1 \pmod 4) - t_0^*(\delta^*, a \equiv 3 \pmod 4)) \\ &= t_0^*(\delta^*/4) - \left(\left(\tfrac{d^*}{2}\right) t_0^*(\delta^*/2) - t_0^*(\delta^*/4)\right) = 2t_0^*(\delta^*/4) \\ &= h(-4d^*) = h(d). \end{aligned} \tag{2.11}$$

Moreover we find that

$$\begin{aligned} t_1(\delta/4) &= t_1^*(\delta^*, a \equiv 1 \pmod 4)) - t_1^*(\delta^*, a \equiv 3 \pmod 4)) \\ &= [-t_1^*(\delta^*, 4 \,|\, a) + \delta^* t_0^*(\delta^*, 4 \,|\, a)] - [-t_1^*(\delta^*, a \equiv 2 \pmod 4)) \\ &\quad + \delta^* t_0^*(\delta^*, a \equiv 2 \pmod 4))] \\ &= -4t_1^*(\delta^*/4) + \delta^* t_0^*(\delta^*/4) + \left[2\left(\tfrac{d^*}{2}\right) t_1^*(\delta^*/2) - 4t_1^*(\delta^*/4)\right] - \\ &\quad - \delta^* \left[\left(\tfrac{d^*}{2}\right) t_0^*(\delta^*/2) - t_0^*(\delta^*/4)\right] \\ &= 2\left(\tfrac{d^*}{2}\right) t_1^*(\delta^*/2) - 8t_1^*(\delta^*/4) + 2\delta^* t_0^*(\delta^*/4). \end{aligned} \tag{2.11a}$$

Consequently, since for $d > 0$, $2 \nmid d$

$$t_1(\delta/2) = -\tfrac{1}{4}\left(4 - \left(\tfrac{d}{2}\right)\right) k_2(d)\beta(d), \tag{2.12}$$

and

$$t_1(\delta/4) = -\tfrac{1}{16}\left(7 + 2\left(\tfrac{d}{2}\right)\right) k_2(d)\beta(d) + \tfrac{1}{8} d h(-4d), \tag{2.13}$$

(see Thm. 1 (i) [2]), by (2.10) in our case for $d \neq -4$ we get

$$t_1(\delta/4) = \left(4 - \left(\tfrac{d^*}{2}\right)\right) k_2(d^*)\beta(d^*). \tag{2.14}$$

Furthermore we have

$$\begin{aligned} t_2(\delta/4) &= t_2^*(\delta^*, a \equiv 1 \pmod 4)) - t_2^*(\delta^*, a \equiv 3 \pmod 4)) \\ &\equiv 2 \sum_{r=1 \text{ or } 5} r t_1^*(\delta^*, a \equiv r \pmod 8)) - \sum_{r=1 \text{ or } 5} r^2 t_0^*(\delta^*, a \equiv r \pmod 8)) - \\ &\quad - 2 \sum_{r=3 \text{ or } 7} r t_1^*(\delta^*, a \equiv r \pmod 8)) + \sum_{r=3 \text{ or } 7} r^2 t_0^*(\delta^*, a \equiv r \pmod 8)) \\ &\equiv 2t_1^{*\prime} - t_0^{*\prime} + 8t_1^*(\delta^*, a \equiv 5 \pmod 8)) - 8t_1^*(\delta^*, a \equiv 3 \pmod 8)) - \\ &\quad - 16t_1^*(\delta^*, a \equiv 7 \pmod 8)) - 24t_0^*(\delta^*, a \equiv 5 \pmod 8)) + \\ &\quad + 10t_0^*(\delta^*, a \equiv 3 \pmod 8)) - 14t_0^*(\delta^*, a \equiv 7 \pmod 8)) \\ &\equiv 2t_1^{*\prime} - t_0^{*\prime} - 24t_0^*(\delta^*, a \equiv 5 \pmod 8)) - 24t_0^*(\delta^*, a \equiv 3 \pmod 8)) + \\ &\quad + 16t_0^*(\delta^*, a \equiv 7 \pmod 8)) - 24t_0^*(\delta^*, a \equiv 5 \pmod 8)) + \\ &\quad + 10t_0^*(\delta^*, a \equiv 3 \pmod 8)) - 14t_0^*(\delta^*, a \equiv 7 \pmod 8)) \\ &\equiv 2t_0^{*\prime} - t_0^{*\prime} + 16t_0^*(\delta^*, a \equiv 5, 7 \pmod 8)) - \\ &\quad - 14t_0^*(\delta^*, a \equiv 3, 7 \pmod 8)) \pmod{64}. \end{aligned} \tag{2.14a}$$

Consequently by $t_0^{*\prime} = 0$ and Lemma 1(i) we find that

$$\begin{aligned} t_2(\delta/4) &\equiv 2t_1^{*\prime} + 16\left[\left(1 + \left(\tfrac{d^*}{2}\right)\right) t_0^*(\delta^*/4) - 2t_0^*(\delta^*/8)\right] + 14t_0^*(\delta^*/4) \\ &\equiv 2t_1^{*\prime} + 32t_0^*(\delta^*/8) + 2\left(15 + 8\left(\tfrac{d^*}{2}\right)\right) t_0^*(\delta^*/4) \pmod{64}. \end{aligned}$$

On the other hand for $d > 0$, $2 \nmid d$ we have

$$t_0(\delta/8) = \frac{1}{4}\left(\frac{d}{2}\right)h(-4d) + \frac{1}{4}h(-8d) \tag{2.15}$$

(see Thm. 7.1 [1]). Therefore by $t_1^* = 0$, together with (2.12), (2.2) and (2.10) in our case for $d \neq -4$ we get

$$t_2(\delta/4) \equiv \left(4\left(\frac{d^*}{2}\right) - 1\right)k_2(d^*)\beta(d^*) - h(d) + 8h(2d) \pmod{64}. \tag{2.16}$$

Now we can apply (2.9). From (2.11), (2.14) and (2.16) we deduce that

$$\begin{aligned} t_3' = t_3 \equiv &\left[3\left(1 - 4\left(\frac{d^*}{2}\right)\right) + \frac{3}{2}\delta\left(4 - \left(\frac{d^*}{2}\right)\right)\right]\delta k_2(d^*)\beta(d^*) + \\ &+ (3 - \delta^2)\delta h(d) - 24\delta h(2d) \pmod{64\delta}. \end{aligned}$$

Hence and from Cor. 1(i) to Thm. 1 [2], (v) of the lemma for $4 \| d$ follows.

Now let $d = \pm 8d^*$. Since for $2 \nmid a$ we have

$$\left(\frac{\pm 8}{a}\right) \pm \left(\frac{d^*}{-1}\right)\left(\frac{\pm 8}{2\delta^* - a}\right) = \left(\frac{\pm 8}{a}\right)\left(1 \pm (-1)^{\frac{a-1}{2}}\right), \tag{2.17}$$

we find that

$$\begin{aligned} t_1(\delta/4) &= t_1(\delta^*) + \sum_{a=0}^{\delta^*}\left(\frac{d}{2\delta^* - a}\right)(2\delta^* - a) \\ &= \sum_{a=0}^{\delta^*}\left[\left(\frac{\pm 8}{a}\right) - \left(\frac{d^*}{-1}\right)\left(\frac{\pm 8}{2\delta^* - a}\right)\right]\left(\frac{d^*}{a}\right)a + 2\delta^* \sum_{\substack{0 \le a \le \delta^* \\ 2 \nmid a}} (-1)^{\frac{a-1}{2}}\left(\frac{d}{a}\right) \\ &= 2t_1(\delta^*, a \equiv 3 \pmod 4) + 2\delta^*[t_0(\delta^*, a \equiv 1 \pmod 4) - \\ &\quad - t_0(\delta^*, a \equiv 3 \pmod 4)]. \end{aligned} \tag{2.17a}$$

Therefore for $d < 0$ we have

$$\begin{aligned} t_1(\delta/4) \equiv\ & 2\left(\frac{d^*}{-1}\right)[3t_0^*(\delta^*, a \equiv 3 \pmod 8) - 7t_0^*(\delta^*, a \equiv 7 \pmod 8)] + \\ &+ 2\delta^*\Big[t_0^*(\delta^*, a \equiv 1 \pmod 8) - t_0^*(\delta^*, a \equiv 5 \pmod 8) - \\ &- \left(\frac{d^*}{-1}\right)t_0^*(\delta^*, a \equiv 3 \pmod 8) + \left(\frac{d^*}{-1}\right)t_0^*(\delta^*, a \equiv 7 \pmod 8)\Big] \\ \equiv\ & 2\left(\frac{d^*}{-1}\right)(3 - \delta^*)t_0^*(\delta^*, a \equiv 3 \pmod 8) + 2\left(\frac{d^*}{-1}\right)(\delta^* - 7)t_0^*(\delta^*, a \equiv 7 \pmod 8) + \\ &+ \quad 2\delta^* t_0^*(\delta, a \equiv 1 \pmod 8) - 2\delta^* t_0^*(\delta^*, a \equiv 5 \pmod 8) \pmod{16}. \end{aligned}$$

Consequently by Lemma 1 if $d = \pm 8d^*$, $d < 0$ then we observe that

$$\begin{aligned} t_1(\delta/4) \equiv\ & 4\left(\frac{d^*}{-1}\right)t_0^*(\delta^*/8) - 2\left(\frac{d^*}{-1}\right)\left(1 + \left(\frac{d^*}{-1}\right) + \left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/4) + \\ &+ 2\left(2 - \left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/2) \pmod{16}. \end{aligned}$$

Now, let $d = 8d^*$, $d^* < 0$. Then since for $d < 0$, $2 \nmid d$

$$t_0(\delta/8) = \frac{1}{4}\left(5 - \left(\frac{d}{2}\right)\right)h(d)\alpha(d) - \frac{1}{4}h(8d) \tag{2.18}$$

(see Thm. 7.1 [1], again), by (2.1) and (2.6) in our case we see that

$$t_1(\delta/4) \equiv 6\left(1-\left(\frac{d^*}{2}\right)\right)h(d^*)\alpha(d^*)+h(d) \pmod{16}. \tag{2.19}$$

Similarly, if $d=-8d^*$, $d^*>0$, $d\neq -8$ then by (2.10) and (2.15) we obtain

$$t_1(\delta/4) \equiv -2h(-4d^*)+h(d) \pmod{16}. \tag{2.20}$$

Now the lemma will be proved as soon as we can find $t_2(\delta/4)$ modulo 64. But applying (2.17) for any d we get

$$\begin{aligned}
t_2(\delta/4) &= t_2(\delta^*)+\sum_{a=0}^{\delta^*}\left(\frac{d}{2\delta^*-a}\right)(2\delta^*-a)^2=\sum_{a=0}^{\delta^*}\left[\left(\frac{\pm 8}{a}\right)+\left(\frac{d^*}{-1}\right)\left(\frac{\pm 8}{2\delta^*-a}\right)\right]\left(\frac{d^*}{a}\right)a^2+\\
&\quad +4\delta^{*2}\sum_{\substack{0\le a\le\delta^*,\\ 2\nmid a}}(-1)^{\frac{a-1}{2}}\left(\frac{d}{a}\right)-4\delta^*\sum_{\substack{0\le a\le\delta^*,\\ 2\nmid a}}(-1)^{\frac{a-1}{2}}\left(\frac{d}{a}\right)a\\
&= 2t_2(\delta^*,a\equiv 1 \pmod 4)+4\delta^{*2}[t_0(\delta^*,a\equiv 1 \pmod 4)-t_0(\delta^*,a\equiv 3 \pmod 4)]-\\
&\quad -4\delta^*\left[t_1(\delta^*,a\equiv 1 \pmod 4)-t_1(\delta^*,a\equiv 3 \pmod 4)\right]\\
&\equiv 2[2t_1^*(\delta^*,a\equiv 1 \pmod 8)-t_0^*(\delta^*,a\equiv 1 \pmod 8)-10t_1^*(\delta^*,a\equiv 5 \pmod 8)+\\
&\quad +25t_0^*(\delta^*,a\equiv 5 \pmod 8)]+4\delta^{*2}[t_0^*(\delta^*,a\equiv 1 \pmod 8)-\\
&\quad -t_0^*(\delta^*,a\equiv 5 \pmod 8)+\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)t_0^*(\delta^*,a\equiv 3 \pmod 8)-\\
&\quad -\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)t_0^*(\delta^*,a\equiv 7 \pmod 8)]-\\
&\quad -4\delta^*\Big[t_1^*(\delta^*,a\equiv 1 \pmod 8)-t_1^*(\delta^*,a\equiv 5 \pmod 8)+\\
&\quad +\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)t_1^*(\delta^*,a\equiv 3 \pmod 8)-\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)t_1^*(\delta^*,a\equiv 7 \pmod 8)\Big]\\
&\equiv 2(1-2\delta^*+2\delta^{*2})t_0^*(\delta^*,a\equiv 1 \pmod 8)-\\
&\quad -2(25-10\delta^*+2\delta^{*2})t_0^*(\delta^*,a\equiv 5 \pmod 8)+\\
&\quad +4\delta^*\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)[-t_1^*(\delta^*,a\equiv 3 \pmod 4)+14t_1^*(\delta^*,a\equiv 7 \pmod 8)]+\\
&\quad +4\delta^{*2}\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)[t_0^*(\delta^*,a\equiv 3 \pmod 4)-2t_0^*(\delta^*,a\equiv 7 \pmod 8)]\\
&\equiv 2\left(2\delta^*+3-4\left(\frac{d^*}{-1}\right)\right)t_0^*(\delta^*,a\equiv 1 \pmod 8)-\\
&\quad -2\left(2\delta^*+11+4\left(\frac{d^*}{-1}\right)\right)t_0^*(\delta^*,a\equiv 5 \pmod 8)-\\
&\quad -8(\delta^*+1)\left(\frac{d}{-1}\right)\left(\frac{d^*}{-1}\right)t_0^*(\delta^*,a\equiv 7 \pmod 8)+\\
&\quad +4\delta^*\left(\frac{d}{-1}\right)\left[\frac{1}{2}\left(1+\left(\frac{d^*}{-1}\right)\right)t_1^*(\delta^*,2\,|\,a)-\left(\frac{d^*}{-1}\right)t_1^*(\delta^*,4\,|\,a)\right]+\\
&\quad +4\delta^{*2}\left(\frac{d}{-1}\right)\left(1-\left(\frac{d^*}{-1}\right)\right)t_0^*(\delta^*,4\,|\,a) \pmod{64},
\end{aligned} \tag{2.20a}$$

because

$$t_1^*(\delta^*,a\equiv\delta^*-3 (\mathrm{mod} 4))=\frac{1}{2}\left(1+\left(\frac{d^*}{-1}\right)\right)t_1^*(\delta^*,2\,|\,a)-\left(\frac{d^*}{-1}\right)t_1^*(\delta^*,4\,|\,a),$$

and for $d^*<0$

$$t_0^*(\delta^*,a\equiv 3(\mathrm{mod} 4))=-t_1^*(\delta^*,4\,|\,a).$$

Consequently for $d<0$ by Lemma 1 we deduce

$$\begin{aligned} t_2(\delta/4) \equiv -4\delta^*\left[\left(1+\left(\tfrac{d^*}{-1}\right)\right)\left(\tfrac{d^*}{2}\right)t_1^*(\delta^*/2)-4\left(\tfrac{d^*}{-1}\right)t_1^*(\delta^*/4)\right]+20\left(\tfrac{d^*}{-1}\right)t_0^*(\delta^*/8)+ \\ +2\left(2\delta^*-5\left(\tfrac{d^*}{2}\right)+2\right)t_0^*(\delta^*/2)+ \\ +2\left[2\delta^*-5\left(\tfrac{d^*}{2}\right)+6-12\left(\tfrac{d^*}{-1}\right)\right]t_0^*(\delta^*/4) \pmod{64}, \end{aligned}$$

because $t_0^*(\delta^*/2)=0$, if $d^*>0$, and by (2.6) we have $t_0^*(\delta^*/4)=0$, if $d^*<0$, $\left(\frac{d^*}{2}\right)=-1$.

We now turn to the cases. Let $d=8d^*$, $d^*<0$. Then from (2.1), (2.6), (2.7) and (2.18) we obtain

$$\begin{aligned} t_2(\delta/4) &\equiv 16t_1^*(\delta^*/4)-20t_0^*(\delta^*/8)+2\,(2\delta^*+13)\,t_0^*(\delta^*/4)+ \\ &\quad +2(2\delta^*-5\left(\tfrac{d^*}{2}\right)+2)t_0^*(\delta^*/2) \\ &\equiv k_2(-4d^*)+5h(d)+2\left(13-5\left(\tfrac{d^*}{2}\right)\right)h(d^*)\alpha(d^*) \pmod{64}, \end{aligned} \tag{2.21}$$

because $t_0^*(\delta^*/4)=0$, if $d^*<0$ and $\left(\frac{d^*}{2}\right)=-1$, again.

Likewise, if $d=-8d^*$, $d^*>0$, $d\neq -8$ then by $t_0^*(\delta^*/2)=0$, (2.10), (2.12), (2.13) and (2.15) we find that

$$\begin{aligned} t_2(\delta/4) &\equiv -8\delta^*\left(\tfrac{d^*}{2}\right)t_1^*(\delta^*/2)+16t_1^*(\delta^*/4)+20t_0^*(\delta^*/8)+ \\ &\quad +2\left(2\delta^*-5\left(\tfrac{d^*}{2}\right)-6\right)t_0^*(\delta^*/4) \\ &\equiv 3\left(1-2\left(\tfrac{d^*}{2}\right)\right)k_2(d^*)\beta(d^*)+5h(d)-2\left(5-4\left(\tfrac{d^*}{2}\right)\right)h(-4d^*) \\ &\quad \pmod{64}. \end{aligned} \tag{2.22}$$

Now to finish the proof of the lemma it suffices to use (2.19), (2.21), if $d=8d^*$, $d^*<0$ or (2.20), (2.22), if $d=-8d^*$, $d^*>0$ together with (2.9). Let us note in the case $8\,|\,d$ we have

$$t_3'=t_3\equiv -3\delta t_2(\delta/4)+\tfrac{3}{2}\delta^2 t_1(\delta/4) \pmod{64\delta}.$$

This gives immediately the congruences modulo 64δ of (v) of the lemma for $8\,|\,d$. The congruences modulo 64 follow easily from them and Cor. 1(i) to Thm. 1, 2 [2]. ■

LEMMA 3. *Let X be be a subset of the set of the odd natural numbers. For given $x\geq 0$ and any d we have:*

$$t_k(x,a\in X)\equiv \tfrac{k-\sigma}{2}t_{\sigma+2}(x,a\in X)-\tfrac{k-\sigma-2}{2}t_\sigma(x,a\in X) \pmod{64},$$

where $k\equiv\sigma \pmod 2$, $\sigma\in\{0,1\}$.

PROOF. It is easily seen that for natural $2\nmid a$ and even k

$$a^k\equiv \tfrac{k}{2}a^2-\tfrac{k}{2}+1 \pmod{64}.$$

Therefore for even k and any discriminant d the lemma follows. Furthermore, the above congruence implies

$$a^k\equiv \tfrac{k-1}{2}a^3-\tfrac{k-1}{2}a+a \pmod{64},$$

if k is odd. Hence the lemma for odd k follows easily. ■

COROLLARY. *For any d we have:*

$$t'_k \equiv \begin{cases} \frac{k}{2}t'_2 - \frac{k-2}{2}t'_0 \pmod{64}, & \text{if } 2\,|\,k, \\ \frac{k-1}{2}t'_3 - \frac{k-3}{2}t'_1 \pmod{64}, & \text{if } 2\nmid k. \end{cases}$$

■

LEMMA 4. *Let d be the discriminant of an imaginary quadratic field. Then we have:*

(i) *If $i \geq 2$ then*

$$t_{2i} \equiv \begin{cases} A_1(d,i)h(d)\alpha(d) + ik_2(-4d) \pmod{64}, & \text{if } 2\nmid d, \\ -id^2h(d)\gamma(d) \pmod{64}, & \text{if } 2\,|\,d, \end{cases}$$

where

$$A_1(d,i) := -3\cdot 2^{2i-1}\left(1-\left(\tfrac{d}{2}\right)\right) + 2i\left(2\left(\tfrac{d}{2}\right) - 1 - \left(\tfrac{d}{2}\right)d\right) - 2\left(\tfrac{d}{2}\right) + 1.$$

(ii) *If $i \geq 1$ then*

$$t_{2i+1} \equiv \begin{cases} A_2(d,i)h(d)\alpha(d) - 2^{2i-2}k_2(-4d) \pmod{64}, & \text{if } 2\nmid d \\ A_2(d,i)h(d) \pmod{64}, & \text{if } 2\,|\,d, \end{cases}$$

where

$$\begin{aligned} A_2(d,i) &:= 2^{2i-1}d\left(1-\left(\tfrac{d}{2}\right)\right) + 2i\left(\left(\tfrac{d}{2}\right) - 2 + \left(\tfrac{d}{2}\right)d\right) + \left(\tfrac{d}{2}\right)d, && \text{if } 2\nmid d, \\ A_2(d,i) &:= i\left[\gamma(d)(60-d)+8\right] + d\gamma(d), && \text{if } 4\,\|\,d, \\ A_2(d,i) &:= i(8\operatorname{sgn} d^* - d) + d, && \text{if } 8\,|\,d,\ d = \pm 8d^*. \end{aligned}$$

PROOF. Let $i \geq 3$. We start with the following obvious congruence:

$$t_i \equiv 2^i\left(\tfrac{d}{2}\right)t_\sigma(\delta/2, 2\nmid a) + t'_i \pmod{64}, \tag{2.23}$$

where $i \equiv \sigma \pmod 2$, $\sigma \in \{0,1\}$. Hence and from Corollary to Lemma 3 for $i \geq 2$ we obtain

$$\begin{aligned} t_{2i} &\equiv 2^{2i}\left(\tfrac{d}{2}\right)t_0(\delta/2, 2\nmid a) + t'_{2i} \\ &\equiv 2^{2i}\left(\tfrac{d}{2}\right)\left[t_0(\delta/2) - \left(\tfrac{d}{2}\right)t_0(\delta/4)\right] + it'_2 - (i-1)t'_0 \pmod{64}. \end{aligned}$$

Now, if $2\nmid d$ then (i) of the lemma follows immediately from the above congruence, (2.1), (2.6) and Lemma 2(i), (iv). If $2\,|\,d$ then it is an easy consequence of Lemma 2(i), (iii).

Similarly from (2.23) and Corollary to Lemma 3 for $i \geq 1$ we get

$$\begin{aligned} t_{2i+1} &\equiv 2^{2i+1}\left(\tfrac{d}{2}\right)t_1(\delta/2, 2\nmid a) + t'_{2i+1} \\ &\equiv 2^{2i+1}\left(\tfrac{d}{2}\right)\left[t_1(\delta/2) - 2\left(\tfrac{d}{2}\right)t_1(\delta/4)\right] + it'_3 - (i-1)t'_1 \pmod{64}. \end{aligned}$$

Now, if $2\nmid d$ then (ii) of the lemma is an obvious consequence of Lemma 2 (i), (v), (2.4) and (2.7). If $2\,|\,d$ then it follows immediately from Lemma 2(i), (v). The lemma is proved. ■

LEMMA 5. *Let d, $2\mid d$, $d \neq -4, -8$ and $k \geq 5$ be the discriminant of an imaginary quadratic field and an odd natural number respectively. Then we have:*

$$t_k \equiv \begin{cases} \frac{7}{2}\left(k+3+4\left(\frac{d^*}{2}\right)\right)\binom{k}{2}\delta k_2(d^*)\beta(d^*) - A_3\delta h(d) + \\ \quad +2\binom{k}{2}(k-7)\delta h(2d) \pmod{64\delta}, & \text{if } d=-4d^*,\ d^*>0, \\ -\binom{k}{2}\delta k_2(-4d^*) + A_3\delta h(d) + \\ \quad +2\left(7-15\left(\frac{d^*}{2}\right)\right)\binom{k}{2}\delta h(d^*)\alpha(d^*) \pmod{64\delta}, & \text{if } d=8d^*,\ d^*<0, \\ -3\left(1-2\left(\frac{d^*}{2}\right)\right)\binom{k}{2}\delta k_2(d^*)\beta(d^*) - A_3\delta h(d) + \\ \quad +10\binom{k}{2}\delta h(-4d^*) \pmod{64\delta}, & \text{if } d=-8d^*,\ d^*>0, \end{cases}$$

where $A_3 := A_3(d,k)$, and $A_3 = \omega\binom{k}{2} + \left(\frac{d^*}{-1}\right)k$, where $\omega = -3(k+3)$, resp. $12+4\left(\frac{d^*}{-1}\right)$, if $4 \parallel d$, resp. $8\mid d$.

PROOF. We begin by proving the lemma in the case $4 \parallel d$, $d \neq -4$. Applying (2.8) for $2\nmid k$, $k \geq 5$ we get

$$\begin{aligned} t_k \equiv -k\delta t_{k-1}(\delta/4) + \tfrac{1}{2}\delta^2\tbinom{k}{2}t_{k-2}(\delta/4) - 16\tbinom{k}{3}\delta t_0(\delta/4) - \\ - 2\tbinom{k}{4}\delta^2 t_1(\delta/4) - 32\lambda_k\tbinom{k}{6}\delta t_0(\delta/4) \pmod{64\delta}. \end{aligned}$$

Thus by Lemma 3 for $2\nmid k$, $k \geq 5$ we deduce that

$$\begin{aligned} t_k &\equiv -k\delta\left[\tfrac{k-1}{2}t_2(\delta/4) - \tfrac{k-3}{2}t_0(\delta/4)\right] + \tfrac{1}{2}\delta^2\tbinom{k}{2}\left[\tfrac{k-3}{2}t_3(\delta/4) - \tfrac{k-5}{2}t_1(\delta/4)\right] - \\ &\quad - 16\tbinom{k}{3}\delta t_0(\delta/4) - 2\tbinom{k}{4}\delta^2 t_1(\delta/4) - 32\lambda_k\tbinom{k}{6}t_0(\delta/4) \\ &\equiv \tfrac{1}{4}(k-3)\tbinom{k}{2}\delta^2 t_3(\delta/4) - \tbinom{k}{2}\delta t_2(\delta/4) - \tfrac{1}{4}\delta^2\left[\tbinom{k}{2}(k-5) + 8\tbinom{k}{4}\right]t_1(\delta/4) + \\ &\quad + \left[\tfrac{k(k-3)}{2} - 16\tbinom{k}{3} - 32\lambda_k\tbinom{k}{6}\right]\delta t_0(\delta/4) \pmod{64\delta}. \end{aligned} \tag{2.24}$$

On the other hand in this case by $t_0^{*\prime} = 0$ and Lemma 1(i) we have

$$\begin{aligned} t_3(\delta/4) &= t_3^*(\delta^*, a \equiv 1 \pmod 4) - t_3^*(\delta^*, a \equiv 3 \pmod 4) \\ &\equiv 3t_1^*(\delta^*, a \equiv 1 \pmod 4) - 2t_0^*(\delta^*, a \equiv 1 \pmod 4) + \\ &\quad + 5t_1^*(\delta^*, a \equiv 3 \pmod 4) - 10t_0^*(\delta^*, a \equiv 3 \pmod 4)) \\ &= 3t_1^{*\prime} - 2t_0^{*\prime} + 2t_1^*(\delta^*, a \equiv 3 \pmod 4) - 8t_0^*(\delta^*, a \equiv 3 \pmod 4)) \\ &= 3t_1^{*\prime} + 6t_0^*(\delta^*, a \equiv 3 \pmod 8) - 2t_0^*(\delta^*, a \equiv -1 \pmod 8) - \\ &\quad - 8t_0^*(\delta^*, a \equiv 3 \pmod 4)) = 3t_1^{*\prime} - 2t_0^*(\delta^*, a \equiv 3 \pmod 4) - \\ &\quad - 8t_0^*(\delta^*, a \equiv -1 \pmod 8)) \equiv 3t_1^{*\prime} + 2t_0^*(\delta^*/4) - \\ &\quad - 4\left(1+\left(\tfrac{d^*}{2}\right)\right)t_0^*(\delta^*/4) + 8t_0^*(\delta^*/8) \equiv 3t_1^{*\prime} - \\ &\quad - 2\left(1+2\left(\tfrac{d^*}{2}\right)\right)t_0^*(\delta^*/4) + 8t_0^*(\delta^*/8) \pmod{16}, \end{aligned}$$

and by (2.12) (together with $t_1^* = 0$ and (2.2)), (2.10) and (2.15) for $d \neq -4$ we obtain

$$t_3(\delta/4) \equiv \tfrac{3}{2}\left(4\left(\tfrac{d^*}{2}\right) - 1\right) k_2(d^*)\beta(d^*) - h(d) + 2h(2d) \pmod{16}.$$

Consequently by (2.24) together with (2.11), (2.14) and (2.16), the lemma for $4 \,\|\, d$ follows, because $2 \mid h(d)$, $h(2d)$, $4 \mid k_2(d^*)$, and $4 \mid h(d)$, if $\left(\frac{d^*}{2}\right) = 1$.

Now, consider the case $8 \mid d$. From (2.8) in this case we have

$$t_k \equiv -k\delta t_{k-1}(\delta/4) + \tfrac{1}{2}\delta^2\tbinom{k}{2} t_{k-2}(\delta/4) \pmod{64\delta},$$

and in consequence by Lemma 3 for $2 \nmid k$, $k \geq 5$ we deduce that

$$\begin{aligned} t_k &\equiv -k\delta\left[\tfrac{k-1}{2}t_2(\delta/4) - \tfrac{k-3}{2}t_0(\delta/4)\right] + \tfrac{1}{2}\delta^2\tbinom{k}{2}\left[\tfrac{k-3}{2}t_3(\delta/4) - \tfrac{k-5}{2}t_1(\delta/4)\right] \\ &\equiv -\tbinom{k}{2}\delta t_2(\delta/4) + \tfrac{1}{2}\delta^2\tbinom{k}{2}t_1(\delta/4) + \tfrac{k(k-3)}{2}\delta t_0(\delta/4) \pmod{64\delta}. \end{aligned} \tag{2.25}$$

On the other hand in view of (2.17), putting $d = \pm 8d^*$ we get

$$\begin{aligned} t_0(\delta/4) &= t_0(\delta^*) + \sum_{a=0}^{\delta^*}\left(\frac{d}{2\delta^* - a}\right) = \sum_{a=0}^{\delta^*}\left[\left(\frac{\pm 8}{a}\right) + \left(\frac{d^*}{-1}\right)\left(\frac{\pm 8}{2\delta^* - a}\right)\right]\left(\frac{d^*}{a}\right) \\ &= 2t_0(\delta^*, a \equiv 1 \pmod 4) = 2t_0^*(\delta^*, a \equiv 1 \pmod 8) - 2t_0^*(\delta^*, a \equiv 5 \pmod 8), \end{aligned}$$

and in consequence by Lemma 1 we see that

$$t_0(\delta/4) = 4\left(\tfrac{d^*}{-1}\right)t_0^*(\delta^*/8) + 2\left(1 - 2\left(\tfrac{d^*}{-1}\right)\right)\left(\tfrac{d^*}{2}\right)t_0^*(\delta^*/4) - 2\left(\tfrac{d^*}{2}\right)t_0^*(\delta^*/2), \tag{2.25a}$$

because $t_0^*(\delta^*/2)$, resp. $t_0^*(\delta^*/4) = 0$, if $d^* > 0$, resp. $d^* < 0$, $\left(\frac{d^*}{2}\right) = -1$ (see (2.6)).

Thus by (2.1), (2.6) and (2.18) in the case $d^* < 0$, and by (2.10) and (2.15) in the case $d^* > 0$, $d \neq -8$ we obtain

$$t_0(\delta/4) = h(d).$$

Thus the lemma for $8 \mid d$ follows from (2.25) together with the above formula and (2.19), (2.21) in the case $d^* < 0$, and (2.20), (2.22) in the case $d^* > 0$, because $4 \mid h(d)$, $h(-4d^*)$, if $\left(\frac{d^*}{2}\right) = 1$. ■

3. Used formulas.

It is known that for $k \geq 0$, a Dirichlet character χ with the conductor f and $f \mid F$ the following formula holds:

$$B_{k,\chi} = F^{k-1}\sum_{a=0}^{F}\chi(a)B_k(a/F) \tag{3.1}$$

(see Proposition 4.1 [3]). Here $B_k(x)$ denotes the kth Bernoulli polynomial. It is known that for $k \geq 0$

$$B_k(x) = \sum_{i=0}^{k}\tbinom{k}{i}B_i x^i,$$

where B_i are ordinary Bernoulli numbers. Hence and from (3.1) for $F = 2f$ and $\chi(-1) = (-1)^k$ we obtain

$$\begin{aligned} B_{k,\chi} &= (2f)^{k-1} \sum_{a=0}^{2f} \chi(a) B_k(a/2f) \\ &= (2f)^{k-1} \Big[\sum_{a=0}^{f} \chi(a) B_k(a/2f) + \sum_{a=0}^{f} \chi(f-2a) B_k\left(\tfrac{2f-a}{2f}\right) \Big] \\ &= (2f)^{k-1} (1 + \chi(-1)(-1)^k) \sum_{a=0}^{f} \chi(a) B_k(a/2f) \\ &= \sum_{i=0}^{k} \tbinom{k}{i} 2 B_{k-i} (2f)^{k-i-1} t_i. \end{aligned}$$

Therefore for $k \geq 2$ we conclude that

$$B_{k,\chi} = \sum_{i=0}^{k-2} \tbinom{k}{i} 2 B_{k-i} (2f)^{k-1-i} - k t_{k-1} + \tfrac{1}{f} t_k. \tag{3.2}$$

4. Proof of Theorem 1.

We start with the formula (3.2). For $k \geq 3$, $2 \nmid k$ and $\chi = \left(\frac{d}{\cdot}\right)$, $d < 0$ it states that

$$B_{k,\left(\frac{d}{\cdot}\right)} = \sum_{i=0}^{\frac{k-3}{2}} \tbinom{k}{2i+1} 2 B_{k-2i-1} (2\delta)^{k-2i-2} t_{2i+1} - k t_{k-1} + \tfrac{1}{\delta} t_k. \tag{4.1}$$

Thus in view of the von Staudt–Clausen theorem for $p = 2$ and Lemma 4 we see that the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral unless $d = -4$. Then we have $2 \,\|\, t_k$ and $\mathrm{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = -1$.

Let us apply the formula (4.1) to the case $2 \nmid d$ and $k \geq 7$. Then we have

$$\delta B_{k,\left(\frac{d}{\cdot}\right)} \equiv \sum_{i=\frac{k-3}{2}-2}^{\frac{k-3}{2}} \tbinom{k}{2i+1} 2 B_{k-2i-1} 2^{k-2i-2} \delta^{k-2i-1} t_{2i+1} - k\delta t_{k-1} + t_k \pmod{64}. \tag{4.2}$$

Hence in virtue of Lemma 4 (in the case $2 \nmid d$) for $k \geq 9$ we get

$$\begin{aligned} \delta B_{k,\left(\frac{d}{\cdot}\right)} \equiv h(d)\alpha(d) &\sum_{i=\frac{k-3}{2}-2}^{\frac{k-3}{2}} \tbinom{k}{2i+1} 2 B_{k-2i-1} 2^{k-2i-2} \delta^{k-2i-1} A_2(d,i) - \\ &- k_2(-4d) 2^{k-4} \sum_{i=\frac{k-3}{2}-2}^{\frac{k-3}{2}} \tbinom{k}{2i+1} 2 B_{k-2i-1} \delta^{k-2i-1} - \\ &- k\delta \left[A_1\left(d, \tfrac{k-1}{2}\right) h(d)\alpha(d) + \tfrac{k-1}{2} k_2(-4d) \right] + A_2\left(d, \tfrac{k-1}{2}\right) h(d)\alpha(d) \pmod{64}. \end{aligned}$$

From this, in view of $4\,|\,k_2(-4d)$ and $k \geq 9$ it follows

$$\delta B_{k,\left(\frac{d}{\cdot}\right)} \equiv A_4(d,k)h(d)\alpha(d) + d\binom{k}{2}k_2(-4d) \pmod{64},$$

where

$$A_4(d,k) := 32\binom{k}{6} + 8\binom{k}{4}\left(\frac{d}{2}\right)d + \frac{2}{3}\left(\frac{d}{2}\right)\binom{k}{2}d^3 + kdA_1\left(d,\frac{k-1}{2}\right) + A_2\left(d,\frac{k-1}{2}\right).$$

The task is now to find A_4 modulo 64. Indeed we have

$$\begin{aligned}
A_4(d,k) &\equiv 32\binom{k}{6} + 8\binom{k}{4}\left(\frac{d}{2}\right)d + 22\left(\frac{d}{2}\right)\binom{k}{2}d^3 + \\
&+ 2\binom{k}{2}d\left(2\left(\frac{d}{2}\right) - 1 - \left(\frac{d}{2}\right)d\right) - kd\left(2\left(\frac{d}{2}\right) - 1\right) + \\
&+ k\left(\left(\frac{d}{2}\right) - 2 + \left(\frac{d}{2}\right)d\right) - \left(\frac{d}{2}\right) + 2 \equiv 32\binom{k}{6} + 8\binom{k}{4}\left(\frac{d}{2}\right)d + \\
&+ 2\binom{k}{2}\left(\frac{d}{2}\right)\left(d + 2 + 8\left(\frac{d}{2}\right)\right) + 2\binom{k}{2}\left(-d + 8 - 7\left(\frac{d}{2}\right)\right) + \\
&+ k\left(d + \left(\frac{d}{2}\right) - 2 - \left(\frac{d}{2}\right)d\right) + 2 - \left(\frac{d}{2}\right) \equiv 32\binom{k}{6} + 8\binom{k}{4}\left(\frac{d}{2}\right)d + \\
&+ 2\binom{k}{2}\left(\left(\frac{d}{2}\right)d - d + 16 - 5\left(\frac{d}{2}\right)\right) + k\left(d + \left(\frac{d}{2}\right) - 2 - \left(\frac{d}{2}\right)d\right) + \\
&+ 2 - \left(\frac{d}{2}\right) \pmod{64}.
\end{aligned}$$

We need consider two cases. If $\left(\frac{d}{2}\right) = 1$ then we see that

$$\begin{aligned}
A_4(d,k) &\equiv 32\binom{k}{6} + 8\binom{k}{4} + 22\binom{k}{2} - k + 1 \\
&\equiv (k-1)\left[2(k-3)(k-5) + 3(1-2k)(k-3) + 11k - 1\right] \\
&\equiv (k-1)\left[(k-3)(4k+1) + 11k - 1\right] \\
&\equiv 4(k-1)^2(k+1) \equiv 0 \pmod{64}.
\end{aligned}$$

Let $\left(\frac{d}{2}\right) = -1$. Then we have

$$A_4(d,k) \equiv 32\binom{k}{6} + 24\binom{k}{4} + 2\binom{k}{2}(21 - 2d) + k(2d - 3) + 3 \pmod{64}.$$

Therefore in the case $k \equiv 1 \pmod 4$ we conclude that

$$\begin{aligned}
A_4(d,k) &\equiv 24\binom{k}{4} - 10\binom{k}{2} - 9(k-1) + 2d \\
&\equiv (k-1)\left[(k-3)(1-2k) - 5k - 9\right] + 2d \equiv 2(k^2-1) + 2d \pmod{64},
\end{aligned}$$

and in the case $k \equiv 3 \pmod 4$ we get

$$\begin{aligned}
A_4(d,k) &\equiv 32\binom{k}{6} + 24\binom{k}{4} + 11k(k-3) - 3(k-3) + 16 + 2d \\
&\equiv (k-3)[2(k-1)(k-5) + (k-1)(4k-9) + 11k - 3] + 16 + 2d \\
&\equiv (k-3)[-(k-1)(2k+3) + 11k - 3] + 16 + 2d \\
&\equiv 2(k-3)(5k-1) + 16 + 2d \equiv 2(k^2-1) + 2d \pmod{64}.
\end{aligned}$$

On account of the above for $k \geq 9$ the following congruence

$$\delta B_{k,(\frac{d}{\cdot})} \equiv (k^2 - \mu' + d)\left(1 - \left(\tfrac{d}{2}\right)\right) h(d)\alpha(d) + \vartheta' d\tbinom{k}{2} k_2(-4d) \pmod{64} \tag{4.3}$$

holds with $\mu' = \vartheta' = 1$, and consequently after an easy computation the theorem for $k \geq 9$ follows.

If $k = 3$ then from (3.1) we have

$$\delta B_{3,(\frac{d}{\cdot})} = 2d^2 t_1 + 3dt_2 + t_3.$$

Therefore by Lemma 4, (2.3) and (2.5) we deduce that

$$\delta B_{3,(\frac{d}{\cdot})} \equiv \left[-d^3 + A_2(d,1)\right] h(d)\alpha(d) - k_2(-4d) \pmod{64}.$$

Consequently the congruence (4.3) with $\mu' = 3 + 2d$ and ϑ' satisfying $3d\vartheta' = -1$ holds, and in consequence the theorem for $k = 3$ follows.

If $k = 5$ then from (3.1) we have

$$\delta B_{5,(\frac{d}{\cdot})} = -\tfrac{8}{3}d^4 t_1 + \tfrac{20}{3}d^2 t_3 + 5dt_4 + t_5.$$

Thus by Lemma 4, (2.3) and in view of $4 \mid k_2(-4d)$

$$\delta B_{5,(\frac{d}{\cdot})} \equiv A_5(d)h(d)\alpha(d) + 10dk_2(-4d) \pmod{64},$$

where

$$A_5(d) := -24d + 28(d-1)A_2(d,1) + 5dA_1(d,2) + A_2(d,2).$$

After a computation this congruence implies (4.3) with $\mu' = 17$ and $\vartheta' = 1$, and so the theorem for $k = 5$ follows.

Finally, if $k = 7$ then from (4.2) we find that

$$\delta B_{7,(\frac{d}{\cdot})} \equiv 32t_1 + 24t_3 + 14d^2 t_5 + 7dt_6 + t_7 \pmod{64}.$$

Therefore by Lemma 4, (2.3) and in virtue of $4 \mid k_2(-4d)$ we observe that

$$\delta B_{7,(\frac{d}{\cdot})} \equiv A_6(d)h(d)\alpha(d) + 5dk_2(-4d) \pmod{64},$$

where

$$A_6(d) := 32 + 24A_2(d,1) + 14d^2 A_2(d,2) + 7dA_1(d,3) + A_2(d,3).$$

This yields (4.3) with $\mu' = \vartheta' = 1$, and consequently the theorem for $k = 7$. The theorem is proved. ■

5. Proofs of Theorems 2 and 3.

We start with the formula (4.1). In the case $2\,|\,d$ and $k \geq 3$ it implies the congruence

$$\delta B_{k,\left(\frac{d}{\cdot}\right)} \equiv 6\binom{k}{2}\delta^2 t_{k-2} - k\delta t_{k-1} + t_k \pmod{64}.$$

But Lemma 4 for $k \geq 5$ and (2.3) for $k = 3$ give

$$64\delta \,|\, 6\binom{k}{2}\delta^2 t_{k-2}.$$

Consequently we get the congruence

$$\delta B_{k,\left(\frac{d}{\cdot}\right)} \equiv -k\delta t_{k-1} + t_k \pmod{64\delta}. \tag{5.1}$$

On the other hand by Lemma 4 we have

$$-k\delta t_{k-1} \equiv \binom{k}{2}\delta^3 h(d) \equiv \begin{cases} 16\binom{k}{2}\delta h(d) \pmod{64\delta}, & \text{if } 4\,\|\,d, \\ 0 \pmod{64\delta}, & \text{if } 8\,|\,d, \end{cases}$$

and consequently, from (5.1) and Lemma 5 for $k \geq 5$, or Lemma 2 for $k = 3$ the theorems follow at once because of the divisibilities $4 \mid k_2(d^*)$, $2 \mid h(d)$, $h(2d)$, and $8\,|\,k_2(d^*)$, $4\,|\,h(d)$, $h(2d)$, if $\left(\frac{d'}{2}\right) = 1$ in the case $d = -4d^*$. ■

6. Corollaries to Theorem 1.

In the corollaries below let us adopt the notation of Theorem 1. The following congruences follow immediately from the above theorem and Cor. 1, 2 to Thm. 2 [2].

COROLLARY 1. *If $\left(\frac{d}{2}\right) = 1$ and $k \geq 3$ then we have:*

$$b_k(d) \equiv -\tfrac{k-1}{2}k_2(-4d) \pmod{64},$$

and

$$b_k(d) \equiv 8(3 + \left(\tfrac{-1}{k}\right))\kappa \equiv 2(k-1)h(8d) \equiv (k-1)k_2(-8d) \pmod{16(3 + \left(\tfrac{-1}{k}\right))},$$

where $\kappa := 1$, if $\delta = p \equiv 7 \pmod{16}$ is a prime number and $k \not\equiv 1 \pmod 8$, and $\kappa := 0$, otherwise.

Moreover if $\delta = p \equiv -1 \pmod 8$ is a prime number then we have

$$b_k(d) \equiv 2(k-1)(p + 1 + h(8d)) \pmod{64}.$$ ■

COROLLARY 2. *If $\left(\frac{d}{2}\right) = 1$ and $k \geq 3$ then we have:*

(i) $\mathrm{ord}_2\, B_{k,\left(\frac{d}{\cdot}\right)} \geq 4$.

(ii) $\mathrm{ord}_2\, B_{k,\left(\frac{d}{\cdot}\right)} = 4 \iff 16\,\|\,k_2(-4d)$ *(or $4\,\|\,h(8d)$) and $k \equiv 3 \pmod 4$.*

(iii) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 5 \iff (16 \,\|\, k_2(-4d)$ *and* $k \equiv -3 \pmod 8)$ *or* $(32 \,\|\, k_2(-4d)$ *and* $k \equiv 3 \pmod 4))$.

(iv) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 6 \iff k \equiv 1 \pmod 8$ *or* $(32 \mid k_2(-4d)$ *and* $k \equiv -3 \pmod 8))$ *or* $64 \mid k_2(-4d)$.

(v) *If* $\delta = p \equiv -1 \pmod 8$ *is a prime number then we have:*

$$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 4 \iff p \equiv 7 \pmod{16} \text{ and } k \equiv 3 \pmod 4,$$

$$\begin{aligned}\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 5 \iff & (p \equiv 7 \pmod{16} \text{ and } k \equiv -3 \pmod 8) \text{ or} \\ & (p \equiv -1 \pmod{16} \text{ and } 8 \,\|\, h(8d) \text{ and } k \equiv 3 \pmod 4)),\end{aligned}$$

$$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 6, \text{ otherwise.}$$ ■

Corollary 3. *If* $\left(\frac{d}{2}\right) = -1$ *and* $k \geq 3$ *then we have:*

$$b_k(d) \equiv -2\lambda(6k-5)h(d)\alpha(d) - 2(k-1)h(8d) \pmod{16(3 + \left(\tfrac{-1}{k}\right))},$$

where $\lambda := \lambda_k(d)$, *and* $\lambda_3 = 2\nu_3 - \mu_3$, $\lambda_5 = \mu_5$, *and* $\lambda_k = 1$, *otherwise.*

Moreover if $\delta = p \equiv 3 \pmod 8$ *is a prime number then*

$$b_k(d) \equiv -2\theta_k h(d)\alpha(d) - 2(k-1)h(8d) \pmod{64},$$

where $\theta_k := \theta_k(d)$, *and*

$$\theta_k = (6k-5)\lambda_k + (p-3)(k-1).$$ ■

The above corollary and Theorem 1 imply the following:

Corollary 4. *If* $\left(\frac{d}{2}\right) = -1$ *and* $k \geq 3$ *then we have:*

(i) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 1$.

(ii) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 1 \iff 2 \nmid h(d)$ $($*or* $4 \,\|\, k_2(-4d)$, *or* $2 \,\|\, h(8d)$, *or* $4 \,\|\, k_2(-8d)) \iff \delta = p \equiv 3 \pmod 8$ *is a prime number.*

(iii) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 2 \iff 2 \,\|\, h(d)$ $($*or* $8 \,\|\, k_2(-4d))$.

(iv) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 3 \iff 4 \,\|\, h(d)$ *and* $[k \equiv 1 \pmod 4$ *or* $(k \equiv 3 \pmod 4)$ *and* $4 \mid h(8d))]$.

(v) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 4 \iff \{8 \,\|\, h(d)$ *and* $[k \equiv 1 \pmod 8$ *or* $(k \equiv -3 \pmod 8$ *and* $4 \mid h(8d))$ *or* $(k \equiv 3 \pmod 4$ *and* $8 \mid h(8d))]\}$ *or* $(16 \mid h(d)$ *and* $k \equiv 3 \pmod 4$ *and* $4 \,\|\, h(8d))$.

(vi) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = 5 \iff [8 \,\|\, h(d)$ *and* $16 \,\|\, k_2(-4d)$ *and* $k \equiv 3 \pmod 4$ *and* $\frac{1}{8}h(d)\alpha(d) \not\equiv \frac{1}{16}\left(\frac{2}{k}\right)k_2(-4d) \pmod 4]$ *or* $\{16 \,\|\, h(d))$ *and* $[(k \equiv 1 \pmod 8$ *and* $16 \mid k_2(-4d))$ *or* $(k \equiv -3 \pmod 8$ *and* $32 \mid k_2(-4d))$ *or* $(k \equiv 3 \pmod 4$ *and* $64 \mid k_2(-4d))]\}$ *or* $\{32 \mid h(d)$ *and* $[(k \equiv -3 \pmod 8$ *and* $16 \,\|\, k_2(-4d))$ *or* $(k \equiv 3 \pmod 4$ *and* $32 \,\|\, k_2(-4d))]\}$.

(vii) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 6 \iff$ [$8 \parallel h(d)$ *and* $16 \parallel k_2(-4d)$ *and* $k \equiv 3 \pmod 4$ *and* $\frac{1}{8}h(d)\alpha(d) \equiv \frac{1}{16}\left(\frac{2}{k}\right) k_2(-4d) \pmod 4$] *or* {$16 \parallel h(d)$ *and* [($k \equiv -3 \pmod 8$ *and* $16 \parallel k_2(-4d)$) *or* ($k \equiv 3 \pmod 4$ *and* $32 \parallel k_2(-4d)$)]} *or* {$32 \mid h(d)$ *and* [($k \equiv 1 \pmod 8$ *and* $16 \mid k_2(-4d)$) *or* ($k \equiv -3 \pmod 8$ *and* $32 \mid k_2(-4d)$) *or* ($k \equiv 3 \pmod 4$ *and* $64 \mid k_2(-4d)$)]}. ■

If $k \equiv 1 \pmod{2^a}$, $a \leq 5$ then the congruence of the theorem implies the congruence (1.2) and so this is a generalization of this congruence. If $k \equiv 1 \pmod{2^a}$ then Theorem 1 leads to

$$b_k(d) \equiv -k\mu\left(1 - \left(\frac{d}{2}\right)\right) h(d)\alpha(d) \pmod{2^{a+f-1}},$$

where $f := \mathrm{ord}_2 k_2(-4d)$(i.e. $f \geq 2$, or $f \geq 4$, if $\left(\frac{d}{2}\right) = 1$, cf. (1.2)).

In the case $\left(\frac{d}{2}\right) = 1$ by the congruences of Cor. 1, for any k we get a fairly straightforward generalization of the formula (1.4) for $a \leq 5$ (see Cor. 2(iv)). Also in this case we get formulas of the type of (1.5) (see Cor. 2(ii), (iii), (v)). In the case $\left(\frac{d}{2}\right) = -1$, Cor. 4 gives an extension of the formula (1.5) for $a \leq 5$.

In the second part of the paper we shall present analogous congruences and formulas to (1.3) and (1.6), (1.7).

7. Corollaries to Theorems 2, 3 and 4.

Applying Cor. 1,2 to Thm. 1, 2 [2] to Theorems 2,3 and 4 in the notation of these theorems we obtain the following:

Corollary 1. *If* $4 \parallel d$ *and* $k \geq 3$ *then we have:*

(i) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = -1$, *if* $d = -4$, *and* $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 1$, *if* $d < -4$.

(ii) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = \nu$, $1 \leq \nu \leq 3 \iff 2^\nu \parallel h(d)$.

(iii) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = 4 \iff$ {$16 \parallel h(d)$ *and* [$k \equiv 1 \pmod 4$ *or* ($k \equiv 3 \pmod 4$ *and* $32 \mid k_2(d^*)$)]} *or* ($32 \mid h(d)$ *and* $k \equiv 3 \pmod 4$ *and* $16 \parallel k_2(d^*)$).

(iv) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = 5 \iff$ [$16 \parallel h(d)$ *and* $16 \parallel k_2(d^*)$ *and* $k \equiv 3 \pmod 4$ *and* $k_2(d^*) + h(d) \equiv 16\left(1 - \left(\frac{2}{k}\right)\right) \pmod{64}$] *or* {$32 \parallel h(d)$ *and* [$k \equiv 1 \pmod 8$ *or* ($k \equiv 5 \pmod 8$ *and* $32 \mid k_2(d^*)$) *or* ($k \equiv 3 \pmod 4$ *and* $64 \mid k_2(d^*)$)]} *or* ($64 \mid h(d)$ *and* $k \equiv 3 \pmod 4$ *and* $32 \parallel k_2(d^*)$),

$\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 6$, *otherwise.*

Corollary 2. *If* $8 \mid d$ *and* $k \geq 3$ *then we have:*

(i) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 0$.

(ii) $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = \nu, 0 \leq \nu \leq 3 \iff 2^\nu \parallel h(d)$ *(i.e.* $\mathrm{ord}_2 B_{k,(\frac{d}{\cdot})} = 0 \iff d = -8$*).*

(iii) *If* $d^* < 0$ *and* $\left(\frac{d^*}{2}\right) = 1$ *then:*

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 4 \iff \{16 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 4 \text{ or } (k \equiv 3 \pmod 4 \text{ and } 4 \mid h(d^*)]\} \text{ or } (32 \mid h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } 2 \,\|\, h(d^*)),$$

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 5 \iff (16 \,\|\, h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } 2 \,\|\, h(d^*) \text{ and } k_2(-4d^*) + 8h(d^*)\alpha(d^*) \equiv -\left(\tfrac{2}{k}\right) h(d) \pmod{64}) \text{ or } \{32 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 8 \text{ or } (k \equiv -3 \pmod 8 \text{ and } 4 \mid h(d^*)) \text{ or } (k \equiv 3 \pmod 4 \text{ and } 64 \mid k_2(-4d^*) + 8h(d^*)\alpha(d^*))]\} \text{ or } \{64 \mid h(d) \text{ and } [(k \equiv -3 \pmod 8 \text{ and } 2 \,\|\, h(d^*)) \text{ or } (k \equiv 3 \pmod 4 \text{ and } 32 \,\|\, k_2(-4d^*) + 8h(d^*)\alpha(d^*))]\},$$

$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} \geq 6$, *otherwise.*

If $d^* < 0$ *and* $\left(\frac{d^*}{2}\right) = -1$ *then:*

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 4 \iff \{16 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 4 \text{ or } (k \equiv 3 \pmod 4 \text{ and } k_2(-4d^*) \equiv 12h(d^*)\alpha(d^*) \pmod{32}))]\} \text{ or } (32 \mid h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } 16 \,\|\, k_2(-4d^*) + 20h(d^*)\alpha(d^*)),$$

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 5 \iff (16 \,\|\, h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } k_2(-4d^*) + 20h(d^*)\alpha(d^*) \equiv -\left(\tfrac{2}{k}\right) h(d) \pmod{64}) \text{ or } \{32 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 8 \text{ or } (k \equiv -3 \pmod 8 \text{ and } k_2(-4d^*) \equiv 12h(d^*)\alpha(d^*) \pmod{32}) \text{ or } (k \equiv 3 \pmod 4 \text{ and } k_2(-4d^*) \equiv -20h(d^*)\alpha(d^*) \pmod{64})]\} \text{ or } \{64 \mid h(d) \text{ and } [(k \equiv -3 \pmod 8 \text{ and } 16 \,\|\, k_2(-4d^*) + 20h(d^*)\alpha(d^*)) \text{ or } (k \equiv 3 \pmod 4 \text{ and } 32 \,\|\, k_2(-4d^*) + 20h(d^*)\alpha(d^*))]\},$$

$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} \geq 6$, *otherwise.*

(iv) *If* $d^* > 0$ *then*

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 4 \iff \{16 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 4 \text{ or } (k \equiv 3 \pmod 4 \text{ and } k_2(d^*)\beta(d^*) \equiv 2h(-4d^*) \pmod{32}))]\} \text{ or } (32 \mid h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } 16 \,\|\, k_2(d^*)\beta(d^*) - 2h(-4d^*)),$$

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = 5 \iff [16 \,\|\, h(d) \text{ and } k \equiv 3 \pmod 4 \text{ and } \left(1 - 2\left(\tfrac{d^*}{2}\right)\right) k_2(d^*)\beta(d^*) + 2h(-4d^*) \equiv \left(\tfrac{2}{k}\right) h(d) \pmod{64}] \text{ or } \{32 \,\|\, h(d) \text{ and } [k \equiv 1 \pmod 8 \text{ or } (k \equiv -3 \pmod 8 \text{ and } \left(1 - 2\left(\tfrac{d^*}{2}\right)\right) k_2(d^*)\beta(d^*) \equiv -2h(-4d^*) \pmod{32}) \text{ or } (k \equiv 3 \pmod 4 \text{ and } \left(1 - 2\left(\tfrac{d^*}{2}\right)\right) k_2(d^*)\beta(d^*) \equiv -2h(-4d^*) \pmod{64})]\} \text{ or } \{64 \mid h(d) \text{ and }$$

$$[(k \equiv -3 \pmod 8) \text{ and } 16 \,\|\, k_2(d^*)\beta(d^*) - 2h(-4d^*))$$
$$\text{or } (k \equiv 3 \pmod 4) \text{ and}$$
$$32 \,\|\, \left(1 - 2\left(\tfrac{d'}{2}\right)\right) k_2(d^*)\beta(d^*) + 2h(-4d^*)]\},$$

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq 6$, *otherwise.*

Corollaries to Theorems 2, 3 and 4 are extensions of the formulas (1.4) and (1.5) for $2 \mid d$ and $a \leq 5$.

REMARK. If $d = -4$ then by Lemma 4 and the congruence (5.1) we get

$$2b_k(d) \equiv 18 + \tfrac{13}{k} \pmod{32},$$

i.e.

$$E_{k-1} \equiv 14 - \tfrac{13}{k} \pmod{32},$$

where E_i denotes the ith Euler number. This congruence completes Theorem 4.

8. Proofs of Corollaries to Theorems.

Corollaries to Theorem 1 follow immediately from this theorem and Cor. 1, 2 to Thm. 2 [2]. It remains to prove Corollaries to Theorems 2, 3 and 4. Indeed, Corollaries 1, 2(i), (ii) (2(ii) for $\nu \leq 2$) are easy consequences of Theorem 4. To prove Corollary 1(iii), (iv) let us notice that by Cor. 2(i), (ii) to Thm. 1 [2], $16 \mid h(d)$ implies $4 \mid h(2d)$ and $16 \mid k_2(d^*)$. Therefore we have

$$\vartheta_1 k_2(d^*)\beta(d^*) + \vartheta_3 h(2d) \equiv \begin{cases} 2(k-1)h(2d) \pmod{64}, & \text{if } k \equiv 1 \pmod 4, \\ -k_2(d^*) - 2(k+1)h(2d) & \\ \qquad \pmod{64}, & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{6.1}$$

This yields (iii) at once. To prove (iv) let us make the following observation. If $a, b, c \in \mathbb{Z}$ then

$$\left.\begin{array}{r} a+c, b \equiv 16 \pmod{32} \\ a+b+c \equiv 32 \pmod{64} \end{array}\right\} \iff \left.\begin{array}{r} a+c \equiv b \pmod{64} \\ b \equiv 16 \pmod{32} \end{array}\right\}. \tag{6.2}$$

Combining this with (6.1) gives (iv) because by Cor. 2(iii) to Thm. 1 [2], $16 \,\|\, h(d)$ and $16 \,\|\, k_2(d^*)$ imply $4 \,\|\, h(2d)$, and if $16 \mid h(d)$ then $8 \mid h(2d)$ if and only if $32 \mid k_2(d^*)$.

We now turn to Corollary 2(ii) (for $\nu = 3$), (iii), (iv). Let $d^* < 0$. If $\left(\frac{d'}{2}\right) = 1$ then by Cor. 2(i) to Thm. 2 [2] we find that $16 \mid k_2(-4d^*)$ and $2 \mid h(d^*)$, if $\left(\frac{d'}{2}\right) = -1$ then by Cor. 2(iii) to the mentioned theorem the divisibility $8 \mid h(d)$ implies $2 \mid h(d^*)$ and $8 \mid k_2(-4d^*)$. Moreover we have

$$\vartheta_1 k_2(-4d^*) + \vartheta_3 h(d^*)\alpha(d^*) \equiv \begin{cases} -\frac{k-1}{2}[k_2(-4d^*) + 8h(d^*)\alpha(d^*)] & \\ \qquad \pmod{64}, & \text{if } \left(\frac{d'}{2}\right) = 1, \\ -\frac{k-1}{2}[k_2(-4d^*) + 20h(d^*)\alpha(d^*)] & \\ \qquad \pmod{64}, & \text{if } \left(\frac{d'}{2}\right) = -1. \end{cases}$$

Hence and from Cor. 1(i) to Thm. 2[2],(ii) for $\nu = 3$ follows immediately. Indeed, $8 \mid h(d)$ yields $16 \mid \vartheta_1 k_2(-4d^*) + \vartheta_3 h(d^*)\alpha(d^*)$ in both the cases. To prove (iii) it suffices to use Cor. 2(i) ((i), resp. (iii), if $\left(\frac{d'}{2}\right) = 1$, resp. -1) to Thm. 2 [2]. In fact, if $\left(\frac{d'}{2}\right) = 1$ and $16 \parallel h(d)$ then by the mentioned corollary we have $32 \mid k_2(-4d^*)$. Now Corollary 2(iii) follows from (6.3) and (6.2).

Now let $d^* > 0$. Then by Cor. 2(i) to Thm. 1 [2], $8 \mid h(d)$ implies $4 \mid h(-4d^*)$ and $8 \mid k_2(d^*)$. Moreover we have

$$\mu_1 k_2(d^*)\beta(d^*) + \mu_3 h(-4d^*) = -\tfrac{k-1}{2}\left[3\left(1 - 2\left(\tfrac{d'}{2}\right)\right) k_2(d^*)\beta(d^*) - 10h(-4d^*)\right]. \quad (6.3)$$

Thus by Cor. 1(i) to Thm. 1 [2] we get (ii) for $\nu = 3$. Then the left hand side of (6.3) is congruent to $-4\left(1 + 3\left(\frac{d'}{2}\right)\right) h(-4d^*) \equiv 0 \pmod{16}$.

To prove (iv) it is sufficient to use (6.3), and also (6.2), if $16 \parallel h(d)$ and $k \equiv 3 \pmod 4$. This completes the proof. ∎

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ON SOME NEW CONGRUENCES

BETWEEN GENERALIZED BERNOULLI NUMBERS, II

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On some new congruences between generalized Bernoulli numbers, II

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On some new congruences between generalized Bernoulli numbers, II

Abstract. The paper is a continuation of my earlier paper on this subject. We prove analogous congruences as in that paper, but for positive discriminants d. Also for each $0 \leq \nu \leq 5$ all positive d and even k satisfying $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \nu + \operatorname{ord}_2 k$ are found.

The proofs are similar in spirit to proofs of [3], and based on ideas of [1] and [2], again.

In the third part of the paper we shall study related problems, but from a p-adic measure point of view.

Key words: Bernoulli numbers, Kummer congruences, class numbers.

1. Notation.

We follow the notation of [3]. Let d stand for the discriminant of a quadratic field. Denote by $\left(\frac{d}{\cdot}\right)$, resp. $B_{k,\chi}$ the Kronecker symbol, resp. the kth generalized Bernoulli number belonging to the Dirichlet character χ. Set $\delta := |d|$. Write $h(d) := -B_{1,\left(\frac{d}{\cdot}\right)}$, if $d < -4$ and $h(-3) = h(-4) := 1$. Put $k_2(d) := B_{2,\left(\frac{d}{\cdot}\right)}$, if $d > 8$ and $k_2(5) = k_2(8) := 4$. Let $x \geq 0$ and $X \subset \mathbb{N} \cup \{0\}$. To simplify the notation we continue to write (as in [3]) $t_k(x)$, $t_k(x, a \in X)$, resp. $t_k^*(x)$, $t_k^*(x, a \in X)$ for sums of the kth powers of natural numbers taken from 0 to x, involving quadratic characters $\left(\frac{d}{\cdot}\right)$, resp. $\left(\frac{d^*}{\cdot}\right)$, where d^* is the discriminant of a quadratic field satisfying $d = -4d^*$ or $\pm 8d^*$. Write $t_k := t_k(\delta)$ and $t'_k := t_k(\delta, 2 \nmid a)$.

Let us recall that $B_{0,\left(\frac{d}{\cdot}\right)} = 0$, and for $k \geq 1$

$$B_{k,\left(\frac{d}{\cdot}\right)} = 0 \tag{1.1}$$

if and only if $\left(\frac{d}{-1}\right) \neq (-1)^k$. Write $\alpha(-3) := \frac{1}{3}$, $\beta(5) := \frac{1}{5}$, $\rho(8) := \frac{1}{2}$, and $\alpha(d)$, $\beta(d)$, $\rho(d) := 1$, otherwise. Put $\eta(5) := \beta(5)$, $\eta(8) := \rho(8)$, and $\eta(d) := 1$, otherwise. Set

$$b_k(d) := \frac{B_{k,\left(\frac{d}{\cdot}\right)}}{k}.$$

2. Theorems.

Our purpose is for each $0 \leq \nu \leq 5$ to find all positive d and even k such that $\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)} = \nu + \operatorname{ord}_2 k$. We prove some new congruences between generalized Bernoulli numbers of the Kummer congruences type modulo 64 but with deleted assumptions on k. For a deeper discussion of it we refer the reader to [3].

In this part we prove the following generalizations of the congruence (1.3) of [3]:

THEOREM 1. *Let d, $2 \nmid d$ and $k \geq 4$ be the discriminant of a real quadratic field and an even natural number respectively. With the above notation, the numbers $b_k(d)$ are 2-integral and the following congruence holds:*

$$b_k(d) \equiv \left(2k\left(\frac{d}{2}\right) + k + 2\right)\mu h(-4d) + \frac{3}{2}\left(-k - 2\left(\frac{d}{2}\right) + 1\right)\vartheta k_2(d)\beta(d) \pmod{64},$$

where $\mu := \mu_k(d)$, $\vartheta := \vartheta_k(d)$, and

$$\mu_4 = -d + 10 + 4\left(\frac{d}{2}\right), \quad \mu_6 = 8 + 5\left(\frac{d}{2}\right),$$
$$\vartheta_4 = 2d + 8\left(\frac{d}{2}\right) + 7, \qquad \vartheta_6 = -4\left(\frac{d}{2}\right) - 11,$$

and μ_k, $\vartheta_k = 1$, otherwise.

The case $2\,|\,d$ is more complicated. We prove the following:

THEOREM 2. *Let $d = -4d^*$, where d^* is the discriminant of an imaginary quadratic field, and let $k \geq 4$ be an even natural number. Then the numbers $b_k(d)$ are 2-integral and we have:*

$$b_k(d) \equiv \vartheta_1 \tfrac{1}{2} k_2(d) + \vartheta_2 h(d^*)\alpha(d^*) + \vartheta_3 h(-2d) \pmod{64},$$

where $\vartheta_i := \vartheta_i(d,k) \in \mathbb{Z}$ $(i = 1,2,3)$ are of the form $\vartheta = pk + q$, and

$$\vartheta_1 = k - 1, \qquad \vartheta_3 = -4(k-2),$$

$$\vartheta_2 = -3\left(1 - \left(\tfrac{d^*}{2}\right)\right)(k-2) + 8\left(1 + \left(\tfrac{d^*}{2}\right)\right)\left(1 - \left(\tfrac{-1}{k-1}\right)\right).$$

THEOREM 3. *Let $d = \pm 8d^*$, $d > 0$, where d^* is the discriminant of a quadratic field (i.e. $d \neq 8$), and let $k \geq 4$ be an even natural number. Set $\lambda := 1$, if $k = 4$, and $\lambda := 0$, otherwise. Then the numbers $b_k(d)$ are 2-integral and we have:*

$$b_k(d) \equiv \begin{cases} \vartheta_1 \frac{1}{2} k_2(d) + \vartheta_2 h(-d) + \vartheta_3 h(-4d^*) \pmod{64}, & \text{if } d^* > 0, \\ \mu_1 \frac{1}{2} k_2(d) + \mu_2 h(-d) + \mu_3 h(d^*)\alpha(d^*) \pmod{64}, & \text{if } d^* < 0, \end{cases}$$

where $\vartheta_i := \vartheta_i(d,k)$, $\mu_i := \mu_i(d,k) \in \mathbb{Z}$ $(i = 1,2,3)$ are of the form $\vartheta, \mu = pk + q$, and

$$\vartheta_1 = k - 1, \qquad \vartheta_2 = 13(k-2) + 16\lambda, \qquad \vartheta_3 = -4\left(\tfrac{d^*}{2}\right)(k-2),$$

$$\mu_1 = k - 1, \qquad \mu_2 = \left(4\left(\tfrac{d^*}{2}\right) + 1\right)(k-2) + 16\lambda, \qquad \mu_3 = 8(k-2).$$

Combining Thm. 2 and 3 with Cor. 1 to Thm. 1, 2 [2] we can get many new congruences for generalized Bernoulli numbers modulo 64 (or 32).

Lemma 6 and the congruence (5.1) give a weaker version of Thm. 2 and 3:

THEOREM 4. *Let $d > 0$, $2\,|\,d$ be the discriminant of a quadratic field, and let $k \geq 4$ be an even natural number. Then the numbers $b_k(d)$ are 2-integral and*

$$b_k(d) \equiv \tfrac{1}{2} k_2(d)\rho(d) \pmod{2^{6-\mathrm{ord}_2 d}}.$$

3. Lemmas.

We shall need Lemma 1 [3]. Likewise in [3], the proofs of the theorems fall naturally into a sequence of lemmas. First we shall prove a lemma of the kind of the above mentioned lemma:

LEMMA 1. *Let d, $2 \nmid d$ be the discriminant of a quadratic field. Then we have:*

(i) *If $d > 0$ then*

$$t_1(\delta, a \equiv \delta \pmod 8) = \left(\tfrac{d}{2}\right)\left[-8t_1(\delta/8) + \delta t_0(\delta/8)\right],$$

$$t_1(\delta, a \equiv \delta + 2 \pmod 8) = 4\left[t_1(\delta/2) - \left(2\left(\tfrac{d}{2}\right) + 1\right) t_1(\delta/4) + 2\left(\tfrac{d}{2}\right) t_1(\delta/8)\right] - \\ - \delta\left[-\left(1 + \left(\tfrac{d}{2}\right)\right) t_0(\delta/4) + \left(\tfrac{d}{2}\right) t_0(\delta/8)\right],$$

$$
\begin{aligned}
t_1(\delta, a \equiv \delta+4 \pmod 8) &= -4\left[t_1(\delta/4) - 2\left(\tfrac{d}{2}\right) t_1(\delta/4)\right] + \\
&\quad + \delta\left[t_0(\delta/4) - \left(\tfrac{d}{2}\right) t_0(\delta/8)\right], \\
t_1(\delta, a \equiv \delta+6 \pmod 8) &= 8\left(\tfrac{d}{2}\right)[t_1(\delta/4) - t_1(\delta/8)] - \left(\tfrac{d}{2}\right)\delta\,[t_0(\delta/4) - t_0(\delta/8)].
\end{aligned}
$$

(ii) *If $d < 0$ then*

$$
\begin{aligned}
t_1(\delta, a \equiv \delta \pmod 8) &= \left(\tfrac{d}{2}\right)[8t_1(\delta/8) - \delta t_0(\delta/8)], \\
t_1(\delta, a \equiv \delta+2 \pmod 8) &= 8\left(\tfrac{d}{2}\right)[t_1(\delta/4) - t_1(\delta/8)] - \left(\tfrac{d}{2}\right)\delta\,[t_0(\delta/4) - t_0(\delta/8)], \\
t_1(\delta, a \equiv \delta+4 \pmod 8) &= 4\left[t_1(\delta/4) - 2\left(\tfrac{d}{2}\right) t_1(\delta/8)\right] + \\
&\quad + \delta\left[-t_0(\delta/4) + \left(\tfrac{d}{2}\right) t_0(\delta/8)\right], \\
t_1(\delta, a \equiv \delta+6 \pmod 8) &= 4\left[t_1(\delta/2) - \left(2\left(\tfrac{d}{2}\right)+1\right) t_1(\delta/4) + 2\left(\tfrac{d}{2}\right) t_1(\delta/8)\right] - \\
&\quad - \delta\left[t_0(\delta/2) - \left(1+\left(\tfrac{d}{2}\right)\right) t_0(\delta/4) + \left(\tfrac{d}{2}\right) t_0(\delta/8)\right].
\end{aligned}
$$

PROOF. First, let us notice that by (2.1a,b) [3] we have

$$
\begin{aligned}
t_1(\delta/2, a \equiv \delta \pmod 4) &= -\left(\tfrac{d}{-1}\right) \sum_{\substack{0 \le a \le \delta/2, \\ a \equiv \delta \pmod 4}} \left(\tfrac{d}{\delta-a}\right)(\delta - a) + \delta t_0(\delta/2, a \equiv \delta \pmod 4) \\
&= -\left(\tfrac{d}{-1}\right) \sum_{\substack{\delta/2 \le a \le \delta, \\ 4 | a}} \left(\tfrac{d}{a}\right) a + \delta t_0(\delta/2, a \equiv \delta \pmod 4) \\
&= -\left(\tfrac{d}{-1}\right)[t_1(\delta, 4\,|\,a) - t_1(\delta/2, 4\,|\,a)] + \delta t_0(\delta/2, a \equiv \delta \pmod 4) \\
&= -4\left(\tfrac{d}{-1}\right)[t_1(\delta/4) - t_1(\delta/8)] + \delta\left(\tfrac{d}{-1}\right)[t_0(\delta/4) - t_0(\delta/8)],
\end{aligned}
$$

and similarly

$$
\begin{aligned}
t_1(\delta/2, a \equiv \delta+2 \pmod 4) &= -\left(\tfrac{d}{-1}\right) \sum_{\substack{\delta/2 \le a \le \delta, \\ a \equiv 2 \pmod 4}} \left(\tfrac{d}{a}\right) a + \delta t_0(\delta/2, a \equiv \delta+2 \pmod 4) \\
&= -\left(\tfrac{d}{-1}\right)[t_1(\delta, 2\,|\,a) - t_1(\delta, 4\,|\,a) - t_1(\delta/2, 2\,|\,a) + \\
&\quad + t_1(\delta/2, 4\,|\,a)] + \delta t_0(\delta/2, a \equiv \delta+2 \pmod 4) \\
&= -2\left(\tfrac{d}{-1}\right)\left[\left(\tfrac{d}{2}\right) t_1(\delta/2) - \left(2+\left(\tfrac{d}{2}\right)\right) t_1(\delta/4) + 2t_1(\delta/8)\right] + \\
&\quad + \delta\left(\tfrac{d}{-1}\right)\left[\left(\tfrac{d}{2}\right) t_0(\delta/2) - \left(1+\left(\tfrac{d}{2}\right)\right) t_0(\delta/4) + t_0(\delta/8)\right].
\end{aligned}
$$

Applying the above and Lemma 1 [3] gives the lemma because

$$
\begin{aligned}
t_1(\delta, a \equiv \delta \pmod 8) &= -\left(\tfrac{d}{-1}\right) t_1(\delta, 8\,|\,a) + \delta t_0(\delta, a \equiv \delta \pmod 8) \\
&= -8\left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_1(\delta/8) + \delta t_0(\delta, a \equiv \delta \pmod 8),
\end{aligned}
$$

$$
\begin{aligned}
t_1(\delta, a \equiv \delta+2 \pmod 8) &= -\left(\tfrac{d}{-1}\right) t_1(\delta, a \equiv -2 \pmod 8) + \delta t_0(\delta, a \equiv \delta+2 \pmod 8)\\
&= -2\left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_1(\delta/2, a \equiv 3 \pmod 4) +\\
&\quad + \delta t_0(\delta, a \equiv \delta+2 \pmod 8),\\
t_1(\delta, a \equiv \delta+4 \pmod 8) &= -\left(\tfrac{d}{-1}\right) t_1(\delta, a \equiv 4 \pmod 8) + \delta t_0(\delta, a \equiv \delta+4 \pmod 8)\\
&= -\left(\tfrac{d}{-1}\right) [t_1(\delta, 4\,|\,a) - t_1(\delta, 8\,|\,a)] + \delta t_0(\delta, a \equiv \delta+4 \pmod 8)\\
&= -4\left(\tfrac{d}{-1}\right) \left[t_1(\delta/4) - 2\left(\tfrac{d}{2}\right) t_1(\delta/8)\right] +\\
&\quad + \delta t_0(\delta, a \equiv \delta+4 \pmod 8),\\
t_1(\delta, a \equiv \delta+6 \pmod 8) &= -\left(\tfrac{d}{-1}\right) t_1(\delta, a \equiv 2 \pmod 8) + \delta t_0(\delta, a \equiv \delta+6 \pmod 8)\\
&= -2\left(\tfrac{d}{-1}\right)\left(\tfrac{d}{2}\right) t_1(\delta/2, a \equiv 1 \pmod 4) +\\
&\quad + \delta t_0(\delta, a \equiv \delta+6 \pmod 8).
\end{aligned}
$$

■

LEMMA 2. *Let d be the discriminant of a real quadratic field. Then we have:*

(i)
$$t'_0 = 0.$$

(ii)
$$t'_1 = \tfrac{1}{2}\left(\tfrac{d}{2}\right)\left(4 - \left(\tfrac{d}{2}\right)\right) k_2(d)\beta(d).$$

(iii)
$$t'_2 = \begin{cases} 2d\left(\tfrac{d}{2}\right) k_2(d)\beta(d), & \text{if } 2 \nmid d,\\ dk_2(d)\rho(d), & \text{if } 2 \,|\, d.\end{cases}$$

(iv)
$$t'_3 = \tfrac{3}{2} d^2 k_2(d)\rho(d), \qquad \text{if } 2 \,|\, d.$$

(v)
$$t'_3 \equiv \tfrac{3}{2} C(d) k_2(d)\beta(d) + 2dh(-4d) \pmod{64}, \qquad \text{if } 2 \nmid d,$$

where

$$C(d) := 2d - 4 + \left(\tfrac{d}{2}\right).$$

PROOF. Since $t_0 = 0$, and for $d > 0$

$$t_0(d/2) = 0,$$

(i) of the lemma follows easily.

For any d and i we have

$$t'_i = t_i - 2^i \left(\tfrac{d}{2}\right) t_i(\delta/2). \tag{3.1}$$

Therefore (ii) follows from $t_1 = 0$ and (2.12) [3] immediately.

On the other hand for $d > 0$ we have

$$t_2 = dk_2(d)\eta(d). \tag{3.2}$$

Therefore to prove (iii) of the lemma it suffices to note that

$$t_2 = 2t_2(\delta/2) - 2dt_1(\delta/2)$$

and to use (2.12) [3] and (3.1). Then we have

$$t_2(d/2) = -\tfrac{1}{4}d\left(2 - \left(\tfrac{d}{2}\right)\right) k_2(d)\eta(d),$$

and

$$t_2' = \left[1 + 2\left(\tfrac{d}{2}\right) - \left(\tfrac{d}{2}\right)^2\right] dk_2(d)\eta(d).$$

Now we prove (iv) and (v) of the lemma. If $d > 0$ then by (1.1) we deduce that

$$B_{3,\left(\frac{d}{\cdot}\right)} = 0.$$

Consequently for $d > 0$ from the formula (3.1) of [3] (with $F = d$) we get

$$t_3 = \tfrac{3}{2}d^2 k_2(d)\eta(d). \tag{3.3}$$

This gives (iv). We now turn to (v). For any d we have

$$t_3(\delta/2) \equiv t_1(\delta/2, 2\nmid a) = t_1(\delta/2) - 2\left(\tfrac{d}{2}\right) t_1(\delta/4) \pmod 8.$$

Hence and from (2.12, 13) [3] for $d > 0$, $2 \nmid d$ we obtain

$$t_3(d/2) \equiv -\tfrac{3}{8}\left(2 - 3\left(\tfrac{d}{2}\right)\right) k_2(d)\beta(d) - \tfrac{1}{4}d\left(\tfrac{d}{2}\right) h(-4d) \pmod 8.$$

Thus (v) follows from (3.1) and (3.3). This completes the proof of the lemma. ∎

Similarly as in [3], combining the above lemma with Corollary to Lemma 3 of [3] yields:

LEMMA 3. *Let d be the discriminant of a real quadratic field. Then we have:*

(i) *If $i \geq 2$ then*

$$t_{2i} \equiv \begin{cases} 2\left[id\left(\frac{d}{2}\right) - 2^{2i-3}\right] k_2(d)\beta(d) \pmod{64}, & \text{if } 2 \nmid d, \\ idk_2(d)\rho(d) \pmod{64}, & \text{if } 2 \mid d. \end{cases}$$

(ii) *if $i \geq 1$ then*

$$t_{2i+1} \equiv \begin{cases} \frac{1}{4}A_1(d,i)k_2(d)\beta(d) + 2(i - 2^{2i-2})dh(-4d) \pmod{64}, & \text{if } 2 \nmid d, \\ \frac{3}{2}id^2 k_2(d)\rho(d) \pmod{64}, & \text{if } 2 \mid d, \end{cases}$$

where

$$A_1(d,i) := 3 \cdot 2^{2i}\left(3 - 2\left(\tfrac{d}{2}\right)\right) - 4i(d+2) + 2\left(4\left(\tfrac{d}{2}\right) - 1\right).$$

PROOF. As in the proof of Lemma 4 of [3], let us note that if $2 \mid d$ then $t_i = t'_i$, and if $2 \nmid d$ and $i \geq 3$ then

$$t_i \equiv 2^i \left(\tfrac{d}{2}\right) t_\sigma \left(d/2, 2 \nmid a\right) + t'_i \pmod{64}, \tag{3.4}$$

where $i \equiv \sigma \pmod 2$, $\sigma \in \{0,1\}$.

Since $t_0(d/2) = t'_0 = 0$, applying Cor. to Lem. 3 [3] to (3.4) for $i \geq 2$ we get

$$t_{2i} \equiv 2^{2i} \left(\tfrac{d}{2}\right) t_0(d/2, 2 \nmid a) + t'_{2i} \equiv -2^{2i} \left(\tfrac{d}{2}\right)^2 t_0(d/4) + it'_2 \pmod{64}. \tag{3.5}$$

Hence and from Lemma 1 (iii), (i) of the lemma for $2 \mid d$ follows immediately. In order to prove (i) in complete, it remains to consider the case $2 \nmid d$. Then in view of (2.10) [3], (3.5) and Lemma 1 (iii) imply

$$t_{2i} \equiv -2^{2i-1} h(-4d) + 2id \left(\tfrac{d}{2}\right) k_2(d)\beta(d) \pmod{64}.$$

Consequently (i) of the lemma for $2 \nmid d$ follows by Cor. 1 (i) to Thm. 1 [3] that implies the congruence

$$k_2(d)\beta(d) \equiv 2h(-4d) \pmod{16}. \tag{3.6}$$

We now turn to (ii) of the lemma. From (3.4) and Cor. to Lem. 3 [4] for $i \geq 1$ we get

$$\begin{aligned} t_{2i+1} &\equiv 2^{2i+1} \left(\tfrac{d}{2}\right) t_1(d/2, 2 \nmid a) + t'_{2i+1} \\ &\equiv 2^{2i+1} \left(\tfrac{d}{2}\right) \left[t_1(d/2) - 2 \left(\tfrac{d}{2}\right) t_1(d/4)\right] + it'_3 - (i-1)t'_1 \pmod{64}. \end{aligned} \tag{3.7}$$

Hence and from Lemma 1 (ii), (iv), (ii) of the lemma for $2 \mid d$ follows easily. If $2 \nmid d$ then by Lemma 1 (ii), (v) and (2.12,13) of [3], and (3.6), (3.7), an easy computation shows that (ii) of the lemma follows. Thus the lemma is proved. ∎

We next prove the following:

LEMMA 4. *Let X be a subset of the set of the odd natural numbers. Put $X(r) := \{a \in X \mid a \equiv r \pmod 8\}$. Then for any $x \geq 0$, d and even k we have:*

$$\begin{aligned} t_k(x, a \in X) \equiv \tfrac{k}{2} t_2(x, a \in X) + \left(1 - \tfrac{k}{2}\right) t_0(x, a \in X(\pm 1)) + \\ + 9\left(3^{k-2} - \tfrac{k}{2}\right) t_0(x, a \in X(\pm 3)) \pmod{2^{\operatorname{ord}_2 k + 6}}. \end{aligned}$$

PROOF. First, let us notice that for any natural k the congruence $a \equiv r \pmod 8$ implies

$$\begin{aligned} a^k = (a - r + r)^k = \sum_{i=0}^{k} \binom{k}{i} (a-r)^i r^{k-i} \equiv r^k + k(a-r)r^{k-1} + \\ + \binom{k}{2}(a-r)^2 r^{k-2} + (a-r)^k \pmod{2^{\operatorname{ord}_2 k + 6}}, \end{aligned}$$

because for $i \geq 3$

$$\operatorname{ord}_2 \frac{(a-r)^i}{i!} \geq \operatorname{ord}_2 \frac{2^{3i}}{i!} > 2i \geq 6.$$

Consequently in the case even k, $k \geq 4$ the congruence $a \equiv r \pmod 8$ with odd r leads to

$$a^k \equiv r^k + k(a-r)r + \tfrac{k}{2}(a-r)^2 \pmod{2^{\operatorname{ord}_2 k + 6}},$$

because for $k \geq 3$ we have

$$\mathrm{ord}_2(a-r)^k \geq 3k \geq \mathrm{ord}_2 k + 6.$$

Thus for even $k \geq 4$ and odd r we get the congruence

$$a^k \equiv r^k - \tfrac{k}{2} r^2 + \tfrac{k}{2} a^2 \pmod{2^{\mathrm{ord}_2 k+6}},$$

if $a \equiv r \pmod 8$. Hence the lemma follows immediately. Indeed we have

$$t_k(x, a \in X) = \sum_{r=\pm1,\pm3} t_k(x, a \in X_r) \equiv \tfrac{k}{2} t_2(x, a \in X) +$$
$$+ \sum_{r=\pm1,\pm3} r^2 \left(r^{k-2} - \tfrac{k}{2}\right) t_0(x, a \in X_r) \pmod{2^{\mathrm{ord}_2 k+6}}.$$

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COROLLARY. *For any d and even k we have:*

$$t'_k \equiv \tfrac{k}{2} t'_2 + \left(1 - \tfrac{k}{2}\right) t_0(\delta, a \equiv \pm 1 \pmod 8) +$$
$$+ 9\left(3^{k-2} - \tfrac{k}{2}\right) t_0(\delta, a \equiv \pm 3 \pmod 8) \pmod{2^{\mathrm{ord}_2 k+6}}.$$

■

LEMMA 5. *Let d and $k \geq 4$ be the discriminants of a real quadratic field and an even natural number respectively. Put λ_k, resp. $\pi_k := 1$, if $k \leq 8$, resp. $k = 4$ and λ_k, $\pi_k := 0$, otherwise. Then we have:*

(i) *If $2 \nmid d$ then*

$$t_k \equiv \tfrac{1}{2}(1 - 3^k - \lambda_k 2^k + 4k) h(-4d) + kd \left(\tfrac{d}{2}\right) k_2(d) \beta(d) \pmod{2^{\mathrm{ord}_2 k+6}}.$$

(ii) *If $d = -4d^*$, where d^* is the discriminant of an imaginary quadratic field then*

$$t_k \equiv \tfrac{3}{2}(3k+5) kdk_2(d) + A_2(d,k) kdh(d^*)\alpha(d^*) - 4(k-2) kdh(-2d)$$
$$\pmod{2^{\mathrm{ord}_2 k+6} d},$$

where

$$A_2(d,k) := \left(2d^* + 15 - \left(\tfrac{d^*}{2}\right)\right) k + 2\left(d^* - 3 + 2\left(\tfrac{d^*}{2}\right)\right)\left(1 + \left(\tfrac{-1}{k-1}\right)\right) +$$
$$+ 2\left(11 + 5\left(\tfrac{d^*}{2}\right)\right).$$

(iii) *If $d = \pm 8d^*$, where d^* is the discriminant of a quadratic field (i.e. $d \neq 8$) then*

$$t_k \equiv \begin{cases} \tfrac{k-1}{2} kdk_2(d) + [13(k-2) + 16\pi_k] kdh(-d) \\ \quad -4\left(\tfrac{d^*}{2}\right)(k-2) kdh(-4d^*) \pmod{2^{\mathrm{ord}_2 k+6} d}, & \text{if } d^* > 0, \\ \tfrac{k-1}{2} kdk_2(d) + \left[\left(4\left(\tfrac{d^*}{2}\right) + 1\right)(k-2) + 16\pi_k\right] kdh(-d) + \\ \quad +8(k-2) kdh(d^*)\alpha(\delta^*) \pmod{2^{\mathrm{ord}_2 k+6} d}, & \text{if } d^* < 0. \end{cases}$$

PROOF. We have

$$t_k = 2^k \left(\tfrac{d}{2}\right) t_k(d/2) + t'_k,$$

and

$$2^k \equiv 0 \pmod{2^{\operatorname{ord}_2 k+6}, 2^{\operatorname{ord}_2 k+5}, \text{ resp. } 2^{\operatorname{ord}_2 k+2}},$$

if $k \geq 10$, $k = 8$ or 6, resp. $k = 4$.

Hence and from (2.10) [3] we get

$$t_k \equiv t'_k - 2^{k-1}\lambda_k h(-4d) \pmod{2^{\operatorname{ord}_2 k+6}},$$

because by $t_0(d/2) = 0$ we have

$$t_k(d/2) \equiv t_0(d/2, 2\nmid a) = -\left(\tfrac{d}{2}\right) t_0(d/4) \pmod 2, \qquad \text{or} \pmod{16}, \text{ if } k = 4.$$

On the other hand applying Lemma 1 (i) [4] gives

$$t_0(d, a \equiv \pm 1 \pmod 8) = -t_0(d, a \equiv \pm 3 \pmod 8) = t_0(d/4).$$

Now to prove (i) of Lemma 5 it suffices to use Corollary to Lemma 4, (2.10) [3] and Lemma 2 (iii).

Our next concern will be the case $2\,|\,d$. Then by (2.7a) [3] for $d > 0$ and even k we obtain

$$\begin{aligned} t_k &= t_k(d/4) - \sum_{i=0}^{k} \binom{k}{i}(d/2)^i(-1)^{k-i}t_{k-i}(d/4) + \sum_{i=0}^{k} \binom{k}{i} d^i(-1)^{k-i}t_{k-i}(d/4) - \\ &\quad - \sum_{i=0}^{k} \binom{k}{i}(d/2)^i t_{k-i}(d/4) \\ &= t_k(d/4) - 2\sum_{\substack{0\leq i\leq k,\\ i \text{ even}}} \binom{k}{i}(d/2)^i t_{k-i}(d/4) + \sum_{i=0}^{k} \binom{k}{i} d^i(-1)^{k-i}t_{k-i}(d/4). \end{aligned}$$

Therefore we have

$$\begin{aligned} t_k &\equiv t_k(d/4) - 2\sum_{\substack{0\leq i\leq \tau_1,\\ i \text{ even}}} \binom{k}{i}(d/2)^i t_{k-i}(d/4) - 2\pi_k (d/2)^k t_0(d/4) + \\ &\quad + \sum_{i=0}^{\tau_2} \binom{k}{i} d^i i(-1)^i t_{k-i}(d/4) + \pi_k d^k t_0(d/4) \pmod{2^{\operatorname{ord}_2 k+6} d}, \end{aligned} \tag{3.8}$$

where $\tau_1 := \min(k, 2(6 - \operatorname{ord}_2 d))$, $\tau_2 := 2(4 - \operatorname{ord}_2 d)$, $\pi_k := 1$, if $k = 4$ and $8\,|\,d$, and $\lambda_k := 0$, otherwise. Indeed, if $4 \,\|\, d$, resp. $8 \mid d$ then each of the following numbers: $\operatorname{ord}_2 \left[2\binom{k}{i}(d/2)^i\right]$ for even $i \geq 10$, resp. $i \geq 8$ and $\operatorname{ord}_2 \left[\binom{k}{i} d^i\right]$ for $i \geq 5$, resp. $i \geq 3$ equals at least $\operatorname{ord}_2 k + \operatorname{ord}_2 d + 6$. So do the numbers $\operatorname{ord}_2[2(d/2)^k]$ for $k \geq 10$, resp. $k \geq 6$ and $\operatorname{ord}_2(d^k)$ for $k \geq 4$ because

$$2^{k+1}, \text{ resp. } 2^{2k} \equiv 0 \pmod{2^{\operatorname{ord}_2 k+8}}$$

for $k \geq 10$, resp. $k \geq 6$, and

$$2^{2k+1},\ 2^{3k} \equiv 0 \pmod{2^{\operatorname{ord}_2 k+9}}$$

for $k \geq 6$.

We need consider the cases. First, let $d = -4d^*$, where d^* is the discriminant of an imaginary quadratic field. Then by (3.8) we have

$$t_k \equiv t_k(d/4) - 2 \sum_{\substack{0 \le i \le \min(k,8),\\ i \text{ even}}} \binom{k}{i} (d/2)^i t_{k-i}(d/4) +$$
$$+ \sum_{i=0}^{4} \binom{k}{i} d^i (-1)^i t_{k-i}(d/4) \pmod{2^{\operatorname{ord}_2 k+6} d}.$$

Therefore putting $\binom{k}{i} := 0$ for $i > k$ and $t_s(x) := 0$ for $s < 0$ we get

$$t_k \equiv -kdt_{k-1}(d/4) + \tfrac{1}{2}d^2\binom{k}{2}t_{k-2}(d/4) - \binom{k}{3}d^3t_{k-3}(d/4) +$$
$$+ \tfrac{7}{8}\binom{k}{4}d^4t_{k-4}(d/4) - \tfrac{1}{32}\binom{k}{6}d^6t_{k-6}(d/4) - \tfrac{1}{128}\binom{k}{8}d^8t_{k-8}(d/4) \pmod{2^{\operatorname{ord}_2 k+6}d}.$$

Hence, by Lemma 3 [3] (applied to the sums $t_{k-1}(d/4)$, $t_{k-2}(d/4)$ and $t_{k-4}(d/4)$) we find that

$$\begin{aligned}
t_k \equiv\ & -kd\left[\tfrac{k-2}{2}t_3(d/4) - \tfrac{k-4}{2}t_1(d/4)\right] + \\
& + \tfrac{1}{2}d^2\binom{k}{2}\left[\tfrac{k-2}{2}t_2(d/4) - \tfrac{k-4}{2}t_0(d/4)\right] - \binom{k}{3}d^3t_1(d/4) + \\
& + \tfrac{7}{8}\binom{k}{4}d^4\left[\tfrac{k-4}{2}t_2(d/4) - \tfrac{k-6}{2}t_0(d/4)\right] - \tfrac{1}{32}\binom{k}{6}d^6t_0(d/4) - \\
& - \tfrac{1}{128}\binom{k}{8}d^8t_0(d/4) \\
\equiv\ & -kd\tfrac{k-2}{2}t_3(d/4) + G_k'(d)t_2(d/4) + \left[kd\tfrac{k-4}{2} - \binom{k}{3}d^3\right]t_1(d/4) - \\
& - G_k''(d)t_0(d/4) \pmod{2^{\operatorname{ord}_2 k+6}d},
\end{aligned}$$

where

$$G_k'(d) := \tfrac{1}{2}d^2\binom{k}{2}\tfrac{k-2}{2} + \tfrac{7}{8}\binom{k}{4}d^4\tfrac{k-4}{2},$$

and

$$G_k''(d) := \tfrac{1}{2}d^2\binom{k}{2}\tfrac{k-4}{2} + \tfrac{7}{8}\binom{k}{4}d^4\tfrac{k-6}{2} + \tfrac{1}{32}\binom{k}{6}d^6 + \tfrac{1}{128}\binom{k}{8}d^8.$$

Consequently we get

$$t_k \equiv -kd\tfrac{k-2}{2}t_3(d/4) + G_k'''(d)kdt_2(d/4) + \left(-15\tfrac{k}{2} + 14\right)kdt_1(d/4) +$$
$$+ G_k^{\mathrm{IV}}(d)kdt_0(d/4) \pmod{2^{\operatorname{ord}_2 k+6}d}, \quad (3.9)$$

where

$$G_k'''(d) := -d^*\left[-\left(\tfrac{-1}{k-1}\right)\left(12 + \left(\tfrac{-1}{k-1}\right) + 8\left(\tfrac{d^*}{2}\right)\right)\tfrac{k}{2} + 12\left(\tfrac{d^*}{2}\right)\left(\tfrac{-1}{k-1}\right) - 4\left(\tfrac{d^*}{2}\right) + 13\right]$$

and

$$G_k^{\mathrm{IV}}(d) := d^*\left[-\left(1 + 4\left(\tfrac{-1}{k-1}\right)\right)\tfrac{k}{2} + \left(-1 + 5\left(\tfrac{-1}{k-1}\right)\right) - 7\left(1 - \left(\tfrac{-1}{k-1}\right)\right)d^{*2}\right].$$

Indeed we have

$$(k-3)(k-5)(k-7) = k-5 \pmod 4,$$

and

$$k(k-2)(k-4)(k-6) \equiv 0 \pmod{2^7},$$

and consequently we find that

$$\begin{aligned}
G_k''(d) &\equiv -\Big[(k-4) + 13d^{*2}(k-2)(k-3)(k-6)- \\
&\quad -4(k-2)(k-4) + 2(k-2)(k-4)(k-6)\Big] dd^* \tbinom{k}{2} \\
&\equiv -dd^*\tbinom{k}{2} \times \begin{cases} k^2+5k+12+28d^{*2} \pmod{2^{\mathrm{ord}_2 k+6}d}, & \text{if } k \equiv 0 \pmod 4, \\ -k^2+k \pmod{2^{\mathrm{ord}_2 k+6}d}, & \text{if } k \equiv 2 \pmod 4. \end{cases}
\end{aligned}$$

Therefore we obtain

$$G_k''(d) \equiv -kdG_k^{\mathrm{IV}}(d) \pmod{2^{\mathrm{ord}_2 k+6}d}.$$

Moreover we see that

$$\begin{aligned}
G_k'(d) &\equiv -dd^*(k-2)\tbinom{k}{2}[1+13d^{*2}(k-3)(k-4)] \\
&\equiv -kdd^* \times \begin{cases} \left(-13-8\left(\frac{d}{2}\right)\right)\left(\frac{k}{2}-1\right) \pmod{2^{\mathrm{ord}_2 k+6}}, & \text{if } k \equiv 2 \pmod 4, \\ \left(11+8\left(\frac{d}{2}\right)\right)\left(\frac{k}{2}-1\right)+8\left(3-\left(\frac{d}{2}\right)\right) & \\ \quad \pmod{2^{\mathrm{ord}_2 k+6}}, & \text{if } k \equiv 0 \pmod 4. \end{cases}
\end{aligned}$$

and consequently

$$G_k'(d) \equiv kdG_k'''(d) \pmod{2^{\mathrm{ord}_2 k+6}d}.$$

On the other hand we have

$$t_i(d/4) = t_i^*(\delta^*, a \equiv 1 \pmod 4)) - t_i^*(\delta^*, a \equiv 3 \pmod 4)), \tag{3.10}$$

and so by (2.1), (2.6) [3] (cf. (2.11) [3]) we deduce that

$$\begin{aligned}
t_0(d/4) &= -t_0^*(\delta^*, a \equiv 2 \pmod 4)) + t_0^*(\delta^*, 4|a) \\
&= \left[-\left(\tfrac{d}{2}\right) t_0^*(\delta^*/2) + t_0^*(\delta^*/4)\right] + t_0^*(\delta^*/4) \\
&= 2t_0^*(\delta^*/4) - \left(\tfrac{d}{2}\right) t_0^*(\delta^*/2) \\
&= \left(2-\left(\tfrac{d}{2}\right)\right) h(d^*)\alpha(d^*),
\end{aligned} \tag{3.11}$$

and by (2.1), (2.4), (2.6), (2.7) of [3] (cf. (2.11a) [3]) we find that

$$\begin{aligned}
t_1(d/4) &= t_1^*(\delta^*, a \equiv 2 \pmod 4)) - t_1^*(d^*, 4|a) - \delta^* t_0^*(\delta^*, a \equiv 2 \pmod 4)) + \\
&\quad + \delta^* t_0^*(\delta^*, 4\,|\,a) = 2\left(\tfrac{d}{2}\right) t_1^*(\delta^*/2) - 8t_1^*(\delta^*/4) + \\
&\quad - \left(\tfrac{d}{2}\right)\delta^* t_0^*(\delta^*/2) + 2\delta^* t_0^*(\delta^*/4) \\
&= -\tfrac{1}{2}k_2(d) + \left(\left(\tfrac{d}{2}\right)-2\right) d^* h(d^*)\alpha(d^*).
\end{aligned} \tag{3.12}$$

Moreover by (2.10) and (2.14a) [3] we get

$$\begin{aligned}
t_2(d/4) &\equiv 2t_1^{*\prime} - t_0^{*\prime} + 16t_0^*(\delta^*, a \equiv 5,7 \pmod 8)) - \\
&\qquad - 14t_0^*(\delta^*, a \equiv 3,7 \pmod 8)) \pmod{64}.
\end{aligned} \tag{3.13}$$

On the other hand in view of Lemma 1(ii) [3] we have

$$16t_0^*(\delta^*, a \equiv 5,7 \pmod 8) - 14t_0^*(\delta^*, a \equiv 3,7 \pmod 8)$$
$$= 8\left(1 + \left(\tfrac{d}{2}\right)\right) t_0^*(\delta^*/2) - 18t_0^*(\delta^*/4), \quad (3.14)$$

and so (3.13) together with (2.1), (2.6) [3] and Lemma 2 (i), (ii) [3] imply the congruence

$$t_2(d/4) \equiv \left(2\left(\tfrac{d}{2}\right) d^* + \left(\tfrac{d}{2}\right) - 2\right) h(d^*)\alpha(d^*) \pmod{64}. \quad (3.15)$$

We are left with the task of determining of $t_3(d/4)$ modulo 64. Since $a \equiv r \pmod 8$ yields the congruence

$$a^3 \equiv 3ar^2 - 2r^3 \pmod{64}, \quad (3.16)$$

it may be concluded by (3.10) that

$$\begin{aligned} t_3(d/4) \equiv\ & 3 \sum_{r=1 \text{ or } 5} r^2 t_1^*(\delta^*, a \equiv r \pmod 8) - 2 \sum_{r=1 \text{ or } 5} r^3 t_0^*(\delta^*, a \equiv r \pmod 8) - \\ & - 3 \sum_{r=3 \text{ or } 7} r^2 t_1^*(\delta^*, a \equiv r \pmod 8) + 2 \sum_{r=3 \text{ or } 7} r^3 t_0^*(\delta^*, a \equiv r \pmod 8) \\ \equiv\ & 3t_1^*(\delta^*, a \equiv 1 \pmod 8) + 11t_1^*(\delta^*, a \equiv 5 \pmod 8) - \\ & - 2t_0^*(\delta^*, a \equiv 1 \pmod 8) + 6t_0^*(\delta^*, a \equiv 5 \pmod 8) - \\ & - 27t_1^*(\delta^*, a \equiv 3 \pmod 8) - 19t_1^*(\delta^*, a \equiv 7 \pmod 8) - \\ & - 10t_0^*(\delta^*, a \equiv 3 \pmod 8) - 18t_0^*(\delta^*, a \equiv 7 \pmod 8) \\ \equiv\ & 3t_1^{*\prime} - 2t_0^{*\prime} - 16t_0^*(\delta^*, a \equiv 5,7 \pmod 8) - \\ & - 22t_1^*(\delta^*, a \equiv 3 \pmod 4) - 32t_0^*(\delta^*, a \equiv 3 \pmod 8) \pmod{64}. \end{aligned} \quad (3.17)$$

But in virtue of Lemma 1 (ii) [3] we have

$$t_0^*(\delta^*, a \equiv 5,7 \pmod 8) = -2t_0^*(\delta^*/4) + \tfrac{1}{2}\left(1 + \left(\tfrac{d}{2}\right)\right) t_0^*(\delta^*/2)$$

(cf. (3.14)), and

$$t_0^*(\delta^*, a \equiv 3 \pmod 8) = t_0^*(\delta^*/4) - \tfrac{1}{2}\left(1 + \left(\tfrac{d}{2}\right)\right) t_0^*(\delta^*/4).$$

Moreover we have

$$t_1^*(\delta^*, a \equiv 3 \pmod 4) = t_1^*(\delta^*, 4|a) - \delta^* t_0^*(\delta^*, 4|a) = 4t_1^*(\delta^*/4) - \delta^* t_0^*(\delta^*/4).$$

Therefore (3.17) implies

$$\begin{aligned} t_3(d/4) \equiv\ & 3t_1^{*\prime} - 2t_0^{*\prime} + 2\left(11\delta^* + 8\left(\tfrac{d}{2}\right) - 8\right) t_0^*(\delta/4) - \\ & - 8\left(1 + \left(\tfrac{d}{2}\right)\right) t_0^*(\delta^*/2) - 24t_1^*(\delta^*/4) - 32t_0^*(\delta^*/8) \pmod{64}. \end{aligned}$$

Now it is sufficient to apply Lemma 1 (i), (ii) [3] and the formulas (2.6), (2.1), (2.7), (2.18) of [3]. Then we get

$$\begin{aligned} t_3(d/4) \equiv\ & -\tfrac{3}{2}k_2(d) + \left[-\left(2 + \left(\tfrac{d}{2}\right)\right) d^* + 16\left(\tfrac{d}{2}\right) - 14\right] h(d^*)\alpha(d^*) + \\ & + 8h(-2d) \pmod{64}. \end{aligned}$$

Applying the above congruence together with (3.11), (3.12) and (3.15) to (3.9) gives

$$t_k \equiv \tfrac{3}{2}(3k+5)kdk_2(d) + A(d,k)kdh(d^*)\alpha(d^*) - \\ - 4(k-2)kdh(-2d) \pmod{2^{\operatorname{ord}_2 k+6} d},$$

where

$$A(d,k) := -\tfrac{k-2}{2}\left[-\left(2+\left(\tfrac{d^{\cdot}}{2}\right)\right)d^* + 16\left(\tfrac{d^{\cdot}}{2}\right) - 14\right] + \\ + \left(2\left(\tfrac{d^{\cdot}}{2}\right)d^* + \left(\tfrac{d^{\cdot}}{2}\right) - 2\right)G_k'''(d) + \left(-15\tfrac{k}{2}+14\right)\left(\left(\tfrac{d^{\cdot}}{2}\right)-2\right)d^* + \\ + G_k^{\mathrm{IV}}(d)\left(2-\left(\tfrac{d^{\cdot}}{2}\right)\right).$$

Now the proof of Lemma 5 in the case $4 \,\|\, d$ will be completed as soon as we can prove that

$$A(d,k) \equiv A_2(d,k) \pmod{64}. \tag{3.18}$$

Indeed putting $G_k''' = m'''\frac{k}{2} + n'''$ and $G_k^{\mathrm{IV}} = m^{\mathrm{IV}}\frac{k}{2} + n^{\mathrm{IV}}$, we have

$$A(d,k) = A'(d,k)\tfrac{k}{2} + A''(d,k),$$

where

$$A'(d,k) = \left[\left(2+\left(\tfrac{d^{\cdot}}{2}\right)\right)d^* - 16\left(\tfrac{d^{\cdot}}{2}\right) + 14\right] + \left(2\left(\tfrac{d^{\cdot}}{2}\right)d^* + \left(\tfrac{d^{\cdot}}{2}\right) - 2\right)m''' + \\ + 15\left(2-\left(\tfrac{d^{\cdot}}{2}\right)\right)d^* + m^{\mathrm{IV}}\left(2-\left(\tfrac{d^{\cdot}}{2}\right)\right),$$

and

$$A''(d,k) = \left[-\left(2+\left(\tfrac{d^{\cdot}}{2}\right)\right)d^* + 16\left(\tfrac{d^{\cdot}}{2}\right) - 14\right] + \left(2\left(\tfrac{d^{\cdot}}{2}\right)d^* + \left(\tfrac{d^{\cdot}}{2}\right) - 2\right)n''' + \\ + 14\left(\left(\tfrac{d^{\cdot}}{2}\right)-2\right)d^* + n^{\mathrm{IV}}.$$

Thus in virtue of

$$A'(d,k) \equiv \begin{cases} 4\left(d^*+3+4\left(\frac{-1}{k-1}\right)\right) \pmod{64}, & \text{if } \left(\frac{d^{\cdot}}{2}\right)=1, \\ 4(d^*+8) \pmod{64}, & \text{if } \left(\frac{d^{\cdot}}{2}\right)=-1, \end{cases}$$

and

$$A''(d,k) \equiv \begin{cases} 2(d^*-1)\left(1+\left(\frac{-1}{k-1}\right)\right)+32 \pmod{64}, & \text{if } \left(\frac{d^{\cdot}}{2}\right)=1, \\ 2(d^*-5)\left(1+\left(\frac{-1}{k-1}\right)\right)+12 \pmod{64}, & \text{if } \left(\frac{d^{\cdot}}{2}\right)=-1, \end{cases}$$

(3.18) follows.

Similar arguments apply to the case $8 \mid d$. Then by (3.8) we have

$$t_k \equiv t_k(d/4) - 2\sum_{\substack{1\le i\le \min(k,6),\\ i \text{ even}}} \binom{k}{i}(d/2)^i t_{k-i}(d/4) - 2\pi_k(d/2)^k t_0(d/4) + \\ + \sum_{i=0}^{2}\binom{k}{i} d^i(-1)^i t_{k-i}(d/4) + \pi_k d^k t_0(d/4) \pmod{2^{\operatorname{ord}_2 k+6} d}.$$

Therefore we get

$$t_k \equiv -kdt_{k-1}(d/4) + \frac{1}{2}\binom{k}{2}d^2t_{k-2}(d/4) - k(k-2)d^2t_0(d/4) - \\ - 8\pi_k d^2 t_0(d/4) \pmod{2^{\mathrm{ord}_2 k+6}d},$$

because

$$\mathrm{ord}_2 \tfrac{1}{32}\binom{k}{6}d^5 \geq \mathrm{ord}_2 k + 9,$$

and

$$t_{k-4}(d/4) \equiv t_0(d/4) \pmod 4.$$

Hence and by Lemma 3 [3] (applied for the sums t_{k-1} and t_{k-2}) we obtain

$$t_k \equiv -kd\left[\tfrac{k-2}{2}t_3(d/4) - \tfrac{k-4}{2}t_1(d/4)\right] + \frac{1}{2}\binom{k}{2}d^2\left[\tfrac{k-2}{2}t_2(d/4) - \tfrac{k-4}{2}t_0(d/4)\right] + \\ + k(k-2)d^2(1-\pi_k)t_0(d/4) \pmod{2^{\mathrm{ord}_2 k+6}d}.$$

Thus we have

$$t_k \equiv -kd\tfrac{k-2}{2}t_3(d/4) + \frac{1}{4}\binom{k}{2}(k-2)d^2t_2(d/4) + \\ + kd\tfrac{k-4}{2}t_1(d/4) + \tfrac{d}{8}kdH'_k t_0(d/4) \pmod{2^{\mathrm{ord}_2 k+6}d}, \quad (3.19)$$

where

$$H'_k = -(k-1)(k-4) + 8(k-2)(1-\pi_k).$$

On the other hand putting $d = \pm 8d^*$, from(2.25a) [3], by (2.10), (2.15) [3] in the case $d^* > 0$ and by (2.18), (2.6), (2.1) [3] in the case $d^* < 0$ it follows that

$$t_0(d/4) = h(-d). \quad (3.20)$$

Moreover by (2.17a) [3] for $d > 0$ we have

$$t_1(d/4) = 2\left(\tfrac{d^{\cdot}}{-1}\right)\left[-t_1^*(\delta^*, a \equiv 3 \pmod 8) + t_1^*(\delta^*, a \equiv -1 \pmod 8)\right] + \\ + 2\delta^*\Big[t_0^*(\delta^*, a \equiv 1 \pmod 8) - t_0^*(\delta^*, a \equiv -3 \pmod 8) + \\ + \left(\tfrac{d^{\cdot}}{-1}\right)t_0^*(\delta^*, a \equiv 3 \pmod 8) - \left(\tfrac{d^{\cdot}}{-1}\right)t_0^*(\delta^*, a \equiv -1 \pmod 8)\Big].$$

Therefore Lemma 1 and Lemma 1 [3] imply

$$\begin{aligned} t_1(d/4) = 2\Big[&-4\left(\tfrac{d^{\cdot}}{2}\right)t_1^*(\delta^*/2) + 4\left(4 + \left(\tfrac{d^{\cdot}}{2}\right)\right)t_1^*(\delta^*/4) - \\ &- 16t_1^*(\delta^*/8) - \delta^*\left(2 + \left(\tfrac{d^{\cdot}}{2}\right)\right)t_0^*(\delta^*/4) + 2\delta^* t_0^*(\delta^*/8)\Big] + \\ &+ 2\delta^*\Big[-2\left(1 + \left(\tfrac{d^{\cdot}}{2}\right)\right)t_0^*(\delta^*/4) + 4t_0^*(\delta^*/8)\Big] \\ = &-8\left(\tfrac{d^{\cdot}}{2}\right)t_1^*(\delta^*/2) + 8\left(4 + \left(\tfrac{d^{\cdot}}{2}\right)\right)t_1^*(\delta^*/4) - 32t_1^*(\delta^*/8) - \\ &- 2\delta^*\left(3\left(\tfrac{d^{\cdot}}{2}\right) + 4\right)t_0^*(\delta^*/4) + 12\delta^* t_0^*(\delta^*/8), \end{aligned}$$

if $d^* > 0$, and

$$t_1(d/4) = -2\left[-4\left(\frac{d^*}{2}\right)t_1^*(\delta^*/4) + 16t_1^*(\delta^*/8) + \delta^*\left(\frac{d^*}{2}\right)t_0^*(\delta^*/4) - 2\delta^* t_0^*(\delta^*/8)\right] +$$
$$+ 2\delta^*\left[-\left(\frac{d^*}{2}\right)t_0^*(\delta^*/2) + 2\left(1 + \left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/4) - 4t_0^*(\delta^*/8)\right]$$
$$= 8\left(\frac{d^*}{2}\right)t_1^*(\delta^*/4) - 32t_1^*(\delta^*/8) - 2\delta^*\left(\frac{d^*}{2}\right)t_0^*(\delta^*/2) +$$
$$+ 2\delta^*\left(2 + \left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/4) - 4\delta^* t_0^*(\delta^*/8),$$

if $d^* < 0$.

Thus since for $d > 0$, $2 \nmid d$

$$t_1(\delta/8) = \tfrac{1}{64}k_2(8d) - \tfrac{1}{64}\left(34 - \left(\tfrac{d}{2}\right)\right)k_2(d)\beta(d) + \tfrac{1}{32}d\left[\left(\tfrac{d}{2}\right)h(-4d) + h(-8d)\right],$$

and for $d < 0$, $2 \nmid d$

$$t_1(\delta/8) = \tfrac{1}{64}k_2(-8d) + \tfrac{1}{64}\left(\tfrac{d}{2}\right)k_2(-4d) - \tfrac{1}{32}d\left[\left(1 - \left(\tfrac{d}{2}\right)\right)h(d)\alpha(d) - h(8d)\right]$$

(see Thm. 1,2 [2]), in our case for $d \neq 8$ we conclude that

$$t_1(d/4) = 2\delta^* h(-d) - \tfrac{1}{2}k_2(d) \tag{3.21}$$

because of (2.10), (2.12), (2.13), (2.15) of [3] in the case $d^* > 0$, and of (2.1), (2.6), (2.7), (2.18) of [3] in the case $d^* < 0$.

Now we shall prove the lemma as soon as we find $t_2(d/4)$, $t_3(d/4)$ modulo 64. But using Lemma 1 and (2.20a)[3] we get

$$t_2(d/4) \equiv 4\delta^*\left[\left(1 + \left(\frac{d^*}{-1}\right)\right)\left(\frac{d^*}{2}\right)t_1^*(\delta^*/2) - 4\left(\frac{d^*}{-1}\right)t_1^*(\delta^*/4)\right] +$$
$$+ 2\left(2\delta^* + 2 - 5\left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/2) +$$
$$+ 2\left(2\delta^* - 13\left(\frac{d^*}{-1}\right)\left(\frac{d^*}{2}\right) - 3 - 11\left(\frac{d^*}{-1}\right)\right)t_0^*(\delta^*/4) +$$
$$+ 4\left(\left(\frac{d^*}{-1}\right) - 4\right)t_0^*(\delta^*/8) \pmod{64},$$

because $t_0^*(\delta^*/2) = 0$, if $d^* > 0$.

We now turn to the cases, again. Let $d = 8d^*$, $d^* > 0$, $d \neq 8$. Then by (2.12), (2.13), (2.10) and (2.15) of [3] we obtain

$$t_2(d/4) \equiv 8\left(2 - \left(\frac{d^*}{2}\right)\right)t_1^*(\delta^*/2) - 16t_1^*(\delta^*/4) +$$
$$+ 2\left(2\delta^* - 13\left(\frac{d^*}{2}\right) - 14\right)t_0^*(\delta^*/4) - 12t_0^*(\delta^*/8)$$
$$\equiv \left(5 - 2\left(\frac{d^*}{2}\right)\right)k_2(d^*)\beta(d^*) - 3h(-d) + 2h(-4d^*) \pmod{64}, \tag{3.22}$$

because $2 \mid h(-4d^*)$ and $4 \mid k_2(d^*)$.

Likewise, if $d = -8d^*$, $d^* < 0$ then by (2.7), (2.1), (2.6) and (2.18) of [3] we deduce that

$$t_2(d/4) \equiv -16t_1^*(\delta^*/4) + 2\left(2\delta^* + 2 - 5\left(\frac{d^*}{2}\right)\right)t_0^*(\delta^*/2) +$$
$$+ 2(2\delta^* - 11)t_0^*(\delta^*/4) - 20t_0^*(\delta^*/8)$$
$$\equiv -k_2(-4d^*) + 5h(-d) + 2\left(1 + 3\left(\frac{d^*}{2}\right)\right)h(d^*)\alpha(d^*) \pmod{64} \tag{3.23}$$

because $t_0^*(\delta^*/4) = 0$, if $\left(\frac{d^*}{2}\right) = -1$, $d^* < 0$.

The same method goes for $t_3(d/4)$. Then by (3.16) we have

$$\begin{aligned}
t_3(d/4) &= \sum_{r=\pm1,\pm3} t_3(d/4, a \equiv r \pmod 8) \\
&\equiv 3 \sum_{r=\pm1,\pm3} r^2 t_1(d/4, a \equiv r \pmod 8) - 2 \sum_{r=\pm1,\pm3} r^3 t_0(d/4, a \equiv r \pmod 8) \\
&= 3\left[t_1(d/4, a \equiv \pm1 \pmod 8) + 9t_1(d/4, a \equiv \pm3 \pmod 8)\right] - \\
&\quad -2\Big[-t_0(d/4, a \equiv -1 \pmod 8) - 27t_0(d/4, a \equiv -3 \pmod 8) + \\
&\quad + t_0(d/4, a \equiv 1 \pmod 8) + 27t_0(d/4, a \equiv 3 \pmod 8)\Big] \\
&\equiv 3t_1(d/4) + 2t_0(d/4, a \equiv -1 \pmod 8) - 2t_0(d/4, a \equiv 1 \pmod 8) - \\
&\quad - 18t_0(d/4, a \equiv -3 \pmod 8) + 18t_0(d/4, a \equiv 3 \pmod 8) \\
&\equiv 3t_1(d/4) - 2t_0(d/4) + L(d) \pmod{64}, \qquad (3.24)
\end{aligned}$$

where

$$\begin{aligned}
L(d) := 4t_0(d/4, a \equiv -1 \pmod 8) - 16t_0(d/4, a \equiv -3 \pmod 8) + \\
+ 20t_0(d/4, a \equiv 3 \pmod 8).
\end{aligned}$$

But by the formula

$$t_0(d/4, a \equiv r \pmod 8) = t_0(\delta^*, a \equiv r \pmod 8) - (-1)^{\frac{r+1}{2}} t_0(\delta^*, a \equiv 2\delta^* - r \pmod 8)$$

(cf. (2.17) [3]) we see that

$$\begin{aligned}
L(d) = &\, 4[t_0(\delta^*, a \equiv -1 \pmod 8) - t_0(\delta^*, a \equiv 2\delta^* + 1 \pmod 8)] - \\
&- 16[t_0(\delta^*, a \equiv -3 \pmod 8) + t_0(\delta^*, a \equiv 2\delta^* + 3 \pmod 8)] + \\
&+ 20[t_0(\delta^*, a \equiv 3 \pmod 8) - t_0(\delta^*, a \equiv 2\delta^* - 3 \pmod 8)],
\end{aligned}$$

and consequently for $d > 0$ we have

$$L(d) = \begin{cases} -16t_0^*(\delta^*, a \equiv -1 \pmod 8) - 16t_0^*(\delta^*, a \equiv 3 \pmod 8) + \\ \quad +32t_0^*(\delta^*, a \equiv -3 \pmod 8)), & \text{if } d^* > 0, \\ 16t_0^*(\delta^*, a \equiv -3 \pmod 8) - 16t_0^*(\delta^*, a \equiv 1 \pmod 8)), & \text{if } d^* < 0. \end{cases}$$

Therefore by Lemma 1 [3] we obtain

$$L(d) \equiv 16\left(\tfrac{d^*}{2}\right) t_0^*(\delta^*/2) - 16\left(\tfrac{d^*}{-1}\right)\left(\tfrac{d^*}{2}\right) t_0^*(\delta^*/4) + 32t_0^*(\delta^*/8) \pmod{64},$$

and consequently by (2.10), (2.15) of [3], if $d^* > 0$, and by (2.1), (2.6), (2.18) of [3], if $d^* < 0$ we conclude that

$$L(d) \equiv 8\left(\tfrac{d^*}{-1}\right) h(-d) \pmod{64}.$$

Thus (3.24) together with (3.20) and (3.21) implies the congruences

$$t_3(d/4) \equiv -\tfrac{3}{2}k_2(d) + 6(d^* + 1)h(-d) \pmod{64}, \qquad (3.25)$$

if $d^* > 0$, and

$$t_3(d/4) \equiv -\tfrac{3}{2}k_2(d) + 2(d^*+7)h(-d) \pmod{64}, \tag{3.26}$$

if $d^* < 0$.

Now to finish the proof of the lemma it remains to substitute (3.20), (3.21), (3.22) or (3.23), and (3.25) or (3.26) into (3.19).

If $d^* > 0$ then we have

$$\begin{aligned} t_k \equiv\ & 3(k-1)(k-2)k_2(d^*)\beta(d^*)kd + H_k''(d)h(-d)kd + \\ & + \tfrac{k-1}{2}k_2(d)kd + 2(k-1)(k-2)d^*h(-4d^*)kd \pmod{2^{\operatorname{ord}_2 k+6}d}, \end{aligned}$$

where

$$\begin{aligned} H_k''(d) &:= -3(d^*+1)(k-2) - 3(k-1)(k-2)d^* + (k-4)d^* + d^*H_k' \\ &\equiv 9(k-2) + 16\pi_k \pmod{32}, \end{aligned}$$

because $4 \mid k_2(d^*)$ and $2 \mid h(-4d^*)$.

Now to get the congruence of the lemma for $d^* > 0$ it is sufficient to use Cor. 1 (ii) to Thm. 1 [2] i.e. the congruence

$$k_2(d^*)\beta(d^*) \equiv 6h(-4d^*) - 4\left(2 - \left(\tfrac{d^*}{2}\right)\right)h(-d) \pmod{32}.$$

Indeed by the divisibilities $2 \mid h(-4d^*)$, $h(-d)$ and $4 \mid h(-d)$, if $\left(\frac{d^*}{2}\right) = 1$, we find that

$$\begin{aligned} & 3(k-1)(k-2)k_2(d^*)\beta(d^*) + H''(d)h(-d) + 2(k-1)(k-2)d^*h(-4d^*) \\ &\equiv 2(k-1)(k-2)(d^*+1)h(-4d^*) + [4(k-2) + 9(k-2) + 16\pi_k]h(-d) \\ &\equiv -4\left(\tfrac{d^*}{2}\right)(k-2)h(-4d^*) + [13(k-2) + 16\pi_k]h(-d) \pmod{64}. \end{aligned}$$

We now turn to the case $d^* < 0$. Then we have

$$\begin{aligned} t_k \equiv\ & (k-1)(k-2)\left(2\left(\tfrac{d^*}{2}\right) - 1\right)k_2(-4d^*)kd + H_k'''(d)h(-d)kd + \\ & + \tfrac{k-1}{2}k_2(d)kd - 2(k-1)(k-2)\left(5 - \left(\tfrac{d^*}{2}\right)\right)h(d^*)\alpha(d^*)kd \pmod{2^{\operatorname{ord}_2 k+6}d}, \end{aligned}$$

where

$$\begin{aligned} H_k'''(d) &:= -(k-2)(d^*+7) - 5(k-1)(k-2)d^* - (k-4)d^* - d^*H_k' \\ &\equiv k - 2 + 16\pi_k \pmod{32}, \end{aligned}$$

because $2 \mid h(-4d^*)$, $h(-d)$, and $4 \mid k_2(-4d^*)$. Now to obtain the congruence of the lemma for $d^* < 0$ it suffices to apply Cor. 1 (ii) to Thm. 2 [2] i.e. the congruence

$$k_2(-4d^*) \equiv 6\left(\tfrac{d^*}{2}\right)\left[7\left(\left(\tfrac{d^*}{2}\right) - 1\right)h(d^*)\alpha(d^*) + 2h(-d)\right] \pmod{32}.$$

In fact by the divisibility $2 \mid h(-d)$ we conclude that

$$\begin{aligned} & (k-1)(k-2)\left(2\left(\tfrac{d^*}{2}\right) - 1\right)k_2(-4d^*) + H_k'''(d)h(-d) - \\ & - 2(k-1)(k-2)\left(5 - \left(\tfrac{d^*}{2}\right)\right)h(d^*)\alpha(d^*) \equiv \left[\left(4\left(\tfrac{d^*}{2}\right) + 1\right)(k-2) + 16\pi_k\right]h(-d) + \\ & \qquad + 8(k-2)h(d^*)\alpha(d^*) \pmod{64}. \end{aligned}$$

The proof of the lemma is complete. ■

Now we shall prove a weaker version of (ii), (iii) of the previous lemma.

LEMMA 6. *Let $d>0$, $2\,|\,d$ be the discriminant of a quadratic field, and let $k\geq 4$ be an even natural number. Then we have:*

$$t_k \equiv \tfrac{1}{2}kdk_2(d)\rho(d) \pmod{2^{\operatorname{ord}_2 k+6}}.$$

PROOF. By Lemma 4 for any d we get

$$\begin{aligned} t_k &\equiv \tfrac{k}{2}t_2 + \left(1-\tfrac{k}{2}\right)t_0(\delta, a\equiv \pm 1 \pmod 8) + 9\left(3^{k-2}-\tfrac{k}{2}\right)t_0(\delta, a\equiv \pm 3 \pmod 8) \\ &\equiv \tfrac{k}{2}t_2 + \left(1-\tfrac{k}{2}\right)t_0 + (3^k-4k-1)t_0(\delta, a\equiv \pm 3 \pmod 8) \pmod{2^{\operatorname{ord}_2 k+6}}. \end{aligned} \tag{3.27}$$

On the other hand we have

$$\begin{aligned} t_0(\delta, a\equiv \pm 3 \pmod 8) &= t_0(\delta/4, a\equiv \pm 3 \pmod 8) + \sum_{\substack{0\leq a\leq \delta/4,\\ a\equiv \delta/2\pm 3 \pmod 8}} \left(\frac{d}{\delta/2-a}\right) + \\ &\quad + \sum_{\substack{0\leq a\leq \delta/4,\\ a\equiv -\delta/2\pm 3 \pmod 8}} \left(\frac{d}{\delta/2+a}\right) + \sum_{\substack{0\leq a\leq \delta/4,\\ a\equiv \delta\pm 3 \pmod 8}} \left(\frac{d}{\delta-a}\right) \\ &= S_1+S_2+S_3+S_4, \end{aligned}$$

where S_i denotes the ith sum (summand) of the left hand side of the above equality.

Thus in view of

$$S_1+S_4 = \begin{cases} t_0(d/4), & \text{if } 4\,\|\,d,\\ 2t_0(d/4, a\equiv \pm 3 \pmod 8), & \text{if } 8\,|\,d, \end{cases}$$

and

$$S_2+S_3 = \begin{cases} t_0(d/4), & \text{if } 4\,\|\,d,\\ 2t_0(d/4, a\equiv \pm 1 \pmod 8), & \text{if } 8\,|\,d, \end{cases}$$

we obtain

$$t_0(\delta, a\equiv \pm 3 \pmod 8) = 2t_0(d/4).$$

Consequently (3.27) implies

$$t_k \equiv \tfrac{k}{2}t_2 + 2(3^k-4k-1)t_0(d/4) \equiv \tfrac{k}{2}t_2 \pmod{2^{\operatorname{ord}_2 k+6}},$$

because

$$\begin{aligned} 3^k-4k-1 &\equiv -2k + 4\tbinom{k}{2} + 8\tbinom{k}{3} + 16\tbinom{k}{4} \\ &\equiv -2k + 2k(k-1) - 4k(k-1)(k-2) - 2k(k-1)(k-2)(k-3) \\ &\equiv -2k + 2k(k-1) + 4k(k-2) - 2k^2(k-2) + \\ &\quad + 2k^2(k-2) - 6k(k-2) \equiv 0 \pmod{2^{\operatorname{ord}_2 k+5}}. \end{aligned}$$

Hence and from Lemma 2, the lemma follows. ∎

4. Proof of Theorem 1.

We start with the formula (3.2) [3]. For $k \geq 2$, $2\,|\,k$ and $\chi = \left(\frac{d}{\cdot}\right)$, $d > 0$ it states that

$$B_{k,\left(\frac{d}{\cdot}\right)} = \sum_{i=0}^{\frac{k}{2}-1} \binom{k}{2i} 2B_{k-2i}(2d)^{k-2i-1} t_{2i} - kt_{k-1} + \tfrac{1}{d} t_k. \tag{4.1}$$

Thus by the von Staudt–Clausen theorem for $p = 2$ and Lemma 3 we see that for any $d > 0$ the numbers $B_{k,\left(\frac{d}{\cdot}\right)}$ are 2-integral and by Lemma 4 so are the numbers $b_k(d)$ because for $2 \nmid d$ we have

$$\operatorname{ord}_2(1 - 3^k - \lambda_k 2^k) \geq \operatorname{ord}_2 k,$$

and for $2\,|\,d$ we have

$$\operatorname{ord}_2 t_k \geq \operatorname{ord}_2 k + \operatorname{ord}_2 d.$$

Let us use the formula (4.1) to the case $2 \nmid d$ and $k \geq 8$. Then we get

$$dB_{k,\left(\frac{d}{\cdot}\right)} \equiv \sum_{i=\frac{k}{2}-2}^{\frac{k}{2}-1} \binom{k}{2i} 2B_{k-2i} 2^{k-2i-1} d^{k-2i} t_{2i} - kdt_{k-1} + t_k \pmod{2^{\operatorname{ord}_2 k+6}},$$

because in view of $8\,|\,t_{2i}$ for $i \geq 2$ and $4\,|\,t_2$, and

$$\operatorname{ord}_2 \frac{2^s}{s} \geq 4$$

for $s = k - 2i \geq 6$, we have

$$\operatorname{ord}_2 \left[\binom{k}{2i} 2^{k-2i-1} t_{2i}\right], \text{ resp. } \operatorname{ord}_2 \left[\binom{k}{2} 2^{k-3}\right] \geq \operatorname{ord}_2 k + 6$$

for $i \leq \frac{k}{2} - 3$, resp. $k \geq 8$.

Hence for $k \geq 8$ we obtain the congruence

$$dB_{k,\left(\frac{d}{\cdot}\right)} \equiv -\tfrac{8}{15}\binom{k}{4} d^4 t_{k-4} + \tfrac{2}{3}\binom{k}{2} d^2 t_{k-2} - kdt_{k-1} + t_k \pmod{2^{\operatorname{ord}_2 k+6}}.$$

Therefore by Lemma 3 for t_{k-4}, t_{k-2}, t_{k-1} and Lemma 4 (i) for t_k we deduce that

$$\begin{aligned} dB_{k,\left(\frac{d}{\cdot}\right)} \equiv & -\tfrac{16}{15}\binom{k}{4} d^4 \left[\tfrac{k-4}{2} d \left(\tfrac{d}{2}\right) - 2^{k-7}\right] k_2(d)\beta(d) + \\ & + \tfrac{4}{3}\binom{k}{2} d^2 \left[\tfrac{k-2}{2} d \left(\tfrac{d}{2}\right) - 2^{k-5}\right] k_2(d)\beta(d) - \\ & - kd\left[\tfrac{1}{4} A_1\left(d, \tfrac{k-2}{2}\right) k_2(d)\beta(d) + 2\left(\tfrac{k-2}{2} - 2^{k-4}\right) dh(-4d)\right] + \\ & + \tfrac{1}{2}(1 - 3^k - \lambda_k 2^k + 4k) h(-4d) + kd\left(\tfrac{d}{2}\right) k_2(d)\beta(d) \pmod{2^{\operatorname{ord}_2 k+6}}. \end{aligned}$$

Consequently by $2\,|\,h(-4d)$ and $4\,|\,k_2(d)$ we find that

$$\begin{aligned} dB_{k,\left(\frac{d}{\cdot}\right)} \equiv & \Big[(k-2)(k-4-2^{k-6}) + 3(k-1)(k-2)\left(\tfrac{d}{2}\right) + \tfrac{1}{2}(k-2)(d+2) - \\ & - \tfrac{1}{2}\left(2\left(\tfrac{d}{2}\right) - 1\right)\Big] kdk_2(d)\beta(d) + \\ & + \left[-(k-2)d + \tfrac{1-3^k-\lambda_k 2^k+4k}{2kd}\right] kdh(-4d) \pmod{2^{\operatorname{ord}_2 k+6}}. \end{aligned}$$

Hence by the divisibility $4\,|\,h(-4d)$, if $\left(\frac{d}{2}\right)=1$ we obtain the congruence

$$b_k(d)\equiv H_k(d)k_2(d)\beta(d)-\tfrac{1}{2}\left(2\left(\tfrac{d}{2}\right)-1\right)k_2(d)\beta(d)+ \\ +\left[-k+20+9\left(\tfrac{1-3^k-\lambda_k 2^k}{2k}\right)\right]dh(-4d)\pmod{64}, \quad (4.2)$$

where

$$H_k(d):=\left[2\left(1+\left(\tfrac{d}{2}\right)\right)+d+8\lambda_k\right]\tfrac{k-2}{2},$$

because for even k

$$\mathrm{ord}_2(1-3^k)=\mathrm{ord}_2\,k+2,$$

and for $k=8$

$$\mathrm{ord}_2\,\frac{2^k}{k}=5.$$

Consequently, since for $k\geq 8$

$$\begin{aligned}\tfrac{1-3^k}{2k}&=-\tfrac{1}{k}\sum_{i=1}^{k}\tbinom{k}{i}2^{i-1}\equiv-\tfrac{1}{k}\sum_{\substack{1\leq i\leq 8,\\ i\neq 7}}\tbinom{k}{i}2^{i-1}\\ &\equiv -k-\tfrac{2}{3}(k-1)(k-2)-\tfrac{1}{3}(k-1)(k-2)(k-3)-\\ &\quad-4(k-2)(k-4)-(k-2)(k-4)(k-6)\\ &\equiv -k+5(k-1)^2(k-2)-(k-2)(k-4)(k-6)\\ &\equiv 2\left(2k-1-4\left(\tfrac{-1}{k-1}\right)\right)\pmod{32}\end{aligned}$$

and

$$H_k(d)\equiv\left(1+\left(\tfrac{d}{2}\right)\right)(k-2)+\tfrac{dk}{2}-d+8\lambda_k\equiv 5\left(\tfrac{k}{2}-1\right)+ \\ +\tfrac{1}{2}\left(\left(\tfrac{-1}{k-1}\right)-1\right)\left(2\left(\tfrac{d}{2}\right)-3+d\right)+8\lambda_k\pmod{16},$$

Theorem 1 for $k\geq 8$ follows from (4.2), the divisibilities $4\mid k_2(d)$, $2\mid h(-4d)$ and the congruence (3.6). Indeed, by the mentioned congruence we have

$$8\lambda_k k_2(d)\beta(d)\equiv 16\lambda_k dh(-4d)\pmod{128},$$

$$2(k-2)k_2(d)\beta(d)\equiv 4(k-2)dh(-4d)\pmod{64},$$

$$\tfrac{1}{2}k(d-1)k_2(d)\beta(d)\equiv k(d-1)dh(-4d)\pmod{64},$$

$$(1-d)k_2(d)\beta(d)\equiv 2(1-d)h(-4d)\pmod{64}.$$

Therefore we get

$$\begin{aligned}b_k(d)&\equiv\tfrac{1}{2}\left[\left(5k-2\left(\tfrac{d}{2}\right)-9\right)+\left(\left(\tfrac{-1}{k-1}\right)-1\right)\left(2\left(\tfrac{d}{2}\right)-3+d\right)\right]k_2(d)\beta(d)+\\ &\quad+\left(3k+2-8\left(\tfrac{-1}{k-1}\right)\right)dh(-4d)\\ &\equiv\tfrac{1}{2}\left[\left(k-2\left(\tfrac{d}{2}\right)-1\right)+(k-2)\left(2\left(\tfrac{d}{2}\right)-3+d\right)\right]k_2(d)\beta(d)+\\ &\quad+(-k+2)dh(-4d)\\ &\equiv\tfrac{1}{2}\left[-k-6\left(\tfrac{d}{2}\right)+3+2k\left(\tfrac{d}{2}\right)\right]k_2(d)\beta(d)+(2-k)h(-4d)\pmod{64},\end{aligned}$$

and consequently Theorem 1 follows from (3.6), i.e. from the congruence

$$k\left(\left(\tfrac{d}{2}\right)+1\right)k_2(d)\beta(d) \equiv 2k\left(\left(\tfrac{d}{2}\right)+1\right)h(-4d) \pmod{64}.$$

Consequently we deduce that

$$b_4(d) \equiv \left[d^2+d+5\left(\tfrac{d}{2}\right)-\tfrac{13}{2}\right]k_2(d)\beta(d)-\tfrac{10}{d}h(-4d) \pmod{64}.$$

Hence Theorem 1 (i) for $k=4$ follows immediately.

If $k=6$ then from (4.1) we get

$$dB_{6,\left(\frac{d}{\cdot}\right)} = -8d^4t_2+10d^2t_4-6dt_5+t_6.$$

Therefore by (3.2), Lemma 3 (used for t_4, t_5) and Lemma 5 (i) (used for t_6) we find that

$$dB_{6,\left(\frac{d}{\cdot}\right)} \equiv 6d\left[5\left(\tfrac{d}{2}\right)-\tfrac{1}{4}A_1(d,2)\right]k_2(d)\beta(d)++24d^2h(-4d) \pmod{2^7d}.$$

Hence Theorem 1 (i) for $k=6$ follows easily, and the proof of the theorem is complete. ■

5. Proofs of Theorems 2 and 3.

The proof starts with the formula (4.1). In the case $2\mid d$ and $k\ge 4$ (in view of $2^{\operatorname{ord}_2 d+2}\mid t_{2i}$) it gives the congruence

$$dB_{k,\left(\frac{d}{\cdot}\right)} \equiv -kdt_{k-1}+t_k \pmod{2^{\operatorname{ord}_2 k+6}d}, \tag{5.1}$$

because for $i\le \frac{k}{2}-1$

$$1+\operatorname{ord}_2\frac{(2d)^{k-2i}}{(k-2i)!} > 1+2(k-2i)\ge 5.$$

But by Lemma 3 (ii) we have

$$-t_{k-1}\equiv\begin{cases}4(k-2)k_2(d) \pmod{64}, & \text{if } 4\,\|\,d,\\ 0 \pmod{64}, & \text{if } 8\mid d,\end{cases}$$

and consequently by Lemma 5 (ii), (iii) the theorems follow at once. ■

6. Corollaries to Theorems.

COROLLARY 1. *Let d and $k \geq 4$ be the discriminant of a real quadratic field and an even natural number respectively. Then we have:*

(i) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq \operatorname{ord}_2 k + 1$, *if* $d \neq 8$, *and* $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k$, *if* $d = 8$.

(ii) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + \nu$, $1 \leq \nu \leq 3 \iff 2^{\nu+1} \,\|\, k_2(d)$.

The next corollary is an immediate consequence of the previous one and Theorem 1.

COROLLARY 2. *Let d, $2 \nmid d$ and $k \geq 4$ be the discriminant of a real quadratic field and an even natural number respectively. Then we have:*

(i) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + 4 \iff 32 \,\|\, k_2(d)$ *and* $[k \equiv 2 \pmod 4$ *or* $(k \equiv 0 \pmod 4)$ *and* $16 \mid h(-4d))]$.

(ii) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + 5 \iff \big[32 \,\|\, k_2(d)$ *and* $k \equiv 0 \pmod 4$ *and* $8 \,\|\, h(-4d)$ *and* $\frac{1}{8}h(-4d) \not\equiv \frac{1}{32}k_2(d)\beta(d) \pmod 4\big]$ *or* $\{64 \,\|\, k_2(d)$ *and* $[k \equiv 2 \pmod 8$ *or* $(k \equiv -2 \pmod 8)$ *and* $16 \mid h(-4d))$ *or* $32 \mid h(-4d)]\}$ *or* $\{128 \,\|\, k_2(d)$ *and* $[(k \equiv 0 \pmod 4)$ *and* $16 \,\|\, h(-4d))$ *or* $(k \equiv -2 \pmod 8)$ *and* $8 \,\|\, h(-4d))]\}$,

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq \operatorname{ord}_2 k + 6$, *otherwise.* ■

COROLLARY 3. *Let $d = -4d^*$, where d^* is the discriminant of an imaginary quadratic field, and let $k \geq 4$ be an even natural number. Then we have:*

(i) $\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + 4 \iff \left\{\left(\frac{d^*}{2}\right) = 1 \text{ and } 32 \,\|\, k_2(d)\right\}$ *or* $\left\{\left(\frac{d^*}{2}\right) = -1 \text{ and } 32 \,\|\, k_2(d) \text{ and } [8 \mid h(d^*) \text{ or } (4 \,\|\, h(d^*) \text{ and } k \equiv 2 \pmod 4))]\right\}$.

(ii) *If* $\left(\frac{d^*}{2}\right) = 1$ *then:*

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + 5 \iff \{64 \,\|\, k_2(d)$ *and* $[k \equiv 2 \pmod 4$ *or* $(k \equiv 0 \pmod 4)$ *and* $2 \mid h(d^*))]\}$ *or* $(128 \mid k_2(d)$ *and* $k \equiv 0 \pmod 4$ *and* $2 \nmid h(d^*))$,

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq \operatorname{ord}_2 k + 6$, *otherwise.*

If $\left(\frac{d^*}{2}\right) = -1$ *then:*

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} = \operatorname{ord}_2 k + 5 \iff \big[32 \,\|\, k_2(d)$ *and* $4 \,\|\, h(d^*)$ *and* $k \equiv 0 \pmod 4$ *and*
$\frac{1}{32}k_2(d) \equiv -\frac{1}{4}\left(\frac{2}{k-1}\right) h(d^*)\alpha(d^*) \pmod 4\big]$ *or*
$\{64 \,\|\, k_2(d)$ *and* $[k \equiv 2 \pmod 8$ *or*
$(k \equiv -2 \pmod 8)$ *and* $8 \mid h(d^*))$ *or*
$(k \equiv 0 \pmod 4)$ *and* $16 \mid h(d^*))]\}$ *or*
$\{128 \mid k_2(d)$ *and* $[(k \equiv -2 \pmod 8)$ *and* $4 \,\|\, h(d^*))$
or $(k \equiv 0 \pmod 4)$ *and* $8 \,\|\, h(d^*))]\}$.

$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq \operatorname{ord}_2 k + 6$, *otherwise.*

COROLLARY 4. *Let $d=-8d^*$, where d^* is the discriminant of an imaginary quadratic field, and let $k\geq 4$ be an even natural number. Then we have:*

(i) *If* $\left(\frac{d^*}{2}\right)=1$ *then:*

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+4 \iff \{32\,\|\,k_2(d) \text{ and } [2\,|\,h(d^*) \text{ or } (2\nmid h(d^*) \text{ and } k\equiv 2 \pmod 4)]\} \text{ or } (64\,|\,k_2(d) \text{ and } 2\nmid h(d^*) \text{ and } k\equiv 0 \pmod 4),$$

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+5 \iff \Big[32\,\|\,k_2(d) \text{ and } 2\,|\,h(d^*) \text{ and } k\equiv 0 \pmod 4 \text{ and } 64\,\|\,k_2(d)+4h(-d)+32\left(\tfrac{2}{k-1}\right)h(d^*)\alpha(d^*)\Big] \text{ or } \{64\,\|\,k_2(d) \text{ and } [k\equiv 2 \pmod 8) \text{ or } (k\equiv -2 \pmod 8) \text{ and } 2\,|\,h(d^*)) \text{ or } (k\equiv 0 \pmod 4) \text{ and } 32\,|\,h(-d)+8h(d^*)\alpha(d^*))]\} \text{ or } \{128\,|\,k_2(d) \text{ and } [(k\equiv -2 \pmod 8) \text{ and } 2\nmid h(d^*)) \text{ or } (k\equiv 0 \pmod 4) \text{ and } 16\,\|\,h(-d)+8h(d^*)\alpha(d^*))]\},$$

$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}\geq \operatorname{ord}_2 k+6$, *otherwise.*

(ii) *If* $\left(\frac{d^*}{2}\right)=-1$ *then:*

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+4 \iff \{32\,\|\,k_2(d) \text{ and } [16\,|\,h(-d) \text{ or } (8\,\|\,h(-d) \text{ and } k\equiv 2 \pmod 4))]\} \text{ or } [64\,|\,k_2(d) \text{ and } 8\,\|\,h(-d) \text{ and } k\equiv 0 \pmod 4)],$$

$$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+5 \iff \Big[32\,\|\,k_2(d) \text{ and } 8\,\|\,h(-d) \text{ and } k\equiv 0 \pmod 4 \text{ and } \tfrac{1}{32}k_2(d)\equiv -\tfrac{1}{8}\left(\tfrac{2}{k-1}\right)h(-d) \pmod 4\Big] \text{ or } \{64\,\|\,k_2(d) \text{ and } [k\equiv 2 \pmod 8) \text{ or } (k\equiv -2 \pmod 8) \text{ and } 16\,|\,h(-d)) \text{ or } 32\,|\,h(-d)]\} \text{ or } \{128\,|\,k_2(d) \text{ and}[(k\equiv 0 \pmod 4) \text{ and } 16\,\|\,dh(-d)) \text{ or } (k\equiv -2 \pmod 8) \text{ and } 8\,\|\,h(-d))]\},$$

$\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}\geq \operatorname{ord}_2 k+6$, *otherwise.*

COROLLARY 5. *Let $d=8d^*$, where d^* is the discriminant of a real quadratic field, and let $k\geq 4$ be an even natural number. Then we have:*

(i) $\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+4 \iff \{32\,\|\,k_2(d)$ *and* $[16\,|\,h(-d)$ *or* $(8\,\|\,h(-d)$ *and* $k\equiv 2 \pmod 4))]\}$ *or* $(64\,|\,k_2(d)$ *and* $8\,\|\,h(-d)$ *and* $k\equiv 0 \pmod 4))$.

(ii) $\operatorname{ord}_2 B_{k,\left(\frac{d}{\cdot}\right)}=\operatorname{ord}_2 k+5 \iff \Big[32\,\|\,k_2(d)$ *and* $8\,\|\,h(-d)$ *and* $k\equiv 0 \pmod 4$ *and* $\frac{1}{32}k_2(d)\equiv -\frac{1}{8}\left(\frac{2}{k-1}\right)h(-d) \pmod 4\Big]$ *or* $\{64\,\|\,k_2(d)$ *and* $[k\equiv 2 \pmod 8)$ *or*

$(k \equiv -2 \pmod 8$ and $16 \mid h(-d))$ or $32 \mid h(-d)]\}$ or $\{128 \mid k_2(d)$ and $[(k \equiv -2 \pmod 8)$ and $8 \| h(-d))$ or $(k \equiv 0 \pmod 4)$ and $16 \| h(-d))]\}$,
$\operatorname{ord}_2 B_{k,(\frac{d}{\cdot})} \geq \operatorname{ord}_2 k + 6$, *otherwise.*

7. Proofs of Corollaries.

Corollary 1(i) for $2 \nmid d$ is an obvious consequence of Theorem 1 and the divisibilities $2 \mid h(-4d)$, $4 \mid k_2(d)$, and for $2 \mid d$ of Theorem 4. In order to prove (ii) of this corollary for $2 \nmid d$ we use the congruence (3.6). In fact, in view of this congruence Theorem 1 implies

$$b_k(d) \equiv \left[\left(k\left(\tfrac{d}{2}\right) + \tfrac{k}{2} + 1\right)\mu + \tfrac{3}{2}\left(-k - 2\left(\tfrac{d}{2}\right) + 1\right)\vartheta\right] k_2(d)\beta(d) \pmod{16},$$

and consequently

$$b_k(d) \equiv -\tfrac{1}{2}\left(3 - 2\left(\tfrac{d}{2}\right)\right) k_2(d)\beta(d) \pmod{16}.$$

Hence Corollary 1(ii) for $2 \nmid d$ follows immediately because μ, resp. $\vartheta \equiv 1$ (mod4, resp. 8).

Corollary 1(ii) for $2 \mid d$ up to the case $\nu = 3$ and $8 \mid d$ is an obvious consequence of Theorem 4. If $8 \mid d$ then we consider two cases. First, let $d^* > 0$. Let us note that if $8 \mid h(-d)$, then we have $16 \mid \vartheta_2 h(-d)$, $\vartheta_3 h(-4d^*)$, and consequently Corollary 1(ii) in the case $\nu = 3$, $d^* > 0$ follows from Cor. 2(iii) to Thm. 1 [2] that states that $16 \| k_2(d)$ if and only if $8 \| h(-d)$ and $8 \mid h(-4d^*)$, or $16 \mid h(-d)$ and $4 \| h(-4d^*)$. Now, let $d^* < 0$. Then $16 \mid \mu_3$. If $\left(\frac{d^*}{2}\right) = 1$ then by Cor. 2(i) to Thm. 2 [2], $16 \| k_2(d)$ if and only if $8 \| h(-d)$, and consequently Corollary 1(ii) for $d^* < 0$, $\left(\frac{d^*}{2}\right) = 1$ follows easily. If $\left(\frac{d^*}{2}\right) = -1$ then $16 \mid \mu_2 h(-d)$, $\mu_3 h(d^*)\alpha(d^*)$ because in view of Cor. 2(ii) to Thm. 2 [2], $16 \| k_2(d)$ if and only if $16 \mid h(-d)$ and $2 \| h(d^*)$, or $8 \| h(-d)$ and $4 \mid h(d^*)$. This completes the proof of Corollary 1.

Now we prove Corollary 3. Let $d = -4d^*$, where $d^* < 0$. We consider two cases, again. If $\left(\frac{d^*}{2}\right) = 1$ then $4 \mid h(-2d)$, and $32 \mid \vartheta_2$, and so $32 \mid \vartheta_2 h(d^*)\alpha(d^*) + \vartheta_3 h(-2d)$. Therefore Corollary 3(i) for $\left(\frac{d^*}{2}\right) = 1$ follows. Also, in the case $\left(\frac{d^*}{2}\right) = 1$ the divisibility $32 \mid k_2(d)$ implies $8 \mid h(-2d)$ (see Cor. 2(i) to Thm. 2 [2]). Consequently we get (ii) of Corollary 3 in this case easily. We turn to the case $\left(\frac{d^*}{2}\right) = -1$. Then by Cor. 2(iii) to Thm. 2 [2], $32 \mid k_2(d)$ if and only if $4 \| h(d^*)$ and $4 \| h(-2d)$, or $8 \mid h(d^*)$ and $8 \mid h(-2d)$. Thus in both the cases we have $\vartheta_2 h(d^*)\alpha(d^*) + \vartheta_3 h(-2d) \equiv 2(k-2)h(d^*)\alpha(d^*) \pmod{32}$. This completes the proof of Corollary 3(i). Likewise, by the used above arguments we get (ii) of the corollary and its proof is complete.

Now we consider the case $8 \mid d$. If $d^* < 0$ and $\left(\frac{d^*}{2}\right) = 1$ then by Cor. 2(i) to Thm. 2 [2], $32 \mid k_2(d)$ if and only if $16 \mid h(-d)$. Consequently we have $\mu_2 h(-d) + \mu_3 h(d^*)\alpha(d^*) \equiv 0$, resp. $16 \pmod{32}$ if and only if $2 \mid h(d^*)$, or $2 \nmid h(d^*)$ and $k \equiv 2 \pmod 4$, resp. $2 \nmid h(d^*)$ and $2 \nmid h(d^*)$. This gives the first part and the beginning of the second one of Corollary 4(i). Similar considerations apply to the remaining one of the corollary. To prove (ii) of it let us note that in virtue of Cor. 2(iii) to Thm. 2 [2], $32 \mid k_2(d)$ if and only if $16 \mid h(-d)$ and $4 \mid h(d^*)$, or $8 \| h(-d)$ and $2 \| h(d^*)$. Thus $2 \mid h(d^*)$ and $\mu_2 h(-d) + \mu_3 h(d^*)\alpha(d^*) \equiv (k-2)h(-d) \equiv 0$, resp. $16 \pmod{32}$ if and only if $16 \mid h(-d)$, or $8 \| h(-d)$ and $k \equiv 2 \pmod 4$, resp. $8 \| h(-d)$ and $k \equiv 0 \pmod 4$. This gives the proof of the first part of Corollary 4(ii) and the beginning of the second one of it. The remaining one may be handled in the similar way. It remains to prove Corollary 5. Then by Cor. 2 to Thm. 1 [2], $32 \mid k_2(d)$ if and only if $8 \parallel h(-d)$ and $4 \parallel h(-4d^*)$, or $16 \nmid h(-d)$ and $8 \nmid h(-4d^*)$. Therefore $\vartheta_2 h(-d) + \vartheta_3 h(-4d^*) \equiv (k-2)h(-d) \equiv 0$, resp. $16 \pmod{32}$ if and only if $16 \mid h(-d)$, or $8 \parallel h(d^*)$ and $k \equiv 2 \pmod 4$, resp. $8 \parallel h(-d)$ and $k \equiv 0 \pmod 4$. This establishes (i) and the beginning of (ii) of Corollary 5. The similar reasoning applies to the remaining part of the corollary, and Corollaries to Theorems are proved. ■

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