Kazuhiko Yamaki

Survey on the geometric Bogomolov conjecture

SURVEY ON THE GEOMETRIC BOGOMOLOV CONJECTURE

by

Kazuhiko Yamaki

To the memory of Alexey Zykin.

Abstract. — This is a survey paper of the developments on the geometric Bogomolov conjecture. We explain the recent results by the author as well as previous works concerning the conjecture. This paper also includes an introduction to the height theory over function fields and a quick review on basic notions on nonarchimedean analytic geometry.

Résumé. — Ce texte est un article de synthèse portant sur la conjecture de Bogomolov géométrique. Nous y expliquons nos résultats récents ainsi que les travaux qui les ont précédés. Cet article contient également une introduction à la théorie des hauteurs sur les corps de fonctions et un exposé rapide des notions de base de géométrique analytique non-archimédienne.

1. Introduction

The Bogomolov conjecture is a problem in Diophantine geometry; it is a conjecture which should characterize the closed subvarieties with a dense set of points of small height. There are several versions of the Bogomolov conjecture. The conjectures for curves and for abelian varieties are widely studied problems among others, and the former is a special case of the latter.

The Bogomolov conjectures is considered both over number fields and over function fields. When we consider the conjecture over function fields with respect to classical heights, we call it the *geometric* Bogomolov conjecture. When we do that over number fields, we sometimes call it the arithmetic Bogomolov conjecture.

The arithmetic Bogomolov conjectures for curves and for abelian varieties have been already established as theorems; that for curves is due to Ullmo [36], and that for abelian varieties is due to Zhang [47]. On the other hand, the geometric Bogomolov conjecture for abelian varieties is still open, but there are some significant partial works, such as [13, 40, 41, 42, 43].

Recently, using those results concerning the conjecture for abelian varieties, we have shown in [43] that the conjecture for curves holds in full generality.

The purpose of this survey is to give an exposition of the recent developments of the geometric Bogomolov conjecture for abelian varieties and to describe the idea of the proof of the geometric Bogomolov conjecture for curves.

1.1. Notation and conventions. — We put together the notation and conventions that will be used throughout this article.

1.1.1. *Sets.* — A *natural number* means a rational integer greater than 0. Let $\mathbb{N}$ denote the set of natural numbers.

By the convention of this paper, the notation $A \subset B$ allows the case of $A = B$.

1.1.2. *Varieties.* — Let $\mathfrak{K}$ be any field. A variety over $\mathfrak{K}$ means a geometrically integral scheme that is separated and of finite type over $\mathfrak{K}$.

For a variety X over $\mathfrak{K}$, let $\mathfrak{K}(X)$ denote the function field of X.

1.1.3. *Abelian varieties.* — Let A be an abelian variety over any field $\mathfrak{K}$. For each $n \in \mathbb{Z}$, let $[n] : A \to A$ denote the n-times endomorphism on A, i.e. the morphism given by $x \mapsto nx$.

Let L be a line bundle on A. We say that L is *even* if $[-1]^*(L) \cong L$. This property is called symmetry in some literatures. We remark that for any line bundle L on A, $L \otimes [-1]^*(L)$ is even.

1.1.4. *Models.* — Let R be any ring. Let X be a scheme over R. Let S be a scheme with a dominant morphism $\mathrm{Spec}(R) \to S$. A *model over S of X* is a morphism of schemes $\mathcal{X} \to S$ equipped with an isomorphism $\mathcal{X} \times_S \mathrm{Spec}(R) \cong X$. A model is said to be *projective* (resp. *proper*, *flat*) if the morphism $\mathcal{X} \to S$ is projective (resp. proper, flat). Furthermore, let L be a line bundle on X. A *model over S of (X, L)* is a pair of a model $\mathcal{X}$ over S of X and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with an identification $\mathcal{L}|_X = L$, where $\mathcal{L}|_X$ is the pull-back of $\mathcal{L}$ by the morphism $X \to \mathcal{X}$ induced from the given identification $\mathcal{X} \times_S \mathrm{Spec}(R) \cong X$.

Let R_0 be a subring of R. We say that *X can be defined over R_0* if there exists a model over R_0 of X.

1.1.5. *Ground fields.* — Let k be an algebraically closed field. Let $\mathfrak{B}$ be a projective normal variety over k with $\dim(\mathfrak{B}) \geq 1$. When $\dim(\mathfrak{B}) \geq 2$, we fix an ample line bundle $\mathcal{H}$ on $\mathfrak{B}$, which is needed to define height in this case; see §2.1

Let K denote the function field of $\mathfrak{B}$ or a number field. We fix an algebraic closure $\overline{K}$ of K. Any finite extension of K should be taken in $\overline{K}$.

1.2. Background and history. — The geometric Bogomolov conjecture is not an isolated topic; there are many related topics and previous results. Let us begin with its background together with brief historical notes.

1.2.1. *Manin–Mumford conjecture.* — According to Lang [19], Manin and Mumford conjectured around 1963 (independently) the following. Let $\mathfrak{K}$ be an algebraically closed field. Assume that $\mathrm{ch}(\mathfrak{K}) = 0$. Let C be a smooth projective curve over $\mathfrak{K}$ of genus $g \geq 2$. Fix a divisor D on C of degree 1 and let $\jmath_D : C \to J_C$ be the embedding of C into its Jacobian J_C given by $\jmath_D(x) = x - D$. Then the set

$$\{x \in C(\mathfrak{K}) \mid \jmath_D(x) \in J_C(\mathfrak{K})_{tor}\}$$

is finite, where $J_C(\mathfrak{K})_{tor}$ is the set of torsion points of $J_C(\mathfrak{K})$. This conjecture is called the *Manin–Mumford conjecture* for curves over $\mathfrak{K}$.

1.2.2. *Bogomolov conjecture for curves.* — Inspired by the conjecture by Manin and Mumford, Bogomolov proposed in 1980 an arithmetic analogue of the conjecture (cf. [6]), which is now called the *Bogomolov conjecture for curves*. Let K be a number field or a function field. Let C be a smooth projective curve over $\overline{K}$ of genus $g \geq 2$. Let h_{NT} be the Néron–Tate height on J_C. It is known that h_{NT} is a positive semidefinite quadratic form on the additive group $J_C(\overline{K})$. For each real number ϵ, set

$$C(\epsilon) := \{x \in C(\overline{K}) \mid h_{NT}(\jmath_D(x)) \leq \epsilon\}.$$

When K is a function field, assume that C is non-isotrivial. Then the Bogomolov conjecture for C asserts that there should exist an $\epsilon > 0$ such that $C(\epsilon)$ is finite.
The Bogomolov conjecture for curves over number fields generalizes the Manin–Mumford conjecture for curves over $\overline{\mathbb{Q}}$. Indeed, since h_{NT} is a quadratic form, we have $h_{NT}(\tau) = 0$ for any $\tau \in J_C(\overline{\mathbb{Q}})_{tor}$. It follows that

$$\{x \in C(\overline{\mathbb{Q}}) \mid \jmath_D(x) \in J_C(\overline{\mathbb{Q}})_{tor}\} \subset C(\epsilon)$$

for any $\epsilon > 0$. This shows that the Manin–Mumford conjecture over $\overline{\mathbb{Q}}$ is deduced from the Bogomolov conjecture over number fields.

1.2.3. *Raynaud's theorem.* — In 1983, Raynaud proved that the Manin–Mumford conjecture holds. In fact, he proved the following.

Theorem 1.1 ([31]). — *Let $\mathfrak{K}$ be an algebraically closed field. Assume that $\mathrm{ch}(\mathfrak{K}) = 0$. Let A be an abelian variety over $\mathfrak{K}$. Let X be a closed subvariety of A of dimension 1. Suppose that X is not a torsion subvariety. Then $X \cap A(\overline{K})_{tor}$ is a finite set.*

Here, *torsion subvariety* is by definition the translate of an abelian subvariety by a torsion point. The above theorem generalizes the Manin–Mumford conjecture for curves, because a smooth projective curve of genus at least 2 embedded in its Jacobian is not a torsion subvariety.
In the same year, Raynaud generalized Theorem 1.1 to obtain the following theorem, which is often also called the Manin–Mumford conjecture.

Theorem 1.2 (**Raynaud's theorem,** [32]). — *Let $\mathfrak{K}$ be an algebraically closed field. Assume that $\mathrm{ch}(\mathfrak{K}) = 0$. Let A be an abelian variety over $\mathfrak{K}$. Let X be a closed subvariety of A. Suppose that X is not a torsion subvariety. Then $X \cap A(\overline{K})_{tor}$ is not Zariski dense in X.*

1.2.4. *Admissible paring and some results.* — While the Manin–Mumford conjecture had been established in a generalized form, the Bogomolov conjecture for curves was still open.
Let K be a number field or the function field of $\mathfrak{B}$ with $\dim(\mathfrak{B}) = 1$. In [45], Zhang defined an *admissible pairing* $(\,\cdot\,,\cdot\,)_a$ on a curve over $\overline{K}$ and introduced the admissible dualizing sheaf ω_a. Further, he proved that the Bogomolov conjecture for a curve C amounts to the positivity of the admissible self-pairing $(\omega_a, \omega_a)_a$ of ω_a on C. In that paper, this admissible pairing is described as the self-intersection of the relative dualizing sheaf of the stable model of the curve minus a positive number that is a combinatorial data arising from the dual graphs of the singular fibers of the stable model.

By using Zhang's admissible pairing, some partial affirmative answers to the Bogomolov conjecture for curves over function fields (of transcendence degree 1) has been obtained, such as [23, 24, 25, 26, 38, 39, 45]. However, the conjecture was not proved in full generality.

1.2.5. *Ullmo's theorem and Zhang's theorem.* — In 1998, Ullmo finally proved that the Bogomolov conjecture over number fields holds.

Theorem 1.3. — *Assume that K is a number field. Let C and $C(\epsilon)$ be as in §1.2.2. Then there exists an $\epsilon > 0$ such that $C(\epsilon)$ is finite.*

The proof is *not* given by showing the positivity of the admissible pairing. It is obtained by a different approach, namely, the equidistribution of small points due to Szpiro, Ullmo, and Zhang [35].

Just after Ullmo established the above theorem, Zhang proved the following generalized version of the Bogomolov conjecture over number fields. Let A be an abelian variety over $\overline{K}$ and let L be an even ample line bundle on A (cf. §1.1.3). Let $\widehat{h}_L$ be the canonical height on A associated to L; see §2. It is a semipositive definite quadratic form on $A(\overline{K})$. Let X be a closed subvariety of A. For any real number ϵ, set $X(\epsilon; L)$ to be $\{x \in X(\overline{K}) \mid \widehat{h}_L(x) \leq \epsilon\}$.

Theorem 1.4 ([47, Cor. 3]). — *Assume that K is a number field. Let A, L, and X be as above. Suppose that $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$. Then X is a torsion subvariety.*

Zhang's theorem (Theorem 1.4) generalizes Ullmo's theorem. Indeed, let C be a smooth projective curve over $\overline{K}$. With the notation in §1.2.2, consider the case where $A = J_C$ and $X = \jmath_D(C)$. Then h_{NT} is the canonical height associated to a symmetric theta divisor (cf. Remark 2.10), and thus one easily sees that Ullmo's theorem is a part of Zhang's theorem. We remark that Zhang's theorem generalizes Raynaud's theorem (Theorem 1.2) over $\overline{\mathbb{Q}}$.

The proof uses equidistribution theory, like the proof of Ullmo's theorem does. We will give an outline of the proof of Zhang in the sequel.

1.2.6. *Moriwaki's generalization and the Manin–Mumford conjecture.* — In 2000, Moriwaki generalized Zhang's theorem. Let F be a finitely generated field over $\mathbb{Q}$. Moriwaki constructed in [27] arithmetic height functions over F. There, a pair of an arithmetic variety with function field F and an arithmetic line bundle on the arithmetic variety is called a polarization of F, and the polarization is said to be big if the arithmetic line bundle is big. Once a polarization of F is fixed, Moriwaki's arithmetic height over F is defined. He proved basic properties of this height, and he constructed the canonical height on abelian varieties. Furthermore, he established the following theorem.

Theorem 1.5 ([27, Thm. B]). — *Let F be a finitely generated field over $\mathbb{Q}$. Let A be an abelian variety over $\overline{F}$ and let L be an even ample line bundle on A. Fix a big polarization on F, and let $\widehat{h}_L^{arith}$ be the canonical height on A associated to L. Let X be a closed subvariety of A. Suppose that for any $\epsilon > 0$, the set*

$$\left\{ x \in X(\overline{F}) \,\middle|\, \widehat{h}_L^{arith}(x) \leq \epsilon \right\}$$

is dense in X. Then X is a torsion subvariety.

Moriwaki's theorem (Theorem 1.5) generalizes Raynaud's theorem over any algebraically closed field of characteristic 0; we omit the detail here but will see an analogous argument in §12.

We notice that the arithmetic height arising from a big polarization, which is used in Moriwaki's theorem above, is different from the classical "geometric" heights over function fields. Therefore, the geometric Bogomolov conjecture, which will be formulated in the sequel, addresses a problem that is not the same as in Moriwaki's theorem.

1.2.7. *Gubler's theorem.* — Let K be any function field (cf. §1.1.5). In 2007, Gubler established a result over a function field which is analogous to Zhang's theorem, under the assumption that the abelian variety is totally degenerate at some place. Let A be an abelian variety over $\overline{K}$. We say that A is *totally degenerate at some place* if there exists a codimension one point v of $\mathfrak{B}$ satisfying the following condition: there exist a discrete valuation ring $R' \subset \overline{K}$ that dominates $\mathcal{O}_{\mathfrak{B},v}$ and a group scheme $\mathscr{A}^\circ \to \mathrm{Spec}(R')$ with geometric generic fiber A such that the spacial fiber $\widetilde{\mathscr{A}^\circ}$ is an algebraic torus; see also §3.2 for details on degeneracy of abelian varieties.

Theorem 1.6 ([13, Thm. 1.1]). — *Let A be an abelian variety over $\overline{K}$ and let L an even ample line bundle on A. Assume that A is totally degenerate at some place of K. Let X be a closed subvariety of A. Suppose that $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$. Then X is a torsion subvariety.*

The proof is given by a nonarchimedean analogue of the proof of Ullmo and Zhang, which is a quite important argument. We will explain it later in detail.

1.2.8. *Cinkir's theorem.* — In 2011, the geometric Bogomolov conjecture for curves was proved by Cinkir under the assumption that $\mathrm{ch}(k) = 0$ and that K is the function field of a curve, i.e. $\dim(\mathfrak{B}) = 1$; see [9, Thm. 2.13]. We remark that Cinkir's theorem is effective in the sense that he gave explicitly a positive number r such that $C(r)$ is finite when C has a semistable model over $\mathfrak{B}$.

To explain Cinkir's proof, let K be the function field of a curve and let C be a smooth projective curve over $\overline{K}$ of genus $g \geq 2$. In 2010, Zhang proved a new description of the admissible pairing of the admissible dualizing sheaf (cf. [48]). This description uses the height of the Gross–Schoen cycle. More precisely, he showed that the admissible pairing of the admissible dualizing sheaf equals the sum of the height of the Gross–Schoen cycle and the "φ-invariants" of the dual graphs of the semistable model of the curve. It is known that if $\mathrm{ch}(k) = 0$, the height of the Gross–Schoen cycle are non-negative, so that the Bogomolov conjecture amounts to the positivity of φ-invariants. In 2009, using Zhang's work, Faber proved in [10] the geometric Bogomolov conjecture for curves over $\overline{K}$ of small genus. In 2011, after contributions on the study of graph invariants by many authors, Cinkir proved in [9] that the φ-invariants for non-trivial graphs are positive. Since the case where every reduction graph is trivial, that is, the case of everywhere good reduction case had been known, this proved the geometric Bogomolov conjecture for curves over such $\overline{K}$.

However, that is not the finial answer to the conjecture. Cinkir's theorem needs the assumption that $\dim(\mathfrak{B}) = 1$ and $\mathrm{ch}(k) = 0$. The first assumption is needed because Zhang's description in [48] of the admissible pairing is obtained under this assumption. The second assumption on the characteristic is more crucial. If $\mathrm{ch}(k) = 0$, the positivity of the Gross–Schoen cycle follows from the Hodge index theorem, but if that is not the case, the Hodge index theorem, which is part of the standard conjectures, is not known. Therefore the positivity of the φ-invariants is not enough for the positivity of the admissible pairing, and thus the Bogomolov conjecture in positive characteristic cannot be deduced in the same way.

1.3. Geometric Bogomolov conjecture for abelian varieties. — It is natural to ask whether or not the same statement as Theorem 1.4 holds for any abelian variety over function fields, but in fact, it does not holds in general. For example, suppose that B is a constant abelian variety, that is, $B = \widetilde{B} \otimes_k \overline{K}$ for some abelian variety $\widetilde{B}$ over k. Let Y be a constant closed subvariety of B, that is, $Y := \widetilde{Y} \otimes_k \overline{K}$ for some closed subvariety $\widetilde{Y} \subset \widetilde{B}$. Then $\widetilde{Y}(k)$, which is naturally a subset of $Y(\overline{K})$, is dense in Y, and for any point y of this set, we have $\widehat{h}_M(y) = 0$, where M is an even ample line bundle on B (cf. Example 2.11). Furthermore, if $\phi : B \to A$ is a homomorphism of abelian varieties, then we see that $\phi(\widetilde{Y}(k))$ is a dense subset of $\phi(Y)$ and that for any $x \in \phi(\widetilde{Y}(k))$ we have $\widehat{h}_L(x) = 0$, where L is an even ample line bundle on A (cf. Remark 2.12). This suggests that an abelian variety over a function field in general has a non-torsion closed subvariety with a dense set of height 0 points.

While we cannot expect the same statements as Zhang's theorem holds over function fields in general, it is still natural to ask how we can characterize the closed subvarieties X such that $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$. In 2013, we proposed in [40] a conjecture that the "special subvarieties" should be the only such closed subvarieties.

The special subvarieties are the subvarieties which are the sum of a torsion subvariety and a closed subvariety which is the image of a constant closed subvariety of a constant abelian variety. To be precise, let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A. We say that X is *special* if there exist a torsion subvariety T of A, a constant abelian variety $B = \widetilde{B} \otimes_k \overline{K}$, a constant closed subvariety $Y = \widetilde{Y} \otimes_k \overline{K}$ of B, and a homomorphism $\phi : B \to A$ such that $X = \phi(Y) + T$. Note that in this definition of special subvarieties, we take any constant abelian variety B, but it is enough to consider the universal one among homomorphisms from constant abelian varieties to A, which is called the $\overline{K}/k$ trace; refer to §3.3 for details.

It is not difficult to see that if X is a special subvariety, then for any $\epsilon > 0$, $X(\epsilon; L)$ is dense in X. (cf. Remark 3.3). The *geometric Bogomolov conjecture for abelian varieties* asserts that the converse should also hold.

Conjecture 1.7 ([40, Conj. 0.3]). — *Let A be an abelian variety over $\overline{K}$, where K is a function field. Let L be an even ample line bundle on A. Let X be a closed subvariety of A. Suppose that for any $\epsilon > 0$, $X(\epsilon; L)$ is dense in X. Then X is a special subvariety.*

The geometric Bogomolov conjecture for abelian varieties is still open, but the study of this conjecture has been developed, and there are significant partial answers. Gubler's theorem is of course an important one. In [37, 40, 41, 42, 43], we have generalized Gubler's theorem, seeing that the conjecture holds for a large class of abelian varieties. This is the main topic of this paper, and we will explain it in detail (cf. Theorems 3.10, 11.7, and 11.9).

By virtue of the development of the study of the geometric Bogomolov conjecture for abelian varieties, we have very recently proved that the geometric Bogomolov conjecture for curves holds in full generality. More generally, we have shown the following.

Theorem 1.8 ([43, Thm. 1.3]). — *Let A be an abelian variety over $\overline{K}$ and let L be an even ample line bundle on A. Let X be a closed subvariety of A. Assume that $\dim(X) = 1$. Suppose that $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$. Then X is a special subvariety.*

The geometric Bogomolov conjecture for curves is deduced from the above theorem. Indeed, in the setting of the conjecture for curves, put $X := \jmath_D(C)$. Then $\dim(X) = 1$, and since C is non-isotrivial, X is non-special. Note that the Néron–Tate height on the Jacobian is the canonical height associated to the theta divisor. By Theorem 1.8, $C(\epsilon)$ is not dense for some $\epsilon > 0$. Thus we obtain the conjecture for curves.

Our result on the geometric Bogomolov conjecture for curves is not effective in contrast to Cinkir's theorem (when K is a function field of transcendence degree 1 over k of characteristic 0 and C has semistable reduction over K). On the other hand, we have an advantage in working over any function field of any characteristic. One benefit is that we have an application to the Manin–Mumford conjecture in positive characteristic (cf. §12).

We give a remark on the proof of Theorem 1.8. To prove this theorem, we first show that Conjecture 1.7 holds under the assumption of $\mathrm{codim}(X, A) = 1$ (cf. Theorem 11.2) by using the recent partial results on the Conjecture 1.7, and then we reduce Theorem 1.8 to this codimension 1 case. It should be remarked that our proof of the geometric Bogomolov conjecture for curves works via partial results of the conjecture for abelian varieties.

1.4. Organization. — This paper consists of twelve sections including the introduction, with an appendix. In §2, we recall the notion of canonical heights on abelian varieties, where we mainly focus on the heights over function fields. In §3, we formulate the geometric Bogomolov conjecture for abelian varieties, and we present some partial results on the conjecture. In a part of the argument, we use a nonarchimedean analogue of the proofs of Ullmo's and Zhang's theorem. Therefore, in §4, we recall the idea of Zhang's proof, and in §5, we recall the basic ideas of nonarchimedean geometry. In §6, we explain the proof of Gubler's theorem. This theorem is the first one where the nonarchimedean analogue of Zhang's proof worked well, and we will use the idea of the proof of this theorem. In §7, we describe the structure of the canonical measures. This structure theorem plays a crucial role in our argument to reduce the geometric Bogomolov conjecture to that for nowhere degenerate abelian varieties. §8 is the first main part of this paper. There, we prove that the geometric Bogomolov conjecture for any abelian variety is reduced to the conjecture for its maximal nowhere degenerate abelian subvariety. By this result, in particular, the conjecture is reduced to that for abelian varieties that is nowhere degenerate. In §9, we recall the notion of canonical heights of closed subvarieties of an abelian variety. In §10, we prove that the geometric Bogomolov conjecture for nowhere degenerate abelian varieties are reduced to the conjecture for those with trivial $\overline{K}/k$-trace. §11 is the second main part of this paper, where we give an outline of the proof of Theorem 1.8. In the last section §12, we give a remark of the Manin–Mumford conjecture in positive characteristic. In the appendix, we give a summary of some ideas in nonarchimedean geometry.

2. Canonical heights on abelian varieties

In this section, we recall basic properties of canonical heights on abelian varieties. For the theory of heights, [21, Ch. 1–6] is a basic reference; see also [7].

2.1. Heights over function fields. — Before describing the notion of canonical heights on abelian varieties, let us recall the idea of heights. The theory of heights is developed over number fields and over function fields. Because our main topics concern the *geometric* Bogomolov conjecture, we mainly focus on the heights over function fields in this subsection. At the end of this subsection, we will give a comment on the heights over number fields.

Let K be the function field of $\mathfrak{B}$ (cf. §1.1.5). We begin with the notion of height arising from a model. Let X be a projective variety over $\overline{K}$ and let L be a line bundle on X. Let K' be a finite extension of K and let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. Let $(\mathcal{X}, \mathcal{L})$ be a model of (X, L) over $\mathfrak{B}'$ satisfying the following conditions: the morphism $\pi : \mathcal{X} \to \mathfrak{B}'$ is proper; there exists an open subset $\mathfrak{U} \subset \mathfrak{B}'$ with $\mathrm{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ over which $\pi : \mathcal{X} \to \mathfrak{B}'$ is flat. When $\dim(\mathfrak{B}) = 1$, the last condition is equivalent to saying that π is flat. There always exists such a model. When we say a model in this subsection, we assume that it satisfies the above conditions.

Let $\mathcal{H}'$ be the pull-back of $\mathcal{H}$ by the finite morphism $\mathfrak{B}' \to \mathfrak{B}$. We take a point $x \in X(\overline{K})$. Let Δ_x be the closure of x in $\mathcal{X}$ and let $K'(x)$ be the function field of Δ_x. Then we set

$$h_{(\mathcal{X}, \mathcal{L})}(x) := \frac{\deg_{\mathcal{H}'} \pi_* \left(c_1(\mathcal{L}) \cdot [\Delta_x] \right)}{[K'(x) : K]}$$

where $c_1(\mathcal{L}) \cdot [\Delta_x]$ is a cycle class on $\mathcal{X}$, $\pi_* \left(c_1(\mathcal{L}) \cdot [\Delta_x] \right)$ is the pushout by π, and $\deg_{\mathcal{H}'}$ means the degree with respect to $\mathcal{H}'$. This defines a function

$$h_{(\mathcal{X}, \mathcal{L})} : X(\overline{K}) \to \mathbb{R},$$

called the *height (function) arising from a model* $(\mathcal{X}, \mathcal{L})$.

The following lemma immediately follows from the definition.

Lemma 2.1. — *Let X and Y be projective varieties over $\overline{K}$. Let $f : Y \to X$ be a morphism of varieties. Let L and L' be line bundles on X. Let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in a finite extension of K and let $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}, \mathcal{L}')$ be models over $\mathfrak{B}'$ of (X, L) and (X, L'), respectively. Let $\mathcal{Y}$ be a model over $\mathfrak{B}'$ of Y and let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism over $\mathfrak{B}'$ which extends f. Then we have the following:*

(1) $h_{(\mathcal{X}, \mathcal{L} \otimes \mathcal{L}')} = h_{(\mathcal{X}, \mathcal{L})} + h_{(\mathcal{X}, \mathcal{L}')}$;

(2) $f^* h_{(\mathcal{X}, \mathcal{L})} = h_{(\mathcal{Y}, \varphi^* \mathcal{L})}$;

(3) *if $\mathcal{L}$ is relatively ample, then there exists a constant $C \in \mathbb{R}$ such that $h_{(\mathcal{X}, \mathcal{L})} \geq C$.*

Indeed, (1) follows from the linearity of the intersection product with respect to the line bundle, and (2) follows from the projection formula. To see (3), first note that for any line bundle $\mathcal{M}$ on $\mathfrak{B}'$, we have

$$(2.1) \qquad h_{(\mathcal{X}, \pi^*(\mathcal{M}))}(x) = \frac{\deg_{\mathcal{H}'} \pi_* \left(c_1(\pi^*(\mathcal{M})) \cdot [\Delta_x] \right)}{[K'(x) : K]} = \frac{\deg_{\mathcal{H}'}(\mathcal{M})}{[K' : K]}.$$

Suppose that $\mathcal{L}$ is relatively ample. Then there exists a line bundle $\mathcal{M}$ on $\mathfrak{B}'$ such that $\mathcal{L} \otimes \pi^*(\mathcal{M})$ is semiample on $\mathcal{X}$, that is, $(\mathcal{L} \otimes \pi^*(\mathcal{M}))^{\otimes a}$ is basepoint free for some positive integer a. Since a semiample line bundle is nef, we obtain $h_{(\mathcal{X}, \mathcal{L} \otimes \pi^*(\mathcal{M}))} \geq 0$. It follows that

$$h_{(\mathcal{X}, \mathcal{L})} = h_{(\mathcal{X}, \mathcal{L} \otimes \pi^*(\mathcal{M}))} + h_{(\mathcal{X}, \pi^*(\mathcal{M}^{\otimes -1}))} \geq \frac{\deg_{\mathcal{H}'}(\mathcal{M}^{\otimes -1})}{[K' : K]},$$

where we use (2.1). Thus we have (3).

For a pair (X, L) of a projective variety X over K and a line bundle L on X, one would hope, naively, to define the height function h_L on X associated to L by using a model. In fact, such a model is not unique, so that the function arising from a model depends on the choice of a model. Indeed, in the argument to show Lemma 2.1 (3), both $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}, \mathcal{L} \otimes \pi^*(\mathcal{M}))$ are models of (X, L), and $h_{(\mathcal{X}, \mathcal{L})} \neq h_{(\mathcal{X}, \mathcal{L} \otimes \pi^*(\mathcal{M}))}$ in general.

However, the following lemma indicates that the notion of heights with respect to line bundles can be defined as functions up to bounded functions:

Lemma 2.2. — *Let X be a projective variety over $\overline{K}$ and let L be line a bundle on X. Let $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$ be normal varieties over k finite over $\mathfrak{B}$, and let $(\mathcal{X}_1, \mathcal{L}_1)$ and $(\mathcal{X}_2, \mathcal{L}_2)$ be proper flat models over $\mathfrak{B}'_1$ and $\mathfrak{B}'_2$, respectively. Then $h_{(\mathcal{X}_1, \mathcal{L}_1)} - h_{(\mathcal{X}_2, \mathcal{L}_2)}$ is a bounded function on $X(\overline{K})$.*

Taking into account Lemma 2.2, we define a height associated to a line bundle L as a representative of an equivalence class of functions on $X(\overline{K})$ modulo bounded functions. To be precise, let $h_L : X(\overline{K}) \to \mathbb{R}$ be a function. We call h_L a *height (function) associated to L* if there exist a normal variety $\mathfrak{B}'$ over k finite over $\mathfrak{B}$ and a proper model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\mathfrak{B}$ such that $\mathcal{X} \to \mathfrak{B}'$ is flat over an open subset whose complement has codimension in $\mathfrak{B}'$ at least 2 and such that $h_L - h_{(\mathcal{X}, \mathcal{L})}$ is a bounded function over $X(\overline{K})$. Note that by Lemma 2.2, this condition is equivalent to say that $h_L - h_{(\mathcal{X}, \mathcal{L})}$ is bounded for any normal variety $\mathfrak{B}'$ over k finite over $\mathfrak{B}$ and any proper model $(\mathcal{X}, \mathcal{L})$ over $\mathfrak{B}'$ as above. In particular, the height arising from a model is a height function.

In the following, when we write h_L, it means a height function associated to L. By definition, it is not a uniquely determined function from L, but it is unique up to a bounded function. For any real-valued functions h_1 and h_2 defined over the same set, let $h_1 \sim h_2$ mean that $h_1 - h_2$ is a bounded function. The following proposition is a consequence of Lemma 2.1.

Proposition 2.3. — *Let X and Y be projective varieties over $\overline{K}$, and let L and L' be line bundles on X.*

(1) *We have $h_{L \otimes L'} \sim h_L + h_{L'}$.*

(2) *Let $f : Y \to X$ be a morphism of varieties over $\overline{K}$. Then $h_{f^*L} \sim f^*h_L$.*

(3) *Suppose that L is ample. Then h_L is positive up to a bounded function, that is, there exists a real constant C such that $h_L \geq C$.*

Proof. — Assertions (1) and (2) follows from Lemma 2.1. To see (3), suppose that L is ample. There exist a finite extension K', a projective variety X' over K', and a line bundle L' on X' such that $X = X' \otimes_{K'} K$ and $L = L' \otimes_{K'} K$. Since L is ample, so is L', and hence there exists a positive integer N such that $(L')^{\otimes N}$ is very ample. Let $\jmath : X' \hookrightarrow \mathbb{P}_{K'}^{N'}$ be the closed

embedding associated to global sections of $(L')^{\otimes N}$. Let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. Noting that $\mathbb{P}_{K'}^{N'}$ is the generic fiber of the canonical projection $\mathbb{P}^{N'} \times \mathfrak{B}' \to \mathfrak{B}'$, we take the closure $\mathcal{X}$ of $\jmath(X')$ in $\mathbb{P}^{N'} \times \mathfrak{B}'$. Then $\mathcal{X}$ is a proper model of X over $\mathfrak{B}'$, and it is flat over any point of codimension 1. Further, let $\mathcal{L}$ be the restriction of the tautological line bundle on $\mathbb{P}^{N'} \times \mathfrak{B}'$ to $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{L})$ is a model of (X, L), and $\mathcal{L}$ is relatively ample. It follows from Lemma 2.1(3) and Lemma 2.2 that h_L is bounded below. Thus (3) holds. □

We remark that there is another approach to the height theory, which begin with the Weil height; see [21, Ch. 3] for the detail. The notion of heights we describe here and that in [21, Ch. 3] are same (cf. [21, Ch. 4]).

Also over number fields, one can define the notion of heights. In introducing the notion of heights over number fields, one can define the heights arising from models by using the arithmetic intersection theory; see [18] for example. Another way, which may be more standard way, is to use the Weil heights; see [21, Ch. 3] for details. We just remark that the heights over number fields also satisfy the properties of Proposition 2.3. Further, the notion of canonical height, which is given in the next subsection, can be well defined not only over function fields but also over number fields.

2.2. Canonical heights on abelian varieties.
— For a given line bundle, a height function associated to the line bundle are only determined up to a bounded function. However, in some cases, we can make a canonical choice of height functions among them. In fact, we have a notion of canonical height over abelian varieties, as we are going to explain.

In this subsection, let K be a function field or a number field. Let A be an abelian variety over $\overline{K}$ and let L be an even line bundle on A. Fix an integer $n > 1$. Since we have $[n]^*(L) \cong L^{\otimes n^2}$ by the theorem of cube (cf. [28, §6, Cor. 3]), Proposition 2.3 gives us

$$(2.2) \qquad\qquad\qquad [n]^* h_L \sim n^2 h_L,$$

where $\sim$ means that they are equal up to bounded functions, namely, an equality modulo bounded functions. The following proposition shows that there exists a unique height associated to L with which (2.2) is actually an equality:

Proposition 2.4. — *Fix an integer $n > 1$. Then there exists a unique height function h_L such that $[n]^* h_L = n^2 h_L$.*

We can construct such an h_L as in the above proposition by limiting process. Fix an integer $n \geq 2$. Let h_1 be any height function associated to L. We define a sequence $(h_m)_{m \in \mathbb{N}}$ of function on $A(\overline{K})$ inductively by $h_{m+1} := \frac{1}{n^2}[n]^* h_m$. Then each h_m is a height function on A associated to L. Furthermore, one can show that this sequence converges to a height function h_L associated to L. By this construction, one also checks $[n]^* h_L = n^2 h_L$.

The uniqueness is shown as follows. Suppose that h_L' is a height function associated to L such that $[n]^* h_L' = n^2 h_L'$. Set $f := h_L - h_L'$. Since h_L and h_L' are height functions associated to L, f is a bounded function. We prove $f = 0$ by contradiction. Suppose that there exists an $x \in X(\overline{K})$ such that $f(x) \neq 0$. Then

$$f(nx) = [n]^* h_L(x) - [n]^* h_L'(x) = n^2 h_L(x) - n^2 h_L'(x) = n^2 f(x).$$

It follows inductively that for any positive integer m, we have $f(n^m x) = n^{2m} f(x)$. Taking $m \to \infty$, we see that this equality indicates that f is not bounded, and that is contradiction. More strongly than Proposition 2.4, the following theorem holds.

Theorem 2.5. — *There exists a unique bilinear form $b_L : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$ such that the function $\widehat{h}_L : A(\overline{K}) \to \mathbb{R}$ defined by $\widehat{h}_L(x) = \frac{1}{2} b_L(x, x)$ is a height function associated to L.*

The above theorem shows that there exists a height function associated to L that is a quadratic form. Since $[n]^* \widehat{h}_L = n^2 \widehat{h}_L$ for any $n \in \mathbb{Z}$, the uniqueness in Theorem 2.5 follows from the uniqueness assertion in Proposition 2.4. Note that by the uniqueness, Theorem 2.5 indicates that the height function in Proposition 2.4 is actually a quadratic form and the heights in Proposition 2.4 and Theorem 2.5 are the same. For the proof of Theorem 2.5, we refer to [21, Ch. 5].

The height $\widehat{h}_L$ in Theorem 2.5 is called the *canonical height associated to L*. For an even line bundle L on an abelian variety, let $\widehat{h}_L$ always denote the canonical height associated to L.

Proposition 2.6. — *For the canonical heights associated to even line bundles on abelian varieties, the following hold.*

(1) *Let L_1 and L_2 be even line bundles on an abelian variety. Then $\widehat{h}_{L_1 \otimes L_2} = \widehat{h}_{L_1} + \widehat{h}_{L_2}$.*

(2) *Let $\phi : B \to A$ be a homomorphism of abelian varieties and let L be an even line bundle on A. Then $\widehat{h}_{\phi^*(L)} = \phi^* \widehat{h}_L$.*

(3) *Let L be an even line bundle on an abelian variety. Suppose that L is ample. Then $\widehat{h}_L \geq 0$.*

Proof. — By Proposition 2.3, all the assertions hold up to bounded functions. By the uniqueness assertion of Theorem 2.5, we see that (1) and (2) are really equalities, and (3) also follows from the fact that the canonical height is a quadratic form (cf. Theorem 2.5). $\square$

Next, we recall basic facts on points of height 0.

Remark 2.7. — Let L be an even line bundle on an abelian variety A. Since $\widehat{h}_L$ is a quadratic form, we have $\widehat{h}_L(a) = 0$ for any torsion point a of $A(\overline{K})$.

Lemma 2.8. — *Let L_1 and L_2 be even ample line bundles on A. Let $a \in A(\overline{K})$. Then $\widehat{h}_{L_1}(a) = 0$ if and only if $\widehat{h}_{L_2}(a) = 0$.*

Proof. — It suffices to show that $\widehat{h}_{L_1}(a) = 0$ implies $\widehat{h}_{L_2}(a) = 0$. Suppose that $\widehat{h}_{L_1}(a) = 0$. Since L_1 is ample, there exists a positive integer N such that $L_2^{\otimes -1} \otimes L_1^{\otimes N}$ is ample. By Proposition 2.6 (3), we have $\widehat{h}_{L_2^{\otimes -1} \otimes L_1^{\otimes N}} \geq 0$, and by Proposition 2.6 (1), we obtain $\widehat{h}_{L_2} \leq N \widehat{h}_{L_1}$. It follows that $\widehat{h}_{L_2}(a) \leq N \widehat{h}_{L_1}(a) = 0$. Since L_2 is ample, $\widehat{h}_{L_2}(a) \geq 0$. This concludes $\widehat{h}_{L_2}(a) = 0$. $\square$

We say that a point $a \in A(\overline{K})$ *has height 0* if $\widehat{h}_L(a) = 0$, where L is an even ample line bundle on A. This is well defined by Lemma 2.8.

Lemma 2.9. — *Let $\phi : B \to A$ be a homomorphism of abelian varieties over $\overline{K}$. Let $b \in B(\overline{K})$ be a point of height 0. Then $\phi(b)$ has height 0.*

Proof. — Let L be an even ample line bundle on A and let M be an even ample line bundle on B. Replacing M with $M^{\otimes a}$ for some positive integer a if necessary, we may and do assume that $M \otimes \phi^*(L^{\otimes -1})$ is ample as well as even. By Proposition 2.6, we have

$$0 \leq \widehat{h}_L(\phi(b)) = \widehat{h}_{\phi^*(L)}(b) \leq \widehat{h}_M(b) = 0,$$

and thus $\phi(a)$ has height 0. $\square$

We end with a remark on the Néron–Tate height on the Jacobian varieties.

Remark 2.10. — Let $J_C^{(g-1)}$ be the Jacobian variety of degree $g-1$ divisor classes on C. Let Θ be the theta divisor on $J_C^{(g-1)}$, that is, the image of $C^{g-1} \to J_C^{(g-1)}$ given by $(p_1, \ldots, p_{g-1}) \mapsto p_1 + \cdots + p_{g-1}$. Let c_0 be a divisor on C of degree 1 such that $(2g-2)c_0$ is a canonical divisor on C. Let $\lambda : J_C \to J_C^{(g-1)}$ be the isomorphism defined by $a \mapsto a + (g-1)c_0$. Let θ be the pullback of Θ by λ. It is known that θ is ample. Since $(2g-2)c_0$ is a canonical divisor on C, the line bundle $L := \mathcal{O}_{J_C}(\theta)$ is even. This θ is called a symmetric theta divisor. Then the Néron–Tate height h_{NT} is defined to be $\widehat{h}_L$. We refer to [21, Ch. 5, §5] for details.

2.3. Canonical height on the generic fiber of abelian scheme. — In this subsection, we assume that K is a function field. Let A be an abelian variety over $\overline{K}$. Assume that there exist a finite covering $\mathfrak{B}' \to \mathfrak{B}$ with $\mathfrak{B}'$ normal and an abelian scheme $\pi : \mathcal{A} \to \mathfrak{B}'$ whose geometric generic fiber equals A. Under this assumption, we describe the canonical height in terms of intersection products on $\mathcal{A}$. Using that description, we furthermore see that in general, a constant abelian variety has a lot of points which are non-torsion but of canonical height 0. The description of the canonical heights given here will be generalized in §10.1.

Let L be an even ample line bundle on A. Then replacing $\mathfrak{B}'$ by a further finite covering, we may assume that there exists a line bundle $\mathcal{L}$ on $\mathcal{A}$ whose restriction to A coincides with L. By tensoring the pull-back by π of a line bundle on $\mathfrak{B}'$, we may take $\mathcal{L}$ such that $0_\pi^*(\mathcal{L})$ is trivial, where 0_π is the zero-section of the abelian scheme $\pi : \mathcal{A} \to \mathfrak{B}'$.

Then we have $[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2}$. Indeed, since $\mathcal{L}|_A = L$ is even, there exists a line bundle on $\mathcal{N}$ on $\mathfrak{B}'$ such that $[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2} \otimes \pi^*(\mathcal{N})$. Note that $0_\pi^*(\pi^*(\mathcal{N})) = \mathcal{N}$, and we only have to show that $0_\pi^*(\pi^*(\mathcal{N})) \cong \mathcal{O}_{\mathfrak{B}'}$. Since $0_\pi^*(\mathcal{L}) \cong \mathcal{O}_{\mathfrak{B}'}$, we have $0_\pi^*(\mathcal{L}^{\otimes n^2}) \cong \mathcal{O}_{\mathfrak{B}'}$. On the other hand, we see

$$0_\pi^*([n]^*(\mathcal{L})) = ([n] \circ 0_\pi)^*(\mathcal{L}) = 0_\pi^*(\mathcal{L}) \cong \mathcal{O}_{\mathfrak{B}'}.$$

Since $[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes n^2} \otimes \pi^*(\mathcal{N})$, we obtain $0_\pi^*(\pi^*(\mathcal{N})) \cong \mathcal{O}_{\mathfrak{B}'}$, as required.

Let $\mathcal{H}'$ be the pull-back of $\mathcal{H}$ by the morphism $\mathfrak{B}' \to \mathfrak{B}$. We prove that for any $x \in A(\overline{K})$, we have

$$(2.3) \qquad \widehat{h}_L(x) = \frac{\deg_{\mathcal{H}'}(\pi_*(c_1(\mathcal{L}) \cdot [\Delta_x]))}{[K'(x) : K]}$$

where Δ_x is the closure of x in $\mathcal{A}$, $[\Delta_x]$ is the cycle that gives Δ_x, and $K'(x)$ is the function field of Δ_x. Note that the right-hand side in (2.3) gives a height function associated to L. By Proposition 2.4. and Theorem 2.5, we only have to show that

$$(2.4) \qquad \frac{\deg_{\mathcal{H}'}(\pi_*(c_1(\mathcal{L}) \cdot [\Delta_{nx}]))}{[K'(nx) : K]} = n^2 \frac{\deg_{\mathcal{H}'}(\pi_*(c_1(\mathcal{L}) \cdot [\Delta_x]))}{[K'(x) : K]}.$$

Since $[n]_*([\Delta_x]) = [K'(x) : K'(nx)][\Delta_{nx}]$, it follows from the projection formula that as cycle classes,

$$[K'(x) : K'(nx)]c_1(\mathcal{L}) \cdot [\Delta_{nx}] = c_1([n]^*(\mathcal{L})) \cdot [\Delta_x].$$

Since

$$c_1([n]^*(\mathcal{L})) \cdot [\Delta_x] = c_1(\mathcal{L}^{\otimes n^2}) \cdot [\Delta_x] = n^2(c_1(\mathcal{L}) \cdot [\Delta_x]),$$

it follows that $[K'(x) : K'(nx)]c_1(\mathcal{L}) \cdot [\Delta_{nx}] = n^2(c_1(\mathcal{L}) \cdot [\Delta_x])$. Thus we obtain (2.4).

2.4. Height 0 points. — As we mentioned in Remark 2.7, torsion points have canonical height 0. Over number fields, it is classically known that the converse also holds; the torsion points are the only points with canonical height 0. Over function fields, however, there are height 0 points other than torsion points, in general. A typical example of an abelian variety which can have non-torsion height 0 points is a constant abelian variety. A *constant abelian variety* is an abelian variety B over $\overline{K}$ endowed with an identification $B = \widetilde{B} \otimes_k \overline{K}$ for some abelian variety $\widetilde{B}$ over k. Note that for a constant abelian variety $B = \widetilde{B} \otimes_k \overline{K}$, we regard $\widetilde{B}(k) \subset B(\overline{K})$ naturally.

Example 2.11. — Let $B := \widetilde{B} \otimes_k \overline{K}$ be a constant abelian variety. Note $\widetilde{B}(k) \subset B(\overline{K})$. Then any point $b \in \widetilde{B}(k)$ has height 0. To see that, we take an even ample line bundle $\widetilde{M}$ on $\widetilde{B}$ and set $M := \widetilde{M} \otimes_k \overline{K}$. Let $\pi : \widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \mathfrak{B}$ and $\mathrm{pr}_{\widetilde{B}} : \widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \widetilde{B}$ be the projections. Then the pair $(\widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B}, \mathrm{pr}_{\widetilde{B}}^*(\widetilde{M}))$ is a proper flat model of (B, M) such that $\pi : \widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B} \to \mathfrak{B}$ is an abelian scheme. (We call such a model a *standard model*.) For any $b \in \widetilde{B}(k)$, the closure Δ_b of b (as a point of $B(\overline{K})$) in $\widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B}$ equals $\mathrm{pr}_{\widetilde{B}}^{-1}(b)$. Note that $c_1(\mathrm{pr}_{\widetilde{B}}^*(\widetilde{M})) \cdot [\mathrm{pr}_{\widetilde{B}}^{-1}(b)] = 0$ as a cycle class on $\widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B}$. By (2.3), we then obtain

$$\widehat{h}_M(b) = \deg_{\mathcal{H}} \pi_* \left(c_1(\mathrm{pr}_{\widetilde{B}}^*(\widetilde{M})) \cdot [\mathrm{pr}_{\widetilde{B}}^{-1}(b)] \right) = 0,$$

as required.

Remark 2.12. — Let $B = \widetilde{B} \otimes_k \overline{K}$ be as in the above example. Let A be an abelian variety over $\overline{K}$ and let $\phi : B \to A$ be a homomorphism. Then any point of $\phi(B(k)) \subset A(\overline{K})$ has height 0 by Lemma 2.9.

To describe the points of height 0 over function fields, we recall the notion of $\overline{K}/k$-trace of an abelian variety. It will also be used to give a definition of special subvariety.

A $\overline{K}/k$-trace is universal among the homomorphisms from a constant abelian variety to the abelian variety. To be precise, let A be an abelian variety over $\overline{K}$. A $\overline{K}/k$-*trace* of A is a pair $\left(\widetilde{A}^{\overline{K}/k}, \mathrm{Tr}_A \right)$ of an abelian variety over k and a homomorphism $\mathrm{Tr}_A : \widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \to A$ having the following universal property: for each constant abelian variety $B = \widetilde{B} \otimes_k \overline{K}$ be and each homomorphism $\phi : B \to A$, there exists a unique homomorphism $\phi^t : \widetilde{B} \to \widetilde{A}^{\overline{K}/k}$ such that $\mathrm{Tr}_A \circ (\phi^t \otimes_k \overline{K}) = \phi$. It is known that there exists a unique $\overline{K}/k$-trace of A, the uniqueness following from the universal property. Further, it is known that the $\overline{K}/k$-trace homomorphism Tr_A is finite and purely inseparable. We refer to [20, Ch. VIII, §3] for details.

Proposition 2.13 ([21, Thm. 5.4 in Ch. 6]). — *With the above notation, we regard*

$$\widetilde{A}^{\overline{K}/k}(k) \subset \left(\widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \right)(\overline{K}).$$

Then we have

$$\left\{ a \in A(\overline{K}) \,\middle|\, \widehat{h}_L(a) = 0 \right\} = \mathrm{Tr}_A\left(\widetilde{A}^{\overline{K}/k}(k) \right) + A(\overline{K})_{tor}.$$

Indeed, The inclusion $\supset$ is shown as follows. We take an even ample line bundle L on A. Since $\widehat{h}_L$ is a quadratic form and since the torsion points have height 0, we only have to show that for $a \in \mathrm{Tr}_A\left(\widetilde{A}^{\overline{K}/k}(k) \right)$, we have $\widehat{h}_L(a) = 0$; in fact it is Remark 2.12. For the proof of the other inclusion, we refer to [21, Ch. 6].

2.5. Density of small points. — In this section, let K be a function field or a number field. Let A be an abelian variety over $\overline{K}$. We introduce the notion of density of small points and give basic facts on this notion.

The notion of density of small points can be defined due to the following lemma.

Lemma 2.14 ([40, proof of Lem. 2.1]). — *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety. Let L_1 and L_2 be even ample line bundles on A. Then the following are equivalent:*

(1) *$X(\epsilon_1; L_1)$ is dense in X for any $\epsilon_1 > 0$;*

(2) *$X(\epsilon_2; L_2)$ is dense in X for any $\epsilon_2 > 0$.*

Proof. — The proof is given by an argument similar to the proof of Lemma 2.8, so that we omit the detail. $\square$

We say that X *has dense small points* if for any $\epsilon > 0$

$$X(\epsilon; L) := \left\{ x \in X(\overline{K}) \,\middle|\, \widehat{h}_L(x) \leq \epsilon \right\}$$

is dense in X. This notion does not depend on the choice of L by Lemma 2.14. If X has a dense subset of points of height 0, then it has dense small points.

Using the terminology of density of small points, Zhang's theorem is restated as follows. Let K be a number field. Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Suppose that X has dense small points. Then X is a torsion subvariety.

We put together some lemmas concerning the density of small points. The following lemma indicates that the image of a closed subvariety with dense small points by a homomorphism has the same property.

Lemma 2.15 ([40, Lem. 2.1]). — *Let $\phi : B \to A$ be a homomorphism of abelian varieties over $\overline{K}$. Let Y be a closed subvariety of B. If Y has dense small points, then $\phi(Y)$ has dense small points.*

Proof. — Let L be an even ample line bundle on A and let M be an even ample line bundle on B. Since M is ample, $M^{\otimes m} \otimes \phi^*(L)^{\otimes -1}$ is ample for some $m \in \mathbb{N}$. By Proposition 2.6(3), we have $\widehat{h}_{M^{\otimes m} \otimes \phi^*(L)^{\otimes -1}} \geq 0$. Further, by Proposition 2.6(1) and (2), we obtain $\widehat{h}_{M^{\otimes m}} \geq \phi^* \widehat{h}_L$. Put $X := \phi(Y)$. The last inequality gives us $\phi(Y(\epsilon; M^{\otimes m})) \subset X(\epsilon; L)$. This shows that if Y has dense small points, then so does X. $\square$

Lemma 2.16. — *Let A_1 and A_2 be abelian varieties over $\overline{K}$ and let X_1 and X_2 be closed subvarieties of A_1 and A_2, respectively. Suppose that X_1 and X_2 have dense small points. Then $X_1 \times X_2$ has dense small points.*

Proof. — This lemma follows from Proposition 2.6 (1). See also [40, Lem. 2.4]. $\qquad\square$

In the argument later, we will use the equidistribution theorem of small points. There, we will use a small generic net of points. Let X be a closed subvariety of A. An element of $(x_i)_{i \in I} \in X(\overline{K})^I$, where I is a directed set, is called a *net* on $X(\overline{K})$. We say that $(x_i)_{i \in I}$ is *generic* if for any proper closed subset Y of X, there exists an $i_0 \in I$ such that $x_i \notin Y$ for any $i \geq i_0$. We say that $(x_i)_{i \in I}$ is *small* if $\lim_i \widehat{h}_L(x_i) = 0$, where L is an even ample line bundle on A. The notion of small does not depend on the choice of L.

We will later use the following lemma, which asserts that there exists a small generic net on a closed subvariety if the closed subvariety has dense small points.

Lemma 2.17. — *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Suppose that X has dense small points. Then there exists a small generic net $(x_i)_{i \in I}$ on $X(\overline{K})$.*

Proof. — If $\dim(X) = 0$, then the assertion is trivial. Therefore we may assume that $\dim(X) > 0$. Let $\mathfrak{S}$ be the set of closed subsets Z of X such that each irreducible component of Z has codimension 1 in X. Let I be an index set of $\mathfrak{S}$, which means there exists a bijective map from I to $\mathfrak{S}$. For each $i \in I$, let Z_i denote the closed subset corresponding to i. We put a partial order on I in such a way that for $i_1, i_2 \in I$, $i_1 \leq i_2$ if and only if $Z_{i_1} \subset Z_{i_2}$. Then I is a directed set with respect to this order.

We construct a net which will be checked to be generic and small. We take an even ample line bundle L on A. For any $i \in I$, let n_i denote the number of irreducible components of Z_i. Since X has dense small points, $X(1/n_i; L)$ is dense in X. It follows that for each $i \in I$ there exists a point $x_i \in X(1/n_i; L) \setminus Z_i$. By the axiom of choice, this constructs a net $(x_i)_{i \in I}$ on $X(\overline{K})$.

We prove that $(x_i)_{i \in I}$ is generic and small. It is generic by construction. Indeed, if we take any closed subset Y of X and an $i_0 \in I$ with $Y \subset Z_{i_0}$, then for any $i \geq i_0$, $x_i \notin Z_i \supset Z_{i_0} \supset Y$, which shows that $(x_i)_{i \in I}$ is generic. To see that it is small, we take any $\epsilon > 0$. There exists $i_0 \in I$ such that $n_{i_0} > \epsilon^{-1}$. For any $i \geq i_0$, we have $Z_i \supset Z_{i_0}$. Since any irreducible component of Z_i and Z_{i_0} has codimension 1 in X, it follows that $n_i \geq n_{i_0}$. Thus for any $i \geq i_0$, we have

$$0 \leq \widehat{h}_L(x_i) \leq 1/n_i \leq 1/n_{i_0} < \epsilon,$$

which shows that $(x_i)_{i \in I}$ is small. $\qquad\square$

3. Geometric Bogomolov conjecture

Throughout this section, let K be a function field, that is, K is the function field of a normal projective variety $\mathfrak{B}$ over a fixed algebraically closed field k (cf. §1.1.5).

3.1. Place of $\overline{K}$. — We define the set $M_{\overline{K}}$ of $\overline{K}$. For a finite extension K' of K, let $\mathfrak{B}'$ denote the normalization of $\mathfrak{B}$ in K. Let $M_{K'}$ be the set of points of $\mathfrak{B}'$ of codimension 1. We call each $v \in M_{K'}$ a *place* of K'. (Note that the notion of place of K' depends not only on K' but also on $\mathfrak{B}$ unless $\dim(\mathfrak{B}) = 1$.) If K'' is a finite extension of K', then it is well known that there exists a natural surjective map $M_{K''} \to M_{K'}$, and thus we have an inverse system $(M_{K'})_{K'}$, where K' runs through the finite extensions of K' in $\overline{K}$. We define $M_{\overline{K}}$ to be $\varprojlim M_{K'}$. We call each element of $M_{\overline{K}}$ a *place of $\overline{K}$.*

Each place $v \in M_{\overline{K}}$ gives a unique equivalence class of nonarchimedean absolute values on $\overline{K}$ such that if $v_{K'}$ denote the natural projection of v to $M_{K'}$, then restriction of the value to K' is equivalent to the value on K' given by the discrete valuation ring $\mathcal{O}_{\mathfrak{B}',v_{K'}}$. Let $\overline{K}_v$ denote the completion of $\overline{K}$ with respect to this value. Let $\overline{K}_v^{\circ}$ denote the ring of integer of $\overline{K}_v$. This has residue field k.

For an algebraic variety X over $\overline{K}$, we write $X_v := X \otimes_{\overline{K}} \overline{K}_v$.

3.2. Degeneration of abelian varieties. — We recall the notion of degeneracy of abelian varieties. Let A be an abelian variety over $\overline{K}$. We take a $v \in M_{\overline{K}}$. Then by the semistable reduction theorem, there exists a unique semiabelian scheme $\mathscr{A}^{\circ} \to \mathrm{Spec}(\overline{K}_v^{\circ})$ whose generic fiber equals A_v. (Note that A can be defined over the quotient field of some discrete valuation ring.) Let $\widetilde{\mathscr{A}^{\circ}}$ be the special fiber. Since it is a semiabelian variety, there exist a nonnegative integer r and an exact sequence

$$(3.1) \qquad\qquad 1 \to (\mathbb{G}_m^r)_k \to \widetilde{\mathscr{A}^{\circ}} \to \widetilde{B} \to 0$$

of algebraic group over k, where $(\mathbb{G}_m^r)_k$ is the algebraic torus over k of dimension r and $\widetilde{B}$ is an abelian variety over k. We call $\widetilde{B}$ the abelian part of the reduction of A_v. Put $b(A_v) := \dim(\widetilde{B})$, which is well defined from A_v.

We say that A is *degenerate at v* if $b(A_v) < \dim(A)$. We say A is *non-degenerate at v* if it is not degenerate at v, i.e. $b(A_v) = \dim(A)$. Further, we say that A is *totally degenerate at v* if $b(A_v) = 0$. Notice that in this terminology, if $\dim(A) = 0$, then A is non-degenerate and totally degenerate at any place.

We restate Gubler's theorem here, as we have just defined the notion of total degeneracy.

Theorem 3.1 **(Gubler's theorem,** [13, Thm. 1.1]**).** — *Assume that A is totally degenerate at some place, that is, there exists $v \in M_{\overline{K}}$ such that A_v is totally degenerate. Let X be a closed subvariety of A. Suppose that X has dense small points. Then X is a torsion subvariety.*

3.3. Statement of the conjecture. — For a general abelian variety A over $\overline{K}$, the statement of Gubler's theorem does not hold: let $(\widetilde{A}^{\overline{K}/k}, \mathrm{Tr}_A)$ be the $\overline{K}/k$-trace of A; then the image of a constant subvariety of $\widetilde{A}^{\overline{K}/k}$ by Tr_A has dense small points, and in general it is not a torsion subvariety. This fact indicates that if we wish to characterize the closed subvarieties with dense small points, we have to define a suitable counterpart of torsion subvariety. The candidate is the class of special subvarieties, which we are going to define.

Let X be a closed subvariety of A. We say that X is *special* if there exist an abelian subvariety G of A, a torsion point $\tau \in A(\overline{K})_{tor}$, and a closed subvariety $\widetilde{Y} \subset \widetilde{A}^{\overline{K}/k}$ such that

$$X = \mathrm{Tr}_A\left(\widetilde{Y} \otimes_k \overline{K}\right) + G + \tau.$$

This definition of special subvariety coincides with the definition in §1.3 by the universal property of the $\overline{K}/k$-trace.

One shows that if A is totally degenerate at some place, then it has trivial $\overline{K}/k$-trace. In the setting of Gubler's theorem, therefore, the notion of special subvarieties is the same as that of torsion subvarieties.

A point $x \in A(\overline{K})$ is said to be *special* if $\{x\}$ is a special subvariety. By definition, x is a special point if and only if

$$x \in \mathrm{Tr}_A \left(\widetilde{A^{\overline{K}/k}}(k) \right) + A(\overline{K})_{tor} \, ,$$

where we regard $\widetilde{A^{\overline{K}/k}}(k) \subset \widetilde{A^{\overline{K}/k}}(\overline{K})$ naturally.

Remark 3.2. — By Proposition 2.13, it follows that a point is special if and only if it has height 0.

Remark 3.3. — Any special subvariety has dense small points. Indeed, with the above expression of X,

$$\mathrm{Tr}_A \left(\widetilde{Y}(k) \right) + G(\overline{K})_{tor} + \tau$$

is a dense subset of X, and by Proposition 2.13, each point of this subset has height 0.

Now, we state the geometric Bogomolov conjecture for abelian varieties, which has been proposed as [40, Conj. 2.9] and asserts that the converse of Remark 3.3 should hold.

Conjecture 3.4 (Geometric Bogomolov conjecture for abelian varieties). — *Let A be an abelian variety over $\overline{K}$, where K is a function field. Let X be a closed subvariety of A. Suppose that X has dense small points. Then X is a special subvariety.*

In view of Conjecture 3.4, Gubler's theorem is a partial answer to the conjecture because a spacial subvariety is a torsion subvariety in the totally degenerate setting.

Remark 3.5. — We keep the setting of Conjecture 3.4.

(1) Assume that $\dim(X) = 0$. Suppose that X has dense small points. Then X is special. Indeed, since $\dim(X) = 0$ and X has dense small points, we can write $X = \{x\}$ with a point x of height 0. Then X is special by Remark 3.2.

(2) Assume that $\dim(A) = 1$. Then the geometric Bogomolov conjecture holds for A, because any proper closed subvariety of A has dimension 0.

3.4. Partial answers to the conjecture. — Although the geometric Bogomolov conjecture is still open, there are some important partial answers. In [40], where Conjecture 3.4 is proposed, we prove the following theorem.

Theorem 3.6 ([40, Thm. 0.5]). — *Let A be an abelian variety over $\overline{K}$. Assume that $b(A_v) \leq 1$ for some $v \in M_{\overline{K}}$. Suppose that X has dense small points. Then X is a special subvariety.*

The above theorem generalizes Gubler's theorem because in his setting, we have $b(A_v) = 0$ for some $v \in M_{\overline{K}}$.

In [41], we generalize Theorem 3.6, where we use the maximal nowhere degenerate abelian subvariety. An abelian variety over $\overline{K}$ is said to be *nowhere degenerate* if it is non-degenerate at any place of $\overline{K}$. For any abelian variety A over $\overline{K}$, there exists a unique maximal nowhere degenerate abelian subvariety $\mathfrak{m}$ of A; $\mathfrak{m}$ is an abelian subvariety that is characterized by the conditions that $\mathfrak{m}$ is nowhere degenerate and that if $\mathfrak{m}'$ is an abelian subvariety of A that is nowhere degenerate, then $\mathfrak{m}' \subset \mathfrak{m}$. By considering the dimension, the existence is obvious; the uniqueness is actually proved in [41, §7.3].

Theorem 3.7 ([41, Thm. E]). — *Let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Then the following are equivalent.*

(1) *The geometric Bogomolov conjecture holds for A.*

(2) *The geometric Bogomolov conjecture holds for $\mathfrak{m}$.*

The essential part in Theorem 3.7 is that (2) implies (1). In fact, the other implication holds because if the geometric Bogomolov conjecture holds for A, then it holds for any abelian subvariety of A.

As in Remark 3.5 (2), the conjecture holds for abelian varieties of dimension at most 1. Thus we have the following corollary.

Corollary 3.8 ([41, Thm. F]). — *With the notation as in Theorem 3.7, suppose that $\dim(\mathfrak{m}) \leq 1$. Then the geometric Bogomolov conjecture holds for A.*

It is not difficult to see that Corollary 3.8 generalizes Theorem 3.6 if you use the Lemma 3.9 below; see [41, Prop. 3.3] for the proof.

Lemma 3.9. — *Let A be an abelian variety over $\overline{K}$ and let A' be an abelian subvariety of A. Then for any $v \in M_{\overline{K}}$, we have $b(A'_v) \leq b(A_v)$.*

Indeed, by this lemma, we have $\dim(\mathfrak{m}) = b(\mathfrak{m}_v) \leq b(A_v)$ for any $v \in M_{\overline{K}}$. Therefore, $b(A_v) \leq 1$ implies $\dim(\mathfrak{m}) \leq 1$, and thus Theorem 3.7 generalizes Theorem 3.6.

By Theorem 3.7, the geometric Bogomolov conjecture is reduced to the conjecture for nowhere degenerate abelian varieties. In [42], furthermore, we show that the conjecture is reduced to the case where the abelian variety has trivial $\overline{K}/k$-trace. To be precise, let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian variety of A. Let $\left(\widetilde{A^{\overline{K}/k}}, \mathrm{Tr}_A \right)$ be the $\overline{K}/k$-trace. We set $\mathfrak{t} := \mathrm{Image}(\mathrm{Tr}_A)$, the image of the trace homomorphism $\mathrm{Tr}_A : \widetilde{A^{\overline{K}/k}} \otimes_k \overline{K} \to A$. One can show that the image of the maximal nowhere degenerate abelian subvariety by a homomorphism is contained the maximal nowhere degenerate abelian subvariety (cf. [41, Lem. 7.8 (2)]). Since the maximal nowhere degenerate abelian subvariety of $\widetilde{A^{\overline{K}/k}} \otimes_k \overline{K}$ equals itself, it follows that $\mathfrak{t} \subset \mathfrak{m}$, and hence we can take the quotient $\mathfrak{m}/\mathfrak{t}$.

Theorem 3.10 ([42, Thm. 1.5]). — *Let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Let $\mathfrak{t}$ be the image of the $\overline{K}/k$-trace homomorphism. Then the following are equivalent.*

(1) *The geometric Bogomolov conjecture holds for A.*

(2) *The geometric Bogomolov conjecture holds for $\mathfrak{m}$.*

(3) *The geometric Bogomolov conjecture holds for $\mathfrak{m}/\mathfrak{t}$.*

The equivalence between the first two statement is nothing but Theorem 3.7. By [41, Lem. 7.7], we see that (2) implies (3), which is not difficult. Thus the main contribution the theorem the assertion that (3) implies (2).

Since the quotient of nowhere degenerate abelian variety by an abelian subvariety is again nowhere degenerate, $\mathfrak{m}/\mathfrak{t}$ is nowhere degenerate (cf. [41, Lem. 7.8 (2)]). Furthermore, $\mathfrak{m}/\mathfrak{t}$ has

trivial $\overline{K}/k$-trace; see [42, Rem. 5.4]. Thus the geometric Bogomolov conjecture for abelian varieties is reduced the conjecture for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace.

In most part of the sequel, we will explain the idea of the proof of Theorems 3.7 and 3.10. As for Theorem 3.7, we give an idea of the proof in §8; we will not prove this theorem but give an outline the proof of a little weaker result. We recall in §6 the proof of Gubler's theorem because the idea of our proof is inspired by the proof of Gubler's theorem. Since the Gubler's proof is a nonarchimedean analogue of the proof of Zhang's theorem, we also recall the idea of Zhang in §4 and fundamental facts on nonarchimedean geometry in §5. We recall the structure of the canonical measures in §7, as these measures are the key ingredients of the proof of Theorem 3.7. As for Theorem 3.10, we give an outline of the proof in §10 after recalling the notion of canonical heights of closed subvarieties in §9.

4. Proof of Zhang's theorem

We recall Zhang's theorem:

Theorem 4.1 **(Zhang's theorem, restated).** — *Assume that K is a number field. Let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A. Suppose that X has dense small points. Then X is a torsion subvariety.*

The proof is based on the equidistribution of small points argued in [35]. In this section, we give an outline of the proof of Zhang's theorem with an emphasis on how to use the equidistribution theorem. The proof of Zhang will be a prototype of the proof of many results concerning the geometric Bogomolov conjecture. The basic reference is Zhang's original paper [47]. The survey paper [18] will be an accessible exposition for Zhang's theorem.

4.1. Archimedean canonical measures. — In this subsection, we work over $\mathbb{C}$. We begin by recalling the notion of canonical metrics on an abelian variety, which is studied in [22]. We refer to [18, §4] for details of this subsection. Let A be the complex analytic space associated to an abelian variety over $\mathbb{C}$ and let L be an even line bundle on A. For an integer $n > 1$, fix an isomorphism $\phi : [n]^*(L) \to L^{\otimes n^2}$. Let $\|\cdot\|$ be a metric on L. We call $\|\cdot\|$ a *canonical metric* on L if $[n]^*\|\cdot\| = \|\cdot\|^{\otimes n^2}$ holds via the isomorphism ϕ. Later we consider a nonarchimedean analogue of this metric, so when we emphasize that we are working over $\mathbb{C}$, we call it an archimedean canonical metric.

Theorem 4.2. — *For a fixed isomorphism $\phi : [n]^*(L) \to L^{\otimes n^2}$, there exists a unique canonical metric $\|\cdot\|$ on L. Furthermore, it is a smooth metric.*

We denote by $\|\cdot\|_{can}$ the canonical metric. Further, we write $\overline{L} = (L, \|\cdot\|_{can})$ in this subsection. Since the canonical metric is a smooth metric, we consider the curvature form $c_1(\overline{L})$.

Remark 4.3. — The canonical metric depends on the choice of the isomorphism $\phi : [n]^*L \to L^{\otimes n^2}$, but $c_1(\overline{L})$ does not. Indeed, one shows that a different choice of ϕ makes the canonical metric change only by positive constant multiple, so that it does not change $c_1(\overline{L})$. Moreover, $c_1(\overline{L})$ does not depend on the choice of $n > 1$, either.

Next, let us give an explicit description of the curvature forms of canonical metrics in terms of the universal covering of A. Let $p : \mathbb{C}^m \to A$ be the universal covering which is a homomorphism with respect to the additive structure on $\mathbb{C}^m$ and the group structure on A. Let $z_1, \ldots, z_m$ be the standard coordinates of $\mathbb{C}^m$. Then there exists a unique hermitian matrix $(c_{ij}) \in M_m(\mathbb{C})$ such that

$$(4.1) \qquad p^*(c_1(\overline{L})) = \sum_{1 \leq i,j \leq m} \sqrt{-1} c_{ij} dz_i \wedge d\overline{z_j} ,$$

where $\overline{z_j}$ denotes the complex conjugate of z_j; see [18, §4] for more details.

Furthermore, suppose that L is ample. Then the matrix (c_{ij}) is positive definite. Note that this shows that $c_1(\overline{L})$ is a positive $(1,1)$-from.

Remark 4.4. — Suppose that L is ample. Let U be a submanifold of A of dimension d. Then $c_1(\overline{L})^{\wedge d}|_U$ is positive in the following sense. Take any $p \in U$. Let $u_1, \ldots, u_d$ be a system of local holomorphic coordinates of U around p. We write

$$c_1(\overline{L})^{\wedge d}|_U = (\sqrt{-1})^d \varphi du_1 \wedge d\overline{u_1} \wedge \cdots \wedge du_d \wedge d\overline{u_d} ,$$

where φ is a smooth function on U around p. Then φ is a positive real valued function. (This follows from the positivity of $c_1(\overline{L})$.)

Finally, we define the canonical measures. Let L be an even line bundle on A. Let $\overline{L}$ denote the line bundle L equipped with a canonical metric on L. Assume that L is ample. Let X be a closed subvariety of A. Then

$$\mu_{X,L} := \frac{c_1(\overline{L})^{\wedge d}|_X}{\deg_L(X)}$$

is naturally a positive regular Borel measure on X with total volume 1. We call this probability measure the *canonical measure associated* to L. By Remark 4.3, the canonical measure does not depend on the choice of a canonical metric on L or $n > 1$, and it is well-defined for L.

4.2. Archimedean equidistribution theorem. — Let K be number field. Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Fix a finite extension K' of K such that A and X can be defined over K'. Set $G_{K'} := \mathrm{Gal}(\overline{K}/K')$. Then $G_{K'}$ acts on $X(\overline{K})$. For any $x \in X(\overline{K})$, let $O(x)$ denote the $G_{K'}$-orbit of x. Let σ be an archimedean place of $\overline{K}$, that is, an embedding $\overline{K} \hookrightarrow \mathbb{C}$ of fields. Let X_σ^{an} be the complex analytic space associated to $X \otimes_{\overline{K}} \mathbb{C}$, where $\mathbb{C}$ is regarded as an $\overline{K}$-algebra via σ. We regard $X(\overline{K}) \subset X_\sigma^{\mathrm{an}}$, and hence $O(x) \subset X_\sigma^{\mathrm{an}}$ for any $x \in X(\overline{K})$.

Theorem 4.5 ([47, Thm. 2.1]). — *With the notation above, let $(x_i)_{i \in I}$ be a small generic net on $X(\overline{K})$. Then we have a weak convergence*

$$\lim_i \frac{1}{\#O(x_i)} \sum_{z \in O(x_i)} \delta_z = \mu_{X_\sigma^{\mathrm{an}}, L} ,$$

where δ_z denotes the Dirac measure of z.

We do not mention the proof of the equidistribution theorem. We refer to the original paper [47] or [18, §6].

4.3. Difference morphism. — Let A be an abelian variety over any algebraically closed field $\mathfrak{K}$. Let X be a closed subvariety of A. Let N be a natural number with $N > 1$. We define $\alpha_N : X^N \to A^{N-1}$ by

$$\alpha_N(x_1, \ldots, x_N) = (x_1 - x_2, \ldots, x_{N-1} - x_N).$$

We call α_N the *difference morphism*. In this subsection, we give remarks on this morphism. Lemma 4.6 below will be significantly used not only in the proof of Zhang but also in the proof of some results on the geometric Bogomolov conjecture.

The *stabilizer* of X, denoted by G_X, is a reduced closed subgroup scheme of A characterized by

$$G_X(\mathfrak{K}) = \{a \in A(\mathfrak{K}) \mid X + a \subset X\}.$$

Lemma 4.6 ([1, Lem. 4.1]). — *Suppose that $G_X = 0$. Then there exists an $N \in \mathbb{N}$ such that $\alpha_N : X^N \to A^{N-1}$ is generically finite.*

Proof. — For a subset S of $X(\mathfrak{K})$, we set

$$G_{X,S} := \{a \in A(\mathfrak{K}) \mid S + a \subset X\}.$$

Further, for any $x_1, \ldots, x_m \in X(\mathfrak{K})$, we write $G_{X,x_1,\ldots,x_m} := G_{X,\{x_1,\ldots,x_m\}}$. We prove that for any $y_1, \ldots, y_N \in X(\mathfrak{K})$, we have

$$\alpha_N^{-1}(\alpha_N(y_1, \ldots, y_N))(\mathfrak{K}) = \{(y_1 + a, \ldots, y_N + a) \in A(\mathfrak{K}) \mid a \in G_{X,y_1,\ldots,y_N}\}.$$

The inclusion "$\supset$" is obvious. To show the other inclusion, we take any $(y_1', \ldots, y_N') \in \alpha_N^{-1}(\alpha_N(y_1, \ldots, y_N))(\mathfrak{K})$. Then for any $i = 1, \ldots, N - 1$, we have $y_i' - y_{i+1}' = y_i - y_{i+1}$, and hence $y_1' - y_1 = \cdots = y_N' - y_N$. Let a denote this element. Then we have $a \in G_{X,y_1,\ldots,y_N}$ and $(y_1', \ldots, y_N') = (y_1 + a, \ldots, y_N + a)$, which shows "$\subset$".

For any $S \subset X(\mathfrak{K})$, we note that $G_X(\mathfrak{K}) \subset G_{X,S}$. Further, we see that $G_X(\mathfrak{K}) = \bigcap_S G_{X,S}$ where S runs through all the finite subset of $X(\overline{K})$. Since $G_X(\mathfrak{K})$ and $G_{X,S}$ are closed subsets of $A(\mathfrak{K})$, it follows that there exist $x_1, \ldots, x_N \in X(\mathfrak{K})$ such that $G_X(\mathfrak{K}) = G_{X,x_1,\ldots,x_N}$. Now, since $G_X = 0$, there exist an $N \in \mathbb{N}$ and $x_1, \ldots, x_N$ such that $G_{X,x_1,\ldots,x_N} = G_X = 0$. It follows from what we have proved above that

$$\dim\left(\alpha_N^{-1}(\alpha_N(x_1, \ldots, x_N))\right) = 0.$$

This shows that α_N is generically finite. (In fact, the above argument shows that α_N is generically injective.) $\qquad\square$

4.4. Proof of Zhang. — We start the proof of Zhang. It is argued by contradiction; suppose that we have a counterexample to Zhang's theorem, that is, suppose that there exist an abelian variety A over $\overline{K}$ and a closed subvariety X of A such that X is not a torsion subvariety but has dense small points. Let G_X be the stabilizer of X and consider the quotient $\phi : A \to A/G_X$. Then X/G_X is a closed subvariety of A/G_X and has trivial stabilizer. Further, X/G_X has dense small points by Lemma 2.15.

We prove that $\dim(X/G_X) > 0$ by contradiction; suppose that $\dim(X/G_X) = 0$. Then $X/G_X = \{\phi(x)\}$ for some point x. Since X/G_X has dense small points, $\phi(x)$ has height 0, and hence it is a torsion point. Since ϕ induces a surjective homomorphism between the subgroups of torsion points (cf. [40, Proof of Lem. 2.10]), we may take x to be a torsion point of A. Then we see that $X = G_X + x$, which means that X is a torsion subvariety. That is a contradiction.

The above argument suggests that replacing A and X by A/G_X and X/G_X respectively if necessary, we have an abelian variety A and a closed subvariety X with $\dim(X) > 0$ such that X has dense small points and has trivial stabilizer. Put $d := \dim(X)$. Since X has trivial stabilizer, there exists, by Lemma 4.6, an integer $N \geq 2$ such that

$$\alpha_N : X^N \to A^{N-1}; \quad (x_1, \ldots, x_N) \mapsto (x_1 - x_2, \ldots, x_{N-1} - x_N)$$

is generically finite. Set $Z := X^N$ and $Y := \alpha_N(Z)$. Let $\alpha : Z \to Y$ denote the restriction of α_N, which is a generically finite surjective morphism. We remark that $\dim(Z) = \dim(Y) = dN$. Since X has dense small points, so does Z by Lemma 2.16. By Lemma 2.17, there exists a small generic net $(z_i)_{i \in I}$ on $Z(\overline{K})$. The image $(\alpha(z_i))_{i \in N}$ is a generic net on $Y(\overline{K})$. Further, if follows from Proposition 2.6 (2) that this net is small as well. Take an archimedean place σ of $\overline{K}$, that is, an embedding $\sigma : \overline{K} \hookrightarrow \mathbb{C}$. Fix even ample line bundles M and L on A^N and A^{N-1}, respectively. Let $\mu_{Z_\sigma^{\mathrm{an}}, M}$ and $\mu_{Y_\sigma^{\mathrm{an}}, L}$ be the canonical measures on Z_σ^{an} and Y_σ^{an}, respectively. By the (archimedean) equidistribution theorem (Theorem 4.5), we have

$$\lim_i \frac{1}{\#O(z_i)} \sum_{u \in O(z_i)} \delta_u = \mu_{Z_\sigma^{\mathrm{an}}, M}$$

and

$$\lim_i \frac{1}{\#O(\alpha(z_i))} \sum_{v \in O(\alpha(z_i))} \delta_v = \mu_{Y_\sigma^{\mathrm{an}}, L} \,.$$

Let $\alpha^{\mathrm{an}} : Z_\sigma^{\mathrm{an}} \to Y_\sigma^{\mathrm{an}}$ denote the morphism of analytic spaces associated to α. Since

$$\alpha_*^{\mathrm{an}} \left(\frac{1}{\#O(z_i)} \sum_{u \in O(z_i)} \delta_u \right) = \frac{1}{\#O(\alpha(z_i))} \sum_{v \in O(\alpha(z_i))} \delta_v \,,$$

we obtain $\alpha_*^{\mathrm{an}}(\mu_{Z_\sigma^{\mathrm{an}}, M}) = \mu_{Y_\sigma^{\mathrm{an}}, L}$.

Let V be the nonsingular locus of X_σ^{an}. Then V^N is a non-empty nonsingular open subset of Z_σ^{an}. Since $\alpha : Z \to Y$ is generically finite and $\dim(Z) = \dim(Y) = dN$, $\alpha_*^{\mathrm{an}}(\mu_{Z_\sigma^{\mathrm{an}}, M}) = \mu_{Y_\sigma^{\mathrm{an}}, L}$ gives us an equality

$$(4.2) \qquad \frac{c_1(\overline{M})^{\wedge dN}|_{V^N}}{\deg_M(Z)} = \frac{(\alpha^{\mathrm{an}}|_{V^N})^* c_1(\overline{L})^{\wedge dN}}{\deg_L(Y)}$$

of smooth (dN, dN)-forms on V^N. Let $p \in V^N$ be a point on the diagonal. Then the left-hand side on (4.2) is positive at p (Remark 4.4). On the other hand, since $\alpha^{\mathrm{an}}|_{V^N}$ is ramified at p, the right-hand side is not positive at p. That is a contradiction. Thus the proof of Zhang's theorem is complete.

5. Nonarchimedean geometry

In the proof of Zhang, the following are crucially used:

(1) analytic spaces over an archimedean place;

(2) canonical measures over analytic spaces;

(3) equidistribution theorem of small points over analytic spaces.

Over function fields, we do not have (1) above: there does not exist an archimedean place over function fields.

Therefore, if we wish to follow the idea of Zhang (and Ullmo), we need counterparts of the above. The counterparts of (1) that we will use are the Berkovich analytic spaces over some nonarchimedean place. Further, over Berkovich analytic spaces, we can consider "canonical measures".

In this section, we briefly review the notions of Berkovich analytic spaces, skeleta, and measures on Berkovich spaces. The reader familiar with those notions can skip this section. Basic references are Berkovich's original papers [2, 3, 4, 5]. For the exposition in this section, we refer to Nicaise's exposition [29]. This is very accessible to non-experts.

5.1. Notation and convention. — Throughout this section, let $\mathbb{K}$ be an algebraically closed field complete with respect to a non-trivial nonarchimedean value $|\cdot| = |\cdot|_{\mathbb{K}}$. Let $\mathbb{K}^{\circ}$ denote the ring of integers of $\mathbb{K}$ and let k denote the residue field.

Let $\mathbf{K}$ be a subfield of $\mathbb{K}$. The restriction $|\cdot|_{\mathbf{K}}$ of $|\cdot|_{\mathbb{K}}$ is an absolute value on $\mathbf{K}$. We always assume that $\mathbf{K}$ is complete with respect to $|\cdot|_{\mathbf{K}}$. Let $\mathbf{K}^{\circ}$ denote the ring of integers of $\mathbf{K}$.

We say that $\mathbf{K}$ is a *complete discretely valued subfield* if $|\cdot|_{\mathbf{K}}$ is a complete discrete absolute value on $\mathbf{K}$. We abbreviate the name to a *CDV subfield*. A subring that is the ring of integers of some CDV subfield is called the *complete discrete valuation subring*, abbreviated to a *CDV subring*.

In this paper, we mainly use nonarchimedean geometry over $\mathbf{K} = \mathbb{K}$. However, we will sometimes consider nonarchimedean geometry over a CDV subfield because it will help us to make more accessible description of basic notions on nonarchimedean geometry.

Let X be a scheme over $\mathbb{K}$. We say that *X can be defined over a discrete valued field* if it has a model over some CDV subfield.

5.2. Berkovich analytic spaces and skeleta. — Let $\mathbf{K}$ be a CDV subfield of $\mathbb{K}$. Let X be an algebraic variety over $\mathbf{K}$. For a point $p \in X$, let $\kappa(p)$ denote the residue field at p. We mean by the *(Berkovich) analytic space* associated to X, which we denote by X^{an}, a topological space given as follows. As a set,

$$X^{\mathrm{an}} := \{(p, |\cdot|) \mid p \in X \text{ and } |\cdot| \text{ is an absolute value of } \kappa(p) \text{ extending } |\cdot|_{\mathbf{K}}\}.$$

We endow X^{an} with the weakest topology such that the map $\iota : X^{\mathrm{an}} \to X, (p, |\cdot|) \mapsto p$ is continuous and such that for any Zariski open subset U of X and any regular function $g \in \mathcal{O}_X(U)$, the map $\iota^{-1}(U) \to \mathbb{R}, (p, |\cdot|) \mapsto |g(p)|$ is continuous. It is known that X^{an} is a Hausdorff, locally compact, and locally path-connected space.

Let $x \in X(\mathbf{K})$ be a $\mathbf{K}$-valued point. Then this gives a point of X^{an}. Indeed, the natural homomorphism $\mathbf{K} \hookrightarrow \kappa(x)$ is isomorphism in this case, so that $\kappa(x)$ has a unique absolute value extending $|\cdot|_{\mathbf{K}}$ via this isomorphism. Thus we have $X(\mathbf{K}) \subset X^{\mathrm{an}}$ naturally. We call a point in $X(\mathbf{K})$ a *classical point.*

Let $f : X \to Y$ be a morphism of algebraic varieties. Then f induces a continuous map $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ naturally. Indeed, since we have a inclusion $\kappa(f(p)) \hookrightarrow \kappa(p)$ for any $p \in X$, we assign $(p, |\cdot|) \in X^{\mathrm{an}}$ to $(f(p), |\cdot|_{\kappa(f(p))})$, where $|\cdot|_{\kappa(f(p))}$ is the restriction of $|\cdot|$ to $\kappa(f(p))$. We make a remark on the base-change. Set $X_{\mathbb{K}} := X \otimes_{\mathbf{K}} \mathbb{K}$. Then we have a natural map $\rho : X_{\mathbb{K}} \to X$. Furthermore, for each $p \in X_{\mathbb{K}}$, we have $\kappa(\rho(p)) \subset \kappa(p)$, and taking the restriction of the absolute value of $\kappa(p)$ to $\kappa(\rho(p))$, we obtain the natural map $X_{\mathbb{K}}^{\mathrm{an}} \to X^{\mathrm{an}}$. One sees that this is continuous and surjective.

5.2.1. *Divisorial points.* — A Berkovich space in general has many other points than classical points. The divisorial points associated to the generic points of irreducible components of the special fiber of a model are very important class of them. To make the explanation simple, we assume that $\mathbb{K} = \overline{K}_v$, where K is a function field over k and $v \in M_{\overline{K}}$ is a fixed place. Note that $k \subset \mathbb{K}$ and the residue field of $\mathbb{K}$ equals k.

First, we define divisorial points in analytic spaces over a CDV subfield. Let $\mathbf{K}$ be a CDV subfield. Let X be an algebraic variety over $\mathbf{K}$. Let $\mathscr{X} \to \operatorname{Spec}(\mathbf{K}^\circ)$ be a flat model of X over $\mathbf{K}^\circ$, where $\mathbf{K}^\circ$ is a discrete valuation ring by the assumption. Let ϖ be a uniformizer of $\mathbf{K}^\circ$. Let $\operatorname{Irr}(\widetilde{\mathscr{X}})$ be the set of irreducible components of the special fiber $\widetilde{\mathscr{X}}$ of $\mathscr{X}$. For each $V \in \operatorname{Irr}(\widetilde{\mathscr{X}})$, let ξ_V denote the generic point of V in $\mathscr{X}$.

Assume that for any $V \in \operatorname{Irr}(\widetilde{\mathscr{X}})$, $\mathscr{X}$ is normal at ξ_V. Take any $V \in \operatorname{Irr}(\widetilde{\mathscr{X}})$. Then the local ring $\mathcal{O}_{\mathscr{X},\xi_V}$ is a discrete valuation ring, whose fraction field equals the function field $\mathbf{K}(X)$ of X. Note that $\mathbf{K}(X)$ is the residue field at the generic point of X. Let $\operatorname{ord}_{\xi_V} : \mathbf{K}(X) \to \mathbb{Z} \cup \{+\infty\}$ be the order function ($\operatorname{ord}_{\xi_V}(0) = +\infty$ by convention). Let m_V denote the multiplicity of V in $\widetilde{\mathscr{X}}$. Then we define the value $|\cdot|_{\xi_V} : \mathbf{K}(X) \to \mathbb{R}$ by $|f|_{\xi_V} := |\varpi|^{\operatorname{ord}_{\xi_V}(f)/m_V}$. It is easy to see that the restriction of this absolute value equals $|\cdot|_{\mathbf{K}}$, and thus it gives a point of X^{an}. We denote this point by $\xi_V^{\operatorname{an}}$ and call it the *divisorial point associated to ξ_V (or V)*. Set $\operatorname{DP}(\mathscr{X}) := \{\xi_V^{\operatorname{an}} \in X^{\operatorname{an}} | \ V \in \operatorname{Irr}(\widetilde{\mathscr{X}})\}$, whose points are called divisorial points with respect to $\mathscr{X}$.

Next, we introduce the divisorial points over $\mathbb{K}$. Although we can define a similar kind of points in general setting, we only consider the models that can be defined over CDV subring, because they are easy to describe and will be enough for our later use. Let X be an algebraic variety over $\mathbb{K}$. Assume that there exist a CDV subfield with $k \subset \mathbf{K}$ and a flat model $\mathscr{X} \to \operatorname{Spec}(\mathbf{K}^\circ)$ of X such that the special fiber is reduced. For any $V \in \operatorname{Irr}(\widetilde{\mathscr{X}})$, $\mathscr{X}$ is then normal at the generic point ξ_V of V. The base-change $\mathscr{X}_{\mathbb{K}^\circ} \to \operatorname{Spec}(\mathbb{K}^\circ)$ is a flat model of X over $\mathbb{K}^\circ$. Let $\rho : \mathscr{X}_{\mathbb{K}^\circ} \to \mathscr{X}$ be the natural map. Let η be the generic point of an irreducible component of the special fiber of $\mathscr{X}_{\mathbb{K}^\circ}$. Then $\rho(\eta)$ is the generic point of some irreducible component of $\widetilde{\mathscr{X}}$. Thus we have a point $\rho(\eta)^{\operatorname{an}} \in (\mathscr{X}_{\mathbf{K}})^{\operatorname{an}}$, where $\mathscr{X}_{\mathbf{K}}$ is the generic fiber of $\mathscr{X} \to \operatorname{Spec}(\mathbf{K}^\circ)$. Since the natural map $X^{\operatorname{an}} \to (\mathscr{X}_{\mathbf{K}})^{\operatorname{an}}$ is surjective, there exists a point in X^{an} that lies over $\rho(\eta)$. By our assumption, any irreducible component of $\widetilde{\mathscr{X}}$ is geometrically irreducible. From that, one can show that a point of X^{an} over $\rho(\eta)$ is unique. This unique point in X^{an} is called the *divisorial point associated to η*, denoted by η^{an}. Let $\operatorname{DP}(\mathscr{X}_{\mathbb{K}^\circ})$ be the set of those divisorial points. One can show that $\operatorname{DP}(\mathscr{X}_{\mathbb{K}^\circ})$ does not depend on the choice of $\mathbf{K}$, depending only on $\mathscr{X}_{\mathbb{K}^\circ}$.

5.3. Skeleta. — In this subsection, we explain skeleta. A skeleton is a compact subset of an analytic space associated to a strictly semistable model. It has a canonical structure of simplicial set.

5.3.1. *Strictly semistable scheme.* — We begin by recalling the notion of strictly semistable schemes over a discrete valuation ring. Let $\mathbf{K}$ be a CDV subfield. Let $f : \mathscr{X} \to \operatorname{Spec}(\mathbf{K}^\circ)$ be a flat morphism of finite type. We call f a *strictly semistable scheme* if it satisfies the following conditions:

(1) $\mathscr{X}$ is regular;

(2) f is generically smooth;

(3) the special fiber $\widetilde{\mathscr{X}}$ is a normal crossing divisor whose irreducible components are geometrically integral and smooth.

Next, we define a strictly semistable scheme over $\mathbb{K}^\circ$. Let $\mathscr{X} \to \mathrm{Spec}(\mathbb{K}^\circ)$ a morphism of scheme. We call it a strictly semistable scheme if there exist a CDV subfield $\mathbf{K}$ and strictly semistable scheme $\mathscr{X}' \to \mathrm{Spec}(\mathbf{K}^\circ)$ such that $\mathscr{X} = \mathscr{X}'_{\mathbb{K}^\circ}$.

The above definition is an expedient definition; we have a more general definition which makes sense for those not necessarily defined over a discrete valuation ring. However, we only consider those which can be defined over a discrete valuation subring, because they are easy to describe and will be enough for our use.

5.3.2. *Monomial points.* — We recall the notion of strata of a reduced algebraic scheme Z over k. We put $Z^{(0)} := Z$. For $r \in \mathbb{Z}_{\geq 0}$, we inductively define $Z^{(r+1)} \subset Z^{(r)}$ to be the set of non-normal points of $Z^{(r)}$. Then we obtain a descending sequence of closed subsets

$$Z = Z^{(0)} \supsetneq Z^{(1)} \supsetneq \cdots \supsetneq Z^{(s)} \supsetneq Z^{(s+1)} = \emptyset.$$

The irreducible components of $Z^{(r)} \setminus Z^{(r+1)}$ are called the *strata of Z of codimension r*. Let $\mathrm{str}(Z)$ be the set of strata of Z.

Let X be a variety over $\mathbf{K}$ with a strictly semistable model $\mathscr{X}$. The monomial points are the points of X^{an} which are determined by $S \in \mathrm{str}(\widetilde{\mathscr{X}})$ and positive real numbers $(u_0, \ldots, u_r)$ with $u_0 + \cdots + u_r = 1$ where r is the codimension of S. To give a more precise description, we first assume that $\mathbf{K}$ is a CDV subfield, and then we explain them for $\mathbb{K}$.

Assume that $\mathbf{K}$ is a CDV subfield. Let ϖ be a uniformizer of $\mathbf{K}^\circ$. Let X be an algebraic variety over $\mathbf{K}$. Assume that X has a strictly semistable model $\mathscr{X}$ over $\mathbf{K}^\circ$, that is, there exists a strictly semistable scheme $\mathscr{X} \to \mathrm{Spec}(\mathbf{K}^\circ)$ with generic fiber X. Let S be a stratum of $\widetilde{\mathscr{X}}$ of codimension r in $\widetilde{\mathscr{X}}$ and let ξ be the generic point of S. Locally at ξ in $\widetilde{\mathscr{X}}$, S is given by the intersection $\bigcap_{i=0}^r V_i$ of $r + 1$ irreducible components $V_0, \ldots, V_r \in \mathrm{Irr}(\widetilde{\mathscr{X}})$. (In other words, the closure of S is an irreducible component of $\bigcap_{i=0}^r V_i$.) Let $\mathcal{O}_{\mathscr{X},\xi}$ be the local ring of $\mathscr{X}$ at ξ. Let $t_0, \ldots, t_r$ be elements of $\mathcal{O}_{\mathscr{X},\xi}$ defining $V_0, \ldots, V_r$ around ξ, respectively. Then since $\mathcal{O}_{\mathscr{X},\xi}$ is regular and $V_0, \ldots, V_r$ are normally crossing, $t_0, \ldots, t_r$ is a regular system of parameters of $\mathcal{O}_{\mathscr{X},\xi}$. Note that there exists a unit λ in $\mathcal{O}_{\mathscr{X},\xi}$ such that $\varpi = \lambda t_0 \cdots t_r$.

Let $\widehat{\mathcal{O}_{\mathscr{X},\xi}}$ be the completion of $\mathcal{O}_{\mathscr{X},\xi}$. Let $k(\xi)$ be the residue field of $\mathcal{O}_{\mathscr{X},\xi}$. By Cohen's structure theorem of complete regular local rings, there exist a section of the quotient homomorphism $\widehat{\mathcal{O}_{\mathscr{X},\xi}} \to k(\xi)$ and an isomorphism

$$k(\xi)[\![t_0, \ldots, t_r]\!] \cong \widehat{\mathcal{O}_{\mathscr{X},\xi}}.$$

We call this isomorphism a Cohen isomorphism for $\widehat{\mathcal{O}_{\mathscr{X},\xi}}$.

Put $\Delta^r := \{\mathbf{u} = (u_0, \ldots, u_r) \in \mathbb{R}_{\geq 0}^{r+1} \mid u_0 + \cdots + u_r = 1\}$. Then under the setting above, each $\mathbf{u} \in \Delta^r$ determines a point of $\mathbf{X}^{\overline{\mathrm{an}}}$ due to the following proposition.

Proposition 5.1 ([29, §2.3]). — *Let $\mathbf{K}(X)$ be the function field of X. For any $\mathbf{u} = (u_0, \ldots, u_r) \in \Delta^r$, there exists a unique absolute value $|\cdot|_{\mathbf{u}} : \mathbf{K}(X) \to \mathbb{R}_{\geq 0}$ with the following property. Fix a Cohen isomorphism for $\widehat{\mathcal{O}_{\mathscr{X},\xi}}$. Take any $f \in \mathcal{O}_{\mathscr{X},\xi}$. Regarding it as a formal power series via the Cohen isomorphism, we write*

$$f = \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{r+1}} c_{\mathbf{m}} t^{\mathbf{m}}.$$

Then we have

$$\log |f|_{\mathbf{u}} = (\log |\varpi|) \min\{\mathbf{u} \cdot \mathbf{m} \mid \mathbf{m} \in \mathbb{Z}_{\geq 0}^{r+1}, c_{\mathbf{m}} \neq 0\}.$$

We remark that in the above proposition, the absolute value $|\cdot|_{\mathbf{u}}$ is independent of the choice of the Cohen isomorphism.

Since $\mathbf{K}(X)$ is the residue field of the generic point of X, the absolute value $|\cdot|_{\mathbf{u}}$ in the above proposition gives us a point of X^{an}. We call it the *monomial point associated to* $(\mathscr{X}, \xi, \mathbf{u})$. The monomial point associated to $(\mathscr{X}, \xi, \mathbf{u})$ for some stratum S and $\mathbf{u} \in \Delta^r$ is called a monomial point with respect to $\mathscr{X}$.

5.3.3. *Skeleta (over CDV subfields).* — By Proposition 5.1, we obtain an injective map $\rho_S : \Delta^r \to X^{\mathrm{an}}$ which maps $\mathbf{u} = (u_0, \ldots, u_r)$ to the monomial point given by $|\cdot|_{\mathbf{u}}$. One sees that this map is continuous. Let Δ_S denote the image of Δ^r by this map. Then it is a compact subset of X^{an}. We set $S(\mathscr{X}) := \bigcup_{S \in \mathrm{str}(\widetilde{\mathscr{X}})} \Delta_S$. This is the set of monomial points with respect to $\mathscr{X}$. It is called the *skeleton* of X^{an} associated to $\mathscr{X}$.

The skeleton is a compact subset. Furthermore, it has a canonical structure of simplicial complex which reflects the incidence relation of $\mathrm{str}(\widetilde{\mathscr{X}})$. First, we put a structure of a simplex on Δ_S via the homeomorphism $\rho : \Delta^r \to \Delta_S$. Next, to see the incidence relation, let S and S' be strata of $\widetilde{\mathscr{X}}$ and suppose $S' \subset \overline{S}$, where $\overline{S}$ is the Zariski closure of S in $\widetilde{\mathscr{X}}$. Then we have $\Delta_{S'} \supset \Delta_S$ and Δ_S is a face of $\Delta_{S'}$. Indeed, let ξ and ξ' be the generic points of S and S', respectively. Let $V_0, \ldots, V_r$ be the irreducible components of $\widetilde{\mathscr{X}}$ such that $S = V_0 \cap \cdots \cap V_r$ around ξ. Since $S' \subset \overline{S}$, there are irreducible components $V_{r+1}, \ldots, V_{r'}$ of $\widetilde{\mathscr{X}}$ such that $S' = V_0 \cap \cdots \cap V_{r'}$ around ξ'. Let $t_0, \ldots, t_{r'} \in \mathcal{O}_{\mathscr{X}, \xi'}$ be local equations of $V_0, \ldots, V_{r'}$. Then $t_0, \ldots, t_{r'}$ form a regular system of parameters of $\mathcal{O}_{\mathscr{X}, \xi'}$, and $t_0, \ldots, t_r$ form that of $\mathcal{O}_{\mathscr{X}, \xi}$. Let $(u_0, \ldots, u_r) \in \Delta^r$. Note $(u_0, \ldots, u_r, 0, \ldots, 0) \in \Delta^{r'}$. Let $\rho_S : \Delta^r \to \Delta_S$ and $\rho_{S'} : \Delta^{r'} \to \Delta_{S'}$ be the homeomorphism defined above. Then by the uniqueness assertion of Proposition 5.1, one can show that $\rho_S(u_0, \ldots, u_r) = \rho(u_0, \ldots, u_r, 0, \ldots, 0)$. This means that $\Delta_{S'}$ contains Δ_S as a face. To the contrary, suppose that $\Delta_{S'} \supset \Delta_S$. Then one can show that Δ_S is a face of $\Delta_{S'}$ and $S' \subset \overline{S}$. Thus $S(\mathscr{X}) := \bigcup_{S \in \mathrm{str}(\widetilde{\mathscr{X}})} \Delta_S$ is a simplicial complex which reflects the incidence relation of $\mathrm{str}(\widetilde{\mathscr{X}})$.

This will not be really used in the sequel, but let us give a remark when $\dim(X) = 1$. In this case, we have a notion of dual graph by configuration of the special fiber of $\mathscr{X}$. The skeleton $S(\mathscr{X})$ is the realization of the dual graph inside the analytic space X^{an}.

5.3.4. *Skeleta (over $\mathbb{K}$).* — Now we consider skeleta over $\mathbb{K}$. Let X be a smooth variety over $\mathbb{K}$. Assume that there exist a CDV subfield $\mathbf{K}$ and a strictly semistable model $\mathscr{X} \to \mathrm{Spec}(\mathbf{K}^\circ)$ of X. The generic fiber $\mathscr{X}_{\mathbf{K}}$ is a smooth variety over $\mathbf{K}$. Let $\rho : X^{\mathrm{an}} \to (\mathscr{X}_{\mathbf{K}})^{\mathrm{an}}$ be the natural surjective map. Then one can show that for any $u \in S(\mathscr{X}) \subset (\mathscr{X}_{\mathbf{K}})^{\mathrm{an}}$, i.e. a monomial point of $(\mathscr{X}_{\mathbf{K}})^{\mathrm{an}}$ with respect to $\mathscr{X}$, $\rho^{-1}(u)$ consists of one point. It follows that $\rho^{-1}(S(\mathscr{X}))$ is a compact subset of X^{an} and has a structure of simplicial complex. Furthermore, one can show that this does not depend on the choice of $\mathbf{K}$ but depends only on the model $\mathscr{X}_{\mathbb{K}^\circ}$ over $\mathbb{K}^\circ$. We denote this set by $S(\mathscr{X}_{\mathbb{K}})$ and call it the *skeleton* of X^{an} associated to $\mathscr{X}$. This skeleton reflects the incidence relation of the special fiber of $\mathscr{X}_{\mathbb{K}^\circ}$.

We can actually define the skeleton of the analytic space associated to any strictly semistable model over $\mathbb{K}$, but we do not do that, because in our use, the strictly semistable model can be defined over a CDV subring.

5.4. Metrics on line bundles. — In this subsection, we recall basic notions of metrics on line bundles. We refer to [46].

Let V be a 1-dimensional vector space over $\mathbb{K}$. A *metric* $\|\cdot\|_V$ *on* V is a nontrivial function $V \to \mathbb{R}_{\geq 0}$ such that $\|\lambda v\|_V = |\lambda| \|v\|_V$ for any $\lambda \in \mathbb{K}$, where $|\lambda|$ is the absolute value of λ.

Let X be an algebraic variety over $\mathbb{K}$ and let L be a line bundle on X. A metric on L is a collection $\{\|\cdot\|_{L(x)}\}_{x \in X(\mathbb{K})}$ of metrics on the 1-dimensional $\mathbb{K}$-vector spaces $L(x)$ for all $x \in X(\mathbb{K})$, where $L(x) = L \otimes_{\mathcal{O}_X} \kappa(x)$ denotes the fiber of L over the $\mathbb{K}$-valued point x.

Among the metrics on line bundles, there are two very important classes of metrics: the algebraic metrics and the semipositive metrics. The algebraic metrics are determined from models over $\mathbb{K}^\circ$, and the semipositive metrics are the "uniform limits" of algebraic metrics arising from vertically nef models.

First, we recall the notion of algebraic metric. Let X be a projective variety over $\mathbb{K}$. Let $(\mathscr{X} \to \mathrm{Spec}(\mathbb{K}^\circ), \mathscr{L})$ be a proper flat model of (X, L). Take any point $x \in X(\mathbb{K})$. By the valuative criterion of properness, x extends to a section $\sigma_x : \mathrm{Spec}(\mathbb{K}^\circ) \to \mathscr{X}$. There exists an open neighborhood $\mathscr{U}$ of the image of σ_x with a trivialization $\varphi : \mathscr{L}|_{\mathscr{U}} \to \mathcal{O}_{\mathscr{U}}$. Restricting this trivialization to the generic fibers, we obtain an isomorphism $\varphi_U : \mathscr{L}|_U \to \mathcal{O}_U$, where U is the generic fiber of $\mathscr{U}$. For any $s(x) \in L(x)$, $\varphi_U(s(x))$ is an element of $\mathcal{O}_U(x) = \mathbb{K}$, and we set $\|s(x)\|_{\mathscr{L}(x)} := |\varphi_U(s(x))|_{\mathbb{K}}$. It is not difficult to see that $\|s(x)\|_{\mathscr{L}(x)}$ does not depend on any choices other than the model $\mathscr{L}$. Thus we obtain a metric $\|\cdot\|_{\mathscr{L}} = \{\|\cdot\|_{\mathscr{L}(x)}\}_{x \in X(\mathbb{K})}$ on L. This metric is called the *algebraic metric arising from* $(\mathscr{X}, \mathscr{L})$.

We sometimes consider a model $(\mathscr{X}, \mathscr{L})$ of (X, L) over a CDV subring and consider the algebraic metric arising from $(\mathscr{X}_{\mathbb{K}^\circ}, \mathscr{L}_{\mathbb{K}^\circ})$, the base-change of $(\mathscr{X}, \mathscr{L})$ to $\mathbb{K}$ over the CDV subring. To ease notation, this algebraic metric is denoted by $\|\cdot\|_{\mathscr{L}}$ instead of $\|\cdot\|_{\mathscr{L}_{\mathbb{K}^\circ}}$. We say that such an algebraic metric *can be defined over a CDV subring*.

A line bundle $\mathscr{L}$ on $\mathscr{X}$ is said to be *vertically nef* if $\widetilde{\mathscr{L}} := \mathscr{L}|_{\widetilde{\mathscr{X}}}$ is nef. A metric that is of the form $\|\cdot\|_{\mathscr{L}}$ for some vertically nef $\mathscr{L}$ is said to be *semipositive*.

The notion of algebraic metric is generalized to the notion of $\mathbb{Q}$-algebraic metric. Let L be a line bundle on a projective variety over $\mathbb{K}$. Let $\|\cdot\|$ be a metric on L. We call $\|\cdot\|$ a $\mathbb{Q}$-*algebraic metric* if there exists a positive integer N such that $\|\cdot\|^{\otimes N}$, which is a metric on $L^{\otimes N}$, is an algebraic metric. Furthermore, if N can be taken in such a way that $\|\cdot\|^{\otimes N}$ can be defined over a CDV subring, we say that $\|\cdot\|$ can be defined over a CDV subring. A $\mathbb{Q}$-algebraic metric $\|\cdot\|$ is said to be *semipositive* if $\|\cdot\|^{\otimes N}$ is an algebraic and semipositive metric for some $N \geq 1$.

We would like to define the notion of uniform limit of a sequence of metrics. Before that, we define a function arising from two metrics on a line bundle. Let $\|\cdot\|$ and $\|\cdot\|'$ be metrics on a line bundle L. We define a function $-\log(\|\cdot\|/\|\cdot\|')$ on $X(\mathbb{K})$ by

$$-\log \frac{\|\cdot\|}{\|\cdot\|'}(x) := -\log \frac{\|s(x)\|}{\|s(x)\|'},$$

where s is a local section of L that is non-zero at x. This does not depend on the choice of s and gives a well-defined function on $X(\mathbb{K})$. Now, let $\|\cdot\|$ be a metric on L and let $(\|\cdot\|_n)_{n \in \mathbb{N}}$ be a sequence of metrics on L. We say that $(\|\cdot\|_n)_{n \in \mathbb{N}}$ *uniformly converges to* $\|\cdot\|$ if $-\log(\|\cdot\|/\|\cdot\|_n)$ is a bounded function on $X(\overline{K})$ and if

$$\lim_{n \to \infty} \sup_{x \in X(\mathbb{K})} \left\{ -\log \frac{\|\cdot\|}{\|\cdot\|_n}(x) \right\} = 0.$$

If this is the case, we call $\|\cdot\|$ is the *uniform limit* of $(\|\cdot\|_n)_{n\in\mathbb{N}}$.

Let $(L, \|\cdot\|)$ be a metrized line bundle. We say that $\|\cdot\|$ is *semipositive*, if it is a uniform limit of a sequence of semipositive $\mathbb{Q}$-algebraic metrics.

Lemma 5.2. — *Let $f : X \to Y$ be a morphism of projective varieties over $\mathbb{K}$. Let $\overline{L}$ and $\overline{M}$ be line bundles on X and Y with semipositive metrics, respectively. Then the metric of $\overline{L} \otimes f^*(\overline{M})$ is semipositive.*

Let us give some comments. In the above, we consider metrics over $X(\mathbb{K})$, but in fact, we can do that over the analytic space X^{an}. Indeed, we can define algebraic metrics as a metric on line bundles on X^{an}. The algebraic metrics are continuous with respect to the topology on X^{an}. It follows that the $\mathbb{Q}$-algebraic metrics are also continuous metrics. Furthermore, a semipositive metric is actually defined to be the uniform limit of a sequence of $\mathbb{Q}$-algebraic metrics over X^{an} in the usual sense. Therefore a semipositive metric is continuous on X^{an} in the usual sense. We refer to [11, §7] for details.

5.5. Chambert-Loir measures. — Let X be a projective variety over $\mathbb{K}$ of dimension d. Let $\overline{L} = (L, \|\cdot\|)$ be a line bundle on X equipped with a semipositive metric. To these data, we can associate a regular Borel measure $c_1(\overline{L})^{\wedge d}$, which we are going to explain.

In this subsection, we assume that X can be defined over a CDV subfield. Further, we only considers semipositive metrics which are the limit of $\mathbb{Q}$-algebraic metrics that can be defined over a CDV subring.

First, we consider the case where the metric $\|\cdot\|$ is algebraic. By definition, there exist a discrete valued subfield $\mathbf{K}$ and a proper flat model $(\mathscr{X}, \mathscr{L})$ of (X, L) over $\mathbf{K}^\circ$ such that $\|\cdot\| = \|\cdot\|_{\mathscr{L}}$. Further, replacing $\mathbf{K}^\circ$ by a finite extension, we can take a proper flat morphism $\varphi : \mathscr{X}' \to \mathrm{Spec}(\mathbf{K}^\circ)$ with reduced special fiber and a generically finite surjective morphism $\nu : \mathscr{X}' \to \mathscr{X}$ over $\mathbf{K}^\circ$. (In fact, de Jong's semistable alteration theorem will give us such $\varphi : \mathscr{X}' \to \mathrm{Spec}(\mathbf{K}^\circ)$; see §7.1 in the sequel.) Set $\mathscr{L}' := \nu^*(\mathscr{L})$. Put $X' := \mathscr{X}' \otimes_{\mathbf{K}^\circ} \mathbf{K}$ and $X'_{\mathbb{K}} := X' \otimes_{\mathbf{K}} \mathbb{K}$. Then the metric of the pull-back $\overline{L}' := \nu|^*_{X'_{\mathbb{K}}}(\overline{L})$ is the algebraic metric associated to the model $(\mathscr{X}', \mathscr{L}')$. Let $\mathrm{Irr}(\widetilde{\mathscr{X}'})$ denote the set of irreducible component of the special fiber of $\mathscr{X}'$. For each $V \in \mathrm{Irr}(\widetilde{\mathscr{X}'})$, let ξ_V denote the generic point of V. Since φ is flat and $\widetilde{\mathscr{X}'}$ is reduced at ξ_V, $\mathscr{X}'$ is normal at ξ_V. Therefore we have a corresponding point $\xi_V^{\mathrm{an}} \in (X')^{\mathrm{an}}$. As we noted in §5.2.1, there exists a unique point η_V which maps to ξ_V^{an} by the canonical map $(X'_{\mathbb{K}})^{\mathrm{an}} \to (X')^{\mathrm{an}}$. With those notation, we define a measure $c_1(\overline{L}')^{\wedge d}$ on $(X'_{\mathbb{K}})^{\mathrm{an}}$ to be

$$\sum_{V \in \mathrm{Irr}(\widetilde{\mathscr{X}'})} \deg_{\mathscr{L}'}(V)\delta_{\eta_V},$$

where δ_{η_V} is the Dirac measure on η_V. It is a regular Borel measure with total mass $\deg_{L'}(X')$. To define a regular Borel measure $c_1(\overline{L})^{\wedge d}$ on X^{an}, let $\nu^{\mathrm{an}} : (X'_{\mathbb{K}})^{\mathrm{an}} \to X^{\mathrm{an}}$ be the morphism of analytic spaces associated to ν. Consider the pushout $\nu^{\mathrm{an}}_*(c_1(\overline{L}')^{\wedge d})$. Then one shows that this does not depend on the choice of $\mathscr{X}'$ or ν, and hence it is well defined for $(X, \overline{L})$. We define $c_1(\overline{L})^{\wedge d}$ to be $\nu^{\mathrm{an}}_*(c_1(\overline{L}')^{\wedge d})$. This measure also has total mass $\deg_L(X)$. It is called the *Chambert-Loir measure* of $\overline{L}$. It is originally defined by Chambert-Loir [8] and is also developed by Gubler [14, §3]

Remark 5.3. — If one works over the framework of admissible formal schemes, one can define the Chambert-Loir measure without using the generically finite morphism ν as above. In fact, one can take a proper flat model of $(X, \overline{L})$ in the category of admissible formal schemes with reduced fiber, and using those models will lead to the definition of the Chambert-Loir measure.

Next, assume that $\overline{L}$ is a $\mathbb{Q}$-algebraically metrized line bundle on X and let N be a positive integer such that $\overline{L}^{\otimes N}$ is a line bundle with an algebraic metric associated to a model that can be defined over a CDV ring. In this case, we define

$$c_1(\overline{L})^{\wedge d} := \frac{1}{N^d} c_1\left(\overline{L}^{\otimes N}\right)^{\wedge d}.$$

This is a regular Borel measure of total mass $\deg_L(X)$.

Finally, let $\|\cdot\|$ be a semipositive metric. We take a sequence $(\|\cdot\|_n)_{n \in \mathbb{N}}$ of semipositive $\mathbb{Q}$-algebraic metrics that can be define over a CDV subring such that $\|\cdot\|$ is the uniform limit of $(\|\cdot\|_n)_{n \in \mathbb{N}}$.

Proposition 5.4 ([8]). — *With the notation above, the sequence of regular Borel measures $(c_1(L, \|\cdot\|_n)^{\wedge d})_{n \in \mathbb{N}}$ weakly converges to a regular Borel measure. Furthermore, the weak limit does not depend on the choice of the sequence $(\|\cdot\|_n)_{n \in \mathbb{N}}$ and depends only on $\|\cdot\|$.*

By the above proposition, we define the *Chambert-Loir measure* $c_1(\overline{L})^{\wedge d}$ of $\overline{L} = (L, \|\cdot\|)$ to be the weak limit of $(c_1(L, \|\cdot\|_n)^{\wedge d})_{n \in \mathbb{N}}$. This has total mass $\deg_L(X)$.

5.6. Nonarchimedean canonical metrics and canonical measures. — Let A be an abelian variety over $\mathbb{K}$. Let L be an even line bundle. A *rigidification of L* is an isomorphism $L(0) \cong \mathbb{K}$, where 0 is the zero element of A.

For a natural number $m > 1$, there exists an isomorphism $\phi : [m]^*(L) \to L^{\otimes m^2}$, where $[m] : A \to A$ is the m times endomorphism. Once we fix a rigidification $L(0) = \mathbb{K}$ of L, the isomorphism which respects the rigidification is unique. Indeed, then $[m]^*(L)$ has the rigidification $[m]^*(L)(0) \cong \mathbb{K}$ by pullback, and $L^{\otimes m^2}$ has the rigidification $L^{\otimes m^2}(0) \cong \mathbb{K}^{\otimes m^2} = \mathbb{K}$ by tensor product, so that we have an isomorphism

$$\mathbb{K} \xrightarrow{\cong} [m]^*(L)(0) \xrightarrow{\phi} L^{\otimes m^2} \xrightarrow{\cong} \mathbb{K}.$$

We have a unique choice of ϕ in such a way that the above isomorphism is an identity. That is an isomorphism ϕ which respects the rigidification.

Proposition 5.5 ([7, Thm. 9.5.4]). — *Let A and L be as above. Fix a rigidification of L. Then there exists a unique metric $\|\cdot\|$ on L with the following property. For any m, let $\phi : [m]^*L \to L^{\otimes m^2}$ be the isomorphism which respects the rigidification. Then ϕ induces an isometry $[m]^*(L, \|\cdot\|) \cong (L^{\otimes m^2}, \|\cdot\|^{\otimes m^2})$ of metrized line bundles. Furthermore, suppose that L is ample. Then this metric is semipositive.*

We call the metric in this proposition the *canonical metric of L (with respect to the rigidification)*. The canonical metric depends on the choice of the rigidification, but the difference is only up to positive constant multiple.

We do not give a precise proof of the above theorem, but when L is ample, we briefly describe how the semipositive canonical metric is constructed on L under the following assumption: the abelian variety A and the even ample line bundle L can be defined over a CDV subfield

$\mathbf{K}$; further, the rigidification we consider can be defined over $\mathbf{K}$. (Suppose $A = A' \otimes_{\mathbf{K}} \mathbb{K}$ and $L = L' \otimes_{\mathbf{K}} \mathbb{K}$. Then we say that the rigidification $L(0) \cong \mathbb{K}$ can be defined over $\mathbf{K}$ if it is the base-change of an isomorphism $L'(0) \cong \mathbf{K}$.) We construct the metric by making a sequence of semipositive $\mathbb{Q}$-algebraic metrics by induction. We start with any proper flat model $(\mathscr{A}_1, \mathscr{L}_1)$ over $\mathbf{K}^{\circ}$ such that $\mathscr{L}_1$ is vertically nef. We can take such a model because L is ample. Fix a natural number $m > 1$. The m-times homomorphism $[m] : A \to A$ does not extend to $\mathscr{A}_1 \to \mathscr{A}_1$ in general, but if we take a suitable proper model $\mathscr{A}_2$ of A, then $[m] : A \to A$ extends to a morphism $f_1 : \mathscr{A}_2 \to \mathscr{A}_1$. Set $\mathscr{L}_2 := f_1^*(\mathscr{L}_1)$ The pair $(\mathscr{A}_2, \mathscr{L}_2)$ is a model of $(A, [m]^*(L))$. Since $\mathscr{L}_1$ is vertically nef, so is $\mathscr{L}_2$. Since we have the isomorphism $[m]^*(L) \cong L^{\otimes m^2}$ which respects the rigidification, that model is naturally regarded as a model of $(A, L^{\otimes m^2})$. This gives us an algebraic metric on $L^{\otimes m^2}$ and hence defines a $\mathbb{Q}$-algebraic metric $\|\cdot\|_2$ on L. Since $f^*(\mathscr{L}_1)$ is vertically nef, $\|\cdot\|_2$ is semipositive.

Next, we take a model $\mathscr{A}_3$ with a morphism $f_2 : \mathscr{A}_3 \to \mathscr{A}_2$ extending $[m] : A \to A$. Set $\mathscr{L}_3 := f_2^*(\mathscr{L}_2)$, which is vertically nef, and consider $(\mathscr{A}_3, \mathscr{L})$. Then via the isomorphism $[m]^*(L) \cong L^{\otimes m^2}$, this is a model of $(A, L^{\otimes m^4})$. Thus this defines a semipositive $\mathbb{Q}$-algebraic metric $\|\cdot\|_3$ on L. Repeating this process, we obtain a sequence

$$(5.1) \qquad\qquad (\pi_n : \mathscr{A}_n \to \operatorname{Spec}(\mathbf{K}^{\circ}), \mathscr{L}_n)_{n \in \mathbb{N}}$$

whose nth term is a model of $(A, L^{\otimes m^{2(n-1)}})$ and which gives a sequence of semipositive $\mathbb{Q}$-algebraic metrics $(\|\cdot\|_n)_{n \in \mathbb{N}}$ on L. One can show that this sequence uniformly converges to a metric $\|\cdot\|$ on L. It is semipositive, and furthermore it has the required property.

Remark 5.6. — Suppose that the abelian variety A over $\mathbb{K}$ is nondegenerate. Then the canonical metric is an algebraic metric. Indeed, fix a rigidification $L(0) = \mathbb{K}$. Since A is nondegenerate, we take a model $(\mathscr{A}, \mathscr{L})$ of (A, L) such that $\pi : \mathscr{A} \to \operatorname{Spec}(\mathbb{K}^{\circ})$ is an abelian scheme. Let 0_{π} be the zero-section of π. Then the identity $L(0) = \mathbb{K}$ extends to an isomorphism $0_{\pi}(\mathscr{L}) \otimes \mathscr{N} \cong \mathbb{K}^{\circ}$ for some line bundle $\mathscr{N}$ on $\operatorname{Spec}(\mathbb{K}^{\circ})$. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes \pi^*(\mathscr{N})$, we assume that the rigidification extends to $0_{\pi}^*(\mathscr{L}) \cong \mathbb{K}^{\circ}$. Then for any $n \in \mathbb{N}$, we see that $[n]^*(\mathscr{L}) \cong \mathscr{L}^{\otimes n^2}$. It follows that the metric $\|\cdot\|_{\mathscr{L}}$ satisfies $[n]^*(\|\cdot\|_{\mathscr{L}}) = \|\cdot\|_{\mathscr{L}}^{\otimes n^2}$ on L. Thus the canonical metric is an algebraic metric.

Finally, we define the canonical measure. Let A and L be as above. Fix a rigidification of L, and let $\overline{L}$ be the line bundle L with the canonical metric. Let X be a closed subvariety of A. Since the canonical metric is semipositive, the metric of the restriction $\overline{L}|_X$ is also semipositive by Proposition 5.4. It follows that we have the Chambert-Loir measure $c_1(\overline{L}|_X)^{\wedge d}$ on X^{an}. By the semipositivity of the canonical metric, this is a positive measure. We can show that this measure does not depend on the choice of the rigidification, and hence it is well defined for L. One often calls it the canonical measure, but in this paper, we call the probability measure

$$\mu_{X_v^{\mathrm{an}}, L} := \frac{1}{\deg_L(X)} c_1\left(\overline{L}|_X\right)^{\wedge d},$$

given by normalization, the *canonical measure on X^{an} associated to L*.

Remark 5.7. — Suppose that the abelian variety A over $\mathbb{K}$ is nondegenerate. Then since the canonical metric is an algebraic metric (cf. Remark 5.6), the canonical measure is a linear combination of Dirac measures of points.

6. Proof of Gubler's theorem

Recall that in the proof of Zhang over number fields, analytic spaces over an archimedean place, canonical measures on analytic spaces, and equidistribution theorem were key ingredients. Over function fields, we have (candidates for) the counterparts of the first two items; they are Berkovich analytic spaces over a nonarchimedean place and the canonical measures on them.

However, we have not yet had that of the third. Indeed, the equidistribution theorem on Berkovich spaces had not been established yet at the time when Gubler proved his theorem. In such a situation, He considered the tropicalization of closed subvarieties of totally degenerate abelian varieties and canonical measures, and he established the equidistribution theorem over tropical varieties with respect to the tropical canonical measure, instead.

In the sequel, we follow [14, §A] for terminology of basic notions on convex geometry.

6.1. Uniformization and tropicalization of totally degenerate abelian varieties. —
In this subsection, let $\mathbb{K}$ be an algebraically closed field that is complete with respect to a non-trivial nonarchimedean value $|\cdot|$. (The most important example in this paper is $\overline{K}_v$.) The subset $\{-\log|a| \mid a \in \mathbb{K}^\times\}$ is called the value group of $|\cdot|$. For simplicity, we assume that the value group equals $\mathbb{Q}$. For an algebraic variety X over $\mathbb{K}$, let X^{an} denote the associated Berkovich analytic space.

Let A be an abelian variety over $\mathbb{K}$ of dimension n. Assume that A is totally degenerate. Then there exists a homomorphism of group analytic spaces $p : (\mathbb{G}_m^n)_{\mathbb{K}}^{\mathrm{an}} \to A^{\mathrm{an}}$ such that $\mathrm{Ker}(p)$ is a free abelian group of rank n. (That is an alternative definition of totally degenerate abelian varieties.) This p is the universal covering of $(A^{\mathrm{an}}, 0)$. Let $x_1, \ldots, x_n$ be the standard coordinates of $\mathbb{G}_m^n$. For any x_i $(i = 1, \ldots, n)$ and $P = (\iota(P), |\cdot|) \in (\mathbb{G}_m^n)_{\mathbb{K}}^{\mathrm{an}}$ (see §5.2 for the notation), write $|x_i(P)| := |x_i(\iota(P))|$ where $x_i(\iota(P))$ the image of x_i in the residue field $\kappa(\iota(P))$ of $\mathcal{O}_{(\mathbb{G}_m^n)_{\mathbb{K}}}$ at $\iota(P)$. We consider the map $\mathrm{val} : (\mathbb{G}_m^n)_{\mathbb{K}}^{\mathrm{an}} \to \mathbb{R}^n$ defined by

$$(6.1) \qquad \mathrm{val}(P) = (-\log|x_1(P)|, \ldots, -\log|x_n(P)|).$$

Then one shows that $\Lambda := \mathrm{val}(\mathrm{Ker}(p))$ is a complete lattice of $\mathbb{R}^n$. Thus val descends to a homomorphism $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$. This homomorphism called the *tropicalization map* of A^{an}. To summarize, we obtain the following commutative diagram in which all the maps are continuous homomorphisms:

$$
\begin{array}{ccc}
(\mathbb{G}_m^n)_{\mathbb{K}}^{\mathrm{an}} & \longrightarrow & A^{\mathrm{an}} \\
\downarrow & & \downarrow \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^n/\Lambda.
\end{array}
$$

The tropicalization maps are compatible with homomorphisms of abelian varieties and direct products. To be precise, let A_1 and A_2 be an abelian varieties over $\mathbb{K}$, and suppose that A_1 and A_2 are totally degenerate. Let $\phi : A_1 \to A_2$ be a homomorphism of abelian varieties. Let $p_1 : (\mathbb{G}_m^{n_1})_{\mathbb{K}}^{\mathrm{an}} \to A_1^{\mathrm{an}}$ and $p_2 : (\mathbb{G}_m^{n_2})_{\mathbb{K}}^{\mathrm{an}} \to A_2^{\mathrm{an}}$ be the uniformizations. Then ϕ lifts to a homomorphism $(\mathbb{G}_m^{n_1})_{\mathbb{K}}^{\mathrm{an}} \to (\mathbb{G}_m^{n_2})_{\mathbb{K}}^{\mathrm{an}}$. Furthermore, this induces a linear map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, which descends to a homomorphism $\phi_{\mathrm{aff}} : \mathbb{R}^{n_1}/\Lambda_1 \to \mathbb{R}^{n_2}/\Lambda_2$ of real tori. Further, consider the direct product $A_1 \times A_2$ and let $\overline{\mathrm{val}} : (A_1 \times A_2)^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ be the tropicalization. Then we have $\mathbb{R}^n/\Lambda = \mathbb{R}^{n_1}/\Lambda_1 \times \mathbb{R}^{n_2}/\Lambda_2$ naturally.

The following lemma is used in Gubler's proof.

Lemma 6.1. — *Let A_1 and A_2 be an abelian varieties over $\mathbb{K}$. Let $\phi : A_1 \to A_2$ be a homomorphism of abelian varieties. Suppose that A_1 is totally degenerate and that ϕ is surjective. Then A_2 is totally degenerate.*

Proof. — See [13, Lem. 6.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following theorem is a fundamental theorem of tropical analytic geometry.

Theorem 6.2 ([14, Thm. 1.1]). — *Let A be a totally degenerate abelian variety over $\mathbb{K}$ and let X be a closed subvariety of A of dimension d. Then the image $\overline{\mathrm{val}}(X^{\mathrm{an}})$ is a rational polytopal subset of $\mathbb{R}^n/\Lambda$ of pure dimension d.*

We call $\overline{\mathrm{val}}(X^{\mathrm{an}})$ the *tropicalization of X^{an}* or the *tropical variety associated to X*.

6.2. Tropical canonical measure. — From here on to the end of this section, let K be a function field unless otherwise specified. Let A be an abelian variety over $\overline{K}$. Let v be a place of $\overline{K}$. Assume that $A_v := A \otimes_{\overline{K}} \overline{K}_v$ is totally degenerate. Let $\overline{\mathrm{val}} : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ be the tropicalization map of A_v, where $n := \dim(A)$.

Let X be a closed subvariety of A of dimension d. We consider the canonical measure $\mu_{X_v^{\mathrm{an}}}$ on X_v^{an} of an even line bundle L on A. The pushout $\overline{\mathrm{val}}_*(\mu_{X_v^{\mathrm{an}}})$ is a measure on $\overline{\mathrm{val}}(X_v^{\mathrm{an}})$, which is called a *tropical canonical measure*. We write $\mu_{X_v^{\mathrm{an}}}^{\mathrm{trop}} := \overline{\mathrm{val}}_*(\mu_{X_v^{\mathrm{an}}})$.

The following theorem describes the tropical canonical measures.

Theorem 6.3. — *Let A, v, $\overline{\mathrm{val}} : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$, and X be as above. Let L be an even line bundle on A and let $\mu_{X_v^{\mathrm{an}}}$ be the canonical measure associated to L. Then there exist rational simplices $\Delta_1, \ldots, \Delta_m$ of $\mathbb{R}^n/\Lambda$ of dimension d and positive real numbers $r_1, \ldots, r_m$ such that $\mathrm{relin}(\Delta_i) \cap \mathrm{relin}(\Delta_j) = \emptyset$ for $i \neq j$, $\overline{\mathrm{val}}(X_v^{\mathrm{an}}) = \bigcup_{i=1}^m \Delta_i$, and such that*

$$\mu_{X_v^{\mathrm{an}}}^{\mathrm{trop}} = \sum_{i=1}^m r_i \delta_{\Delta_i}$$

where $\mathrm{relin}(\Delta_i)$ is the relative interior of Δ_i and δ_{Δ_i} is the relative Lebesgue measure on Δ_i. Furthermore, if L is ample, then $r_i > 0$ for all $i = 1, \ldots, m$.

Remark 6.4. — Let Δ be a simplex with $\Delta \subset \overline{\mathrm{val}}(X_v^{\mathrm{an}})$. Suppose that L is ample. By Theorem 6.3, if $\dim(\Delta) = d$, then $\mu_{X_v^{\mathrm{an}}}^{\mathrm{trop}}(\Delta) > 0$; if $\dim(\Delta) < d$, then $\mu_{X_v^{\mathrm{an}}}^{\mathrm{trop}}(\Delta) = 0$.

6.3. Tropical equidistribution theorem. — Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Let K' be a finite extension of K over which A and X can be defined. Let $\mathrm{Aut}_{K'}(\overline{K})$ be the group of automorphisms of $\overline{K}$ over K'. For a point $x \in X(\overline{K})$, let $O(x)$ denote the $\mathrm{Aut}_{K'}(\overline{K})$-orbit of x in $X(\overline{K})$. Let $v \in M_{\overline{K}}$ be a place and regard $X(\overline{K}) \subset X_v^{\mathrm{an}}$.

Theorem 6.5 ([15, Thm. 5.5]). — *Let A, X, and $O(x)$ for $x \in X(\overline{K})$ be as above. Let $(x_i)_{i \in I}$ be a generic small net on $X(\overline{K})$ (cf. §2.5). Let $\mu_{X_v^{\mathrm{an}}}$ be the canonical measure on X_v^{an} of an even ample line bundle on A. Assume that A_v is totally degenerate. Let $\overline{\mathrm{val}} : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ be the tropicalization map. Then we have a weak convergence*

$$\lim_i \frac{1}{\#O(x_i)} \sum_{z \in O(x_i)} \delta_{\overline{\mathrm{val}}(z)} = \mu_{X_v^{\mathrm{an}}}^{\mathrm{trop}}$$

of regular Borel measures on $\overline{\mathrm{val}}(X_v^{\mathrm{an}})$, *where* δ_z *is the Dirac measure on* z.

6.4. Proof of Gubler. — Let us start the proof of Gubler. To argue by contradiction, suppose that there exist an abelian variety A over $\overline{K}$ that is totally degenerate at some place v and a non-torsion subvariety X of A with dense small points. Let G_X be the stabilizer of X. Then A/G_X is totally degenerate at v by Lemma 6.1, and X/G_X has dense small points by Lemma 2.15. Replacing A and X with A/G_X and X/G_X respectively, we may assume that X is a non-torsion subvariety of A, has dense small points, and has trivial stabilizer. Since the theorem holds for 0-dimensional subvarieties (cf. Remark 3.5 (1)), we have $d := \dim(X) > 0$. For a positive integer N, let $\alpha_N : X^N \to A^{N-1}$ be the difference morphism, given by $(x_1, \ldots, x_N) \mapsto (x_1 - x_2, \ldots, x_{N-1} - x_N)$. Set $Z := X^N \subset A^N$ and $Y := \alpha_N(Z)$. Since X has trivial stabilizer, there exists an N such that the restriction $\alpha : Z \to Y$ is generically finite by Lemma 4.6.

Note that $(A^N)_v$ and $(A^{N-1})_v$ are totally degenerate. Let $\overline{\mathrm{val}}_1 : (A^N)_v^{\mathrm{an}} \to \mathbb{R}^{Nn}/\Lambda_1$ and $\overline{\mathrm{val}}_2 : (A^{N-1})_v^{\mathrm{an}} \to \mathbb{R}^{(N-1)n}/\Lambda_2$ be the tropicalization maps, where $n := \dim(A)$. The induced affine homomorphism $\mathbb{R}^{Nn}/\Lambda_1 \to \mathbb{R}^{(N-1)n}/\Lambda_2$ restricts to a surjective piecewise affine map $\overline{\alpha}_{\mathrm{aff}} : \overline{\mathrm{val}}_1(Z_v^{\mathrm{an}}) \to \overline{\mathrm{val}}_2(Y_v^{\mathrm{an}})$ between Nd-dimensional polytopal sets. Let $\mu_{Z_v^{\mathrm{an}}}$ and $\mu_{Y_v^{\mathrm{an}}}$ be the canonical measures on Z_v^{an} and Y_v^{an} of even ample line bundles, respectively. We put $\mu_{Z_v^{\mathrm{an}}}^{\mathrm{trop}} := (\overline{\mathrm{val}}_1)_*(\mu_{Z_v^{\mathrm{an}}})$ and $\mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}} := (\overline{\mathrm{val}}_2)_*(\mu_{Y_v^{\mathrm{an}}})$. Since X has dense small points, so does Z. By Lemma 2.17, there exists a generic small net $(z_i)_{i \in I}$ of $Z(\overline{K})$. Since $\alpha : Z \to Y$ is surjective, $(\alpha(z_i))_{i \in I}$ is a generic net on $Y(\overline{K})$, and it follows from Proposition 2.6 (2) that $(\alpha(z_i))_{i \in I}$ is small. Since we have the equidistribution theorem (Theorem 6.5), the same argument as Zhang's proof using the equidistribution theorem gives us $(\overline{\alpha}_{\mathrm{aff}})_*(\mu_{Z_v^{\mathrm{an}}}^{\mathrm{trop}}) = \mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}}$.

Let $\overline{\mathrm{val}}_0 : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda_0$ be the tropicalization map. Since the tropicalization is compatible with direct products, we have $\mathbb{R}^{Nn}/\Lambda_1 = (\mathbb{R}^n/\Lambda_0)^N$ and $\overline{\mathrm{val}}_1(Z_v^{\mathrm{an}}) = (\overline{\mathrm{val}}_0(X_v^{\mathrm{an}}))^N$. Since $d = \dim(X) > 0$, the diagonal of $\overline{\mathrm{val}}_1(Z_v^{\mathrm{an}})$ has positive dimension (cf. Theorem 6.2). Since $\overline{\alpha}_{\mathrm{aff}}$ is linear and contracts the diagonal to a point, there exists an Nd-dimensional simplex $\Delta \subset \overline{\mathrm{val}}_1(X_v^{\mathrm{an}})$ such that $\dim(\overline{\alpha}_{\mathrm{aff}}(\Delta)) < Nd$. Since $\mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}}$ is a piecewise relative Lebesgue measure of dimension Nd, it follows that $\mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}}(\overline{\alpha}_{\mathrm{aff}}(\Delta)) = 0$ (cf. Remark 6.4). On the other hand, since $(\overline{\alpha}_{\mathrm{aff}})_*(\mu_{Z_v^{\mathrm{an}}}^{\mathrm{trop}}) = \mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}}$, we have

$$\mu_{Y_v^{\mathrm{an}}}^{\mathrm{trop}}(\overline{\alpha}_{\mathrm{aff}}(\Delta)) \geq \mu_{Z_v^{\mathrm{an}}}^{\mathrm{trop}}(\Delta).$$

Since $\dim(\Delta) = \dim(Z) = Nd$, the right-hand sides of this inequality should be positive (cf. Remark 6.4). However, that is a contradiction. Thus the proof is complete.

6.5. Nonarchimedean equidistribution theorem. — Gubler had to establish the tropical equidistribution theorem (Theorem 6.5) because no equidistribution theorem that could be applied to his setting was known when he proved the theorem. However, inspired by Yuan's work [44], he established in [15] a nonarchimedean equidistribution theorem, afterwards.

Proposition 6.6 ([15, Thm. 1.1]). — *Let* A *be any abelian variety over* $\overline{K}$ *and let* X *be a closed subvariety of* A. *Let* $O(x)$ *for* $x \in X(\overline{K})$ *be as in §6.3. Let* $v \in M_{\overline{K}}$ *be a place and regard* $X(\overline{K}) \subset X_v^{\mathrm{an}}$. *Let* $\mu_{X_v^{\mathrm{an}}}$ *be the canonical measure on* X_v^{an} *of an even ample line bundle*

on A. Let $(x_i)_{i \in I}$ be a generic small net on $X(\overline{K})$. Then we have a weak convergence

$$\lim_i \frac{1}{\#O(x_i)} \sum_{z \in O(x_i)} \delta_z = \mu_{X_v^{\mathrm{an}}} \, ,$$

where δ_z is the Dirac measure on the point $z \in X_v^{\mathrm{an}}$.

6.6. Naive imitation of Zhang's proof. — In the last subsection, we have finally obtained candidates for counterparts of the three ingredients of Zhang's proof. Now, let us try to apply Zhang's proof to the setting of the geometric Bogomolov conjecture. To show the conjecture by contradiction, suppose that there exists a counterexample to the geometric Bogomolov conjecture for abelian varieties. Then by the same argument as in the proofs of Zhang's theorem and Gubler's theorem, there exist an abelian variety A over $\overline{K}$ and a non-special closed subvariety X of A such that X has dense small points, $d := \dim(X) > 0$, and X has trivial stabilizer. For a positive integer N, we consider the difference homomorphism $\alpha_N : X^N \to A^{N-1}$, and let $\alpha : Z \to Y$ be the restriction from $Z := X^N$ to $Y := \alpha_N(Z)$. For large N, α is generically finite by Lemma 4.6, as well as surjective. Since X has dense small points, so does Z. Fix a $v \in M_{\overline{K}}$. Let $\alpha^{\mathrm{an}} : Z_v^{\mathrm{an}} \to Y_v^{\mathrm{an}}$ be the associated map between analytic spaces. Since we have Proposition 6.6, the same argument gives us

$$(6.2) \qquad\qquad\qquad\qquad \alpha_*^{\mathrm{an}}(\mu_{Z_v^{\mathrm{an}}}) = \mu_{Y_v^{\mathrm{an}}} \, ,$$

where $\mu_{Z_v^{\mathrm{an}}}$ and $\mu_{Y_v^{\mathrm{an}}}$ are the canonical measures on Z_v^{an} and Y_v^{an} associated to even ample line bundles, respectively.

Then the problem is how to deduce a contradiction from equality (6.2). In Gubler's totally degenerate setting, we can deduce a contradiction from (6.2). Indeed, assume that A_v is totally degenerate. Then one shows that the support of $\mu_{Z_v^{\mathrm{an}}}$ and that of $\mu_{Y_v^{\mathrm{an}}}$ have structures of polytopal set of pure dimension $Nd = \dim(Z) = \dim(Y)$. Further, for a suitable simplicial decompositions, $\mu_{Z_v^{\mathrm{an}}}$ and $\mu_{Y_v^{\mathrm{an}}}$ are linear combinations of relative Lebesgue measures of simplices of dimension Nd. Roughly speaking, this says that we have the same situation as Gubler's proof without passing to the tropicalization. Therefore, we can get a contradiction by the same way.

However, equality (6.2) does not necessarily lead to a contradiction, in general. For example, assume that A_v is nondegenerate. Then A_v^N and A_v^{N-1} are also nondegenerate. In the nondegenerate case, the canonical measures are linear combinations of Dirac measure of points, as is noted in Remark 5.7. Thus there is nothing strange with equality (6.2).

That observation suggests that if A is nowhere degenerate, then the equidistribution method is not efficient; it is not almighty. If we wish to apply this method to the geometric Bogomolov conjecture, we have to restrict the setting where the method works. In what follows, we will apply the equidistribution argument to the proof of a weaker version of Theorem 3.7 (cf. Proposition 8.2). In that setting, a detailed analysis of the canonical measures gives us enough information to lead to a contradiction in (6.2).

7. Structure of canonical measures

In this section, we recall the structure of canonical measure due to Gubler. We refer to [16] for the detail.

Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Further, fix a $v \in M_{\overline{K}}$ and put $\mathbb{K} := \overline{K}_v$. We write $X_v := X \otimes_{\overline{K}} \mathbb{K}$. Note that X_v can be defined over a CDV subfield (of $\mathbb{K}$).

7.1. Semistable alteration.

— The canonical measures on X^{an} are described by Gubler by using a semistable alteration. Let A_v and X_v be as above. By a *semistable alteration* for X, we mean a pair $(\mathscr{X}', f : X' \to X_v)$ consisting of a strictly semistable scheme $\mathscr{X}'$ over $\mathbb{K}$ that can be defined over a CDV subring and a generically finite surjective morphism $f : X' \to X$ where X' is the generic fiber of $\mathscr{X}'$. By de Jong's semistable alteration theorem in [17], there always exists a semistable alteration for X_v.

To describe the canonical measures on X^{an}, actually, one needs to use a semistable alteration $(\mathscr{X}', f : X' \to X_v)$ with an additional property. Indeed, Gubler gives a description of the canonical measure on X^{an} in terms of the skeleton of $\mathscr{X}'$, which we are going to explain in this section, under the condition that the semistable alteration $(\mathscr{X}', f : X' \to X_v)$ for X_v is "compatible with some Mumford model of A_v".

A Mumford model of A_v is proper admissible formal scheme $\mathscr{A} \to \mathrm{Spf}(\mathbb{K}^\circ)$ with "Raynaud generic fiber" A_v^{an} that is constructed from a (rational) polytopal decomposition of a real torus via the valuation map. In the appendix, we give a quick review of admissible formal schemes, their Raynaud generic fibers, and the valuation maps for abelian varieties. However, we do not give the definition of Mumford models in this paper. In fact, the readers do not need to know the definition in the following arguments if they assume Lemmas 7.1 and 7.2 as black boxes.

Lemma 7.1. — *Let A_1 and A_2 be abelian varieties over $\overline{K}$, and let $\phi : (A_1)_v \to (A_2)_v$ be a homomorphism of abelian varieties. Then there exist Mumford models $\mathscr{A}_1$ and $\mathscr{A}_2$ of $(A_1)_v$ and $(A_2)_v$, respectively, and a morphism $\varphi : \mathscr{A}_1 \to \mathscr{A}_2$ whose restriction $(A_1)_v^{\mathrm{an}} \to (A_2)_v^{\mathrm{an}}$ to the Raynaud generic fibers coincides with the analytification $\phi^{\mathrm{an}} : (A_1)_v^{\mathrm{an}} \to (A_2)_v^{\mathrm{an}}$ of ϕ.*

A semistable alteration $(\mathscr{X}', f : X' \to X_v)$ for X_v is said to be *compatible with some Mumford model of A_v* if there exists a Mumford model $\mathscr{A}$ of A_v^{an} such that the morphism $f : X' \to X_v \hookrightarrow A_v$ extends to a morphism $\widehat{\mathscr{X}'} \to \mathscr{A}$ of admissible formal schemes, where $\widehat{\mathscr{X}'}$ is the formal completion of $\mathscr{X}'$ with respect to an element $\varpi \in \mathbb{K}$ with $0 < |\varpi| < 1$. The following lemma says that for a given Mumford model, there exists a semistable alteration that is compatible with the Mumford model.

Lemma 7.2. — *Let $\mathscr{A} \to \mathrm{Spf}(\mathbb{K}^\circ)$ be a Mumford model of A_v. Then there exists semistable alteration $(\mathscr{X}', f : X' \to X_v)$ for X_v such that the morphism f extends to a morphism $\widehat{\mathscr{X}'} \to \mathscr{A}$ of admissible formal schemes.*

Remark 7.3. — In the sequel, we only consider semistable alterations that are compatible with some Mumford model.

7.2. Gubler's description of canonical measures.

— Let L be an even ample line bundle on A. Let L_v denote the pullback of L to A_v. Put a canonical metric $\|\cdot\|$ (with respect to a rigidification of L) on L_v, and we have a canonically metrized line bundle $\overline{L_v} = (L_v, \|\cdot\|)$. The canonical metric is a semipositive metric (cf. Proposition 5.5), and hence the restriction $\overline{L_v}|_{X_v}$ is also a semipositive metric by Lemma 5.2.

Let $(\mathscr{X}', f : X' \to X_v)$ be a semistable alteration (cf. Remark 7.3). By Lemma 5.2, $f^*(\overline{L_v})$ is a semipositively metrized line bundle on $(X')^{\mathrm{an}}$, so that we consider the (normalized) Chambert-Loir measure $\frac{1}{\deg_{f^*(L)}(X')} c_1(f^*(\overline{L_v}))^{\wedge d}$ on $(X')^{\mathrm{an}}$, where $d := \dim(X) = \dim(X')$. For a stratum S of $\widetilde{\mathscr{X}'}$, let Δ_S denote the canonical simplex corresponding to S. Recall that the skeleton of $(X')^{\mathrm{an}}$ associated to $\mathscr{X}'$ is $S(\mathscr{X}') = \bigcup_{S \in \mathrm{str}(\widetilde{\mathscr{X}'})} \Delta_S$.

Proposition 7.4 ([16, Cor. 6.9]). — *With the above notation, we can express*

$$\frac{1}{\deg_{f^*(L)}(X')} c_1(f^*(\overline{L_v}))^{\wedge d} = \sum_{S \in \mathrm{str}(\widetilde{\mathscr{X}'})} r_S \delta_{\Delta_S},$$

where r_S is a nonnegative real number, Δ_S is the canonical simplex corresponding to S, and δ_{Δ_S} is the relative Lebesgue measure on Δ_S. Furthermore, whether r_S is positive or not does not depend on the choice of the even ample line bundle L on A.

Remark 7.5. — (1) In [16], Gubler defines the notion that Δ_S is *non-degenerate with respect to f*. We omit the Gubler's definition in this paper but remark that this notion is determined by $(\mathscr{X}', f : X' \to X_v)$ and does not depend on the choice of L. He proves that Δ_S is nondegenerate if and only if $r_S > 0$ in the description of Proposition 7.4 for any even ample line bundle L. Therefore, if we say that Δ_S is non-degenerate with respect to f, this means that $r_S > 0$ in Proposition 7.4 for one and hence any even ample line bundle L.

(2) Let $f^{\mathrm{an}} : (X')^{\mathrm{an}} \to X_v^{\mathrm{an}}$ be the morphism of analytic spaces associated to f. If Δ_S is non-degenerate with respect to f, then f^{an} restricts to an homeomorphism from Δ_S to $f^{\mathrm{an}}(\Delta_S)$. In fact, this property is a part of the definition of non-degeneracy due to Gubler.

Let $\mathrm{str}_{f-\mathrm{nd}}(\widetilde{\mathscr{X}'})$ be the set of strata of $\widetilde{\mathscr{X}'}$ that are non-degenerate with respect to f. It is known that the projection formula holds for Chambert-Loir measures (cf. [16, Prop. 3.8]), so that

$$(7.1) \qquad\qquad \mu_{X_v^{\mathrm{an}}, L} = f_* \left(\frac{1}{\deg_{f^*(L)}(X')} c_1(f^*(\overline{L_v}))^{\wedge d} \right).$$

By Proposition 7.4, we obtain

$$(7.2) \qquad\qquad \mathrm{Supp}(\mu_{X_v^{\mathrm{an}}, L}) = \bigcup_{S \in \mathrm{str}_{f-\mathrm{nd}}(\widetilde{\mathscr{X}'})} f^{\mathrm{an}}(\Delta_S).$$

Noting Remark 7.5 (1), we see that $\mathrm{Supp}(\mu_{X_v^{\mathrm{an}}, L})$ does not depend on the choice of L. We set $S_{X_v^{\mathrm{an}}} := \mathrm{Supp}(\mu_{X_v^{\mathrm{an}}, L})$ and call it the *canonical subset of X_v^{an}*.

Gubler proved that $S_{X_v^{\mathrm{an}}}$ has a canonical piecewise rational affine structure such that for any semistable alteration $(\mathscr{X}', f : X' \to X_v)$ (cf. Remark 7.3) and for any stratum S non-degenerate with respect to f, the homeomorphism $\Delta_S \to f^{\mathrm{an}}(\Delta_S)$ (cf. Remark 7.5 (2)) given by f^{an} is a rational piecewise linear map (cf. [16, Thm. 6.12]). By Proposition 7.4 and (7.1), it follows that $\mu_{X_v^{\mathrm{an}}, L}$ is a linear combination of relative Lebesgue measures on polytopes on $S_{X_v^{\mathrm{an}}}$ with this piecewise linear structure of $S_{X_v^{\mathrm{an}}}$.

Remark 7.6. — If Δ_S is non-degenerate with respect to f, then $f^{\mathrm{an}}(\Delta_S)$ is regarded as a polytopal set of pure dimension $\dim(\Delta_S)$.

8. Reduction to the nowhere degenerate case

Under the preparation so far, we discuss Theorem 3.7. In this theorem, the nontrivial part is to show that (2) implies (1), that is: let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A; if the geometric Bogomolov conjecture holds for $\mathfrak{m}$, then it holds for A. This assertion is obtained by showing the following theorem.

Theorem 8.1. — *Let A be an abelian variety over $\overline{K}$ and let $\phi : A \to \mathfrak{m}$ be a surjective homomorphism, where $\mathfrak{m}$ is the maximal nowhere degenerate abelian subvariety of A. Let X be a closed subvariety of A. Suppose that X has dense small points and that $\phi(X)$ is a special subvariety. Then X is a special subvariety.*

Let us explain how Theorem 3.7 follows from Theorem 8.1. As is noted before, we only have to show that if the geometric Bogomolov conjecture holds for $\mathfrak{m}$, then it holds for A. First we note that there exists a surjective homomorphism $\phi : A \to \mathfrak{m}$. Indeed, by the Poincaré complete reducibility theorem, there exists an abelian subvariety G of A such that $\mathfrak{m} + G = A$ and $\mathfrak{m} \cap G$ is finite. Since the natural homomorphism $\mathfrak{m} \times G \to A$ is an isogeny, there exists an isogeny $A \to \mathfrak{m} \times G$ (cf. [28, p. 157]). This homomorphism composed with the natural projection to $\mathfrak{m}$ gives a desired ϕ. Now, suppose that X has dense small points. Then $\phi(X)$ also has dense small points by Lemma 2.15. Assume that the geometric Bogomolov conjecture holds for $\mathfrak{m}$. It follows that $\phi(X)$ is a special subvariety. By Theorem 8.1, we conclude that X is a special subvariety.

We do not give a complete proof of Theorem 8.1 in this paper. Instead, we give the idea of the proof of the following weaker version of the theorem. Here we call $\dim(\mathfrak{m})$ the nowhere degeneracy rank of A and denote it by nd-rk(A).

Proposition 8.2. — *Let A be an abelian variety over $\overline{K}$. Assume that $\mathrm{nd\text{-}rk}(A) = 0$. Let X be a closed subvariety of A. Suppose that X has dense small points. Then X is a torsion subvariety.*

The above proposition generalizes Gubler's theorem (cf. Theorem 3.1). Let A be an abelian over $\overline{K}$, and assume that it is totally degenerate at some place v. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Since $\mathfrak{m}_v$ is a non-degenerate abelian subvariety of A_v and since this is totally degenerate, $\mathfrak{m}_v$ is trivial. It follows that $\mathfrak{m} = 0$, namely, nd-rk(A) = 0. Thus one can apply the proposition to A to obtain the conclusion.

We remark that we may have nd-rk(A) = 0 even if A is not *totally* degenerate at any place. Indeed, there exists a simple abelian variety A that is not totally degenerate at any place but is degenerate at some place; since A is simple and degenerate, we have nd-rk(A) = 0.

8.1. Strict supports. — In our proof of Proposition 8.2, we apply the method of Zhang. We hope to get a contradiction from the equality (6.2). To do that, we define the notion of strict supports of measures and investigate the structure of the strict supports.

As we noted in §7.2, the canonical subset $S_{X_v^{\mathrm{an}}}$ has a piecewise affine structure, so that we have a notion of polytopal decomposition of $S_{X_v^{\mathrm{an}}}$. We say that a polytopal decomposition Σ

of $S_{X_v^{\mathrm{an}}}$ is f-*subdivisional* if

$$f^{\mathrm{an}}(\Delta_S) \cap \Sigma := \{\sigma \in \Sigma \mid f^{\mathrm{an}}(\Delta_S) \cap \sigma \neq \emptyset\}$$

is a polytopal decomposition of $f^{\mathrm{an}}(\Delta_S)$. Note that for any rational polytopal decomposition Σ_0 of $S_{X_v^{\mathrm{an}}}$, there exists a rational subdivision Σ of Σ_0 that is f-subdivisional.

Proposition 8.3. — *Let L be an even ample line bundle on A. Let Σ be an f-subdivisional rational polytopal decomposition of $S_{X_v^{\mathrm{an}}}$. Then*

$$\mu_{X_v^{\mathrm{an}},L} = \sum_{\sigma \in \Sigma} r'_\sigma \delta_\sigma$$

for some non-negative real numbers r'_σ, where δ_σ is the relative Lebesgue measure on σ.

Proof. — This follows from Proposition 7.4, (7.1), and the fact that $f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to f^{\mathrm{an}}(\Delta_S)$ is a piecewise linear map for Δ_S non-degenerate with respect to f. $\qquad\square$

Remark 8.4. — It follows from equality (7.1) and Remark 7.5 (1) that $r'_\sigma > 0$ if and only if there exists a non-degenerate stratum S of $\widetilde{\mathscr{X}'}$ such that $\sigma \subset \Delta_S$ and $\dim(\sigma) = \dim(\Delta_S)$.

Let σ be a polytope in the canonical subset $S_{X_v^{\mathrm{an}}}$ of X_v^{an}. Let $\mu_{X_v^{\mathrm{an}}}$ be the canonical measure on X_v^{an} associated to an even ample line bundle on A. We say that σ is a *strict support of* $\mu_{X_v^{\mathrm{an}}}$ if there exists an $\epsilon > 0$ such that $\mu_{X_v^{\mathrm{an}}} - \epsilon\delta_\sigma$ is a positive measure, where δ_σ is a relative Lebesgue measure on σ. By the last assertion of Proposition 7.4, this notion does not depend on the choice of the even ample line bundle on A and well defined for X_v.
Let $(\mathscr{X}', f : X' \to X_v)$ be a semistable alteration (cf. Remark 7.3). Let Σ be an f-subdivisional polytopal decomposition of $S_{X_v^{\mathrm{an}}}$.
By Proposition 8.3, we write

$$\mu_{X_v^{\mathrm{an}}} = \sum_{\sigma \in \Sigma} r'_\sigma \delta_\sigma$$

with $r'_\sigma \geq 0$. Then $\sigma \in \Sigma$ is a strict support of $\mu_{X_v^{\mathrm{an}}}$ if and only if $r'_\sigma > 0$.
We would like to see how $f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to X^{\mathrm{an}}$ behaves. If Δ_S is nondegenerate with respect to f, then the map $f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to S_{X_v^{\mathrm{an}}}$ is an isomorphism onto its image, which follows from the definition of the piecewise affine structure for $S_{X_v^{\mathrm{an}}}$. On the other hand, if Δ_S is degenerate (i.e. not non-degenerate), then the behavior of $f^{\mathrm{an}}|_{\Delta_S}$ may be complicated. Nevertheless, when $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$, which does not necessarily means that Δ_S is nondegenerate with respect to f, we can describe $f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to X^{\mathrm{an}}$ in the following sense.

Lemma 8.5. — *Take an $S \in \mathrm{str}(\widetilde{\mathscr{X}'})$. Suppose that $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$. Then we have the following.*

(1) *The map $f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to S_{X_v^{\mathrm{an}}}$ is a piecewise linear map. Furthermore, there exists a polytope $P \subset \Delta_S$ with $\dim(P) = \dim(\Delta_S)$ such that $f^{\mathrm{an}}(P)$ is a polytope and $f^{\mathrm{an}}|_P : P \to f^{\mathrm{an}}(P)$ is an affine map for some rational polytopal decomposition of $S_{X_v^{\mathrm{an}}}$.*

(2) *There exists a non-negative integer r with the following property. Let P be a polytope in Δ_S such that $\dim(P) = \dim(\Delta_S)$ and such that $f^{\mathrm{an}}|_P : P \to S_{X_v^{\mathrm{an}}}$ is an affine map for some rational polytopal decomposition of $S_{X_v^{\mathrm{an}}}$. Then $\dim(f^{\mathrm{an}}(P)) = r$.*

Proof. — It is known that there exists a continuous map $\overline{\mathrm{val}} : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$ for some n determined from A_v, where Λ is a complete lattice of $\mathbb{R}^n$ with $\Lambda \subset \mathbb{Q}^n$; see §A.2. Note that $\mathbb{R}^n/\Lambda$ has a natural piecewise affine structure induced from $\mathbb{R}^n$. The map $\overline{\mathrm{val}}$ has the following properties:

(a) The restriction $\overline{\mathrm{val}}|_{S_{X_v^{\mathrm{an}}}} : S_{X_v^{\mathrm{an}}} \to \mathbb{R}^n/\Lambda$ is a finite piecewise affine map;

(b) For any $S \in \mathrm{str}(\widetilde{\mathscr{X}'})$, the map $\overline{\mathrm{val}} \circ f^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to \mathbb{R}^n/\Lambda$ is an affine map.

We refer to [16, §4].

By (a) above, $f^{\mathrm{an}}(\Delta_S)$ is a finite union of polytopes of dimension r if and only if so is $\overline{\mathrm{val}}(f^{\mathrm{an}}(\Delta_S))$, and in fact, $f^{\mathrm{an}}(\Delta_S)$ is a polytope by (b). Thus we have the first assertion of (1). The second assertion is obvious.

To show (2), put $r := \dim((\overline{\mathrm{val}} \circ f^{\mathrm{an}})(\Delta_S))$ by (b). Then by (a), $\dim(f^{\mathrm{an}}(P)) = r$. This shows the assertion. $\square$

Suppose that $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$. Let r be the integer as in the above lemma. Then $f^{\mathrm{an}}(\Delta_S)$ is a finite union of r-dimensional polytopes. Thus $\dim(f^{\mathrm{an}}(\Delta_S))$ makes sense, i.e. $\dim(f^{\mathrm{an}}(\Delta_S)) = r$.

The following is an immediate consequence of Gubler's description of canonical measures.

Lemma 8.6. — *Let* $(\mathscr{X}', f : X' \to X_v)$ *be a semistable alteration (cf. Remark 7.3). Let* Σ *be an f-subdivisional polytopal decomposition of $S_{X_v^{\mathrm{an}}}$. Let* $\sigma \in \Sigma$ *be a strict support of $\mu_{X_v^{\mathrm{an}}}$. Then there exists a stratum $S \in \mathrm{str}(\widetilde{\mathscr{X}'})$ with the following properties:*

(1) $\sigma \subset f^{\mathrm{an}}(\Delta_S)$;

(2) $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$;

(3) $\dim(\sigma) = \dim(\Delta_S)$.

Proof. — By Remark 8.4, there exists a non-degenerate stratum $S \in \widetilde{\mathscr{X}'}$ such that $\sigma \subset f^{\mathrm{an}}(\Delta_S)$ and $\dim(\sigma) = \dim(\Delta_S)$. By (7.2), we also have $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$. $\square$

Note that if σ and Δ_S satisfy the three conditions in Lemma 8.6, then $\dim(\sigma) = \dim(f^{\mathrm{an}}(\Delta_S))$. Take $\sigma \in \Sigma$. We say tentatively that Δ_S *is over σ* if Δ_S satisfies conditions (1) and (2) in Lemma 8.6 as well as $\dim(\sigma) = \dim(f^{\mathrm{an}}(\Delta_S))$. With this word, Lemma 8.6 means that if σ is a strict support, then there exists Δ_S over σ such that $\dim(\sigma) = \dim(\Delta_S)$, which is condition (3).

Conversely, the following proposition shows that if σ is a strict support, then any canonical simplex Δ_S over σ should satisfy $\dim(\sigma) = \dim(\Delta_S)$.

Proposition 8.7. — *Let* $(\mathscr{X}', f : X' \to X_v)$ *be a semistable alteration (cf. Remark 7.3). Let* Σ *be a f-subdivisional polytopal decomposition of $S_{X_v^{\mathrm{an}}}$. Let* $\sigma \in \Sigma$ *be a strict support of the canonical measure $\mu_{X_v^{\mathrm{an}}}$ of an even ample line bundle on A. Let S be a stratum of $\widetilde{\mathscr{X}'}$ with the following properties:*

(1) $\sigma \subset f^{\mathrm{an}}(\Delta_S)$;

(2) $f^{\mathrm{an}}(\Delta_S) \subset S_{X_v^{\mathrm{an}}}$;

(3) $\dim(\sigma) = \dim f^{\mathrm{an}}(\Delta_S)$.

Then $\dim(\sigma) = \dim(\Delta_S)$.

The above proposition is essentially given in [41, Prop. 5.12, Lem. 5.13]. In the proof, we need detailed analysis on the non-degenerate strata by using Mumford models, subdivision of polytopes, toric method, etc. We omit the proof and refer to [41].

8.2. Proof of Proposition 8.2. — In this subsection, we give an outline of the proof of Proposition 8.2. Let A be an abelian variety over $\overline{K}$, where K is a function field. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Set nd-rk$(A) := \dim(\mathfrak{m})$, called the nowhere degeneracy rank of A.

We recall basic properties of the nowhere degeneracy rank.

Lemma 8.8 ([41, Cor. 7.12]). — *Let* $\phi : A \to A'$ *be a surjective homomorphism of abelian varieties. Then* nd-rk$(A) \geq$ nd-rk(A').

In Proposition 8.2, we assume that nd-rk$(A) = 0$. The following lemma shows that under this assumption, the support of the canonical measure has positive dimension. It is essentially [41, Prop. 7.16].

Lemma 8.9. — *Let* A *be an abelian variety over* $\overline{K}$. *Assume that* nd-rk$(A) = 0$. *Let* X *be a closed subvariety of* A. *Suppose that* $\dim(X) > 0$ *and that* X *has trivial stabilizer. Then there exists a place* v *of* $\overline{K}$ *such that* A *is degenerate at* v *and such that* $\dim(S_{X_v^{\mathrm{an}}}) > 0$, *namely,* $S_{X_v^{\mathrm{an}}}$ *contains a positive dimensional polytope.*

Proof. — Recall that for any $v \in M_{\overline{K}}$, the valuation map $\overline{\mathrm{val}} : A_v \to \mathbb{R}^n/\Lambda$ exists (cf. §A.2). Recall also the fact that this restricts to a finite surjective map $\overline{\mathrm{val}}|_{S_{X_v^{\mathrm{an}}}} : S_{X_v^{\mathrm{an}}} \to \overline{\mathrm{val}(X_v)}$, which is a part of [16, Thm. 1.1]. By the implication from (b) to (a) in [41, Prop. 7.16], we see that if $\dim(X) > 0$, then $\dim(\overline{\mathrm{val}(X_v)}) > 0$ for some $v \in M_{\overline{K}}$. Thus $\dim(S_{X_v^{\mathrm{an}}}) > 0$ for some $v \in M_{\overline{K}}$. □

Let us prove Proposition 8.2. We argue by contradiction; suppose that there exist an abelian variety over $\overline{K}$ of non-degeneracy rank 0 and a closed subvariety that is not a torsion subvariety but has dense small points. We take the quotient of the abelian variety by the stabilizer of the closed subvariety. Taking into account Lemma 8.8, we then construct an abelian A with nd-rk$(A) = 0$ and a closed subvariety X that is non-torsion, has trivial stabilizer, has dense small points, and has positive dimension. For some large natural number N, the difference morphism $\alpha_N : X^N \to A^{N-1}$ given $\alpha_N(x_1, \ldots, x_N) = (x_1 - x_2, \ldots, x_{N-1} - x_N)$ is generically finite (cf. Lemma 4.6). Put $Z := X^N$ and $Y := \alpha_N(Z)$, and let $\alpha : Z \to Y$ be the morphism induced by α_N. Note that α is a generically finite surjective morphism. Let L be an even ample line bundle on A. Then $L^{\boxtimes N}$ is an even ample line bundle on A^N. Let v be a place at which A is degenerate. By Lemma 8.9, we may take v in such a way that $S_{X_v^{\mathrm{an}}}$ has positive dimension. Let $\mu_{X_v^{\mathrm{an}}}$ be the canonical measure on X_v^{an} of L. Let $\mu_{Z_v^{\mathrm{an}}}$ be the canonical measure on Z_v^{an} of $L^{\boxtimes N}$ and let $\mu_{Y_v^{\mathrm{an}}}$ be the canonical measures on Y_v^{an} of an even ample line bundle on A^{N-1}. Then the argument using the equidistribution theorem as in the proof of Zhang and the proof of Gubler gives us

$$\alpha_*^{\mathrm{an}}(\mu_{Z_v^{\mathrm{an}}}) = \mu_{Y_v^{\mathrm{an}}}.$$

Note in particular that $\alpha^{\mathrm{an}}(S_{Z_v^{\mathrm{an}}}) = S_{Y_v^{\mathrm{an}}}$.

By Lemma 7.1, we take Mumford models of $(A_v^N)^{\mathrm{an}}$ and $(A_v^{N-1})^{\mathrm{an}}$ respectively such that the morphism α^{an} extends to a morphism ψ between the Mumford models. We take a semistable alteration $(\mathscr{Z}', h : Z' \to Z_v)$ compatible with this Mumford model of $(A_v^N)^{\mathrm{an}}$ (cf. Lemma 7.2). Since $\alpha : Z \to Y$ is a generically finite proper surjective morphism, setting $g := \alpha \circ h$, we see that $(\mathscr{Z}', g : Z' \to Y_v)$ is also a semistable alteration compatible with a Mumford model.

Note that the restricted morphism $\alpha^{\mathrm{an}} : S_{Z_v^{\mathrm{an}}} \to S_{Y_v^{\mathrm{an}}}$ is a piecewise linear map. Indeed, let Δ_S be any non-degenerate stratum with respect to h. It suffices to show that α^{an} is piecewise linear on $h^{\mathrm{an}}(\Delta_S)$. By the definition of the piecewise linear structure on $S_{Z_v^{\mathrm{an}}}$, this is equivalent to $\alpha^{\mathrm{an}} \circ h^{\mathrm{an}}|_{\Delta_S} : \Delta_S \to S_{Y_v^{\mathrm{an}}}$ being piecewise linear. Since $(\mathscr{Z}', g : Z' \to Y_v)$ is a semistable alteration compatible with a Mumford model and $g^{\mathrm{an}}(\Delta_S) \subset \alpha^{\mathrm{an}}(S_{Z_v^{\mathrm{an}}}) = S_{Y_v^{\mathrm{an}}}$, this follows from Lemma 8.5 (1).

Let $|X_v^{\mathrm{an}}|^N$ denote the direct product of N-copies of X_v^{an} in the category of topological spaces. Since Z_v^{an} is the direct product of N-copies of X_v^{an} in the category of Berkovich spaces, we have a natural continuous map $\beta : |Z_v^{\mathrm{an}}| \to |X_v^{\mathrm{an}}|^N$. By [40, Prop. 4.5], we have $\beta_*(\mu_{Z_v^{\mathrm{an}}}) = \mu_{X_v^{\mathrm{an}}}^N$, the product measure of N-copies of $\mu_{X_v^{\mathrm{an}}}$. Further, we have $\beta(S_{Z_v^{\mathrm{an}}}) = S_{X_v^{\mathrm{an}}}^N$.

We show that there exists a strict support τ of $\mu_{Z_v^{\mathrm{an}}}$ such that

$$(8.1) \qquad\qquad \dim(\alpha^{\mathrm{an}}(\tau)) < \dim(\tau).$$

Here, this is not precise, but we argue as if the above β were an isomorphism, that is, $S_{Z_v^{\mathrm{an}}} = S_{X_v^{\mathrm{an}}}^N$ and $\mu_{Z_v^{\mathrm{an}}} = \mu_{X_v^{\mathrm{an}}}^N$, to explain the idea; see §8.3 below for more precise argument. Since $\dim(S_{X_v^{\mathrm{an}}}) > 0$, there exists a strict support σ of $\mu_{X_v^{\mathrm{an}}}$ with $\dim(\sigma) > 0$. Then σ^N is a strict support of $\mu_{X_v^{\mathrm{an}}}^N$ with $\dim(\sigma^N) > 0$. Since α contracts the diagonal of $Z = X^N$ to a point, the diagonal of σ^N contracts to a point. Furthermore, since $\alpha^{\mathrm{an}}|_{\sigma^N}$ is piecewise linear, we can therefore take a polytope τ such that $\tau \subset \sigma^N$, $\dim(\tau) = \dim(\sigma^N)$, $\alpha^{\mathrm{an}}|_\tau$ is linear, and such that $\alpha^{\mathrm{an}}(\tau)$ is a polytope with $\dim(\alpha^{\mathrm{an}}(\tau)) < \dim(\tau)$. By the choice of τ, this is a strict support of $\mu_{Z_v^{\mathrm{an}}}$.

Since τ is a strict support of $\mu_{Z_v^{\mathrm{an}}}$, Lemma 8.6 give us an $S \in \mathrm{str}(\widetilde{\mathscr{Z}'})$ such that $\tau \subset h^{\mathrm{an}}(\Delta_S)$, $h^{\mathrm{an}}(\Delta_S) \subset S_{Z_v^{\mathrm{an}}}$ and such that

$$(8.2) \qquad\qquad \dim(\tau) = \dim(\Delta_S).$$

Since $\dim(\tau) \le \dim(h^{\mathrm{an}}(\Delta_S)) \le \dim(\Delta_S) = \dim(\tau)$, we remark that

$$(8.3) \qquad\qquad \dim(\tau) = \dim(h^{\mathrm{an}}(\Delta_S)).$$

Recall that $(\mathscr{Z}', g : Z' \to Y_v)$ is also a semistable alteration for Y_v compatible with a Mumford model. We would like to apply Proposition 8.7 in place of σ and f with $\alpha^{\mathrm{an}}(\tau)$ and g, respectively, so that we need to check the conditions in this proposition. Since τ is a strict support of $\mu_{Z_v^{\mathrm{an}}}$ and $\alpha_*^{\mathrm{an}}(\mu_{Z_v^{\mathrm{an}}}) = \mu_{Y_v^{\mathrm{an}}}$, $\alpha^{\mathrm{an}}(\tau)$ is a strict support of $\mu_{Y_v^{\mathrm{an}}}$. Since $\tau \subset h^{\mathrm{an}}(\Delta_S)$, we have

$$\alpha^{\mathrm{an}}(\tau) \subset \alpha^{\mathrm{an}}(h^{\mathrm{an}}(\Delta_S)) = g^{\mathrm{an}}(\Delta_S),$$

which is condition (1) in Proposition 8.7. Since $g^{\mathrm{an}}(\Delta_S) = \alpha^{\mathrm{an}}(h^{\mathrm{an}}(\Delta_S)) \subset \alpha^{\mathrm{an}}(S_{Z_v^{\mathrm{an}}}) = S_{Y_v^{\mathrm{an}}}$, condition (2) in Proposition 8.7 is satisfied. Further, we take by Lemma 8.5 (1) a polytope $P \subset \Delta_S$ such that $\dim(P) = \dim(\Delta_S)$, $h^{\mathrm{an}}(P) \subset \tau$, and such that $h^{\mathrm{an}}|_P$ is an affine map. By Lemma 8.5 (2), we note $\dim(h^{\mathrm{an}}(P)) = \dim(h^{\mathrm{an}}(\Delta_S))$, which equals $\dim(\tau)$ by (8.3). Since

$\alpha^{\mathrm{an}}|_\tau$ is affine, it follows that

$$\dim(g^{\mathrm{an}}(P)) = \dim(\alpha^{\mathrm{an}}(h^{\mathrm{an}}(P))) = \dim(\alpha^{\mathrm{an}}(\tau)).$$

By Lemma 8.5(2), this means that $\dim(\alpha^{\mathrm{an}}(\tau)) = \dim(g^{\mathrm{an}}(\Delta_S))$, which is condition (3) in Proposition 8.7. Thus applying Proposition 8.7, we obtain $\dim(\alpha^{\mathrm{an}}(\tau)) = \dim(\Delta_S)$.

On the other hand, it follows from (8.1) and (8.2) that $\dim(\alpha^{\mathrm{an}}(\tau)) < \dim(\Delta_S)$, which contradicts what we have show above. Thus we complete the proof of Proposition 8.2.

8.3. Complement. — In the above proof, we took a strict support τ of $\mu_{Z_v^{\mathrm{an}}}$ such that $\dim(\alpha^{\mathrm{an}}(\tau)) < \dim(\tau)$. In that argument, we pretended that we had $\mu_{Z_v^{\mathrm{an}}} = \mu_{X_v^{\mathrm{an}}}^N$, which was not precise. Here, we give a precise argument to obtain τ above. We use the valuation map $\overline{\mathrm{val}} : A_v^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$, where n is the dimension of the torus part of A_v^{an} (cf. §A.2). Since the valuation map is compatible with direct product, the map $\overline{\mathrm{val}}^N : (A_v^N)^{\mathrm{an}} \to (\mathbb{R}^n/\Lambda)^N$ and $\overline{\mathrm{val}}^{N-1} : (A_v^{N-1})^{\mathrm{an}} \to (\mathbb{R}^n/\Lambda)^{N-1}$ are the valuation maps. By [40, Lem. 4.1 and Prop. 4.5], we have $(\overline{\mathrm{val}}^N)_*(\mu_{Z_v^{\mathrm{an}}}) = \overline{\mathrm{val}}_*(\mu_{X_v^{\mathrm{an}}})^N$.

Let σ be a positive dimensional strict support of $\mu_{X_v^{\mathrm{an}}}$. Then $\overline{\mathrm{val}}(\sigma)$ is a strict support of $\overline{\mathrm{val}}_*(\mu_{X_v^{\mathrm{an}}})$ of positive dimension. It follows that $(\overline{\mathrm{val}}(\sigma))^N$ is a strict support of $(\overline{\mathrm{val}}^N)_*(\mu_{Z_v^{\mathrm{an}}})$ positive dimension. On the other hand, since the difference map induces the difference map from $(\mathbb{R}^n/\Lambda)^N$ to $(\mathbb{R}^n/\Lambda)^{N-1}$, we see that $\overline{\alpha}_{\mathrm{aff}}$ contracts the diagonal of $(\overline{\mathrm{val}}(\sigma))^N$ to a point. Since $\overline{\alpha}_{\mathrm{aff}}$ is piecewise linear, it follows that there exists a polytope $\tau' \subset (\overline{\mathrm{val}}(\sigma))^N$ such that $\dim(\tau') = \dim((\overline{\mathrm{val}}(\sigma))^N)$ and $\dim(\overline{\alpha}_{\mathrm{aff}}(\tau')) < \dim(\tau')$. Since $(\overline{\mathrm{val}}(\sigma))^N$ is a strict support of $(\overline{\mathrm{val}}^N)_*(\mu_{Z_v^{\mathrm{an}}})$, so is τ'. Since the restriction $S_{Z_v^{\mathrm{an}}} \to \overline{\mathrm{val}}^N(S_{Z_v^{\mathrm{an}}})$ of $\overline{\mathrm{val}}^N$ is a finite surjective piecewise linear map, there exists a strict support τ of $\mu_{Z_v^{\mathrm{an}}}$ such that $\overline{\mathrm{val}}^N(\tau) \subset \tau'$ and $\dim(\tau) = \dim(\tau')$. Noting that $\overline{\mathrm{val}}^{N-1} \circ \alpha^{\mathrm{an}} = \overline{\alpha}_{\mathrm{aff}} \circ \overline{\mathrm{val}}^N$, we see that

$$\overline{\mathrm{val}}^{N-1} \circ \alpha^{\mathrm{an}}(\tau) = \overline{\alpha}_{\mathrm{aff}}(\overline{\mathrm{val}}^N(\tau)) \subset \overline{\alpha}_{\mathrm{aff}}(\tau') < \dim(\tau') = \dim(\tau).$$

Again since $\overline{\mathrm{val}}^{N-1}|_{S_{Y_v^{\mathrm{an}}}}$ is a finite piecewise linear map, that shows $\dim(\alpha^{\mathrm{an}}(\tau)) < \dim(\tau)$. Thus we have a required τ.

Remark 8.10. — This is a remark for the proof of [41, Thm. 6.2]. In the proof of Proposition 8.2, we take an even ample line bundle L on A and consider the even ample line bundle $L^{\boxtimes N}$ on A^N. However, we can argue with any even ample line bundle on A^N. (In the proof of [41, Thm. 6.2], we actually did so.) Indeed, in the argument we assume that Z has dense small points, so that by the equidistribution theorem (Theorem 6.6), the canonical measure does not depend on the choice of the even ample line bundle on A^N. (But in fact, we should also begin with $L^{\boxtimes N}$ in the proof of [41, Thm. 6.2] for the sake of logical simplicity.)

9. Canonical height of closed subvarieties

9.1. Height of a closed subvariety. — Let A be an abelian variety over $\overline{K}$ and let L be an even ample line bundle on A. Let Z be a cycle on A of dimension d. Then the canonical height of Z with respect to L is defined. We briefly explain what it is when K is the function field of a variety $\mathfrak{B}$ with $\dim(\mathfrak{B}) = 1$. The construction is similar to the construction of the canonical metric; we construct a global version of a sequence of models as in (5.1). Fix an integer m with $m > 1$ and fix a rigidification of L. Let K' be a finite extension of K over which

Z can be defined. Let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. We begin with any proper flat model $(\pi_1 : \mathcal{A}_1 \to \mathfrak{B}', \mathcal{L}_1)$ such that $\mathcal{L}_1$ is nef. There exist such π_1 and $\mathcal{L}_1$ because L is ample. Next we take a proper flat model $\pi_2 : \mathcal{A}_2 \to \mathfrak{B}'$ of A such that the morphism $[m] : A \to A$ extends to $f_1 : \mathcal{A}_2 \to \mathcal{A}_1$. Set $\mathcal{L}_2 := f_1^*(\mathcal{L}_1)$. Then the pair $(\mathcal{A}_2, \mathcal{L}_2)$ is a model of $(A, L^{\otimes m^2})$ via the isomorphism $[m]^*(L) \cong L^{\otimes m^2}$ that respects the rigidification. Again, we take a model $\pi_3 : \mathcal{A}_3 \to \mathfrak{B}'$ of A such that the morphism $[m] : A \to A$ extends to $f_2 : \mathcal{A}_3 \to \mathcal{A}_2$, and set $\mathcal{L}_3 := f_2^*(\mathcal{L}_2)$. Then the pair $(\mathcal{A}_3, \mathcal{L}_3)$ is a model of $(A.L^{\otimes m^4})$. Repeating this process, we obtain a sequence $(\pi_n : \mathcal{A}_n \to \mathfrak{B}', \mathcal{L}_n)_{n \in \mathbb{N}}$ whose nth term is a model of $(A, L^{\otimes m^{2(n-1)}})$ with $\mathcal{L}_n$ nef on $\mathcal{A}_n$.

Let $\mathcal{Z}_n$ be the closure of Z in $\mathcal{A}_n$. Note that $\mathcal{Z}_n$ is a model of Z. We consider the sequence

$$(9.1) \qquad \left(\frac{\deg(c_1(\mathcal{L}_n)^{\cdot(d+1)} \cdot [\mathcal{Z}_n])}{m^{2(n-1)(d+1)}[K' : K]} \right)_{n \in \mathbb{N}}$$

of rational numbers. Then one shows that this sequence converges to a real number and that this limit depends only on (Z, L). This number is the *canonical height* of Z with respect to L, denoted by $\widehat{h}_L(Z)$. When X is a closed subvariety of A, we regard X as a cycle and define the canonical height $\widehat{h}_L(X)$ to be the canonical height of this cycle.

Remark 9.1. — In the above construction, each model $\mathcal{L}_n$ is nef on $\mathcal{A}_n$, and hence the sequence in (9.1) is nonnegative. It follows that $\widehat{h}_L(X) \geq 0$. This inequality holds for $\mathfrak{B}$ of arbitrary dimension, in fact.

One finds in the following proposition an important property of the canonical height in view of the density of small points.

Proposition 9.2 ([13, Cor. 4.4]). — *Let X be a closed subvariety of A. Then it has dense small points if and only if $\widehat{h}_L(X) = 0$.*

Remark 9.3. — Since A is an abelian variety over $\overline{K}$, A has dense small points, and hence $\widehat{h}_L(A) = 0$ by the above proposition.

We explain the idea of the proof of Proposition 9.2, which is essentially due to Zhang. We consider the essential minima

$$e_1(X, L) := \sup_Y \inf_{s \in X(\overline{K}) \setminus Y} \widehat{h}_L(x),$$

where Y runs through all closed subsets of codimension 1 in X. Then in general, one shows that

$$\deg_L(X)e_1(X, L) \leq \widehat{h}_L(X) \leq (\dim(X) + 1)\deg_L(X)e_1(X, L).$$

By the definition of $e_1(X, L)$, X has dense small points if and only if $e_1(X, L) = 0$. It follows from the above inequality that X has dense small points if and only if $\widehat{h}_L(X) = 0$.

9.2. Zhang's admissible pairing and the canonical height. — We mentioned in §1.2.4 and §1.2.8 that some partial answers to the geometric Bogomolov conjecture for curves have been obtained before, and that those results are obtained by showing the positivity of the admissible pairing $(\omega_a, \omega_a)_a$ of the admissible dualizing sheaf ω_a of C. This argument using the positivity of the admissible pairing concerns Proposition 9.2. Here, we explain why $(\omega_a, \omega_a)_a > 0$ implies the conjecture for the curve.

Let us recall the setting of the geometric Bogomolov conjecture for curves. Let C be a smooth projective curve of genus $g \geq 2$ over $\overline{K}$ and let J_C be the Jacobian variety of C. Fix a divisor D on C of degree 1 and let $\jmath_D : C \hookrightarrow J_C$ be the embedding defined by $\jmath_D(x) := x - D$. Put $X := \jmath_D(C)$. Assume that C is non-isotrivial, i.e. C cannot be defined over k. What we should show is that X does not have dense small points.

Recall that h_{NT} equals the canonical height $\widehat{h}_L$ associated to $L = \mathcal{O}_{J_C}(\theta)$, where θ is a symmetric theta divisor (cf. Remark 2.10). In [45], it is shown that $\widehat{h}_L(X) \geq \alpha(\omega_a, \omega_a)_a$ for some $\alpha > 0$, and the equality holds if $(2g-2)D_1$ is a canonical divisor on C. It follows that if one shows $(\omega_a, \omega_a)_a > 0$, then $\widehat{h}_L(X) > 0$, so that by Proposition 9.2, X does not have dense small points. That is the reason why $(\omega_a, \omega_a)_a > 0$ suffices for the Bogomolov conjecture for curves.

10. Proof of Theorem 3.10

In this section, we describe the idea of the proof of Theorem 3.10.

10.1. Model for a nowhere degenerate abelian variety. — For a nowhere degenerate abelian variety, we have a good model as follows.

Proposition 10.1 ([42, Prop. 2.5])**.** — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ and let L be a line bundle on A. Then there exist a finite extension K' of K, a proper morphism $\pi : \mathcal{A} \to \mathfrak{B}'$, where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', and a line bundle $\mathcal{L}$ on $\mathcal{A}$ satisfying the following conditions.*

(1) *The pair $(\pi, \mathcal{L})$ is a model of (A, L).*

(2) *There exists an open subset $\mathfrak{U} \subset \mathfrak{B}'$ with $\mathrm{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ such that the restriction $\pi' : \pi^{-1}(\mathfrak{U}) \to \mathfrak{U}$ of π is an abelian scheme.*

(3) *Let $0_{\pi'}$ be the zero-section of the abelian scheme π' in (2). Then $0_{\pi'}^*(\mathcal{L}) \cong \mathcal{O}_{\mathfrak{U}}$.*

Remark 10.2. — Consider the case where $\dim(\mathfrak{B}) = 1$. Then Proposition 10.1 claims the following. Let A and L be as in this proposition. Then, there exist a finite extension K' of K and a model $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$ of (A, L), where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', satisfying the following conditions.

(1) The morphism π is an abelian scheme.

(2) Let 0_π be the zero-section of the abelian scheme π. Then $0_\pi^*(\mathcal{L}) \cong \mathcal{O}_{\mathfrak{B}'}$.

Assume that A is nowhere degenerate. The next lemma shows that the canonical height $\widehat{h}_L(X)$ is given by intersection on a suitable model.

Lemma 10.3. — *Let A be an abelian variety over $\overline{K}$ and let L be an even ample line bundle. Let X be a closed subvariety. Assume that A is nowhere degenerate. Let $(\pi : \mathcal{A} \to \mathfrak{B}', \mathcal{L})$ be a model of (A, L) as in Proposition 10.1 and let K' denote the function field of $\mathfrak{B}'$. Assume that X can be defined over K' and let $\mathcal{X}$ be the closure of X in $\mathcal{A}$. Then we have*

$$\widehat{h}_L(X) = \frac{\deg_{\mathcal{H}'} \pi_*(c_1(\mathcal{L})^{\cdot(d+1)} \cdot [\mathcal{X}])}{[K' : K]},$$

where $\mathcal{H}'$ is the pullback of $\mathcal{H}$ to $\mathfrak{B}'$.

Let us explain how Lemma 10.3 is verified under the assumption that $\dim(\mathfrak{B}) = 1$. Let A, L, and X be as in Lemma 10.3. Let $(\mathcal{A}, \mathcal{L})$ be a model of (A, L) as in Proposition 10.1; see also Remark 10.2. Recall that in the limiting process to define the canonical height of subvarieties, we consider the sequence (9.1). Here, we begin with $(\mathcal{A}_1, \mathcal{L}_1) := (\mathcal{A}, \mathcal{L})$. Then we see from the condition of $(\mathcal{A}, \mathcal{L})$ that $(\mathcal{A}_2, \mathcal{L}_2)$ coincides with $(\mathcal{A}_1, \mathcal{L}_1^{\otimes m^2})$. By repeating the argument, for any $n \in \mathbb{N}$, we see that $(\mathcal{A}_n, \mathcal{L}_n)$ coincides with $(\mathcal{A}_1, \mathcal{L}_1^{\otimes m^2(n-1)})$. This shows that in this case the sequence (9.1) is constant. Thus Lemma 10.3 holds.

Remark 10.4. — Under the setting of Remark 10.2, if X can be defined over K', then we have

$$\widehat{h}_L(X) = \frac{\deg(c_1(\mathcal{L})^{\cdot(d+1)} \cdot [\mathcal{X}])}{[K' : K]}$$

by Lemma 10.3, where $d := \dim(X)$.

10.2. Idea of the proof of Theorem 3.10. — We describe the idea of the proof of the assertion that (3) implies (2) in Theorem 3.10. It suffices to show the following. Let A be a nowhere degenerate abelian variety. Let $\mathfrak{t}$ be the image of the $\overline{K}/k$-trace homomorphism. Suppose that the geometric Bogomolov conjecture holds for $A/\mathfrak{t}$. Then it holds for A.

10.2.1. *Case of constant abelian varieties.* — First, we describe the idea of the proof of the above assertion under the assumption that the abelian variety is a constant abelian variety. Let B be a constant abelian variety. Let Y be a closed subvariety of B. Suppose that Y has dense small points. Then we want to show that Y is a constant subvariety. Note that since B is a constant abelian variety, any torsion subvariety is a constant abelian variety (cf. [42, Prop. 3.7]).
By definition, we may write $B = \widetilde{B} \otimes_k \overline{K}$ with some abelian variety $\widetilde{B}$ over k. Let $\widetilde{M}$ be an even ample line bundle on $\widetilde{B}$. Set $M := \widetilde{M} \otimes_k \overline{K}$, which is an even ample line bundle on B. We consider the standard model $(\pi : \mathcal{B} \to \mathfrak{B}, \mathcal{M})$ of (B, M) (cf. Example 2.11). Then $(\mathcal{B}, \mathcal{M})$ satisfies the conditions of Proposition 10.1. Since Y has dense small points, Proposition 9.2 tells us that $\widehat{h}_M(Y) = 0$. By Lemma 10.3, it follows that $\deg_{\mathcal{H}} \pi_*(c_1(\mathcal{M})^{\cdot(d+1)} \cdot [\mathcal{Y}]) = 0$, where $\mathcal{Y}$ is the closure of Y in $\mathcal{B}$ and $d := \dim(Y)$. Using this, we can show that Y is a constant subvariety without difficulties; see [42, §3] for the details.

10.2.2. *Reduction to the case of a product of a constant abelian variety and a nowhere degenerate abelian variety with trivial trace.* — In this subsection, we remark that we may assume that A is the direct product of a constant abelian variety and a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace.
We begin by recalling the following basic fact.

Lemma 10.5 ([41, Cor. 6.7]). — *Let $\psi : A \to B$ be an isogeny of abelian varieties over $\overline{K}$. Then the geometric Bogomolov conjecture holds for A if and only if it holds for B.*

Let A be a nowhere degenerate abelian variety over $\overline{K}$. Then it is isogenous to the direct product of a constant abelian variety and an abelian variety with trivial $\overline{K}/k$-trace. Indeed, by the Poincaré complete reducibility theorem, there exists a closed subvariety C of A such that the natural homomorphism $\mathfrak{t} \times C \to A$ is an isogeny. Note that C is isogenous to $A/\mathfrak{t}$.

Furthermore, since the $\overline{K}/k$-trace homomorphism is finite, we see that A is isogenous to $\left(\widetilde{A^{\overline{K}/k}} \otimes_k \overline{K} \right) \times A/\mathfrak{t}$. Since $A/\mathfrak{t}$ has trivial $\overline{K}/k$-trace (cf. [42, Rem. 5.4]), it follows that A is isogenous to the product of a constant abelian variety and a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace.

Thus by Lemma 10.5, our goal is the following assertion. Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace and let $B = \widetilde{B} \otimes_k \overline{K}$ be a constant abelian variety. Let X be a closed subvariety of $B \times A$. Assume that the geometric Bogomolov conjecture holds for A. Suppose that X has dense small points. Then there exist a constant abelian subvariety Y of B and a torsion subvariety T of A such that $X = Y \times T$.

Let $\mathrm{pr}_B : B \times A \to B$ be the projection and set $Y := \mathrm{pr}_B(X)$. Since X has dense small points, so does Y by Lemma 2.15. By §10.2.1, there exists a closed subvariety $\widetilde{Y}$ such that $Y = \widetilde{Y} \otimes_k \overline{K}$. Thus we are reduced to showing the following proposition.

Proposition 10.6. — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace. Let $Y = \widetilde{Y} \otimes_k \overline{K}$ be a closed constant subvariety of a constant abelian variety $B = \widetilde{B} \otimes_k \overline{K}$. Let $p : Y \times A \to Y$ be the projection. Let X be a closed subvariety of $Y \times A$ with $p(X) = Y$. Assume that the geometric Bogomolov conjecture holds for A. Suppose that X has dense small points. Then there exists a torsion subvariety T of A such that $X = Y \times T$.*

10.2.3. *Relative height.* — We keep the notation in Proposition 10.6: A is a nowhere degenerate abelian variety, $B = \widetilde{B} \otimes_k \overline{K}$ is a constant abelian variety, X is a closed subvariety of $B \times A$, $Y = \widetilde{Y} \otimes_k \overline{K}$ is a constant closed subvariety of B, and Y equals the image of X by the projection $B \times A \to B$.

In the proof of Proposition 10.6, we use the relative height. Let L be an even ample line bundle on A. The relative height is a function $\mathbf{h}^L_{X/Y}$ defined over a dense open subset of $Y(k)$ in the following way. (In fact, it is defined over a dense open subset of Y, but we omit that; see [42, §4.2] for the detail.) Take a point $\tilde{y} \in \widetilde{Y}(k)$. Then $\tilde{y}$ can be naturally regarded as a point of $Y(\overline{K})$, and we denote by $\overline{y}_K$ the corresponding point in $Y(\overline{K})$. Let $X_{\overline{y}_K} := (p|_X)^{-1}(\overline{y}_K)$. This is a closed subscheme of an abelian variety $p^{-1}(\overline{y}_K) = A$. If $\tilde{y} \in \widetilde{Y}(k)$ is general, $X_{\overline{y}_K}$ is of pure dimension $d - e$, where $d := \dim(X)$ and $e := \dim(Y)$. Therefore, we consider the canonical height $\widehat{h}_L(X_{\overline{y}_K})$, and we set $\mathbf{h}^L_{X/Y}(\tilde{y}) := \widehat{h}_L(X_{\overline{y}_K})$.

Since the relative height at $\tilde{y}$ is given by the height of the fiber $X_{\overline{y}_K}$, we can describe it in terms of intersection products. We take K', $\mathfrak{B}'$ and a model $(\mathcal{A}, \mathcal{L})$ over $\mathfrak{B}'$ of (A, L) as in Proposition 10.1. Replacing K' by a finite extension if necessary, we may and do assume that X can be defined over K'. Let $\mathcal{B} \to \mathfrak{B}'$ be the standard model of B, that is, $\mathcal{B} := \widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B}'$ and $\mathcal{B} \to \mathfrak{B}'$ is the canonical projection (cf. Example 2.11). Then $\mathcal{B} \times_{\mathfrak{B}'} \mathcal{A}$ is a model of $B \times A$, and it satisfies the conditions in Proposition 10.1. Let $\mathcal{X}$ be the closure of X in $\mathcal{B} \times_{\mathfrak{B}'} \mathcal{A}$. Set $\mathcal{Y} := \widetilde{Y} \times_{\mathrm{Spec}(k)} \mathfrak{B}'$. Then $\mathcal{Y}$ is the model of Y and the canonical projection gives a surjective morphism $\mathcal{X} \to \mathcal{Y}$. Take a general $\tilde{y} \in \widetilde{Y}(k)$. Note that $\overline{y}_K$ is the geometric generic point of $\{\tilde{y}\} \times_{\mathrm{Spec}(k)} \mathfrak{B}' = \mathfrak{B}'$. Then one sees that the fiber $\mathcal{X}_{\tilde{y}} \subset \{\tilde{y}\} \times \mathcal{A} = \mathcal{A}$ of the composite $\mathcal{X} \to \mathcal{Y} \to \widetilde{Y}$ is a model of $X_{\overline{y}_K} \subset \{\overline{y}_K\} \times A = A$. By Lemma 10.3, we express the height of $X_{\overline{y}_K}$ by using the intersection with $\mathcal{X}_{\tilde{y}}$; indeed

$$(10.1) \qquad \mathbf{h}^L_{X/Y}(\tilde{y}) = \widehat{h}_L(X_{\overline{y}_K}) = \frac{\deg_{\mathcal{H}'} \pi_* \left(c_1(\mathcal{L})^{\cdot (d-e+1)} \cdot [\mathcal{X}_{\tilde{y}}] \right)}{[K' : K]},$$

where $\mathcal{H}'$ is the pullback of $\mathcal{H}$ by $\mathfrak{B}' \to \mathfrak{B}$.
The relative height is used via the following lemma.

Lemma 10.7 ([42, Prop. 4.6]). — *Under the setting above, suppose that there exists a dense subset S of $\widetilde{Y}(k)$ that satisfies the following two conditions.*

(1) *For any $\tilde{y} \in S$, we have $\mathbf{h}^L_{X/Y}(\tilde{y}) = 0$.*

(2) *Any irreducible component of $X_{\overline{\tilde{y}_K}}$ with its induced reduced subscheme structure is a torsion subvariety of $\{\tilde{y}\} \times A = A$.*

Then there exists a torsion subvariety T of A such that $X = Y \times T$.

We omit the proof Lemma 10.7, and we refer to [42] for the detail. We just remark that this lemma is obtained by showing a kind of rigidity of torsion subvarieties in a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace.

10.2.4. *Proof of Proposition 10.6.* — We give an outline of the proof of Proposition 10.6. First, assuming Lemma 10.8 below, we prove Proposition 10.6. Afterwards, we prove this lemma.

Lemma 10.8 ([42, Prop. 5.1]). — *Under the setting in Proposition 10.6, suppose that X has dense small points. Then there exists a dense subset S of $\widetilde{Y}(k)$ such that $\mathbf{h}^L_{X/Y}(\tilde{y}) = 0$ for any $\tilde{y} \in S$.*

We deduce Proposition 10.6 from Lemmas 10.8. Suppose that X has dense small points. Then by Lemma 10.8, there exists a dense subset S of $\widetilde{Y}(k)$ such that $\mathbf{h}^L_{X/Y}(\tilde{y}) = 0$ for any $\tilde{y} \in S$. This means that for any irreducible component Z of $X_{\overline{\tilde{y}_K}}$, we have $\widehat{h}_L(Z) = 0$ (cf. Remark 9.1). It follows that Z has dense small points.
Now assume that the geometric Bogomolov conjecture holds for A. Since A has trivial $\overline{K}/k$-trace, it follows that Z is a torsion subvariety. Thus we see that this S satisfies conditions (1) and (2) in Lemma 10.7. Therefore by this lemma, we conclude the existence of a torsion subvariety T as required.

Remark 10.9. — We keep the setting of Proposition 10.6. In the above argument, we show that for general $\tilde{y} \in \widetilde{Y}(k)$, any irreducible component Z of the fiber $X_{\overline{\tilde{y}_K}} \subset \{\tilde{y}_K\} \times A = A$ has canonical height 0.

Finally, we give an idea of the proof of Lemma 10.8. To avoid technical difficulties, we assume that $\dim(\mathfrak{B}) = 1$. The proof of Lemma 10.8 uses the expression of the canonical heights in terms of intersections. Recall that $(\mathcal{A}, \mathcal{L})$ is a model over $\mathfrak{B}'$ of (A, L) as in Proposition 10.1. Let $\widetilde{M}$ be an even ample line bundle on $\widetilde{B}$ and set $M := \widetilde{M} \otimes_k \overline{K}$, which is an even ample line bundle on $B = \widetilde{B} \otimes_k \overline{K}$. We assume that $\widetilde{M}$ is very ample, here. Let $(\mathcal{B}, \mathcal{M})$ be the standard model over $\mathfrak{B}'$ of (B, M), that is, $\mathcal{B} := \widetilde{B} \times_{\mathrm{Spec}(k)} \mathfrak{B}'$ and $\mathcal{M} := \widetilde{M} \otimes_k \mathcal{O}_{\mathcal{B}'}$ (cf. Example 2.11). Note that $M \boxtimes L$ is an even ample line bundle on $B \times A$, and $(\mathcal{B} \times_{\mathfrak{B}'} \mathcal{A} \to \mathfrak{B}', \mathcal{M} \boxtimes \mathcal{L})$ is a model over $\mathfrak{B}'$ of $(B \times A, M \boxtimes L)$. Note also that the closure $\mathcal{X}$ of X in $\mathcal{B} \times_{\mathfrak{B}'} \mathcal{A}$ is a model of X. Further, $\widetilde{Y} \times_{\mathrm{Spec}(k)} \mathfrak{B}'$ is a model of Y and equals the image of $\mathcal{X}$ by the projection $\mathcal{B} \times_{\mathfrak{B}'} \mathcal{A} \to \mathcal{B}$.

Suppose that X has dense small points. Then by Proposition 9.2, we have $\widehat{h}_{M \boxtimes L}(X) = 0$. By Lemma 10.3, it follows that

$$(10.2) \qquad\qquad \deg \pi_*(c_1(\mathcal{M} \boxtimes \mathcal{L})^{\cdot(d+1)} \cdot [\mathcal{X}]) = 0,$$

where $d := \dim(X)$.

We prove that for any integer e with $0 \leq e \leq d+1$,

$$(10.3) \qquad\qquad \deg \pi_*(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot (c_1(\mathrm{pr}_\mathcal{B}^*(\mathcal{M}))^e \cdot [\mathcal{X}])) = 0,$$

where $\mathrm{pr}_\mathcal{A} : \mathcal{B} \times_{\mathfrak{B}'} \mathcal{A} \to \mathcal{A}$ and $\mathrm{pr}_\mathcal{B} : \mathcal{B} \times_{\mathfrak{B}'} \mathcal{A} \to \mathcal{B}$ are the canonical projections. First, note that the left-hand side in (10.3) is non-negative. Indeed, it is, up to a positive multiple, the canonical height of X with respect to $(d-e+1)$-copies of $\mathcal{O}_B \boxtimes L$ and e-copies of $M \boxtimes \mathcal{O}_A$ (cf. [15, Thm. 3.5 (d)]), and this canonical height is non-negative by ([12, Thm. 11.18 (e)]). Here, we have

$$\deg \pi_*(c_1(\mathcal{M} \boxtimes \mathcal{L})^{\cdot(d+1)} \cdot [\mathcal{X}]) = \sum_{e=0}^{d+1} \binom{d+1}{e} \deg \pi_*(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot (c_1(\mathrm{pr}_\mathcal{B}^*(\mathcal{M}))^e \cdot [\mathcal{X}])),$$

which equals 0 by (10.2). Thus we obtain (10.3).

We consider the composite morphism $\varphi : \mathcal{B} \times_{\mathfrak{B}'} \mathcal{A} \to \mathcal{B} \to \widetilde{B}$. This gives us $\varphi|_\mathcal{X} : \mathcal{X} \to \widetilde{Y}$ by restriction. We note $\mathrm{pr}_\mathcal{B}^*(\mathcal{M}) = \varphi^*(\widetilde{M})$. Here, recall that $\widetilde{M}$ is very ample. Then there exists a dense open subset $S \subset \widetilde{Y}(k)$ such that for any $\tilde{y} \in S$, there exist a finite number of points $\tilde{y}_1, \ldots, \tilde{y}_m$ with $\tilde{y}_1 = \tilde{y}$ such that

$$c_1(\widetilde{M})^e \cdot [\widetilde{Y}] = \sum_{i=1}^m [\tilde{y}_i]$$

as cycles classes and such that $\varphi|_\mathcal{X}$ is flat over $\tilde{y}_i$ for any $i = 1, \ldots, m$. Put $\mathcal{X}_{\tilde{y}_i} := (\varphi|_\mathcal{X})^{-1}(\tilde{y}_i) = \varphi^{-1}(\tilde{y}_i) \cap \mathcal{X}$. Then by the choice of $\tilde{y}_1, \ldots, \tilde{y}_m$,

$$c_1(\mathrm{pr}_\mathcal{B}^*(\mathcal{M}))^e \cdot [\mathcal{X}] = \sum_{i=1}^m [\mathcal{X}_{\tilde{y}_i}],$$

and hence by (10.3),

$$\deg \pi_* \left(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot \sum_{i=1}^m [\mathcal{X}_{\tilde{y}_i}] \right) = 0.$$

Since

$$\deg \pi_*(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot [(\mathcal{X}_{\tilde{y}_i}]) \geq 0$$

by the same reason as in the proof of (10.3), it follows that

$$\deg \pi_*(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot [(\mathcal{X}_{\tilde{y}_i}]) = 0$$

for any $i = 1, \ldots, m$. Since $\tilde{y}_1 = \tilde{y}$,

$$\deg \pi_*(c_1(\mathrm{pr}_\mathcal{A}^*(\mathcal{L}))^{\cdot(d-e+1)} \cdot [\mathcal{X}_{\tilde{y}}]) = 0$$

holds, in particular. Thus by (10.1), we obtain $\mathbf{h}_{X/Y}^L(\tilde{y}) = 0$. This proves Lemma 10.8.

11. Proof of the conjecture for curves

In this section, we give an idea of the proof of the geometric Bogomolov conjecture for curves. We also give a remark on the conjecture for abelian varieties.

11.1. For curves and for divisors. — As we noted before, we actually prove in [43] the following more general result.

Theorem 11.1 ([43, Thm. 1.3]). — *Let A be an abelian variety over $\overline{K}$ and let X be a closed subvariety of A. Assume that $\dim(X) = 1$. Suppose that X has dense small points. Then X is a special subvariety.*

It should be remarked that the above theorem follows from the following.

Theorem 11.2 ([43, Thm. 1.4]). — *Let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A. Assume that $\operatorname{codim}(X, A) = 1$. Suppose that X has dense small points. Then X is a special subvariety.*

The principle to connect Theorem 11.1 to Theorem 11.2 is as follows: if we take the sum of some copies of a curve in an abelian variety, then it will be a divisor of the abelian variety. Let us describe this idea a little more precisely. Some argument using Theorem 3.10 shows that we may assume that A has trivial $\overline{K}/k$-trace to show Theorem 11.1. Now suppose that X is a curve in A with dense small points. Since the $\overline{K}/k$-trace of A is trivial, our goal is to show that X is the translate of an abelian subvariety of dimension 1 by a torsion point. Consider $X - \tau$ for any $\tau \in A(\overline{K})_{tor}$. We take a τ_1 in such a way that the dimension of the minimal abelian subvariety containing $X - \tau_1$ is minimal. Set $X_1 := X - \tau_1$ and let A_1 be the minimal abelian subvariety containing X_1. Note that X_1 has dense small points, since τ_1 is a torsion point. For a nonnegative integer l, we consider $Z_l := \sum_{j=0}^{l} (-1)^j X_1$, where the sum means the addition of the abelian variety. Then Z is a closed subvariety of A_1, and since X_1 has dense small points, so does Z. Furthermore, one sees that $\operatorname{codim}(Z_l, A_1) = 1$ for some l. Thus we reach the setting of Theorem 11.2, and a few more arguments leads to the conclusion. See [43, Proof of Thm. 5.11] for the detail.

11.2. Three steps of the proof. — By the argument in §11.1, the geometric Bogomolov conjecture for curves is reduced to Theorem 11.2. In this section, we give an outline of the proof of this theorem. The proof consists of three steps, and each of them essentially corresponds to [41], [42], and [43], respectively.

Step 1. — We reduce Theorem 11.2 to the case where the abelian variety is nowhere degenerate, i.e. to the following proposition.

Proposition 11.3. — *Assume that A is nowhere degenerate. Let X be a closed subvariety of A of codimension 1. Suppose that X has dense small points. Then X is a special subvariety.*

We explain how the theorem follows from Theorem 8.1 and Proposition 11.3. Let A be any abelian variety over $\overline{K}$. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Recall that there exists a surjective homomorphism $\phi : A \to \mathfrak{m}$; see the argument just after Theorem 8.1. Let X be a closed subvariety of A of codimension 1 with dense small points. Put $Y := \phi(X)$. Then $Y = \mathfrak{m}$ or $\operatorname{codim}(Y, \mathfrak{m}) = 1$. If $Y = \mathfrak{m}$, then Y is special. If $\operatorname{codim}(Y, \mathfrak{m}) = 1$,

then applying Proposition 11.3 to $\mathfrak{m}$ and Y (in place of A and X respectively), we find that Y is special. In any case, Theorem 8.1 shows that X is a special subvariety.

Step 2. — By the argument in Step 1, we are reduced to showing Proposition 11.3. In this step, we reduce Proposition 11.3 to the assertion for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace. As we noted in §10.2.2, any nowhere degenerate abelian variety is isogenous to the product of a constant abelian variety and a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace. Therefore, by Lemma 10.5, it suffices to show the following:

Proposition 11.4. — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace, B a constant abelian variety, and let $X \subset B \times A$ be a closed subvariety of codimension 1. Suppose that X has dense small points. Then X is a special subvariety.*

Proposition 11.4 follows from the proposition below.

Proposition 11.5. — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace and let X be a closed subvariety of A of codimension 1. Then X does not have dense small points.*

We show that Proposition 11.4 is deduced form Proposition 11.5. Let Y be the image of X by the projection $B \times A \to B$. Since X has dense small points, do does Y, and hence by the argument of §10.2.1, Y is a constant subvariety of B.

We prove that $\dim(Y) < \dim(B)$ by contradiction. Suppose that $\dim(Y) = \dim(B)$, i.e. $Y = B$. Write $B = \widetilde{B} \otimes_k \overline{K}$. As is noted in Remark 10.9, for a general $\tilde{y} \in \widetilde{B}(k)$, any irreducible component Z of $X_{\overline{\tilde{y}_K}}$ has canonical height 0. Thus by Proposition 9.2, Z has dense small points. On the other hand, the dimension counting shows that Z is a divisor on $\{\overline{\tilde{y}_K}\} \times A = A$ for general $\tilde{y}$. That contradicts Proposition 11.5.

Thus we have $\dim(Y) < \dim(B)$. Then Y has codimension 1 in B. Since X has codimension 1 in $B \times A$, it follows that $X = Y \times A$. Since Y is a constant subvariety, this proves that X is a special subvariety of $B \times A$.

Step 3. — Now, our goal is Proposition 11.5. We see that this follows from the proposition below.

Proposition 11.6. — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace, and let X be an irreducible effective ample divisor on A of codimension 1. Set $D := X + [-1]^*(X)$ and $L := \mathcal{O}_A(D)$, where $+$ is the addition of divisors. Then L is an even ample line bundle on A, and we have $\widehat{h}_L(X) > 0$.*

Let us check that Proposition 11.5 is deduced from Proposition 11.6. Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace and let X be a closed subvariety of A of codimension 1. Note that any effective divisor is the pullback of an ample divisor by some homomorphism, which follows from [28, p. 88, Remarks on effective divisors by Nori]; see [43, Step 1 of the proof of Thm. 5.7] for the precise argument. This means that there exists a surjective homomorphism $\phi : A \to A'$ and an effective ample divisor X' on A' such that $X = \phi^{-1}(X')$. By Proposition 11.6 together with Proposition 9.2, X' does not have dense small points. By Lemma 2.15, it follows that X does not have dense small points.

We explain the idea of the proof of Proposition 11.6. It is obvious that L in the proposition is even and ample. Now, to avoid technical difficulties, we make the following assumptions:

(1) $\dim(\mathfrak{B}) = 1$;

(2) there exists an abelian scheme $\pi : \mathcal{A} \to \mathfrak{B}$ with zero-section 0_π and with geometric generic fiber A, and X is defined over K;

(3) $0 \notin X$;

(4) $\#k > \aleph_0$, i.e. k has uncountably infinite cardinality.

In fact, it is not very difficult to see that we may assume (2), (3), and (4) without loss of generality; we make assumption (1) to avoid technical difficulties.

Let us give a sketch of the proof of Proposition 11.6. (This is based on the argument in [43, §1.3].) Since L is even, we have $\widehat{h}_L(D) = 2\widehat{h}_L(X)$. Let $\mathcal{D}$ be the closure of D in $\mathcal{A}$. Since $0 \notin D$, $0_\pi^*(\mathcal{D})$ is a well-defined effective divisor on $\mathfrak{B}$. Set $\mathcal{N} := \mathcal{O}_\mathfrak{B}(0_\pi^*(\mathcal{D}))$ and $\mathcal{L} := \mathcal{O}_\mathcal{A}(\mathcal{D}) \otimes \pi^*(\mathcal{N}^{\otimes -1})$. Then $0_\pi^*(\mathcal{L}) = \mathcal{O}_\mathfrak{B}$. Put $n := \dim(A)$. Then one sees from Remark 10.4 that

$$\widehat{h}_L(D) = \deg\left(c_1(\mathcal{L})^{\cdot n} \cdot [\mathcal{D}]\right) = \deg\left(c_1(\mathcal{L})^{\cdot n} \cdot c_1\left(\mathcal{O}_\mathcal{A}(\mathcal{D})\right) \cdot [\mathcal{A}]\right).$$

Since A has canonical height 0 (cf. Remark 9.3), we have $\deg\left(c_1(\mathcal{L})^{\cdot(n+1)} \cdot [\mathcal{A}]\right) = 0$ (cf. Remark 10.4). It follows that

$$\deg\left(c_1(\mathcal{L})^{\cdot n} \cdot c_1\left(\mathcal{O}_\mathcal{A}(\mathcal{D})\right) \cdot [\mathcal{A}]\right) = \deg\left(c_1(\mathcal{L})^{\cdot(n+1)} \cdot [\mathcal{A}]\right) + \deg\left(c_1(\mathcal{L})^{\cdot n} \cdot \pi^* c_1(\mathcal{N}) \cdot [\mathcal{A}]\right)$$
$$= \deg_L(A) \cdot \deg(\mathcal{N}).$$

Since $\deg_L(A) > 0$ by the ampleness of L on A, it remains to show $\deg(\mathcal{N}) > 0$. In fact, we will see below that $\mathcal{N}$ is non-trivial. Since $0_\pi^*(\mathcal{D})$ is an effective divisor, this concludes $\deg(\mathcal{N}) > 0$.

The outline of the proof of the non-triviality of $\mathcal{N}$ is as follows. We prove the non-triviality by contradiction. Suppose that it is trivial. Then we can show that there exists a finite covering $\mathfrak{B}' \to \mathfrak{B}$ such that the complete linear system $|2\mathcal{D}'|$ on $\mathcal{A}'$ is base-point free (cf. [43, Prop. 4.4]), where $\pi' : \mathcal{A}' \to \mathfrak{B}'$ and $\mathcal{D}'$ are the base-change of π and $\mathcal{D}$ by this $\mathfrak{B}' \to \mathfrak{B}$, respectively. Let $\varphi : \mathcal{A}' \to Z$ be the surjective morphism associated to $|2\mathcal{D}'|$, where Z is a closed subvariety of the dual space of $|2\mathcal{D}'|$. Note that for any curve $\gamma \subset \mathcal{A}'$, $\deg\left(c_1(\mathcal{L}') \cdot \gamma\right) = 0$ if and only if $\varphi(\gamma)$ is a point, where $\mathcal{L}' = \mathcal{O}_{\mathcal{A}'}(\mathcal{D}')$, which is the pull-back of $\mathcal{L}$ to $\mathcal{A}'$. Further, note that for any $a \in A(\overline{K})$, $\widehat{h}_L(a) = 0$ if and only if $\deg\left(c_1(\mathcal{L}') \cdot \Delta_a\right) = 0$, where Δ_a is the closure of a in $\mathcal{A}'$ (cf. (2.3)).

Let Γ be the set of irreducible curves in $\mathcal{A}'$ such that $\dim(\varphi(\gamma)) = 0$. We set

$$A(0; L) := \left\{a \in A(\overline{K}) \mid \widehat{h}_L(a) = 0\right\}.$$

The above argument shows that $\Gamma \supset \{\Delta_a \mid a \in A(0; L)\}$. Since $L^{\otimes 2}$ is ample and π' is an abelian scheme, $(\mathcal{L}')^{\otimes 2}$ is relatively ample with respect to π'. It follows that for any curve γ in a fiber of π', we have $\deg(c_1(\mathcal{L}') \cdot \gamma) > 0$, and hence φ is finite on any fiber of π'. This means that if $\gamma \in \Gamma$, then γ is flat over $\mathfrak{B}'$. Noting that an irreducible curve in $\mathcal{A}'$ that is flat over $\mathfrak{B}'$ is the closure of some point in $A(\overline{K})$, we have $\Gamma \subset \{\Delta_a \mid a \in A(\overline{K})\}$. Further, since $\dim(\varphi(\gamma)) = 0$ for any $\gamma \in \Gamma$, it follows from what we note in the previous paragraph that $\Gamma \subset \{\Delta_a \mid a \in A(0; L)\}$. Thus $\Gamma = \{\Delta_a \mid a \in A(0; L)\}$.

On the other hand, since $A(0; L)$ is dense in A, the set $\Gamma = \{\Delta_a \mid a \in A(0; L)\}$ is dense in $\mathcal{A}'$. Since $\dim(\varphi(\gamma)) = 0$ for any $\gamma \in \Gamma$, it follows that φ is not generically finite on $\mathcal{A}'$.

Here, recall that $\#k > \aleph_0$. Since φ is not generically finite, we have $\#\Gamma > \aleph_0$. Note that the map $A(\overline{K}) \to \{\Delta_a \mid a \in A(\overline{K})\}$ given by $a \mapsto \Delta_a$ induces a bijection $A(0; L) \to \{\Delta_a \mid a \in A(0; L)\} = \Gamma$. Then the above shows that $A(\overline{K})$ has uncountably many points of height 0. However, since A has trivial $\overline{K}/k$-trace, a point of $A(\overline{K})$ has height 0 if and only if it is torsion (cf. Proposition 2.13), and there are only countably many such points. This is a contradiction. Thus Proposition 11.6 is proved.

11.3. Application to the conjecture for abelian varieties. — Here are remarks on some contributions to the geometric Bogomolov conjecture for abelian varieties. Using Theorems 11.2 and 11.1, we obtain the following results on the conjecture for abelian varieties.

Theorem 11.7 ([43, Cor. 6.4]). — *Let A be an abelian variety over $\overline{K}$ and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Let $\mathfrak{t}$ be the image of the $\overline{K}/k$-homomorphism. Assume that $\dim(\mathfrak{m}/\mathfrak{t}) \leq 3$. Then the geometric Bogomolov conjecture holds for A.*

Indeed, since $\dim(\mathfrak{m}/\mathfrak{t}) \leq 3$, it follows by Remark 3.5, Theorems 11.2 and 11.1 that the geometric Bogomolov conjecture holds for $\mathfrak{m}/\mathfrak{t}$. By Theorem 3.10, the geometric Bogomolov conjecture holds for A.

In conclusion, the geometric Bogomolov conjecture is reduced to the following conjecture.

Conjecture 11.8. — *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace. Let X be a closed subvariety of A. Assume that $2 \leq \dim(X) \leq \dim(A) - 2$. Suppose that X has dense small points. Then X is a torsion subvariety.*

This is not what we discuss in this paper, but we have shown in [37] that the geometric Bogomolov conjecture holds for nowhere degenerate abelian varieties of dimension 5 with trivial $\overline{K}/k$-trace (cf. [37, Thm. 1.3]), and thus we have the following.

Theorem 11.9 ([37, Thm. 1.4]). — *Let A, $\mathfrak{m}$, and $\mathfrak{t}$ be as in Theorem 11.7. Assume that $\dim(\mathfrak{m}/\mathfrak{t}) = 5$. Then the geometric Bogomolov conjecture holds for A.*

It would be interesting to obtain the result for an abelian variety A as above with $\dim(A) = 4$ and X with $\dim(X) = 2$.

12. Manin–Mumford conjecture in positive characteristic

In this section, we give a remark on the relationship between the geometric Bogomolov conjecture and the Manin–Mumford conjecture in positive characteristic.

Let $\mathfrak{K}$ be an algebraically closed field. Let A be an abelian variety over $\mathfrak{K}$. Manin–Mumford conjecture, which is Raynaud's theorem, asserts that if $\mathrm{ch}(\mathfrak{K}) = 0$, then any closed subvariety of A that has dense torsion points is a torsion subvariety (cf. §1.2.3).

In the case of $\mathrm{ch}(\mathfrak{K}) > 0$, on the other hand, the same assertion does not hold. If $\mathfrak{K}$ is an algebraic closure of a finite field and X is any closed subvariety of A, then any $\mathfrak{K}$-point of X is a torsion point, and thus X always has dense torsion points.

However, up to such influence that stems from finite fields, one may expect that a similar statement should also hold in positive characteristics. In fact, the following precise result is known.

Theorem 12.1. — *Assume that* $\mathrm{ch}(\mathfrak{K}) > 0$. *Let k be the algebraic closure in $\mathfrak{K}$ of the prime field of $\mathfrak{K}$. Let A be an abelian variety over $\mathfrak{K}$. Let X be a closed subvariety of A. If X has dense torsion points, then there exist an abelian subvariety G of A, a closed subvariety $\widetilde{Y}$ of $\widetilde{A}^{\mathfrak{K}/k}$ and a torsion point $\tau \in A(\mathfrak{K})$ such that $X = G + \mathrm{Tr}_A^{\mathfrak{K}/k}(\widetilde{Y} \otimes_k \mathfrak{K}) + \tau$, where $(\widetilde{A}^{\mathfrak{K}/k}, \mathrm{Tr}_A^{\mathfrak{K}/k})$ is the $\mathfrak{K}/k$-trace of A.*

Here, note that the $\mathfrak{K}/k$-trace of A is defined by the same way as the $\overline{K}/k$-trace in §2.4 (in place of $\overline{K}$ with $\mathfrak{K}$); it is known that there exists a unique $\mathfrak{K}/k$-trace of A (cf. [20, Ch. VIII, §3]).

This theorem is due to the following authors: In 2001, Scanlon gave a sketch of the model-theoretic proof of this theorem ([33]). In 2004, Pink and Roessler gave an algebro-geometric proof ([30]). In 2005, Scanlon gave a detailed model-theoretic proof in [34] based on the argument in [33]. Note that in those papers, they prove a generalized version for semiabelian varieties A, in fact.

Here, we explain that Theorem 12.1 can be deduced from the geometric Bogomolov conjecture for A, as Moriwaki mentioned in [27] in the case of characteristic 0. Let A be an abelian variety over $\mathfrak{K}$ and let X be a closed subvariety of A. Then there exist $t_1, \ldots, t_n \in \mathfrak{K}$ such that A and X can be defined over $K := k(t_1, \ldots, t_n)$, that is, there exist an abelian variety A_0 over K and a closed subvariety X_0 of A_0 such that $A = A_0 \otimes_K \mathfrak{K}$ and $X = X_0 \otimes_K \mathfrak{K}$. Further, there exists a normal projective variety $\mathfrak{B}$ over k with function field K. Let $\overline{K}$ be the algebraic closure of K in $\mathfrak{K}$ and set $A_{\overline{K}} := A_0 \otimes_K \overline{K}$ and $X_{\overline{K}} := X_0 \otimes_K \overline{K}$. Let $\mathcal{H}$ be an ample line bundle on $\mathfrak{B}$. Then we have a notion of height over K, and we can consider the canonical height on $A_{\overline{K}}$ associated to an even ample line bundle. Suppose that X has dense torsion points. Then $X_{\overline{K}}$ has dense torsion points and hence has dense small points. If we assume the geometric Bogomolov conjecture, it follows that $X_{\overline{K}}$ is a special subvariety, and this implies the conclusion of Theorem 12.1.

Since the geometric Bogomolov conjecture is still open, the above argument does not give a new proof of Theorem 12.1. However, we can actually deduce the theorem in the following special cases:

(1) $\dim(X) = 1$ or $\mathrm{codim}(X, A) = 1$ (from Theorems 11.2 and 11.1);

(2) $\dim(A) \leq 3$ (as a consequence above).

In particular, we have recovered the positive characteristic version of Theorem 1.1.

Appendix A. Admissible formal schemes and the Raynaud generic fibers

In this appendix, we give a brief summary on admissible formal schemes and the Raynaud generic fibers. Further, we explain the Raynaud extension and the valuation map for an abelian variety.

A.1. Admissible formal schemes and Raynaud generic fibers. — First of all, we recall the notion of affinoid algebras and associated affinoid spaces in the sense of Berkovich. Let $\mathbb{K}$ be an algebraically closed field complete with respect to a nontrivial nonarchimedean absolute value. Let $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ be the Tate algebra over $\mathbb{K}$, that is, the completion of the

polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with respect to the Gauss norm. By definition, $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ equals

$$\left\{ \sum_{\mathbf{m}=(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n} a_{\mathbf{m}} x_1^{m_1} \cdots x_n^{m_n} \in \mathbb{K}[\![x_1, \ldots, x_n]\!] \,\middle|\, \lim_{m_1 + \cdots + m_n \to \infty} |a_{\mathbf{m}}| = 0 \right\}.$$

A $\mathbb{K}$-*affinoid algebra* is a $\mathbb{K}$-algebra isomorphic to $\mathbb{K}\langle x_1, \ldots, x_n \rangle / I$ for some ideal I of $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. Let $\mathrm{Max}(R)$ be the maximal spectrum of a $\mathbb{K}$-affinoid algebra R, that is, the set of maximal ideals of R. For each $p \in \mathrm{Max}(R)$, the residue field at p is canonically isomorphic to $\mathbb{K}$, and thus it is endowed with a norm. Therefore, we can consider the supremum semi-norm $|\cdot|_{\sup} : R \to \mathbb{R}$ over $\mathrm{Max}(R)$. The *Berkovich spectrum* $\mathcal{M}(R)$ *of* R is the set of multiplicative seminorms R bounded with $|\cdot|_{\sup}$ endowed with the weakest topology such that for any $f \in R$, the function $\mathcal{M}(R) \to \mathbb{R}$ given by $p \mapsto p(f) =: |f(p)|$ is continuous. The Berkovich spectrum of R is also called the Berkovich affinoid space associated to R. A Berkovich affinoid $\mathcal{M}(R)$ have a nonarchimedean analytic structure whose ring of functions R, but we do not explain it here.

Let $\mathbb{K}^\circ$ be the ring of integers of $\mathbb{K}$. A $\mathbb{K}^\circ$-algebra is called an *admissible* $\mathbb{K}^\circ$-*algebra* if it does not have any $\mathbb{K}^\circ$-torsions and it is isomorphic to $\mathbb{K}^\circ \langle x_1, \ldots, x_n \rangle / I$ for some $n \in \mathbb{N}$ and for some ideal I of $\mathbb{K}^\circ \langle x_1, \ldots, x_n \rangle$. Note that an admissible $\mathbb{K}^\circ$-algebra is flat over $\mathbb{K}^\circ$. The formal spectrum of an admissible $\mathbb{K}^\circ$-algebra is called an *affine admissible formal scheme*.

For an admissible $\mathbb{K}^\circ$-algebra $\mathcal{R}$, we can associate an affinoid algebra $\mathcal{R} \otimes_{\mathbb{K}^\circ} \mathbb{K}$. Thus to an affine admissible formal scheme $\mathscr{U} = \mathrm{Spf}(\mathcal{R})$, one associates an Berkovich affinoid space $\mathscr{U}^{\mathrm{an}} = \mathcal{M}(\mathcal{R} \otimes_{\mathbb{K}^\circ} \mathbb{K})$.

A formal scheme over $\mathbb{K}^\circ$ is called an *admissible formal scheme* if it has a locally finite open atlas of affine admissible formal schemes. Let $\mathscr{X}$ be an admissible formal scheme. We take an affine covering $\{\mathscr{U}_\lambda\}$ of $\mathscr{X}$. Then we have a family $\{\mathscr{U}_\lambda^{\mathrm{an}}\}$ of Berkovich affinoid spaces. In fact, the patching data of the covering $\{\mathscr{U}_\lambda\}$ give rise to patching data of the family $\{\mathscr{U}_\lambda^{\mathrm{an}}\}$ of Berkovich affinoid spaces, and hence those Berkovich affinoid spaces patch together to be a topological space. Further, this topological space does not depend on the choice of the affine covering $\{\mathscr{U}_\lambda\}$ and depends only on $\mathscr{X}$. We denote this topological space by $\mathscr{X}^{\mathrm{an}}$ and call it the *Raynaud generic fiber of* $\mathscr{X}$.

The terminology of Berkovich spaces here is compatible with that in §5.2 when we consider proper schemes. Let $\mathscr{X}$ be a proper flat scheme over $\mathbb{K}^\circ$ with irreducible and reduced generic fiber X. Since X is a variety over $\mathbb{K}$, one associates a Berkovich space X^{an} in the sense of §5.2. On the other hand, the formal completion $\widehat{\mathscr{X}}$ with respect to an element $\varpi \in \mathbb{K}$ with $0 < |\varpi| < 1$ is an admissible formal scheme, and hence one can associate the Raynaud generic fiber $\widehat{\mathscr{X}}^{\mathrm{an}}$, which is a Berkovich analytic space in the above sense. Then one checks that $X^{\mathrm{an}} = \widehat{\mathscr{X}}^{\mathrm{an}}$ holds. We remark, however, that if $\mathscr{X}$ is not proper, then X^{an} and $\widehat{\mathscr{X}}^{\mathrm{an}}$ are different.

Example A.1. — Let $x_1, \ldots, x_n$ be indeterminates, and consider $\mathrm{Spec}(\mathbb{K}^\circ[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$. Then the generic fiber equals the algebraic torus $\mathbb{G}_m^n = \mathrm{Spec}(\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$, and we have its Berkovich analytification $(\mathbb{G}_m^n)^{\mathrm{an}}$. On the other hand, the formal completion of $\mathrm{Spec}(\mathbb{K}^\circ[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$ is $\mathrm{Spf}(\mathbb{K}^\circ \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1} \rangle)$, called the *formal torus*, and one sees that the Raynaud generic fiber $\left(\mathrm{Spf}(\mathbb{K}^\circ \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1} \rangle) \right)^{\mathrm{an}} =: (\mathbb{G}_m^n)_1^{\mathrm{an}}$ equals $\{ p \in (\mathbb{G}_m^n)^{\mathrm{an}} \mid |x_1(p)| = \cdots = |x_n(p)| = 1 \}$.

A.2. Raynaud extensions and valuation maps. — Let A be an abelian variety over $\mathbb{K}$. Then there exists a unique admissible formal group scheme $\mathscr{A}^{\circ}$ over $\mathbb{K}^{\circ}$ with a homomorphism $i : (\mathscr{A}^{\circ})^{\mathrm{an}} \to A^{\mathrm{an}}$ having the following properties.

- The image A° of i is an analytic subdomain of A^{an}, and i is an isomorphism to its image.

- There exist a formal abelian scheme $\mathscr{B}$ and an exact sequence

$$1 \to \mathscr{T} \to \mathscr{A}^{\circ} \to \mathscr{B} \to 0$$

of admissible formal group schemes, where $\mathscr{T}$ is a formal torus.

We remark that the exact sequence in (3.1) arises by restricting the above exact sequence of formal group schemes to the special fibers.

Taking the Raynaud generic fiber of the above exact sequence and identifying $\mathscr{T}^{\mathrm{an}}$ with $(\mathbb{G}_m^n)_1^{\mathrm{an}}$, we obtain an exact sequence

$$1 \to (\mathbb{G}_m^n)_1^{\mathrm{an}} \to (\mathscr{A}^{\circ})^{\mathrm{an}} \to \mathscr{B}^{\mathrm{an}} \to 0$$

of Berkovich analytic group spaces. Pushing out this exact sequence by the natural inclusion $(\mathbb{G}_m^n)_1^{\mathrm{an}} \hookrightarrow (\mathbb{G}_m^n)_1^{\mathrm{an}}$, we obtain an exact sequence

$$1 \to (\mathbb{G}_m^n)^{\mathrm{an}} \to E \to \mathscr{B}^{\mathrm{an}} \to 0.$$

These two exact sequences of analytic groups are called the *Raynaud extensions of A^{an}*. One shows that the injective homomorphism $(\mathscr{A}^{\circ})^{\mathrm{an}} \to A^{\mathrm{an}}$ extends to a unique homomorphism $p : E \to A^{\mathrm{an}}$. Further, $M := \mathrm{Ker}(p)$ is a discrete subgroup of E, and p induces an isomorphism $E/M \cong A^{\mathrm{an}}$. The nonnegative integer n above is called the dimension of the torus part of A^{an}. The number r in (3.1) equals this number n for $A = A_v$.

Recall that we have a homomorphism $(\mathbb{G}_m^n)^{\mathrm{an}} \to \mathbb{R}^n$ (cf. (6.1)). In fact, one shows that this homomorphism extends to a unique continuous homomorphism $\mathrm{val} : E \to \mathbb{R}^n$, called the valuation map. Set $\Lambda := \mathrm{val}(M)$. Then one also shows that Λ is a complete lattice of $\mathbb{R}^n$. (If we denote by Γ the value group of $\mathbb{K}$, then $\Lambda \subset \Gamma^n$.) Further, val induces a continuous homomorphism $\overline{\mathrm{val}} : A^{\mathrm{an}} \to \mathbb{R}^n/\Lambda$.

References

[1] A. ABBES, "Hauteurs et discrétude (d'après Szpiro, L., Ullmo, E., Zhang, S.)", in *Séminaire Bourbaki 1996/97*, Astérisque, vol. 245, Société Mathématique de France, 1997, p. 141-166.

[2] V. G. BERKOVICH, *Spectral theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, 1990, ix+169 pages.

[3] ———, "Etale cohomology for non-archimedean analytic spaces", *Publ. Math., Inst. Hautes Étud. Sci.* **78** (1993), p. 5-161.

[4] ———, "Vanishing cycles for formal schemes", *Invent. Math.* **115** (1994), no. 3, p. 539-571.

[5] ———, "Smooth p-adic analytic spaces are locally contractible", *Invent. Math.* **137** (1999), no. 1, p. 1-84.

[6] F. A. BOGOMOLOV, "Points of finite order on Abelian Variety", *Izv. Akad. Nauk SSSR, Ser. Mat.* **44** (1980), no. 4, p. 782-804.

[7] E. BOMBIERI & W. GUBLER, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, 2006, xvi+652 pages.

[8]   A. CHAMBERT-LOIR, "Mesures et équidistribution sur les espaces de Berkovich", *J. Reine Angew. Math.* **595** (2006), p. 215-235.

[9]   Z. CINKIR, "Zhang's conjecture and the effective Bogomolov conjecture over function fields", *Invent. Math.* **183** (2011), no. 3, p. 517-562.

[10] X. W. C. FABER, "The geometric Bogomolov conjecture for curves of small genus", *Exp. Math.* **18** (2009), no. 3, p. 347-367.

[11] W. GUBLER, "Local heights of subvarieties over non-archimedean fields", *J. Reine Angew. Math.* **498** (1998), p. 61-113.

[12] ———, "Local and canonical heights of subvarieties", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **2** (2003), no. 4, p. 711-760.

[13] ———, "The Bogomolov conjecture for totally degenerate abelian varieties", *Invent. Math.* **169** (2007), no. 2, p. 377-400.

[14] ———, "Tropical varieties for non-archimedean analytic spaces", *Invent. Math.* **169** (2007), no. 2, p. 321-376.

[15] ———, "Equidistribution over function fields", *Manuscr. Math.* **127** (2008), no. 4, p. 485-510.

[16] ———, "Non-archimedean canonical measures on abelian varieties", *Compos. Math.* **146** (2010), no. 3, p. 683-730.

[17] A. DE JONG, "Smoothness, semi-stability and alterations", *Publ. Math., Inst. Hautes Étud. Sci.* **83** (1996), p. 51-93.

[18] S. KAWAGUCHI, A. MORIWAKI & K. YAMAKI, "Introduction to Arakelov geometry", in *Algebraic geometry in East Asia (Kyoto, 2001)*, World Scientific, 2002, p. 1-74.

[19] S. LANG, "Division points on curves", *Ann. Mat. Pura Appl.* **70** (1965), p. 229-234.

[20] ———, *Abelian varieties*, Springer, 1983, xii+256 pages.

[21] ———, *Fundamentals of Diophantine Geometry*, Springer, 1983, xviii+370 pages.

[22] L. MORET-BAILLY, "Métriques permises", in *Séminaire sur les pinceaux arithmétiques: La Conjecture de Mordell*, Astérisque, vol. 127, Société Mathématique de France, 1985, p. 29-87.

[23] A. MORIWAKI, "Bogomolov conjecture for curves of genus 2 over function fields", *J. Math. Kyoto Univ.* **36** (1996), no. 4, p. 687-695.

[24] ———, "A sharp slope inequality for general stable fibrations of curves", *J. Reine Angew. Math.* **480** (1996), p. 177-195.

[25] ———, "Bogomolov conjecture over function fields for stable curves with only irreducible fibers", *Compos. Math.* **105** (1997), no. 2, p. 125-140.

[26] ———, "Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves", *J. Am. Math. Soc.* **11** (1998), no. 3, p. 569-600.

[27] ———, "Arithmetic height functions over finitely generated fields", *Invent. Math.* **140** (2000), no. 1, p. 101-142.

[28] D. MUMFORD, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Oxford University Press, 1970, viii+242 pages.

[29] J. NICAISE, "Berkovich skeleta and birational geometry", https://arxiv.org/abs/1409.5229, 2014.

[30] R. Pink & D. Roessler, "On ψ-invariant subvarieties of semiabelian varieties and the Manin–Mumford conjecture", *J. Algebr. Geom.* **13** (2004), no. 4, p. 771-798.

[31] M. Raynaud, "Courbes sur une variété abélienne et points de torsion", *Invent. Math.* **71** (1983), p. 207-233.

[32] ———, "Sous-variétés dúne variété abélienne et points de torsion", in *Arithmetic and geometry, vol. I: Arithmetic*, Progress in Mathematics, vol. 35, Birkhäuser, 1983, p. 327-352.

[33] T. Scanlon, "Diophantine geometry from model theory", *Bull. Symb. Log.* **7** (2001), no. 1, p. 37-57.

[34] ———, "A positive characteristic Manin–Mumford theorem", *Compos. Math.* **141** (2005), no. 6, p. 1351-1364.

[35] L. Szpiro, E. Ullmo & S.-W. Zhang, "Équirépartition des petits points", *Invent. Math.* **127** (1997), no. 2, p. 337-347.

[36] E. Ullmo, "Positivité et discrétion des points algébriques des courbes", *Ann. Math.* **147** (1998), no. 1, p. 167-179.

[37] K. Yamaki, "Geometric Bogomolov conjecture for nowhere degenerate abelian varieties of dimension 5 with trivial trace", to appear in *Math. Res. Lett.*

[38] ———, "Geometric Bogomolov's conjecture for curves of genus 3 over function fields", *J. Math. Kyoto Univ.* **42** (2002), no. 1, p. 57-81.

[39] ———, "Effective calculation of the geometric height and the Bogomolov conjecture for hyperelliptic curves over function fields", *J. Math. Kyoto Univ.* **48** (2008), no. 2, p. 401-443.

[40] ———, "Geometric Bogomolov conjecture for abelian varieties and some results for those with some degeneration (with an appendix by Walter Gubler: The minimal dimension of a canonical measure)", *Manuscr. Math.* **142** (2013), no. 3-4, p. 273-306.

[41] ———, "Strict supports of canonical measures and applications to the geometric Bogomolov conjecture", *Compos. Math.* **152** (2016), no. 5, p. 997-1040.

[42] ———, "Trace of abelian varieties over function fields and the geometric Bogomolov conjecture", *J. Reine Angew. Math.* (2016), `https://doi.org/10.1515/crelle-2015-0086`.

[43] ———, "Non-density of small points on divisors on abelian varieties and the Bogomolov conjecture", *J. Am. Math. Soc.* **30** (2017), no. 4, p. 1133-1163.

[44] X. Yuan, "Big line bundles over arithmetic varieties", *Invent. Math.* **173** (2008), no. 3, p. 603-649.

[45] S.-W. Zhang, "Admissible pairing on a curve", *Invent. Math.* **112** (1993), no. 1, p. 171-193.

[46] ———, "Small points and adelic metrics", *J. Algebr. Geom.* **4** (1995), no. 2, p. 281-300.

[47] ———, "Equidistribution of small points on abelian varieties", *Ann. Math.* **147** (1998), no. 1, p. 159-165.

[48] ———, "Gross-Schoen cycles and dualising sheaves", *Invent. Math.* **179** (2010), no. 1, p. 1-73.

Kazuhiko Yamaki, Institute for Liberal Arts and Sciences, Kyoto University, Kyoto, 606-8501, Japan
E-mail : `yamaki.kazuhiko.6r@kyoto-u.ac.jp`