

# Publications mathématiques de Besançon

## ALGÈBRE ET THÉORIE DES NOMBRES

Fabien Pazuki

### Heights and regulators of number fields and elliptic curves

2014/2, p. 47-62.

<[http://pmb.cedram.org/item?id=PMB\\_2014\\_\\_\\_\\_2\\_47\\_0](http://pmb.cedram.org/item?id=PMB_2014____2_47_0)>

© Presses universitaires de Franche-Comté, 2014, tous droits réservés.

L'accès aux articles de la revue « Publications mathématiques de Besançon » (<http://pmb.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://pmb.cedram.org/legal/>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Publication éditée par le laboratoire de mathématiques  
de Besançon, UMR 6623 CNRS/UFC*

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

---

# HEIGHTS AND REGULATORS OF NUMBER FIELDS AND ELLIPTIC CURVES

*by*

Fabien Pazuki

---

**Abstract.** — We compare general inequalities between invariants of number fields and invariants of elliptic curves over number fields. On the number field side, we remark that there is only a finite number of non-CM number fields with bounded regulator. On the elliptic curve side, assuming the height conjecture of Lang and Silverman, we obtain a Northcott property for the regulator on the set of elliptic curves with dense rational points over a number field. This amounts to say that the arithmetic of CM fields is similar, with respect to the invariants considered here, to the arithmetic of elliptic curves over a number field having a non Zariski dense Mordell-Weil group, *i.e.* with rank zero.

**Résumé.** — *Hauteurs et régulateurs de corps de nombres et de courbes elliptiques.* On compare des inégalités entre invariants classiques de corps de nombres et de courbes elliptiques définies sur un corps de nombres. Dans le cas des corps de nombres, on remarque qu'il n'y a qu'un nombre fini de corps de nombres non-CM avec régulateur borné. On obtient alors comme conséquence d'une conjecture de Lang et Silverman une propriété de Northcott pour le régulateur sur l'ensemble des courbes elliptiques sur un corps de nombres dont les points rationnels sont denses. Cela indique que l'arithmétique d'un corps CM est similaire, au sens des invariants considérés ici, à celle des courbes elliptiques sur un corps de nombres dont le groupe de Mordell-Weil n'est pas dense au sens de Zariski, donc de rang nul.

## 1. Introduction

Comparisons between number fields and abelian varieties defined over number fields are used to help in the understanding of the arithmetic of both. As a recent example of such a

---

**Mathematical subject classification (2010).** — 11G50, 14G40.

**Key words and phrases.** — Heights, abelian varieties, regulators, Mordell-Weil.

**Acknowledgements.** — The author wishes to thank the Laboratoire Mathématiques de l'Université de Franche-Comté, in Besançon and specially Aurélien Galateau for the invitation to give a course in October 2013 on the Lang and Silverman conjecture. One goal of the present text is to explain in details a new possible consequence of this conjecture. Many thanks to Gaël Rémond, Qing Liu, Pascal Autissier, Marc Hindry, Eduardo Friedman and the anonymous referee for their comments. The author is supported by ANR-10-BLAN-0115 Hamot and ANR-10-JCJC-0107 Arivaf.

comparison one can read [Hin07], where a conjectural asymptotic in the spirit of the Brauer-Siegel theorem for number fields is given for abelian varieties. In this work we will focus on inequalities in both settings, but restricting ourselves to abelian varieties of dimension 1, *i.e.* elliptic curves. Firstly, we gather some useful inequalities between classical invariants — discriminant, degree, regulator — of number fields. Secondly we want to understand how the inequality of Theorem 3.6 between the regulator and the discriminant of a number field finds its elliptic counterpart in Theorem 4.8, assuming the Lang-Silverman height conjecture. Let us stress here that Hindry and Silverman showed in [HiSi88] that the Lang-Silverman conjecture for elliptic curves is implied by the ABC conjecture, see Remark 4.6 in the sequel. For some inspiring references about regulators of number fields, one can consult various texts, including Friedman [Fri89], Friedman and Skoruppa [FrSk99], Bergé and Martinet [BeMa90], Silverman [Si84a], Zimmert [Zim81], Remak [Rem52], Odlyzko [Odl77, Odl90]. Let us now state some consequences of the studied inequalities. On the number field side, one has the following. First recall that a CM field is a totally imaginary degree 2 extension of a totally real extension of  $\mathbb{Q}$ .

**Theorem 1.1.** — *There exists only a finite number of non-CM number fields with bounded regulator.*

The proof is relatively short and essentially relies on existing inequalities. It will be given in section 3. On the elliptic curve side, one obtains the following result as a corollary of Theorem 4.8.

**Theorem 1.2.** — *Assume the Lang-Silverman height Conjecture 4.5. Then, the set of  $\overline{\mathbb{Q}}$ -isomorphism classes of elliptic curves  $E$ , defined over a fixed number field  $K$  with  $E(K)$  Zariski dense in  $E$  and bounded regulator is finite.*

The Zariski density of  $E(K)$  is equivalent to  $E$  having positive Mordell-Weil rank over  $K$  because  $E$  is simple.

Focusing on elliptic curves defined over number fields, one also obtains (without assuming any height conjecture) an inequality between the Faltings height of Definitions 2.3 and 4.2, the primes of bad reduction and the injectivity diameter, which is of independent interest for two reasons: there is no assumption of semi-stability and all constants are explicit.

**Theorem 1.3.** — *Let  $K$  be a number field of degree  $d$ . Then for any elliptic curve  $E$  defined over  $K$ , one has*

$$h_{\mathbb{F}^+}(E/K) \geq \frac{1}{3 \cdot 128d} \log N_{E/K}^0 + \frac{1}{3d} \sum_{v \in M_K^\infty} d_v \rho(E_v, L_v)^{-2} - \frac{1}{3} \log \frac{\pi}{\pi - 3}$$

where  $N_{E/K}^0$  is the product of the norms of the primes of  $K$  of bad reduction for  $E$  and  $\rho(E_v, L_v)$  is the injectivity diameter of  $E_v(\mathbb{C})$  polarized by  $L_v$ , the set of archimedean places of  $K$  is denoted by  $M_K^\infty$  and  $d_v = [K_v : \mathbb{Q}_v]$  is the local degree.

In Section 2 we give the definitions of the regulator of a number field, of the regulator of an elliptic curve and of the Faltings height, also called the differential height, of an elliptic curve. In Section 3 we gather the inequalities concerning number fields. In Section 4 we prove the inequalities concerning elliptic curves and we compare these inequalities with the number field case. We conclude in Section 5 with a comparative table.

Another way of reading the inequalities in this text lies in the following remark: in the number field case, both the discriminant and the regulator are volumes that carry arithmetic information, but the regulator is a weak invariant on CM-fields. For elliptic curves over a number field, the regulator and the differential height are both arithmetic volumes, but the regulator is a weak invariant on elliptic curves with non Zariski dense Mordell-Weil group (*i.e.* on rank zero elliptic curves). The inequalities of Theorem 3.6 and Theorem 4.8 compare these volumes precisely enough to reveal the special cases. An interesting question would be to obtain similar results in the case of general abelian varieties. This will be the object of a future work.

## 2. Definitions

If  $K$  is a number field, we denote by  $d$  its degree over  $\mathbb{Q}$  and by  $M_K$  the set of all places of  $K$ . Let  $M_K^\infty$  be the set of archimedean places. For any  $v \in M_K$ , we have  $K_v$  the completion of  $K$  with respect to the valuation  $|\cdot|_v$ . One normalizes  $|p|_v = p^{-1}$  for any finite place  $v$  above a prime number  $p$ . Then the local degree will be  $d_v = [K_v : \mathbb{Q}_v]$ . We will write  $\mathfrak{p}$  for a prime ideal. We will denote by  $D_K$  the absolute discriminant of  $K$ .

Let  $S$  be a set. We will say that a function  $f : S \rightarrow \mathbb{R}$  verifies a Northcott property on  $S$  if for any real number  $B$ , the set  $\{P \in S \mid f(P) \leq B\}$  is finite.

### 2.1. Regulators of number fields and elliptic curves. —

2.1.1. *Number fields.* — Let  $K$  be a number field of degree  $d = r_1 + 2r_2$ , with  $r_1$  real embeddings  $\sigma_1, \dots, \sigma_{r_1}$  and  $r_2$  pairs of complex conjugate embeddings denoted  $(\sigma_{r_1+1}, \overline{\sigma_{r_1+1}}), \dots, (\sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}})$ . We denote by  $U_K$  the group of units in  $K$ . By Dirichlet's theorem,  $U_K$  is a group of rank  $r = r_K = r_1 + r_2 - 1$ . Let  $\varepsilon_1, \dots, \varepsilon_r$  be a fundamental base, *i.e.* any unit  $\varepsilon \in U_K$  can be uniquely written as  $\varepsilon = z\varepsilon_1^{n_1} \cdots \varepsilon_r^{n_r}$  with  $z$  a root of unity in  $K$  and  $n_1, \dots, n_r \in \mathbb{Z}$ . The norm of any  $\alpha \in K$  can be written as

$$N_{K/\mathbb{Q}}(\alpha) = \sigma_1(\alpha) \cdots \sigma_d(\alpha) = \sigma_1(\alpha) \cdots \sigma_{r_1}(\alpha) |\sigma_{r_1+1}(\alpha)|^2 \cdots |\sigma_{r_1+r_2}(\alpha)|^2.$$

Let  $\lambda$  be the logarithmic map given by

$$\begin{aligned} \lambda : K &\rightarrow \mathbb{R}^{r_1+r_2} \\ \alpha &\mapsto (\log |\sigma_1(\alpha)|, \dots, \log |\sigma_{r_1}(\alpha)|, 2 \log |\sigma_{r_1+1}(\alpha)|, \dots, 2 \log |\sigma_{r_1+r_2}(\alpha)|), \end{aligned}$$

Moreover let  $d_1 = \dots = d_{r_1} = 1$  for the real embeddings and  $d_{r_1+1} = \dots = d_{r_1+r_2} = 2$  for the complex embeddings – so  $d_i = d_{v_i} = [\overline{K_{v_i}} : \mathbb{Q}]$  for each  $i \in \{1, \dots, r_1 + r_2\}$  and the first  $r_1$  embeddings are the real ones. Let  $H = \{(x_1, \dots, x_{r_1+r_2}) \in \mathbb{R}^{r_1+r_2} \mid x_1 + \dots + x_{r_1+r_2} = 0\}$ . We remark that  $\lambda(U_K) \subset H$  because the norm of any unit is  $\pm 1$ , hence the logarithm of the absolute norm is 0. We are interested in the volume of the lattice spanned by the image of the units of  $K$  in  $H$ . The vector  $e = (1, \dots, 1)$  is orthogonal to the hyperplane  $H$  with respect to the canonical euclidean structure of  $\mathbb{R}^{r_1+r_2}$ , hence one has the following volume equality

$$\text{Vol}(\mathbb{R}^{r_1+r_2} / \lambda(U_K) \oplus \mathbb{Z}e) = \text{Vol}(H / \lambda(U_K)) \sqrt{r_1 + r_2}.$$

The volume of a fundamental domain of the lattice  $\lambda(U_K)$  in the hyperplane  $H$  is given by  $\text{Vol}(H / \lambda(U_K))$ , where the volume is induced by the euclidean structure on  $\mathbb{R}^{r_1+r_2}$ . This motivates the following definition.

**Definition 2.1.** — The regulator  $R_K$  of  $K$  is defined in the following way

$$R_K = \left| \det \begin{bmatrix} d_1 \log |\sigma_1(\varepsilon_1)| & d_2 \log |\sigma_2(\varepsilon_1)| & \dots & d_{r_1+r_2} \log |\sigma_{r_1+r_2}(\varepsilon_1)| \\ d_1 \log |\sigma_1(\varepsilon_2)| & d_2 \log |\sigma_2(\varepsilon_2)| & \dots & d_{r_1+r_2} \log |\sigma_{r_1+r_2}(\varepsilon_2)| \\ \dots & \dots & \dots & \dots \\ d_1 \log |\sigma_1(\varepsilon_r)| & d_2 \log |\sigma_2(\varepsilon_r)| & \dots & d_{r_1+r_2} \log |\sigma_{r_1+r_2}(\varepsilon_r)| \\ (r_1+r_2)^{-1} & (r_1+r_2)^{-1} & \dots & (r_1+r_2)^{-1} \end{bmatrix} \right|,$$

where the square matrix is of size  $(r_1 + r_2)$ .

Remark that one has  $R_K = \text{Vol}(\mathbb{R}^{r_1+r_2}/\lambda(U_K) \oplus \mathbb{Z}e)(r_1+r_2)^{-1} = \text{Vol}(H/\lambda(U_K))(r_1+r_2)^{-1/2}$ , so the regulator is a volume obtained in the whole space, divided by the square root of the dimension of the whole space. An equivalent definition used in the literature is

$$R_K = \left| \det \begin{bmatrix} d_1 \log |\sigma_1(\varepsilon_1)| & d_2 \log |\sigma_2(\varepsilon_1)| & \dots & d_r \log |\sigma_r(\varepsilon_1)| \\ d_1 \log |\sigma_1(\varepsilon_2)| & d_2 \log |\sigma_2(\varepsilon_2)| & \dots & d_r \log |\sigma_r(\varepsilon_2)| \\ \dots & \dots & \dots & \dots \\ d_1 \log |\sigma_1(\varepsilon_r)| & d_2 \log |\sigma_2(\varepsilon_r)| & \dots & d_r \log |\sigma_r(\varepsilon_r)| \end{bmatrix} \right|,$$

where one deletes the last column and the last line of the former matrix. To show equality of the two determinants, take the  $(r_1 + r_2)^2$  determinant from above and sum over all the columns to get a last column full of zeros (because the logarithm of the absolute norm of a unit is 0) except for the last term at the bottom line where one gets  $(r_1 + r_2)(r_1 + r_2)^{-1} = 1$ , and develop the determinant. By the same argument one knows that the final value does not depend on the column one erases, the only difference would be the sign which is removed by the absolute value outside the determinant.

This definition is the same as the one found in [Neu99] page 43, if one remarks that his  $\lambda(\cdot)$  is defined through a factorization of his  $l(\cdot)$  page 34 that has the same normalization for the metrics given here.

We end this section by recalling that a CM field  $K$  is a totally imaginary quadratic extension of a totally real number field.

2.1.2. *Elliptic curves.* — Let  $E/K$  be an elliptic curve over a number field  $K$  polarized by  $L = 3(O)$  where  $O$  is the neutral element. Let  $m_K$  be the Mordell-Weil rank of  $E(K)$ . Let  $\hat{h} = \hat{h}_{E,L}$  be the Néron-Tate height associated with the pair  $(E, L)$ . Let  $\langle \cdot, \cdot \rangle$  be the associated bilinear form, given by

$$\langle P, Q \rangle = \frac{1}{2} \left( \hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q) \right)$$

for any  $P, Q \in E(\bar{\mathbb{Q}})$ .

**Definition 2.2.** — Let  $P_1, \dots, P_{m_K}$  be a basis of the Mordell-Weil group  $E(K)$  modulo torsion. The regulator of  $E/K$  is defined by

$$\text{Reg}(E/K) = \left| \det(\langle P_i, P_j \rangle_{1 \leq i, j \leq m_K}) \right|,$$

where by convention an empty determinant is equal to 1.

**2.2. The Faltings height.** — Let  $E$  be an elliptic curve defined over a number field  $K$ . Let  $S = \text{Spec}(\mathcal{O}_K)$ , where  $\mathcal{O}_K$  is the ring of integers of  $K$  and let  $\pi: \mathcal{E} \rightarrow S$  be the Néron model of  $E$  over  $S$ . Denote by  $\varepsilon: S \rightarrow \mathcal{E}$  the zero section of  $\pi$  and by  $\omega_{\mathcal{E}/S}$  the sheaf of relative differentials

$$\omega_{\mathcal{E}/S} := \varepsilon^* \Omega_{\mathcal{E}/S} \simeq \pi_* \Omega_{\mathcal{E}/S}.$$

For any archimedean place  $v$  of  $K$ , denote by  $\sigma$  an embedding of  $K$  in  $\mathbb{C}$  associated to  $v$ , then the corresponding line bundle

$$\omega_{\mathcal{E}/S, \sigma} = \omega_{\mathcal{E}/S} \otimes_{\mathcal{O}_K, \sigma} \mathbb{C} \simeq H^0(\mathcal{E}_\sigma(\mathbb{C}), \Omega_{\mathcal{E}_\sigma}(\mathbb{C}))$$

can be equipped with a natural  $L^2$ -metric  $\|\cdot\|_v$  defined by

$$\|\alpha\|_v^2 = \frac{i}{2} \int_{\mathcal{E}_\sigma(\mathbb{C})} \alpha \wedge \bar{\alpha}.$$

The  $\mathcal{O}_K$ -module of rank one  $\omega_{\mathcal{E}/S}$ , together with the hermitian norms  $\|\cdot\|_v$  at infinity defines an hermitian line bundle  $\bar{\omega}_{\mathcal{E}/S} = (\omega_{\mathcal{E}/S}, (\|\cdot\|_v)_{v \in M_K^\infty})$  over  $S$ , which has a well defined Arakelov degree  $\widehat{\text{deg}}(\bar{\omega}_{\mathcal{E}/S})$ . Recall that for any hermitian line bundle  $\bar{\mathcal{L}}$  over  $S$ , the Arakelov degree of  $\bar{\mathcal{L}}$  is defined as

$$\widehat{\text{deg}}(\bar{\mathcal{L}}) = \log \text{Card}(\mathcal{L}/s\mathcal{O}_K) - \sum_{v \in M_K^\infty} d_v \log \|s\|_v,$$

where  $s$  is any non zero section of  $\mathcal{L}$  (the result does not depend on the choice of  $s$  in view of the product formula).

We now give the definition of the classical Faltings height.

**Definition 2.3.** — The Faltings height of  $E/K$  is defined as

$$h_F(E/K) := \frac{1}{[K:\mathbb{Q}]} \widehat{\text{deg}}(\bar{\omega}_{\mathcal{E}/S}).$$

We like to think of the Faltings height as a volume, in particular as a positive real number. An inequality of Bost reads  $h_F(E/K) \geq -\frac{1}{2} \log 2\pi^2$  (a detailed proof being accessible in [GaRé14a]), we thus would like to rescale the height slightly in the following way:

**Notation 2.4.** — Let  $h_{F^+}(E/K) = h_F(E/K) + \frac{1}{2} \log(2\pi^2)$ .

We now obtain a non-negative number that can be interpreted as a volume. This little change of normalization also helps simplifying some formulas in the sequel. We will continue to call this number the Faltings height or the differential height.

The Faltings height does not depend on any choice of polarization on  $E$ . When  $E/K$  is semi-stable, this height only depends on the  $\mathbb{Q}$ -isomorphism class of  $E$ . If  $E/K$  is not semi-stable, one may use Chai's base change conductor, take formula (4) in the sequel as a complementary definition. The definition can be extended to abelian varieties of any fixed dimension. To see that it is really a height (meaning verifying the Northcott property), see for instance [Fa83] Satz 1, page 356 and 357. This can also be derived from the comparison of heights in [Paz12] (an article based on former work by Bost and David) and the work of Mumford on theta structures.

If  $K'/K$  is a number field extension, one has  $h_{F^+}(E/K') \leq h_{F^+}(E/K)$ . If  $E/K$  is semi-stable, one defines the stable height by  $h_{F^+}(E/\mathbb{Q}) := h_{F^+}(E/K)$ , which is invariant by number field

extension. Let us state the following formula (proved in [CoSi86] page 254) which will be crucial in the sequel:

**Theorem 2.5.** — (Silverman) Let  $E/K$  be a semi-stable elliptic curve over a number field  $K$  of degree  $d$ . One has

$$h_{\mathbb{F}^+}(E/K) = \frac{1}{12d} \left[ \log N_{K/\mathbb{Q}}(\Delta_E) - \sum_{v \in M_K^\infty} d_v \log \left( |\Delta(\tau_v)| (2 \operatorname{Im} \tau_v)^6 \right) \right],$$

where  $\Delta_E$  is the minimal discriminant of  $E$ ; at archimedean places one chooses  $\tau_v$  in the upper half plane such that  $E(\overline{K}_v) \simeq \mathbb{C}/\mathbb{Z} + \tau_v \mathbb{Z}$  and  $\Delta(\tau_v) = q \prod_{n=1}^{+\infty} (1 - q^n)^{24}$  is the modular discriminant, with  $q = \exp(2\pi i \tau_v)$ .

We recall that  $|\Delta(\tau_v)| (\operatorname{Im} \tau_v)^6$  is independent of the choice of  $\tau_v$  in his orbit under the classical action of  $\operatorname{SL}_2(\mathbb{Z})$  on the upper half plane. At finite places we focus on the bad reduction locus with the following classical quantities.

**Definition 2.6.** — Let  $E$  be an elliptic curve over a number field  $K$ . Let us denote by  $N_{E/K}^0$  the norm of the product of the bad reduction primes of  $E$ .

**Definition 2.7.** — Let  $E$  be an elliptic curve over a number field  $K$ . We say that a prime  $\mathfrak{p}$  of  $\mathcal{O}_K$  is of *stable bad reduction* if for any number field extension  $K'/K$  and any prime  $\mathfrak{p}'$  of  $\mathcal{O}_{K'}$  above  $\mathfrak{p}$ , the elliptic curve  $E/K'$  has bad reduction at  $\mathfrak{p}'$ . If the prime  $\mathfrak{p}$  is a prime of bad reduction for  $E/K$  that is not a stable one, it will be referred to as an *unstable bad prime* of  $E/K$ .

Regarding archimedean places, let us recall what the injectivity diameter is.

**Definition 2.8.** — Let  $E$  be a complex elliptic curve. Let  $L$  be a polarization on  $E$ . Let  $T_E$  be the tangent space of  $E$  and  $\Gamma_E$  its period lattice and  $H_L$  the associated Riemann form on  $T_E$ . The injectivity diameter is the positive number  $\rho(E, L) = \min_{\gamma \in \Gamma \setminus \{0\}} \sqrt{H_L(\gamma, \gamma)}$ , i.e. the first minimum in the successive minima of the period lattice of  $E$ .

If one writes  $(E, L) \simeq (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, H_\tau)$  as polarized complex abelian varieties of dimension 1, where  $\tau$  is a complex period verifying  $\operatorname{Im} \tau > \sqrt{3}/2$  and  $|\operatorname{Re} \tau| \leq 1/2$ , then one has

$$\rho(E, L) = (\operatorname{Im} \tau)^{-1/2}.$$

We will now carry on a comparison between inequalities relating the number field invariants and inequalities relating elliptic curve invariants.

### 3. Number fields

We start this paragraph by gathering some unconditional inequalities between different classical invariants of number fields, the degree, the absolute discriminant and the regulator. One may consult [Sam03] for a reference. We will then give the proof of Theorem 1.1.

**Theorem 3.1.** — (Hermite-Minkowski) Let  $K$  be a number field of discriminant  $D_K$  and degree  $d$ . Then

$$|D_K|^{1/2} \geq \left(\frac{\pi}{4}\right)^{r_2} \frac{d^d}{d!}$$

and if  $d \geq 2$

$$|D_K| \geq \frac{\pi}{3} \left(\frac{3\pi}{4}\right)^{d-1}.$$

**Remark 3.2.** — For a better bound when  $d$  is large, see Odlyzko [Odl77] who gives

$$|D_K|^{1/d} \geq 60^{r_1/d} 22^{2r_2/d} + o(1)$$

when  $d \rightarrow +\infty$ .

The following classical theorem will be used in the sequel.

**Theorem 3.3.** — (Hermite) There is only a finite number of number fields with bounded discriminant.

The following two theorems compare the regulator with the degree of the number field. The second statement is even more precise because it gives a lower bound on a *relative* regulator.

**Theorem 3.4.** — (Friedman, [Fri89] page 620, Corollary) Let  $K$  be a number field of regulator  $R_K$ , degree  $d$  and number of roots of unity  $w_K$ . Let  $r_1$  be the number of real embeddings of  $K$ . Then

$$\frac{R_K}{w_K} \geq 0.0031 \exp(0.241d + 0.497r_1).$$

**Theorem 3.5.** — (Friedman, Skoruppa, [FrSk99]) There exists absolute constants  $c_1 > 0$  and  $c_2 > 1$  such that for any number field extension  $L/K$  the ratio of their regulators satisfies

$$\frac{R_L}{R_K} \geq (c_1 c_2^{[L:K]})^{[K:\mathbb{Q}]},$$

and one can take  $c_1 = 1/(11.5)^{39}$  and  $c_2 = 1.15$ .

The next step is important, it provides a lower bound for the regulator in terms of the discriminant and of the degree of the field. One needs to pay attention, the inequality is trivial in some cases that are analyzed in the sequel.

**Theorem 3.6.** — (Silverman [Si84a], Friedman [Fri89]) Let  $K$  be a number field of discriminant  $D_K$ , regulator  $R_K$  and degree  $d$ . Let  $r_K$  denote the unit rank of  $K$  and  $r_0$  the maximum of the unit ranks of proper subfields of  $K$ . Then there exists a universal constant  $c_3 > 0$  such that

$$R_K \geq c_3 d^{-2d} \left( \log \frac{|D_K|}{d^d} \right)^{r_K - r_0}.$$

The next proposition explains the case  $r_K = r_0$ .

**Proposition 3.7.** — One has  $r_K = r_0$  if and only if  $K$  is a CM field. Moreover, if  $K$  is a CM field and  $K_0$  its maximal totally real subfield, then

$$\frac{R_K}{R_{K_0}} = 2^s, \quad \text{with } r_0 - 1 \leq s \leq r_0.$$



*Proof.* — Let  $K$  be a number field of rank  $r_K$  and  $K_0$  be a strict subfield realizing  $r_0 = \text{rank}(K_0)$ . Then one has  $r_K = r_1(K) + r_2(K) - 1$  and  $r_0 = r_1(K_0) + r_2(K_0) - 1$ . If  $r_K = r_0$ , it gives

$$(1) \quad r_1(K) + r_2(K) = r_1(K_0) + r_2(K_0)$$

on the one hand, and by the degree formula  $r_1(K) + 2r_2(K) = [K : K_0](r_1(K_0) + 2r_2(K_0))$  on the other hand. It implies by substitution that

$$\begin{aligned} r_1(K) + 2r_2(K) &= [K : K_0](r_1(K) + r_2(K) + r_2(K_0)) \\ &= [K : K_0](r_1(K) + 2r_2(K) + r_2(K_0) - r_2(K)), \end{aligned}$$

hence,

$$(2) \quad [K : K_0](r_2(K) - r_2(K_0)) = ([K : K_0] - 1)(r_1(K) + 2r_2(K)) > 0$$

as  $[K : K_0] \geq 2$  for the last inequality and so  $r_2(K) > 0$ . Next, using  $r_2(K_0) \geq 0$  and  $r_1(K) \geq 0$  one obtains in (2)

$$[K : K_0]r_2(K) \geq (2[K : K_0] - 2)r_2(K)$$

which implies  $2 \geq [K : K_0]$ , so  $[K : K_0] = 2$ . Back to (2), one obtains

$$2r_2(K) - 2r_2(K_0) = r_1(K) + 2r_2(K),$$

so  $-2r_2(K_0) = r_1(K)$  and both numbers  $r_1$  and  $r_2$  are non-negative, hence must both be zero. So from (1) one obtains  $r_2(K) = r_1(K_0)$ . Finally we have  $K_0$  with only real embeddings and  $K$  a purely imaginary degree two extension of  $K_0$ , *i.e.*  $K$  is a CM field.

Reciprocally, if  $K$  is a CM field, let  $K_0$  be its maximal totally real subfield. Then we will show below that any base of  $U_{K_0}$  modulo  $\{\pm 1\}$  lifts to a base of a subgroup of index 1 or 2 in  $U_K$  modulo roots of unity, hence the ranks are equal, which implies  $r_K = r_0$ .

Take a unit  $\varepsilon$  in  $K$ . One will denote by a bar the complex conjugation from  $K/K_0$ . Consider  $\bar{\varepsilon}/\varepsilon$ . It is also a unit and its image by any of the complex embeddings of  $K$  is of module 1. As a unit, it is also an algebraic integer. Hence just by using the definition of the height of an algebraic number, one has  $h(\bar{\varepsilon}/\varepsilon) = 0$ . By a theorem of Kronecker, one obtains that  $\bar{\varepsilon}/\varepsilon$  is a root of unity in  $K$ . Let us denote by  $\mu_K$  the set of roots of unity of  $K$ . Writing  $\varepsilon\bar{\varepsilon} = \varepsilon^2 \frac{\bar{\varepsilon}}{\varepsilon} \in K_0$ , one obtains that up to a torsion element,  $\varepsilon^2 \in K_0$ . Moreover if  $\bar{\varepsilon}/\varepsilon = \zeta^2$  where  $\zeta \in \mu_K$ , then  $\bar{\varepsilon}\zeta^{-1} = \varepsilon\zeta$ , so  $\bar{\varepsilon}\zeta = \pm\varepsilon\zeta$ . Hence the mapping  $\varepsilon \mapsto \bar{\varepsilon}/\varepsilon$  is an injection

$$U_K/\mu_K U_{K_0} \hookrightarrow \mu_K/\mu_K^2,$$

so the index of  $U_K$  modulo torsion over  $U_{K_0}$  modulo torsion is bounded by 2. Now if  $\varepsilon_1, \dots, \varepsilon_r$  is a basis of  $U_{K_0}$  modulo  $K_0$ -roots of unity, it will lift to a basis of a subgroup of index 1 or 2 of  $U_K$  modulo  $K$ -roots of unity. In the determinant defining the regulator given in Definition 2.1 all embeddings are real for  $K_0$ , hence  $d_i = 1$  for all  $i \in \{1, \dots, r_0\}$ , whereas all embeddings are complex for  $K$ , hence  $d_i = 2$  for all  $i \in \{1, \dots, r_K\}$ . Now as  $r_0 = r_K$  we get the result by applying Lemma 4.15 page 41 of [Wash97].  $\square$

One finds the comparison of regulators in [Wash97] page 41, with the  $(r_1 + r_2 - 1)^2$ -matrix to define the regulator (*loc.cit.* same page). The second part of the proof here essentially follows his strategy.

**Remark 3.8.** — Proposition 3.7 shows that the regulator of a CM field is bounded in terms of its maximal totally real subfield, hence one cannot expect to obtain  $R_K \gg \log |D_K|$  in that case, because  $|D_K|$  may not be bounded. Take for instance a family of quadratic imaginary extensions of  $\mathbb{Q}$ , then  $|D_K|$  tends to infinity but  $R_K$  is bounded.

We thus obtain the following proof of Theorem 1.1. In a nutshell: the regulator verifies a Northcott property on the collection of all non-CM fields.

*Proof.* — Let us take a family of non-CM number fields  $K$  with bounded regulator. Then using Theorem 3.3 one has that the degree  $d$  is bounded. Moreover using Theorem 3.6 and the fact that  $r_K - r_0 \neq 0$  by Proposition 3.7, then the discriminant is also bounded because we have just seen that the degree  $d$  is bounded. This gives finiteness by Hermite's Theorem 3.1.  $\square$

If one needs an asymptotic result concerning the discriminant and the regulator of number fields, let us also quote the classical Brauer-Siegel theorem.

**Theorem 3.9.** — (*Brauer-Siegel*) Consider a family of number fields  $K$  of bounded degree and let  $h_K$  denote the class number of  $K$ . Then as  $|D_K|$  tends to infinity, one has for any  $\varepsilon > 0$  the inequalities

$$|D_K|^{1/2-\varepsilon} \ll h_K R_K \ll |D_K|^{1/2+\varepsilon}.$$

If the reader is interested in a comparison between this theorem and the behavior of some invariants of abelian varieties, a detailed study is accessible in [Hin07].

## 4. Elliptic curves

We give in this section a lower bound of the differential height by the norm of the product of the bad reduction primes in the semi-stable case, then we obtain the result in the general case, hence deriving a proof of Theorem 1.3. This implies an upper bound on the Mordell-Weil rank of elliptic curves over number fields in terms of the differential height. Using this and the Lang-Silverman conjecture, we obtain a proof of Theorem 4.8, which implies Theorem 1.2.

**4.1. Bad reduction primes.** — Our aim in this paragraph is to compare the height of  $E$  and the norm of the product of the bad reduction primes of  $E$  over the base field  $K$ . We will first prove the inequality in the semi-stable case and then derive it in general using some base change properties in the next paragraph. The following proposition gives the result in the semi-stable case.

**Proposition 4.1.** — Let  $E/K$  be a semi-stable elliptic curve defined over a number field  $K$  of degree  $d$ . Then

$$h_{\mathbb{F}^+}(E/K) \geq \frac{1}{12d} \log N_{E/K}^0.$$

*Proof.* — One has the exact formula from Theorem 2.5

$$h_{\mathbb{F}^+}(E/K) = \frac{1}{12d} \left[ \log N_{K/\mathbb{Q}}(\Delta_E) - \sum_{v \in M_K^\infty} d_v \log \left( |\Delta(\tau_v)| (2 \operatorname{Im} \tau_v)^6 \right) \right],$$

where  $\Delta_E$  is the minimal discriminant of the curve, one may choose  $\tau_v$  a period in the fundamental domain such that  $E(\bar{K}_v) \simeq \mathbb{C}/\mathbb{Z} + \tau_v\mathbb{Z}$  and  $\Delta(\tau_v) = q \prod_{n=1}^{+\infty} (1 - q^n)^{24}$  is the

modular discriminant, with  $q = \exp(2\pi i\tau_v)$ . A direct analytic estimate using  $\text{Im } \tau_v \geq \sqrt{3}/2$  provides us with

$$\sum_{n=1}^{+\infty} \log |1 - e^{2i\pi\tau_v n}| \leq \sum_{n=1}^{+\infty} \log(1 + e^{-\sqrt{3}\pi n}) \leq 0.005$$

hence

$$-2\pi \text{Im } \tau_v + 24 \sum_{n=1}^{+\infty} \log |1 - e^{2i\pi\tau_v n}| + 6 \log(2 \text{Im } \tau_v) \leq 0,$$

hence  $-\log(|\Delta(\tau_v)|(2 \text{Im } \tau_v)^6) \geq 0$  and

$$(3) \quad h_{\mathbb{F}^+}(E/K) \geq \frac{1}{12d} \log N_{K/\mathbb{Q}}(\Delta_E) \geq \frac{1}{12d} \log N_{E/K}^0,$$

because the bad reduction primes divide the discriminant.  $\square$

**4.2. Reducing to the semi-stable case.** — We explain in this section how to use base change properties to jump from the semi-stable case to the general case. We will use the following definition.

**Definition 4.2.** — Let  $E$  be an elliptic curve defined over a discrete valuation field  $K_{\mathfrak{p}}$  and let  $K'_{\mathfrak{p}'}$  be a finite extension of  $K_{\mathfrak{p}}$  where  $E$  has semi-stable reduction, with ramification index  $e_{K'_{\mathfrak{p}'}/K_{\mathfrak{p}}}$ , where  $\mathfrak{p}'$  is a prime above  $\mathfrak{p}$ , and  $\omega_{E/K_{\mathfrak{p}}}$  the sheaf of differentials. Let

$$c(E) = \frac{1}{e_{K'_{\mathfrak{p}'}/K_{\mathfrak{p}}}} \text{length}_{\mathcal{O}_{K'_{\mathfrak{p}'}}} \frac{\Gamma(\text{Spec}(\mathcal{O}_{K_{\mathfrak{p}}}), \omega_{E/K_{\mathfrak{p}}}) \otimes \mathcal{O}_{K'_{\mathfrak{p}'}}}{\Gamma(\text{Spec}(\mathcal{O}_{K'_{\mathfrak{p}'}}), \omega_{E/K'_{\mathfrak{p}'}})},$$

where  $\Gamma(\cdot, \cdot)$  stands for global sections. For any prime  $\mathfrak{p}$  in  $\mathcal{O}_K$ , we define the local base change conductor at  $\mathfrak{p}$  by setting  $c(E, \mathfrak{p}) = c(E_{K_{\mathfrak{p}}})$ .

This conductor was defined by Chai in [Cha00]. It verifies the two following key properties.

**Proposition 4.3.** — *Let  $E$  be an elliptic curve defined over a discrete valuation field  $K_{\mathfrak{p}}$  and let  $K'_{\mathfrak{p}'}$  be a finite extension of  $K_{\mathfrak{p}}$  where  $E$  has semi-stable reduction with base change conductor  $c(E, \mathfrak{p})$ . Then*

1. *one has  $c(E, \mathfrak{p}) = 0$  if and only if  $E/K_{\mathfrak{p}}$  has semi-stable reduction,*
2. *if  $c(E, \mathfrak{p}) \neq 0$ , then  $c(E, \mathfrak{p}) \geq 1/[K'_{\mathfrak{p}'} : K_{\mathfrak{p}}]$ .*

*Proof.* — To show the first point, let us start by assuming that  $E/K_{\mathfrak{p}}$  has semi-stable reduction. Then if one denotes by  $\mathcal{E}_{\mathcal{O}_{K_{\mathfrak{p}}}}^0$  the identity component of the Néron model of  $E$  over  $K_{\mathfrak{p}}$  one has  $\mathcal{E}_{\mathcal{O}_{K_{\mathfrak{p}}}}^0 \otimes \mathcal{O}_{K'_{\mathfrak{p}'}} \simeq \mathcal{E}_{\mathcal{O}_{K'_{\mathfrak{p}'}}}^0$  by Corollaire 3.3 page 348 of SGA 7.1 [SGA72], hence the differentials are the same and  $c(E, \mathfrak{p}) = 0$ .

Reciprocally, one still has a map  $\Phi : \mathcal{E}_{\mathcal{O}_{K_{\mathfrak{p}}}}^0 \otimes \mathcal{O}_{K'_{\mathfrak{p}'}} \rightarrow \mathcal{E}_{\mathcal{O}_{K'_{\mathfrak{p}'}}}^0$  by the Néron property. As  $c(E, \mathfrak{p}) = 0$ , the Lie algebras are the same and as  $\Phi$  is an isomorphism on the generic fibers,  $\Phi$  is birational. On the special fiber,  $\Phi$  has finite kernel and is thus surjective because the dimensions are equal, here again because  $c(E, \mathfrak{p}) = 0$ .

Overall,  $\Phi$  is quasi-finite and birational. As  $\mathcal{E}_{\mathcal{O}'_{K'_p}}$  is normal, by Zariski's Main Theorem found in Corollary 4.6 page 152 of [Liu02],  $\Phi$  is an open immersion. Hence  $\Phi$  is surjective and an open immersion, hence an isomorphism. This implies that  $E/K_p$  is semi-abelian. For the second point, as the length is non zero it must be at least one and we may upper bound  $e_{K'_p/K_p} \leq [K'_p : K_p]$ .  $\square$

Let  $Uns$  denote the set of unstable primes of  $E$  over  $K$ . Let  $K'$  be a number field extension of  $K$  over which  $E$  has semi-stable reduction everywhere. One then has

$$(4) \quad h_{F^+}(E/K') - h_{F^+}(E/K) = -\frac{1}{[K' : K]} \sum_{\mathfrak{p} \in Uns} c(E, \mathfrak{p}) \log N_{K/\mathbb{Q}}(\mathfrak{p}).$$

**Proposition 4.4.** — *Let  $K$  a number field of degree  $d$ . For any elliptic curve (not necessarily semi-stable)  $E$  defined over  $K$  one has*

$$h_{F^+}(E/K) \geq \frac{1}{12^8 d} \log N_{E/K}^0.$$

*Proof.* — Let  $N_{E/K}^{st}$  be the product of the norms of primes with semi-stable bad reduction. Let  $N_{E/K}^{uns}$  be the product of the norms of primes with unstable bad reduction. By definition one has  $N_{E/K}^0 = N_{E/K}^{st} N_{E/K}^{uns}$ . Let  $K'$  be a number field extension of  $K$  such that  $E$  acquires semi-stable reduction everywhere over  $K'$ . Using equality (4), one gets

$$h_{F^+}(E/K) \geq h_{F^+}(E/K') + \frac{1}{[K' : K]^2} \log N_{E/K}^{uns}.$$

As  $E/K'$  has semi-stable reduction everywhere, one gets by Proposition 4.1 that

$$h_{F^+}(E/K') \geq \frac{1}{12^{d'}} \log N_{E/K'}^{st}.$$

Recall (use Theorem 6.2 page 413 of [SiZa95] or [Se72] page 294) that one may choose  $K' = K[E[12]]$ , hence the degree  $d' = [K' : \mathbb{Q}]$  is controlled by the degree  $d = [K : \mathbb{Q}]$  in the following way: use Lemme 4.7 of [GaRé14b] to get  $[K' : K] \leq 12^4$ , hence  $d' \leq 12^4 d$ . Now by definition  $N_{E/K'}^{st} = N_{E/K}^{st [K':K]}$  and so putting everything together it gives

$$h_{F^+}(E/K) \geq \frac{1}{12^5 d} \log N_{E/K}^{st} + \frac{1}{12^8} \log N_{E/K}^{uns} \geq \frac{1}{12^8 d} \log N_{E/K}^0.$$

$\square$

We obtain a proof of Theorem 1.3 as a consequence of Proposition 4.4 in the following way.

*Proof.* — Apply Proposition 4.4 to get

$$(5) \quad h_{F^+}(E/K) \geq \frac{1}{12^8 d} \log N_{E/K}^0$$

and apply the Matrix Lemma of Autissier's [Aut13] in his simplified version to the polarized elliptic curve  $(E, L)$ . For any  $\varepsilon \in ]0, 1[$  Autissier gives (note that  $2\pi^2$  disappeared with the non-negative normalization, and that  $g = 1$ ):

$$h_{F^+}(E/K) + \frac{1}{2} \log \frac{1}{\varepsilon} \geq \frac{(1 - \varepsilon)\pi}{6d} \sum_{v \in M_K^\infty} d_v \rho(E_v, L_v)^{-2}.$$

Choose  $\varepsilon = 1 - \frac{3}{\pi}$  to get

$$(6) \quad 2h_{\mathbb{F}^+}(E/K) + \log \frac{\pi}{\pi - 3} \geq \frac{1}{d} \sum_{v \in M_K^\infty} d_v \rho(E_v, L_v)^{-2}.$$

Then sum the two inequalities (5) and (6). □

**4.3. Regulator.** — We start this paragraph by recalling the original conjecture of Lang-Silverman in dimension 1 (Conjecture 9.9 of [Si86] page 233, see also [Si84b] page 396 for a generalized version by Silverman for abelian varieties, originally the conjecture was a question of Lang about elliptic curves).

**Conjecture 4.5.** — (*Lang-Silverman*) For any number field  $K$ , there exists a positive constant  $c_4 = c_4(K)$  such that for any elliptic curve  $E/K$ , for any point  $P \in A(K)$ , if  $\mathbb{Z} \cdot P$  is Zariski dense one has

$$\widehat{h}(P) \geq c_4 \max \left\{ h_{\mathbb{F}^+}(E/K), 1 \right\},$$

where  $\widehat{h}(\cdot) = \widehat{h}_{E,L}(\cdot)$  is the Néron-Tate height associated to  $L = 3(O)$  and  $h_{\mathbb{F}^+}(E/K)$  is the (relative) differential height of the elliptic curve  $E/K$ .

As  $E$  is simple, the Zariski density of  $\mathbb{Z} \cdot P$  is equivalent to  $P$  being a non-torsion point.

**Remark 4.6.** — In the article [HiSi88], Hindry and Silverman show that the conjecture is true for a large class of elliptic curves (namely, with bounded Szpiro quotient). They furthermore show that the ABC conjecture implies the Lang-Silverman conjecture for elliptic curves.

Firstly let us show (unconditionally) that the Mordell-Weil rank  $m_K$  is bounded by the height  $h_{\mathbb{F}^+}(E/K)$ .

**Lemma 4.7.** — Let  $E$  be an elliptic curve defined over a number field  $K$ . Let  $m_K$  be the rank of  $E(K)$ . There exists a constant  $c_5 = c_5(K) > 0$  such that

$$m_K \leq c_5 \max\{1, h_{\mathbb{F}^+}(E/K)\},$$

and one may take  $c_5 = (2^{26}3^8 + 2^8 \log 16)d^3 + 2^8 d \log |\Delta_{K/\mathbb{Q}}|$ .

*Proof.* — We will use as a pivot the quantity  $N_{E/K}^0$  defined as the product of the norms of the bad reduction primes of the elliptic curve  $E$  over  $K$ . Applying Theorem 5.1 of [Rém10] page 775, there exists constants  $c_6 = c_6(K) > 0$  and  $c_7 = c_7(K) \geq 0$  such that  $m_K \leq c_6(K) \log N_{E/K}^0 + c_7(K)$ . The constants are given explicitly and depend on the degree and the discriminant of the base field here. We may take the slightly bigger explicit constants  $c_6(K) = d^2 2^{12}$  and  $c_7(K) = d^{28}(\log |\Delta_{K/\mathbb{Q}}| + d^2 \log 16)$ . This last inequality doesn't require semi-stability of  $E$ . Applying Proposition 4.4 of the present text one obtains  $\log N_{E/K}^0 \leq c_8(K) \max\{h_{\mathbb{F}^+}(E/K), 1\}$ , also valid in general with  $c_8(K) = 12^8 d$ . One concludes by

$$m_K \leq c_6(K) \log N_{E/K}^0 + c_7(K) \leq c_6(K)c_8(K) \max\{h_{\mathbb{F}^+}(E/K), 1\} + c_7(K),$$

hence

$$m_K \leq (c_6(K)c_8(K) + c_7(K)) \max\{h_{\mathbb{F}^+}(E/K), 1\},$$

and so  $c_5 = 12^8 2^{12} d^3 + 2^8 d(\log |\Delta_{K/\mathbb{Q}}| + d^2 \log 16) = (2^{26}3^8 + 2^8 \log 16)d^3 + 2^8 d \log |\Delta_{K/\mathbb{Q}}|$ . □

**Theorem 4.8.** — Assume the Lang-Silverman Conjecture 4.5. Let  $K$  be a number field. There exists a constant  $c_{10} = c_{10}(K) > 0$  such that for any elliptic curve  $E$  defined over  $K$ ,

$$\text{Reg}(E/K) \geq \left( c_{10} \max\{h_{\mathbb{F}^+}(E/K), 1\} \right)^{m_K/2}.$$

Thus the set of elliptic curves defined over a fixed number field  $K$  such that  $E(K)$  is Zariski dense and with bounded regulator is finite (up to isomorphisms).

*Proof.* — For the duration of the proof, let us denote  $h = \max\{h_{\mathbb{F}^+}(E/K), 1\}$ . The inequality is trivial for  $m_K = 0$  because in that case the regulator is 1. From now on, let us assume  $m_K \neq 0$ . Let  $L = 3(O)$  and let  $\hat{h} = \hat{h}_{E,L}$  be the associated canonical height on  $E$  and consider the euclidean space  $(E(K) \otimes \mathbb{R}, \hat{h}) \simeq (\mathbb{R}^{m_K}, \hat{h})$ . Apply Minkowski's successive minima inequality to the Mordell-Weil lattice  $\Lambda_K = E(K)/E(K)_{\text{tors}}$  viewed as a lattice inside this euclidean space,

$$\lambda_1(\Lambda_K) \cdots \lambda_{m_K}(\Lambda_K) \leq m_K^{m_K/2} \text{Reg}(E/K).$$

Now apply  $m_K$  times the inequality of Conjecture 4.5 to get

$$(7) \quad \text{Reg}(E/K) \geq \frac{c_4^{m_K} h^{m_K}}{m_K^{m_K/2}},$$

then applying Lemma 4.7 one obtains

$$(8) \quad \sqrt{m_K} \leq \sqrt{c_5} \sqrt{h}.$$

Hence we have in (7)

$$\text{Reg}(E/K) \geq \left( c_4 c_5^{-1/2} \sqrt{h} \right)^{m_K}.$$

Finally, if the regulator is bounded then the height is bounded as soon as  $m_K \neq 0$ , hence the claimed finiteness.  $\square$

**Remark 4.9.** — If the rank  $m_K$  is zero, then the regulator  $\text{Reg}(E/K)$  is 1, so the inequality of Theorem 4.8 is optimal in this respect. If one takes for instance the family of curves  $E_p$  over  $\mathbb{Q}$  defined by the affine model  $y^2 = x^3 + p^2$ , where  $p$  is a prime congruent to 5 modulo 9, then  $\text{rank}_{\mathbb{Q}}(E_p)$  is zero by [CoPa09] page 399 and  $h_{\mathbb{F}^+}(E_p/\mathbb{Q}) \gg \log p$ . The situation is then similar to the CM-fields case as described in the Remark 3.8. One cannot expect the height to be upper bounded for rank zero elliptic curves, even if the regulator is upper bounded for trivial reasons. Then when the rank is non-zero we have two cases. If the Mordell-Weil rank  $m_K$  is one the inequality is exactly Lang-Silverman's. If  $m_K \geq 2$  the inequality of Theorem 4.8 is *a priori* weaker than the Lang-Silverman conjecture.

We thus obtain Theorem 1.2 as a corollary which may be expressed in other words by: the Lang-Silverman conjecture implies that the regulator  $\text{Reg}(E/K)$  verifies a Northcott property on the set of elliptic curves  $E$  defined over a fixed number field  $K$  with  $E(K)$  Zariski dense. This would be the elliptic curve counterpart to Theorem 1.1.

## 5. Conclusion

Starting from the comparison of the residue theorem for the Dedekind zeta function and the strong form of the conjecture of Birch and Swinnerton-Dyer, one gets a way to put number fields invariants and abelian varieties invariants in link. Studying furthermore the Brauer-Siegel theorem for number fields, Hindry gives in [Hin07] a conjectural equivalent in the case of abelian varieties. He then comes with a dictionary linking the invariants. Based on the previous inequalities, let us now add three lines to the (slightly reformulated) dictionary of [Hin07]. Note that we only focus on the case of elliptic curves here.

	Number field $K$	$\leftrightarrow$	Elliptic curve $E/K$	
zeta function	$\zeta_K(s)$	$\leftrightarrow$	$L(E, s)$	$L$ function
log of discriminant	$\log  D_K $	$\leftrightarrow$	$h_{\mathbb{F}^+}(E)$	Faltings height
regulator	$R_K$	$\leftrightarrow$	$\text{Reg}(E/K)$	regulator
class number	$h_K$	$\leftrightarrow$	$ \text{III}(E/K) $	Tate – Shafarevitch group
torsion	$(U_K)_{\text{tors}}$	$\leftrightarrow$	$(E \times \hat{E})(K)_{\text{tors}}$	torsion of $E$ and dual $\hat{E}$
degree	$d$	$\leftrightarrow$	$g$	dimension
CM field	$r_K = r_0$	$\leftrightarrow$	$m_K = 0$	$E(K)$ non Zariski dense
non – CM field	$r_K > r_0$	$\leftrightarrow$	$m_K > 0$	$E(K)$ Zariski dense

## References

- [Aut13] AUTISSIER, P., *Un lemme matriciel effectif*. *Mathematische Zeitschrift* **273** (2013), p. 355–361.
- [BeMa90] BERGÉ, A.-M. AND MARTINET, J., *Sur les minorations géométriques des régulateurs*. *Séminaire de Théorie des Nombres, Paris 1987–88*, *Progr. Math.*, Birkhäuser Boston, Boston, MA, **81** (1990), 23–50.
- [Cha00] CHAI, C.-L., *Néron models for semiabelian varieties: congruence and change of base field*. *Asian J. Math.* **4** (2000), 715–736.
- [CoPa09] COHEN, H. AND PAZUKI, F., *Elementary 3-descent with a 3-isogeny*. *Acta Arith.* **140.4** (2009), 369–404.
- [CoSi86] CORNELL, G. ET SILVERMAN, J. H. (EDITORS), *Arithmetic geometry*. Springer-Verlag (1986).
- [Cus84] CUSICK, T. W., *Lower bounds for regulators*. *Noordwijkerhout 1983 Proceedings*, *Lect. Notes Math.* **1068** (1984), 63–73.
- [Cus91] CUSICK, T. W., *The regulator spectrum for totally real cubic fields*. *Monat. Math.* **112.3** (1991), 217–220.
- [Fa83] FALTINGS, G., *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*. *Invent. Math.* **73** (1983), 349–366.
- [Fri89] FRIEDMAN, E., *Analytic formulas for the regulator of a number field*. *Invent. Math.* **98** (1989), 599–622.

- [FrSk99] FRIEDMAN, E. AND SKORUPPA, N.-P., *Relative regulators of number fields*. Invent. Math. **135** (1999), 115–144.
- [GaRé14a] GAUDRON, E. ET RÉMOND, G., *Théorème des périodes et degrés minimaux d’isogénies*. Comment. Math. Helvet. **89.2** (2014), 343–403.
- [GaRé14b] GAUDRON, E. AND RÉMOND, G., *Polarisations et isogénies*. Duke Math. (To appear, 2014).
- [Hin07] HINDRY, M., *Why is it difficult to compute the Mordell-Weil group?* Diophantine geometry, CRM Series, Ed. Norm., Pisa **4** (2007), 197–219.
- [HiSi88] HINDRY, M. AND SILVERMAN, J., *The canonical height and integral points on elliptic curves*, Invent. Math. **93** (1988), 419–450.
- [Liu02] LIU, Q., *Algebraic Geometry and Arithmetic Curves*. Oxford Graduate Texts in Mathematics, Oxford Science Publications **6** (2002).
- [Neu99] NEUKIRCH, J., *Algebraic number theory*. Grundlehren der Mathematischen Wissenschaften, Springer-Verlag **322** (1999).
- [Odl77] ODLYZKO, A. M., *Lower bounds for discriminants of number fields. II*. Tôhoku Math. J. **29.2** (1977), 209–216.
- [Odl90] ODLYZKO, A. M., *Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results*. Sémin. Théor. Nombres Bordeaux **2.2.1** (1990), 119–141.
- [Paz10] PAZUKI, F., *Remarques sur une conjecture de Lang*. Journal de Théorie des Nombres de Bordeaux **22** no.1 (2010), 161–179.
- [Paz12] PAZUKI, F., *Theta height and Faltings height*. Bull. Soc. Math. France **140.1** (2012), 19–49.
- [Rem52] REMAK, R., *Über Größenbeziehungen zwischen Diskriminante und Regulator eines algebraischen Zahlkörpers*. Compositio Math. **10** (1952), 245–285.
- [Ré05] RÉMOND, G., *Inégalité de Vojta généralisée*. Bull. Soc. Math. France **133.4** (2005), 459–495.
- [Ré10] RÉMOND, G., *Nombre de points rationnels des courbes*. Proc. Lond. Math. Soc. **101.3** (2010), 759–794.
- [Sam03] SAMUEL, P., *Théorie algébrique des nombres*. Hermann, Paris, édition revue et corrigée (2003).
- [SGA72] GROTHENDIECK, A., *Groupes de monodromie en géométrie algébrique*. SGA 7.1, Lecture Notes in Mathematics, Springer-Verlag, **288** (1972).
- [Se72] SERRE, J.-P., *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*. Invent. Math. **15** (1972), 259–331.
- [SiZa95] SILVERBERG, A. AND ZARHIN, YU., *Semistable reduction and torsion subgroups of abelian varieties*. Ann. Inst. Fourier **45** (1995), 403–420.
- [Si86] SILVERMAN, J. H., *Arithmetic of elliptic curves*. Springer GTM **106** (1986), second printing of the first edition.
- [Si84b] SILVERMAN, J. H., *Lower bounds for height functions*. Duke Math. J. **51** (1984), 395–403.
- [Si84a] SILVERMAN, J. H., *An inequality relating the regulator and the discriminant of a number field*. Journal of Number Theory **19.3** (1984), 437–442.



- [Wash97] WASHINGTON, L., *Introduction to cyclotomic fields*. Springer, GTM **83** (1997), second edition.
- [Zim81] ZIMMERT, R., *Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung*. Invent. Math. **62.3** (1981), 367–380.

---

*May 31, 2014*

FABIEN PAZUKI, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark, and IMB, Université de Bordeaux, 351 cours de la Libération, 33405 Talence.  
*E-mail* : `fabien.pazuki@math.u-bordeaux.fr`