# ON I-DUO RINGS

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*This paper is dedicated to professor Souleymane NIANG*

## 1. Introduction

Let $R$ be a non commutative associative ring with unit 1. A left $R$- module $M$ is said to satisfy property $(I)$ if every injective endomorphism of $M$ is an automorphism. It is well know that every artinian module satisfies property $(I)$, and that the converse is false. A ring $R$ is called left (resp right) I-ring if every left (right)$R$-module with property $(I)$ is artinian. Recall that a ring $R$ is a left (resp right) pure-semi-simple ring if every left (resp right) $R$-module is a direct sum of indecomposables left (right ) $R$-modules . $R$ is of finite representation type if $R$ is left (right) artinian and has finite many non isomorphic indecomposable left (right) $R$-modules . We recall that the concept of finite representation type is left-right symmetric. Following [7] a left $R$-module $M$ is said to have property (S) if every surjective endomorphism of M is an automorphism; $R$ is called left (right) S-ring if every left(right) $R$-module with property (S) is noetherian . An $R$-module is said to be uniserial if its submodules are linearly ordered by inclusion . $R$ is left (right) serial if it is a direct sum of left (right) uniserial $R$-modules and $R$ is serial if it is both left and right serial. A duo-ring is a ring in which every one sided ideal is two sided. Definitions and notations used in this paper can be found ind [8].

In this paper we prove that for a duo-ring $R$ the following conditions are equivalent.

1. $R$ is a left $I$ -ring.
2. $R$ is of finite representation type.
3. $R$ is left pure-semi-simple.
4. $R$ is an artinian principal ideal ring (uniserial).
5. $R$ is a left $S$-ring .
6. $R$ is a right I-ring .
7. $R$ is right pure semi-simple.
8. $R$ is a right S-ring .

## 2. Preliminary

**Definition 2.1.** *Let $R$ be a ring. A left $R$-module ${}_RM$ is said to satisfy property $(I)$ if all of its injective endomorphisms are automorphisms. $R$ is called left (right) $I$-ring if every left (right) R-module with property $(I)$ is artinian. $R$ is an $I$-ring if it is a left and right $I$-ring. We recall that a ring $R$ is a left (right) duo-ring if every left (right) ideal of $R$ is a two sided ideal. A left and right duo-ring is called duo-ring.*

1991 *Mathematics Subject Classification.* 16P20, 16A35.

**Proposition 2.1.** *If $R$ is an $I$-duo ring then every prime ideal of $R$ is a maximal ideal and further more the set of all prime ideals is finite .*

*Proof.* Let $P$ be a prime ideal of $R$. the factor ring $R/P$ is an $I$-duo ring which is a domain. Let $K$ be the classical ring of fraction of $R/P$; $K$ is a division ring, hence the $R/P$-module $K$ satisfies property $(I)$. It follows that the module $K$ is artinian and that $K = R/P$. Let $\{ P_l/l \in L \}$ the set of all prime ideals. If $l \neq m$ then $Hom(R/P_l, R/P_m) = \{0\}$, it follows that the $R$-module $M = \bigoplus_{l\in L} R/P$ satisfies property $(I)$ and hence $L$ is a finite set. □

**Corollary 2.1.** *The Jacobson radical $J$ of any $I-$duo ring $R$ is a nil ideal .*

**Proposition 2.2.** *Let $R$ be a semi prime duo ring , then every one sided regular element of $R$ is two sided regular.*

*Proof.* Let $x$ a one sided regular element of $R$ . Assume that $x$ is a right regular; let $y \in R$ such that $yx = 0$ than

$$(xy)^2 = x(yx)y = 0$$

since $R$ is semi prime and $xy$ is nilpotent we have $xy = 0$ and hence $y = 0$. □

**Proposition 2.3.** *Let $R$ be a semi prime $I-$duo ring; then $R$ is artinian.*

*Proof.* Let $R' = S^{-1}R$ be the ring of fraction of $R$ where $S$ is the set of all regular elements of $R$. Any endomorphism of the left$R$–module $R'$ is obtained by multiplication by a element of $R'$ ; it follows that the $R$-module $R'$ satisfies property $(I)$. Since $R$ is an$I-$ring, then $R\prime$ is an artinian $R$–module and hence $R = R'$. So $R$ is artinian. □

**Theorem 2.1.** *Let $R$ be an $I$–duo ring ; then $R$ is artinian.*

*Proof.* It follows from corollary 2.1 that the Jacobson radical $J$ of $R$ is a nil ideal. Then every idempotent of $R/J$ can be lifted to a idempotent of $R$. Since $R/J$ is a semi prime $I-$duo ring, it result from proposition 2.3is semi simple; and that $R$ is a semi-perfect ring. So $R$ can be written

$$R = Re_1 \bigoplus Re_2 \bigoplus ..... \bigoplus R\, e_n,$$

where the $e_i\,'s$ are central idempotents and each $Re_i$ is a local projectif R-module. To prove that $R$ is artinian, it is sufficient to prove that each $Re_i$ is artinian as left $R$-module. Let $f \neq 0$ be an non surjective endomorphism of $Re_i$. We have $f(Re_i) \subseteq Je_i$, let us put $f(e_i) = re_i; r \in J$. As $J$ is a nil ideal, there exists an integer $n \in N^*$ such that $r^n e_i \neq 0$ and $r^{n+1}e_i = 0$. For this integer we have $f(r^n e_i) = r^n f(e_i) = r^{n+1}e_i = 0$. It follows that f is not monic. So the $R-$module $Re_i$ satisfies property $(I)$. Hence $Re_i$ is artinian. □

**Remark 2.1.** *(a) Every artinian duo-ring is a finite direct product of artinian local duo-rings (b) It is proved in* [3] *that if $R$ is an artinian local duo ring with*

*Jacobson radical $J$, then $R$ is uniserial or $R/J$ is a field.*

## 3. Characterization of I-duo rings

In what follows $R$ will denote an artinian local duo-ring with Jacobson radical $J$ satisfying $J^2 = 0$. It results then from remarks 2.1 that if $R$ has a non principal ideal, then $R/J$ is a field. We have then two cases : Case 1 : $R/J$ is an infinite field and $dim_{R/J} J/J^2 = 2$ Let $H$ be a complete set of representants of $(R/J) \setminus \{\bar{0}\}$ $H$ is a infinite set. For $h \in H$ , set $I = R(x_1 - hx_2)$ where $\{\overline{x}_1, \overline{x}_2\}$ is a basis of $J/J^2$ over $R/J$ ; and $M_h = R/I_h$.

**Lemma 3.1.** *If $h \neq h'$ are in $H$ then $x_1 - hx_2 \notin I_{h'}$*

*Proof.* Assume that $x_1 - hx_2 \in I_{h'} = R(x_1 - h'x_2)$. Let $\alpha \in R\backslash J$ such that $x_1 - hx_2 = \alpha(x_1 - h'x_2)$ then $(1 - \alpha)x_1 - (h - \alpha h') = 0$, it follows that $1 - \alpha \in J$ and $h - \alpha h' \in J$. Let $m \in J$ such that $\alpha = 1 + m$. Since $h - (1 + m)h' \in J$, we have $h - h' \in J$ which contradicts the choice of $H$. □

**Lemma 3.2.** *Let $h, h' \in H, h \neq h'$. If $g : M_h \longrightarrow M_{h'}$ is an homomorphism of $R-$modules then $g(1+I_h)$ is note invertible in the ring $M_{h'}$. So $g(1+I_h) \in J/I_{h'}$*

**Notation 3.1.** *If $x \in R$ and $h \in H$, we set $x + I_h = x_{M_h}$.*

*Proof of lemma* 3.2. We have

$$0_{M_{h'}} = g(0_{M_h}) = g[(x_1 - hx_2) + I_h] = (x_1 - hx_2)g(1 + I_h),$$

hence $g(1 + I_h)$ is not invertible in $M_{h'}$ so $g(1 + I_h) \in J/I_{h'}$ .

**Corollary 3.1.** *Let $f : \bigoplus_{h\in H} M_h \longrightarrow \bigoplus_{h\in H} M_h$ be an endomorphism of the $R-$module $\bigoplus_{h\in H} M_h$. If $i_h$ and $p_{h'}$ are respectively the canonical injection of $M_h$ in $\bigoplus_{k\in H} M_k$ and the canonical projection of $\bigoplus_{h\in H} M_h$ on $M_{h'}$, then $p_h \circ f \circ i_h\ (1 + I_h) \in J/I_{h'}$.*

If $x \in \bigoplus_{h\in H} M_h = M$, we note $x = \sum_{h\in H} \alpha_h e_h$ where $e_h = 1 + I_h$ and $\alpha_h \in R$, and $f(e_h) = \sum_{h'\in H} \beta_{h'} e_{h'}$ where $\beta_{h'} e_{h'} = p_{h'} \circ f \circ i_h(e_h)$. So $\beta_{h'} \in J$ if $h \neq h'$. Let f be an injective endomorphism of $M = \bigoplus_{h\in H} M_h$ we have the following lemmas.

**Lemma 3.3.** *For every $h \in H, f(e_h) = \beta_h e_h + \sum_{h\neq h'} \beta_{h'} e_{h'}$; where $\beta_h \notin J$.*

*Proof.* Let $h \in H$. If $h' \in H$ and $h' \neq h$ then, by lemma 3.1, we have

$$0_M \neq f[(x_1 - h'x_2)e_h] = (x_1 - h'x_2)\beta_h e_h,$$

it follows that $\beta_h \notin J$ . □

**Lemma 3.4.** *$J.M \subseteq Imf$ .*

*Proof.* Let $m$ be an element of $J$. For $h \in H$, we have $f(me_h) = m\beta_h e_h$. Since $R$ is a duo ring and $\beta_h \notin J$, there exists $\beta'_h \in R \setminus J$ such that $m\beta_h = \beta'_h m$; we have then $me_h = f({\beta'_h}^{-1} me_h)$. So $me_h \in Imf$ and hence $J.M \subseteq Imf$. □

**Lemma 3.5.** *For every $h \in H$ , $e_h \in Imf$.*

*Proof.* Let $h \in H$. By lemma 3.3 we have

$$f(e_h) = \beta_h e_h + \sum_{h\neq h'} \beta_{h'} e_{h'} \beta_h \notin J \text{ and } \beta_{h'} \in J, for h \neq h'.$$

Then $\beta_h e_h = f(e_h) - \sum_{h\neq h} \beta_{h'} e_{h'}$, so $e_h = f(\beta_h^{-1} e_h) - \sum_{h\neq h'} \beta_h^{-1}\beta'_h e_{h'}$. Since $f(\beta_h^{-1} e_h)$ and $\sum_{h\neq h'} \beta_h^{-1}\beta'_h e_{h'}$ are in $Imf$ then $e_h \in Imf$. ☐

We can now state the following assertion.

**Theorem 3.1.** *Let $R$ be a local artinian duo ring with maximal ideal $J$ such that $J^2 = 0$ . If $R/J$ is an infinite field and $dim_{R/J} J/J^2 \geq 2$ then there exists an non artinian $R$-module $M$ with property $(I)$.*

Case2: We assume that $R/J$ is a finite field and that $dim_{R/J} J/J^2 = 2$. In this

case the characteristic of the field $R/J$ is a prime number $p$ and the characteristic of $R$ is $p$ or $p^2$. and hence $R/J$ is a separable finite extension of $Z(R)/J \cap Z(R)$ where $Z(R)$ is the center of $R$. It follows from[6] that there exists an artinian principal ideal subring $B$ of $R$ such that $R = B \bigoplus Bc$ as $B-$ modules, where $c \in J$. So let us set $Bb$ the Jacobson radical of $B$, we have $b^2 = bc = c^2 = 0$. In what follows homomorphism will be in the opposite side of the scalars. Let $M_R = R_R^{(N^*)} = \bigoplus_{i\in N*} e_i R$ where $e_i = (\delta_i^j)_{j\in N^*}$ and

$$\delta_i^j = \begin{cases} 1_R, \, if\, i = j \\ \\ i_{0_R}, \, if\, i \neq j \end{cases}$$

and let $\sigma : M_R \longrightarrow M_R$ be the endomorphism of $M_R$ given by :

$$\sigma(e_i) = \begin{cases} 0, \, if\, i = 1 \\ \\ e_{i-1}, \, if\, i \geq 2 \end{cases}$$

If $z \in R$ we denote $L_z$ the endomorphism of $M_R$ defined for $m \in M_R$ by $L_z(m) = zm$ . Let $\Lambda$ be the subring of $EndM_R$ generated by $d = L_c \circ \sigma$ and the elements $L_x$, $x \in B$ . By the ring homomorphism

$$R = B \bigoplus Bb \longrightarrow \Lambda$$

$$x + yb \longrightarrow L_x + L_y \circ d.$$

M has a structure of left $R$- module defined as follows

$$(x + yc)m = (L_x + L_y \circ d)(m).$$

Let now $f$ be an injective endomorphism of ${}_R M$, we have $(d.m)f = d.(m)f$ for $m \in M$. We shall prove the following lemmas.

**Lemma 3.6.** *For every $n \in \mathbb{N}^*$, we have $d(e_n)f = (ce_{n-1})f$, for $n \geq 2$, and $0 = d(e_1)f$.*

**Lemma 3.7.** *For every $n \in \mathbb{N}^*$, we have*

$$(e_n)f = \sum_{in} \alpha_{k,n}\, e_k$$

*where $\alpha_{n,n}$ is invertible in $R$, and $\alpha_{k,n} \in J$ for $k > n$.*

*Proof.* Set $(e_1)f = \alpha_{1,1}e_1 + \sum_{i>1} \alpha_{i,1}e_i$. Since $ce_1 \neq 0$, then

$$c(\alpha_{1,1} + \sum_{i>1} \alpha_{i,1}e_i) = (ce_1)f \neq 0 \quad (1).$$

But $c(\sum_{i>1} \alpha_{i,1}e_{i-1}) = c\sigma(\alpha_{1,1}e_1 + \sum_{1<1} \alpha_{i,1}e_i) = c\sigma[(e_1)f] = (c\sigma e_1)f = (0)f = 0$ (2). So by (2), $\alpha_{i,1} \in J for i > 1$ and by (1)$\alpha_{1,1} \notin J$.

Suppose now that

$$(e_{n-1})f = \sum_{i<n-1} \alpha_{i,n-1}e_i + \alpha_{n-1,n-1}e_{n-1} + \sum_{i>n-1} \alpha_{i,n-1}e_i,$$

where $\alpha_{n-1,n-1} \notin J$. and $\alpha_{i,n-1} \in J$ for $i > n-1$ ; and let us set $(e_n)f = \sum_{i\geq 1} \alpha_{i,n}e_i$. Then

$$c\sigma(\sum_{i\geq 1} \alpha_{i,n}e_i.) = c\sigma(e_n)f = c(e_{n-1})f = (ce_{n-1})f \neq 0.$$

Since $c(\sum_{i\geq 2} \alpha_{i,n}e_{i-1}.) = c\sigma(e_n)f = c(e_{n-1})f = \sum_{i<n-1} c\alpha_{i,n-1}e_i + c\alpha_{n-1,n-1}e_{n-1}$, where $c\alpha_{n-1,n-1}e_{n-1} \neq 0$, then $c\alpha_{n,n} \neq 0$ and $c\alpha_{i,n} = 0$ for $i > n$. It follows then that $\alpha_{n,n} \notin J$ and $\alpha_{i,n} \in J$ for $i > n$. □

**Lemma 3.8.** *For every $n \in \mathbb{N}^\star$, we have $J.e_n \subseteq Imf$.*

*Proof.* Let $m \in J$ we have$(me_1)f = m(e_1)f = m\alpha_{1,1}e_1$. Let $\alpha'_{1,1} \in R \setminus J$ such that $m\alpha_{1,1} = a'_{1,1}m$, then $(me_1)f = \alpha'_{1,1}me_1$ hence $me_1 = (\alpha'_{1,1})^{-1}(me_1)f \in Imf$.

Suppose that $Je_k \subseteq Imf$ for $k \leq n-1$ and let $m \in J$, we have :

$$(me_n)f = \sum_{i,1} c_{i,1}e_i$$

where $\alpha_{1,1} \notin J$ and$\alpha_{i,1} \in J$ for $i > 1$.

Assume that for every $k < n$, $e_k \in \operatorname{Im} f$. Since

$$(e_n)f = \sum_{\in} c_{k,n}e_k$$

where $\alpha_{n,n}$ is invertible and $c_{k,n} \in J$, we have

$$\alpha_{n,n}e_n = (e_n)f - \sum_{\in} c_{k,n}e_k \in \operatorname{Im} f$$

and so

$$e_n(\alpha_{n,n}^{-1}e_n)f - \sum_{n>i} \alpha_{i,n}^{-1}e_i - \bigoplus_{nn} Je_i$$

of ${}_RM$ is strictly decreasing. □

We have proved the following result :

**Theorem 3.2.** *Let $R$ be an artinian local duo ring with maximal ideal $J$ such that $J^2 = (0)$. If $R/J$ is a finite field and* $\dim_{R/J} J/J^2 \geq 2$ *then there exists a non artinian $R$-module with property $(I)$.*

We have the following theorem :

**Theorem 3.3.** *Let $R$ be a duo ring. The following statements are equivalent.*

1. *$R$ is a left $I-$ring.*
2. *$R$ is an uniserial ring.*
3. *$R$ is a left $S-$ring.*
4. *$R$ is a left pure semi-simple ring.*
5. *$R$ has a finite representation type.*
6. *$R$ is a right $I-$ring.*
7. *$R$ is a right $S-$ring.*
8. *$R$ is a right pure semi-simple ring.*

*Proof.* It suffices to proved the equivalence $1) \Longleftrightarrow 2)$.

$1) \Longrightarrow 2)$. By theorem 2.1 $R$ is Artinian and by theorem 3.1 and theorem 3.2 $R$ is necessarily a principal ideal ring .

$(2) \Longrightarrow (1)$ If $R$ is an uniserial ring than every left $R-$module is a direct sum of cyclic modules . Let $M$ be an artinian $R-$ module , since there is only finite non isomorphic cyclic $R-$modules we can write $M = K^{(N^*)} \bigoplus L$ where $K$ is cyclic submodule of $M$ . Since $K^{(N^*)}$ does not satisfy property $(I)$, it follows that $M$ does not satisfy property $(I)$. □

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