
THE SECOND WEIGHTED MOMENT OF ζ

by

S. Bettin & J.B. Conrey

Abstract. — We give an explicit formula for the second weighted moment of $\zeta(s)$ on the critical line tailored for fast computations with any desired accuracy.

Résumé. — Nous donnons une formule explicite pour le second moment pondéré de la fonction ζ sur la droite critique permettant une évaluation rapide pour n'importe quel degré de précision.

1. Introduction

For $\Re z > 0$ let

$$E_1(z) = 1 - 4 \sum_{n=1}^{\infty} d(n)e(nz)$$

and let

$$E_1(z) - (1/z)E_1(-1/z) =: \psi(z).$$

Then ψ is analytic in \mathbb{C}' , the complex plane minus the negative real-axis and is given in that region by

$$\psi(z) = -2 \frac{\log(2\pi z) - \gamma}{\pi iz} - \frac{2}{\pi} \int_{(-1/2)} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s} ds.$$

Moreover, ψ satisfies a 3-term relation

$$\psi(z+1) = \psi(z) - \frac{1}{z+1} \psi\left(\frac{z}{z+1}\right)$$

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and for $|z| \leq 1$ satisfies

$$\frac{\pi i}{2} \psi(1+z) = \frac{-\log(1+z) - 1}{1+z} + \frac{1}{1+z} \sum_{m=1}^{\infty} a_m (-1)^m z^m$$

where, for $m \geq 1$,

$$a_m = \frac{-1}{m+1} + 2b_m + 2 \sum_{j=0}^{m-2} \binom{m-1}{j} b_{j+2}$$

with $b_1 = 0$ and

$$b_n = \frac{\zeta(n)B_n}{n}$$

for $n \geq 2$. The a_m are extremely small:

$$a_m = 2^{5/4} \pi^{3/4} \frac{e^{-2\sqrt{\pi m}}}{m^{3/4}} \left(\sin(2\sqrt{\pi m} + \frac{3}{8}\pi) + O\left(\frac{1}{\sqrt{m}}\right) \right)$$

so that the series $\sum a_m z^m$ converges everywhere on its circle of convergence $|z| = 1$. All of these results can be found in [BC].

2. The second moment of ζ

Theorem 1. — *Let*

$$I(\delta) = \int_0^{\infty} |\zeta(1/2 + it)|^2 e^{-\delta t} dt.$$

Then, for $0 < \Re(\delta) < \pi$ and $\Im\delta \leq 0$ we have

$$I(\delta) = C_0(\delta) + C_1(\delta) + C_2(\delta) + C_3(\delta)$$

where

$$C_0(\delta) = -\frac{(\log(1 - e^{-i\delta}) + 1)}{2 \sin \frac{\delta}{2}} + \frac{\frac{\pi}{2} e^{-i\delta/2}}{1 - e^{i\delta}} + e^{i\delta/2} (-\delta + \pi/2 + i\gamma - i \log 2\pi);$$

$$C_1(\delta) = \frac{1}{2 \sin \frac{\delta}{2}} \sum_{m=1}^{\infty} a_m e^{-im\delta},$$

where $a_m = \frac{-1}{m+1} + 2b_m + 2 \sum_{j=0}^{m-2} \binom{m-1}{j} b_{j+2}$ with $b_1 = 0$ and $b_n = \frac{\zeta(n)B_n}{n}$ for $n \geq 2$;

$$C_2(\delta) = -\frac{i\pi}{\sin \delta/2} \sum_{n=1}^{\infty} (-1)^n d(n) e^{-\pi n \cot \frac{\delta}{2}};$$

and

$$C_3(\delta) = \int_0^{\infty} |\zeta(1/2 + it)|^2 \frac{-e^{-\pi t} \sinh t\delta + i \cosh t\delta}{\cosh \pi t} dt.$$

If δ is real this simplifies to

$$I(\delta) = -\frac{1 + \log(2 \sin \frac{\delta}{2})}{2 \sin \frac{\delta}{2}} + (\pi - \delta) \cos \frac{\delta}{2} + (\log 2\pi - \gamma) \sin \frac{\delta}{2} \\ + \frac{1}{2 \sin \frac{\delta}{2}} \sum_{m=1}^{\infty} a_m \cos m\delta - \int_0^{\infty} |\zeta(1/2 + it)|^2 \frac{e^{-\pi t} \sinh t\delta}{\cosh \pi t} dt.$$

We can use the fact that

$$\sum_{m=1}^{\infty} a_m = \gamma + 1 - \log 2\pi$$

and that

$$\frac{\cos m\delta - 1}{2 \sin \frac{\delta}{2}} = -U_{m-1}(\cos \frac{\delta}{2}) \sin \frac{m\delta}{2},$$

where U_m is the m th Chebyshev polynomial, to rewrite this further.

Corollary 1. — *Suppose that $0 < \delta < 2\pi$. Then*

$$I(\delta) = \frac{\gamma - \log 2\pi - \log(2 \sin \frac{\delta}{2})}{2 \sin \frac{\delta}{2}} + (\pi - \delta) \cos \frac{\delta}{2} + (\log 2\pi - \gamma) \sin \frac{\delta}{2} \\ - \sum_{m=1}^{\infty} a_m U_{m-1}(\cos \frac{\delta}{2}) \sin \frac{m\delta}{2} - \int_0^{\infty} |\zeta(1/2 + it)|^2 \frac{e^{-\pi t} \sinh t\delta}{\cosh \pi t} dt.$$

In this version the first term above is the readily recognized usual main term. Note that the integral in the right-hand side of this formula can be rewritten in terms of the original integral I :

$$- \int_0^{\infty} |\zeta(1/2 + it)|^2 \frac{e^{-\pi t} \sinh t\delta}{\cosh \pi t} dt = \sum_{n=1}^{\infty} (-1)^{n+1} (I(2\pi n + \delta) - I(2\pi n - \delta)).$$

Another way to put it is if we let

$$w(\delta, t) := e^{-\delta t} + \frac{e^{-\pi t} \sinh \delta t}{\cosh \pi t} = \frac{\cosh(\pi - \delta)t}{\cosh \pi t},$$

then

$$\int_0^{\infty} |\zeta(1/2 + it)|^2 w(\delta, t) dt = \frac{\gamma - \log 2\pi - \log(2 \sin \frac{\delta}{2})}{2 \sin \frac{\delta}{2}} + (\frac{\pi}{2} - \delta) \cos \frac{\delta}{2} + (\log 2\pi - \gamma) \sin \frac{\delta}{2} \\ + \sum_{m=1}^{\infty} a_m U_{m-1}(\cos \frac{\delta}{2}) \sin \frac{m\delta}{2}.$$

Since $a_m \ll e^{-\sqrt{m}}$ we have the following estimate for a precise evaluation of $I(\delta)$ as $\delta \rightarrow 0$.

Corollary 2. — *Given $\delta > 0$ and $N \geq 1$ we can compute $I(\delta)$ to an accuracy of 10^{-N} in time $t(\delta, N) \ll N^2$.*

This is fairly remarkable in that it doesn't depend on δ ! By contrast if one considers the associated integral

$$\int_0^{\frac{1}{\delta}} |\zeta(1/2 + it)|^2 dt$$

then the time to calculate will depend on δ in a significant way, say $\delta^{-\theta}$ for some $\theta > 0$. Finally, using the fact that when δ is real, the imaginary part of $C_3(\delta)$ involves the weight $w(\pi - \delta, t)$ we deduce a formula for the divisor sum in $C_2(\delta)$ in terms of the coefficients a_m :

Corollary 3. — For $0 < \delta < \pi$

$$\begin{aligned} \frac{\pi}{\sin \frac{\delta}{2}} \sum_{n=1}^{\infty} d(n) (-1)^n e^{-n\pi \cot \frac{\delta}{2}} &= -\frac{1}{\sin \delta} \sum_{m=1}^{\infty} (a_{2m} + a_{2m-1}) \sin((2m - \frac{1}{2})\delta) \\ &\quad + \frac{\delta}{4 \sin \frac{\delta}{2}} - \frac{1 + \log(2 \cos \frac{\delta}{2})}{2 \cos \frac{\delta}{2}}. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \sum_{m=1}^{\infty} (a_{2m} + a_{2m-1}) \sin((2m - \frac{1}{2})\delta) &= \frac{\delta \cos \frac{\delta}{2}}{2} - \sin \frac{\delta}{2} \left(1 + \log(2 \cos \frac{\delta}{2}) \right) \\ &\quad - 2\pi \cos \frac{\delta}{2} \sum_{n=1}^{\infty} d(n) (-1)^n e^{-n\pi \cot \frac{\delta}{2}}. \end{aligned}$$

If $\delta = \frac{1}{1000}$, then the first term of the divisor sum on the right-hand side of the formula is $-1.12 \dots \times 10^{-2729}$, whereas the first two terms on the right-hand side are $-0.000159155 \dots$. That means that the sum on the left-hand side is $-0.000159155 \dots - 1.12 \dots \times 10^{-2729}$ up to an error of around $-1.99 \dots \times 10^{-5458}$, which is the second term of the divisor sum. Since the terms a_{2m} are around $e^{-2\sqrt{2\pi m}}$ it would take more than 6 million terms of the series in m (each with thousands of digits of accuracy) to numerically check this.

3. Proof of Theorem 1

Assume first of all that δ is real with $0 < \delta < \pi$. We have

$$I(\delta) = \frac{1}{i} \int_{1/2}^{1/2+i\infty} \zeta(s) \zeta(1-s) e^{i\delta(s-1/2)} ds = \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} \chi(1-s) \zeta(s)^2 e^{i\delta s} ds.$$

Now

$$\chi(1-s) = (2\pi)^{-s} \Gamma(s) (e^{\pi is/2} + e^{-\pi is/2})$$

so that

$$I(\delta) = I_1(\delta) + I_2(\delta)$$

where

$$I_1(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} (2\pi)^{-s} \Gamma(s) e^{-\pi is/2} \zeta(s)^2 e^{i\delta s} ds$$

and

$$I_2(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} (2\pi)^{-s} \Gamma(s) e^{\pi is/2} \zeta(s)^2 e^{i\delta s} ds.$$

Now

$$I_1(\delta) = I_0(\delta) - I_3(\delta)$$

where

$$I_0(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2+i\infty} (2\pi)^{-s} \Gamma(s) e^{-\pi is/2} \zeta(s)^2 e^{i\delta s} ds$$

and

$$I_3(\delta) = \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2} (2\pi)^{-s} \Gamma(s) e^{-\pi is/2} \zeta(s)^2 e^{i\delta s} ds.$$

Thus, $I(\delta) = I_0(\delta) + I_2(\delta) - I_3(\delta)$. Note that $I_2(\delta)$ and $I_3(\delta)$ are analytic for $|\delta| < \pi/2$. We rewrite I_2 and I_3 as integrals over t as

$$\begin{aligned} I_2(\delta) &= \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} \zeta(s) \zeta(1-s) \frac{e^{\pi is/2}}{2 \cos \pi s/2} e^{i\delta s} ds \\ &= \int_0^\infty |\zeta(1/2 + it)|^2 e^{-\pi t/2} \frac{e^{\pi i/4} e^{-\delta t}}{2 \cos \frac{\pi}{2}(1/2 + it)} dt \end{aligned}$$

and

$$\begin{aligned} I_3(\delta) &= \frac{e^{-i\delta/2}}{i} \int_{1/2-i\infty}^{1/2} \zeta(s) \zeta(1-s) \frac{e^{-\pi is/2}}{2 \cos \pi s/2} e^{i\delta s} ds \\ &= \frac{e^{-i\delta/2}}{i} \int_{1/2}^{1/2+i\infty} \zeta(s) \zeta(1-s) \frac{e^{-\pi i(1-s)/2}}{2 \cos \pi(1-s)/2} e^{i\delta(1-s)} ds \\ &= \int_0^\infty |\zeta(1/2 + it)|^2 e^{-\pi t/2} \frac{e^{-\pi i/4} e^{\delta t}}{2 \cos \frac{\pi}{2}(1/2 - it)} dt. \end{aligned}$$

Next we write $e^{\delta t} = \cosh \delta t + \sinh \delta t$ and $e^{-\delta t} = \cosh \delta t - \sinh \delta t$. Also, for real t ,

$$\frac{e^{\pi i/4}}{2 \cos \frac{\pi}{2}(1/2 + it)} + \frac{e^{-\pi i/4}}{2 \cos \frac{\pi}{2}(1/2 - it)} = \frac{e^{-\pi t/2}}{\cosh \pi t}$$

and

$$\frac{e^{\pi i/4}}{2 \cos \frac{\pi}{2}(1/2 + it)} - \frac{e^{-\pi i/4}}{2 \cos \frac{\pi}{2}(1/2 - it)} = i \frac{e^{\pi t/2}}{\cosh \pi t}.$$

Thus,

$$I_2(\delta) - I_3(\delta) = \int_0^\infty |\zeta(1/2 + it)|^2 \frac{-e^{-\pi t} \sinh t\delta + i \cosh t\delta}{\cosh \pi t} dt.$$

Returning to I_0 we have

$$\begin{aligned} I_0(\delta) &= -2\pi e^{-i\delta/2} \operatorname{Res}_{s=1} (2\pi)^{-s} \Gamma(s) e^{-\pi is/2} \zeta(s)^2 e^{i\delta s} + J(\delta) \\ &= e^{i\delta/2} (-\delta + \pi/2 + i\gamma - i \log 2\pi) + J(\delta) \end{aligned}$$

where

$$J(\delta) = \frac{e^{-i\delta/2}}{i} \int_{2-i\infty}^{2+i\infty} (2\pi)^{-s} \Gamma(s) e^{-\pi i s/2} \zeta(s)^2 e^{i\delta s} ds;$$

note that this is a place where we need the (temporary) assumption that δ is real to ensure convergence of the integral on the new path. Expanding $\zeta(s)^2 = \sum_{n=1}^{\infty} d(n)n^{-s}$ into its Dirichlet series and interchanging the summation and integration, we obtain

$$\begin{aligned} J(\delta) &= 2\pi e^{-i\delta/2} \sum_{n=1}^{\infty} d(n) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) (2\pi i n e^{-i\delta})^{-s} ds \\ &= 2\pi e^{-i\delta/2} \sum_{n=1}^{\infty} d(n) e^{-2\pi i n e^{-i\delta}} \\ &= \frac{\pi}{2} e^{-i\delta/2} (1 - E_1(-e^{-i\delta})). \end{aligned}$$

Now

$$E_1(-e^{-i\delta}) = E_1(1 - e^{-i\delta}) = \frac{1}{1 - e^{-i\delta}} E_1\left(\frac{-1}{1 - e^{-i\delta}}\right) + \psi(1 - e^{-i\delta})$$

and this is

$$\frac{1}{1 - e^{-i\delta}} \left(1 - 4 \sum_{n=1}^{\infty} d(n) e\left(\frac{-n}{1 - e^{-i\delta}}\right)\right) + \psi(1 - e^{-i\delta}).$$

Thus,

$$\begin{aligned} J(\delta) &= \frac{\pi}{2} e^{-i\delta/2} - \frac{\pi}{2} \frac{e^{-i\delta/2}}{1 - e^{-i\delta}} \left(1 - 4 \sum_{n=1}^{\infty} d(n) e\left(\frac{-n}{1 - e^{-i\delta}}\right)\right) - \frac{\pi}{2} e^{-i\delta/2} \psi(1 - e^{-i\delta}) \\ &= \frac{\pi}{2} \frac{e^{-i\delta/2}}{1 - e^{i\delta}} - \frac{i\pi}{\sin \delta/2} \sum_{n=1}^{\infty} (-1)^n d(n) e^{-\pi n \cot \frac{\delta}{2}} - \frac{\pi}{2} e^{-i\delta/2} \psi(1 - e^{-i\delta}). \end{aligned}$$

Altogether we now have

$$\begin{aligned} \int_0^{\infty} |\zeta(1/2 + it)|^2 e^{-\delta t} dt &= \frac{\pi}{2} \frac{e^{-i\delta/2}}{1 - e^{i\delta}} + e^{i\delta/2} (-\delta + \pi/2 + i\gamma - i \log 2\pi) \\ &\quad - \frac{\pi}{2} e^{-i\delta/2} \psi(1 - e^{-i\delta}) \\ &\quad - \frac{i\pi}{\sin \delta/2} \sum_{n=1}^{\infty} (-1)^n d(n) e^{-\pi n \cot \frac{\delta}{2}} \\ &\quad + \int_0^{\infty} |\zeta(1/2 + it)|^2 \frac{e^{-\pi t} \sinh t\delta + i \cosh t\delta}{\cosh \pi t} dt. \end{aligned}$$

Recall that

$$\psi(z) = -\frac{2(\log z + 1)}{\pi i z} + \frac{2}{\pi i z} \sum_{m=1}^{\infty} a_m (-1)^m (z - 1)^m$$

so that

$$\psi(1 - e^{-i\delta}) = -\frac{2(\log(1 - e^{-i\delta}) + 1)}{\pi i(1 - e^{-i\delta})} + \frac{2}{\pi i(1 - e^{-i\delta})} \sum_{m=1}^{\infty} a_m e^{-im\delta}$$

and

$$-\frac{\pi}{2} e^{-i\delta/2} \psi(1 - e^{-i\delta}) = -\frac{(\log(1 - e^{-i\delta}) + 1)}{2 \sin \frac{\delta}{2}} + \frac{1}{2 \sin \frac{\delta}{2}} \sum_{m=1}^{\infty} a_m e^{-im\delta}.$$

The assertion of the theorem now follows for real δ . But both sides are analytic in the region $0 < \Re < \pi$ and $\Im \delta < 0$. Therefore, by analytic continuation the identity of the theorem holds in this larger region of the complex plane.

References

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S. BETTIN, Centre de Recherches Mathématiques Université de Montréal, P. O Box 6128, CentreVille Station, Montreal, Quebec H3C 3J7 • *E-mail* : bettin@crm.umontreal.ca

J.B. CONREY, American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 USA and School of Mathematics, University of Bristol, Bristol, BS8 1TW, United Kingdom
E-mail : conrey@aimath.org