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Structure of small cancellation rings

Agatha Atkarskaya, Alexei Kanel-Belov, Eugene Plotkin & Eliyahu Rips

(Recommended by Alexander Olshanskiy)

Abstract. The theory of small cancellation groups is well known. In this paper we study the notion of the Group-like Small Cancellation Ring. We define this ring axiomatically, by generators and defining relations. The relations must satisfy three types of axioms. The major one among them is called the Small Cancellation Axiom. We show that the obtained ring is non-trivial and enjoys a global filtration that agrees with relations, find a basis of the ring as a vector space and establish the corresponding structure theorems. It turns out that the defined ring possesses a kind of Gröbner basis and a greedy algorithm. Finally, this ring can be used as a first step towards the iterated small cancellation theory, which hopefully plays a similar role in constructing examples of rings with exotic properties as small cancellation groups do in group theory.

1. Introduction

The Small Cancellation Theory for groups is well known (see [13]). The similar theory exists also for semigroups and monoids (see [10, 9, 22]). However, the construction of such a theory for systems with two operations faces severe difficulties.

In the present paper we develop a small cancellation theory for associative algebras with a basis of invertible elements. In fact, in the course of studying the question:

“What is a small cancellation associative ring?"

we axiomatically define a ring that can reasonably be called a ring with small cancellation properties and conditions. We also determine the structure and properties of this ring.

1.1. Motivation, objectives, results. The motivation for developing a ring-theoretical analogue of small cancellation comes from the fact that small cancellation for groups and, especially, its more far-reaching versions, provide a very powerful technique for constructing groups with unusual, and even exotic, properties, like for example, infinite Burnside groups [15, 16, 17, 1, 19, 11, 14], Tarski monster [18], finitely generated infinite divisible groups [8], and many others, see, e.g., [20].

On the other hand, there is a conceptual desire to understand what negative curvature could mean for ring theory.

For any group with fixed system of generators, its Cayley graph can be considered as a metric space. This leads to Gromov's program “Groups as geometric objects" [6], see also [7]. In particular, a finitely generated group is word-hyperbolic when its Cayley graph is δ-hyperbolic for $\delta > 0$ (see [4, 5] for modern exposition and references).

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So far, we do not know a way to associate a geometric object to a ring. Thus, having in mind the negative curvature as a heuristic and indirect hint for our considerations, we, nevertheless, follow a more accessible combinatorial line of studying rings. Therefore, small cancellation groups appear naturally at the stage.

Finitely generated small cancellation groups turned out to be word hyperbolic (when every relation needs at least 7 pieces). So, if we could generalize small cancellation to the ring-theoretic situation, it would provide examples to the yet undefined concept of a ring with a negative curvature. Another source of potential examples are group algebras of hyperbolic groups.

Following this reasoning, we introduce the three types of axioms for rings called Compatibility Axiom, Small Cancellation Axiom, and Isolation Axiom. We study rings $\mathscr{A}$ with the basis of invertible elements that satisfy these axioms with respect to a fixed natural constant $\tau \geqslant 10$. We show the following:

- Such rings $\mathscr{A}$ are non-trivial;
- Such rings $\mathscr{A}$ enjoy a global filtration that agrees with the relations;
- An explicit basis of $\mathscr{A}$ as a linear space is constructed and the corresponding structure theorems are proved;
- These rings possess algorithmic properties similar to the ones valid for groups with small cancellation. In particular they have solvable equality problem and enjoy a greedy algorithm;
- These rings also possess a Gröbner basis with respect to some sophisticated linear order on monomials.

The list of facts above can be viewed as a major result of the paper. In what follows we describe and illuminate all these items. The detailed exposition of these results is contained in the paper [3]. Note that the axiomatic theory presented in this paper is modeled after a particular case we have treated in [2].

1.2. Small cancellation groups, background. Consider a group presentation $G = \langle \mathscr{X} \mid \mathscr{R} \rangle$ where we assume that the set of relations $\mathscr{R}$ is closed under cyclic permutations and inverses and that all elements of $\mathscr{R}$ are cyclically reduced. The interaction between the defining relations is described in terms of small pieces. A word s is called a *small piece* with respect to $\mathscr{R}$ (in generalized group sense, see [21, 13]) if there are relations of the form sr_1 and sr_2 in $\mathscr{R}$ such that $r_1 r_2^{-1} \neq 1$ and $r_1 r_2^{-1}$ is not conjugate to a relator from $\mathscr{R}$ in the corresponding free group, even after possible cancellations.

Remark 1.1. *The geometric way to think about small pieces is seeing them as words that may appear on the common boundary between two cells in the van Kampen diagram* [20, 13]. *In particular, if $r_1 r_2^{-1} \in \mathscr{R}$, then we can substitute these cells by a simple cell, so we are entitled to assume from the beginning that $r_1 r_2^{-1} \notin \mathscr{R}$.*

The *small cancellation condition* says that any relation in $\mathscr{R}$ cannot be written as a product of too few small pieces. For most purposes seven small pieces suffice since the discrete Euler characteristic per cell becomes negative [13, 12].

To ensure this, we can assume that the length of any small piece is less than one sixth of the length of the relation in which it appears. The Main Theorem of Small Cancellation Theory can be stated as follows.

Let w_1, w_2 be two words that do not contain occurrences of more than a half of a relation from $\mathscr{R}$. They represent the same element of G if and only if they can be connected by a one-layer diagram ([13], especially see Greendlinger's Lemma). The transition from w_1 to w_2 can be divided into a sequence of elementary steps called *turns* [15, 16, 17]. Each turn reverses just one cell.

2. Small cancellation axioms for the ring case

First of all, given a field k and the free group $\mathscr{F}$, denote by $k\mathscr{F}$ the corresponding group algebra. Elements of $\mathscr{F}$ and $k\mathscr{F}$ are called *monomials or words* and *polynomials*, respectively. Let a set of polynomials $\mathscr{R}$ from $k\mathscr{F}$ be fixed. Define $\mathscr{I}$ to be the ideal generated by the elements of $\mathscr{R}$.

Let the free group $\mathscr{F}$ be freely generated by an alphabet S. Assume

$$\mathscr{R} = \left\{ p_i = \sum_{j=1}^{n(i)} \alpha_{ij} m_{ij} \;\middle|\; \alpha_{ij} \in k, m_{ij} \in \mathscr{F}, i \in I \right\}$$

is a (finite or infinite) set of polynomials that generates the ideal $\mathscr{I}$ (as an ideal). We denote this way of generating by $\langle \mid \rangle_i$. So,

$$\mathscr{I} = \langle \mathscr{R} \rangle_i = \left\langle p_i = \sum_{j=1}^{n(i)} \alpha_{ij} m_{ij} \;\middle|\; \alpha_{ij} \in k, m_{ij} \in \mathscr{F}, i \in I \right\rangle_i .$$

We assume that the monomials m_{ij} are reduced, the polynomials p_i are additively reduced, I is some index set. In particular, we assume that all coefficients α_{ij} are non-zero. Denote the set of all monomials m_{ij} of $\mathscr{R}$ by $\mathscr{M}$. Throughout the paper we reserve small Greek letters for non-zero elements of the field k.

Condition 1 (Compatibility Axiom)**.** *The axiom consists of the following two conditions.*

(1) *If* $p = \sum_{j=1}^{n} \alpha_j m_j \in \mathscr{R}$, *then* $\beta p = \sum_{j=1}^{n} \beta\alpha_j m_j \in \mathscr{R}$ *for every* $\beta \in k, \beta \neq 0$.

(2) *Let* $x \in S \cup S^{-1}$ *where* S *is an alphabet which freely generates* $\mathscr{F}$, $p = \sum_{j=1}^{n} \alpha_j m_j \in \mathscr{R}$. *Suppose there exists* $j_0 \in \{1, \dots, n\}$ *such that* x^{-1} *is the initial symbol of* m_{j_0}. *Then*

$$xp = \sum_{j=1}^{n} \alpha_j x m_j \in \mathscr{R}$$

(*after the cancellations in the monomials* xm_j).

We require the same condition from the right side as well.

From the second condition of Compatibility Axiom it immediately follows that the set $\mathscr{M}$ is closed under taking subwords. In particular, the empty word always belongs to $\mathscr{M}$.

Now we state a definition of *a small piece* with respect to $\mathscr{R}$ in the algebra $k\mathscr{F}$. The definition of a small piece given in (see [13]) can be viewed as follows. A word is called a small piece if it occurs in two different relations which are not a cyclic shift of each other or in one relation in two essentially different places. The later means that the occurrences of this subword are not obtained by a shift of a period of the corresponding relation. A straightforward way to generalize this for the ring case is to say that if $c \in \mathscr{F}$ and we have two different polynomials $p = \sum_{j=1}^{n_1} \alpha_j a_j + \alpha c$ and $q = \sum_{j=1}^{n_2} \beta a_j + \beta c$, then c is a small piece. However, it turns out that the this way does not work for our needs (see Section 5 with examples). So, we need a special intuition in order to see what "essentially different places in relations" means for rings. This is reflected in our Definition 2.1 of a small piece. This definition plays a central role in the further argument.

Definition 2.1. *Let* $c \in \mathscr{M}$. *Assume there exist two polynomials*

$$p = \sum_{j=1}^{n_1} \alpha_j a_j + \alpha a \in \mathscr{R}, \qquad q = \sum_{j=1}^{n_2} \beta_j b_j + \beta b \in \mathscr{R},$$

such that c is a subword of a and a subword of b. Namely,

$$a = \widehat{a}_1 c \widehat{a}_2, \qquad b = \widehat{b}_1 c \widehat{b}_2,$$

where $\widehat{a}_1$, $\widehat{a}_2$, $\widehat{b}_1$, $\widehat{b}_2$ are allowed to be empty. Assume that

$$\widehat{b}_1 \widehat{a}_1^{-1} p = \widehat{b}_1 \widehat{a}_1^{-1} \left(\sum_{j=1}^{n_1} \alpha_j a_j + \alpha \widehat{a}_1 c \widehat{a}_2 \right) = \sum_{j=1}^{n_1} \alpha_j \widehat{b}_1 \widehat{a}_1^{-1} a_j + \alpha \widehat{b}_1 c \widehat{a}_2 \notin \mathscr{R}$$

(even after the cancellations), or

$$p \widehat{a}_2^{-1} \widehat{b}_2 = \left(\sum_{j=1}^{n_1} \alpha_j a_j + \alpha \widehat{a}_1 c \widehat{a}_2 \right) \widehat{a}_2^{-1} \widehat{b}_2 = \sum_{j=1}^{n_1} \alpha_j a_j \widehat{a}_2^{-1} \widehat{b}_2 + \alpha \widehat{a}_1 c \widehat{b}_2 \notin \mathscr{R}$$

(even after the cancellations). Then the monomial c is called a small piece.

We denote the set of all small pieces by $\mathscr{S}$. Clearly, $\mathscr{S} \subseteq \mathscr{M}$. From the definition it follows that the set $\mathscr{S}$ is closed under taking subwords. In particular, if the set $\mathscr{S}$ is non-empty, the empty word is always a small piece. If the set $\mathscr{S}$ is turned out to be empty, then *we still assume that the empty word is a small piece.*

Let $u \in \mathscr{M}$. Then either $u = p_1 \cdots p_k$, where $p_1, \dots, p_k$ are small pieces, or u cannot be represented as a product of small pieces. We introduce a measure on monomials of $\mathscr{M}$ (aka Λ-*measure*). *We say that $\Lambda(u) = k$ if u can be represented as a product of small pieces and minimal possible number of small pieces in such representation is equal to k. We say that $\Lambda(u) = \infty$ if u can not be represented as a product of small pieces.*

We fix a constant $\tau \in \mathbb{N}$, $\tau \geqslant 10$.

Condition 2 (Small Cancellation Axiom)**.** *Assume $p_1, \dots, p_n \in \mathscr{R}$ and a linear combination $\sum_{s=1}^{n} \gamma_s p_s$ is non-zero after additive cancellations. Then there exists a monomial a in $\sum_{s=1}^{n} \gamma_s p_s$ with a non-zero coefficient after additive cancellations such that either a can not be represented as a product of small pieces or every representation of a as a product of small pieces contains at least $\tau + 1$ small pieces. That is, $\Lambda(a) \geqslant \tau + 1$, including $\Lambda(a) = \infty$.*

Definition 2.2. *Let $p = \sum_{j=1}^{n} \alpha_j a_j \in \mathscr{R}$. Then we call the monomials a_{j_1}, a_{j_2}, $1 \leqslant j_1$, $j_2 \leqslant n$, incident monomials (including the case $a_{j_1} = a_{j_2}$). Recall that $\alpha_j \neq 0$, $j = 1, \dots, n$.*

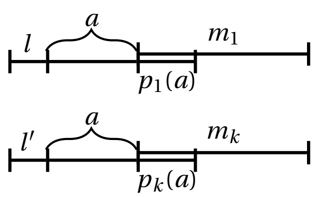
Now we introduce the last condition; we call it Isolation Axiom. Unlike the two previous axioms this is entirely a ring-theoretic condition. Here we use the notions of maximal occurrence of a monomial of $\mathscr{M}$ and of overlap.

Let U be a word and $\widehat{U}$ be its subword. We call the triple that consists of U, $\widehat{U}$ and the position of $\widehat{U}$ in U *an occurrence of $\widehat{U}$ in U*. In fact, we consider occurrences of the form $a \in \mathscr{M}$ in U, that is, $U = LaR$, where L, R can be empty. Since $a \in \mathscr{M}$, there exists a polynomial $p \in \mathscr{R}$ such that a is a monomial of p. An *overlap* is defined as a common part of two occurrences. Under maximal occurrence we mean an occurrence of a monomial of $\mathscr{M}$ which is not contained in a bigger such occurrence. We shall underline that the a common part of two maximal occurrences is a small piece.

The complexity of formulation of Isolation Axiom may perflex the reader. This axiom works in the transition from monomials to tensor products and, thus, to structure theory of rings with small cancellation. It imposes essential constraints on rings under consideration. That is why we have chosen its weakest form to make the corresponding class of rings wider. This resulted in a somewhat cumbersome definition.

Condition 3 (Isolation Axiom, left-sided)**.** *Let $m_1, m_2, \dots, m_k$ be a sequence of monomials of $\mathscr{M}$ such that $m_1 \neq m_k$ and m_i, m_{i+1} are incident monomials for all $i = 1, \dots, k-1$, and $\Lambda(m_i) \geqslant \tau - 2$ for all $i = 1, \dots, k$. Let us take a monomial $a \in \mathscr{M}$ with the following properties.*

1. $\Lambda(a) \geqslant \tau - 2$;
2. $am_1, am_k \notin \mathcal{M}$, am_1 *has no cancellations,* am_k *has no cancellations;*
3. m_1 *is a maximal occurrence in* am_1, m_k *is a maximal occurrence in* am_k.
4. *Let* $ap_1(a)$ *be a maximal occurrence in* am_1 *that contains* a, *let* $ap_k(a)$ *be a maximal occurrence in* am_k *that contains* a (*that is,* $p_1(a)$ *is the overlap of* $ap_1(a)$ *and* m_1, $p_1(a)$ *may be empty, and* $p_k(a)$ *is the overlap of* $ap_k(a)$ *and* m_k, $p_k(a)$ *may be empty*)*. Assume that there exist monomials* l, $l' \in \mathcal{M}$ *such that*
   - l, l' *are small pieces;*
   - $la, l'a \in \mathcal{M}$, la *has no cancellations,* $l'a$ *has no cancellations;*
   - *there exists a sequence of monomials* $b_1, \ldots, b_n$ *from* $\mathcal{M}$ *such that* $b_1 = lap_1(a)$, $b_n = l'ap_k(a)$, b_i, b_{i+1} *are incident monomials for all* $i = 1, \ldots, n-1$, *and* $\Lambda(b_i) \geqslant \tau - 2$ *for all* $i = 1, \ldots, n$.



Notice that since a *is not a small piece, then we get that* $lap_1(a), l'ap_k(a) \in \mathcal{M}$, *and* $lap_1(a)$ *is a maximal occurrence in* $lap_1(a)m_1$, $l'ap_k(a)$ *is a maximal occurrence in* $l'ap_k(a)m_k$.

Then we require that $p_1(a)^{-1} \cdot m_1 \neq p_k(a)^{-1} \cdot m_k$ *for every such* $a \in \mathcal{M}$.

The right-sided Isolation Axiom is formulated symmetrically.

Let us notice that in the examples that we consider (see Section 5 and [2]) we have special properties of the list of defining relations $\mathcal{R}$ that help us to check Isolation Axiom. In particular, in these cases it is enough to check Isolation Axiom for sequences of monomials $m_1, \ldots, m_k$ of length $k = 2$ and this yields Isolation Axiom for sequences of monomials of arbitrary length.

Definition 2.3. *We say that* $\mathcal{A} = k\mathcal{F}/\mathcal{I}(\mathcal{R})$ *is* $C(\tau)$*-small cancellation ring if it satisfies Compatibility Axiom, Small Cancellation Axiom (with respect to* $\tau + 1$ *small pieces) and at least one of Isolation Axioms.*

In the further argument we assume that $\tau \geqslant 10$ (recall that in a small cancellation group we require that every relator is a product of not less than 7 small pieces, see [13]).

3. How we study the structure of small cancellation rings

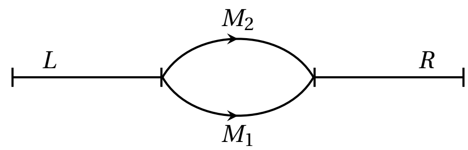
3.1. Towards a filtration on $k\mathcal{F}$: multi-turns, replacements, virtual members of the chart and numerical characteristics of monomials. For the remainder we will study the ring $\mathcal{A} = k\mathcal{F}/\mathcal{I}$, with $\mathcal{R}$ subject to three small cancellation conditions.

Now we indicate a ring-theoretic counterpart of the notion of turn.

Definition 3.1. *Let* U *be a monomial. We define the chart of* U *as the set of all maximal occurrences of monomials of* $\mathcal{M}$ *in* U *and call them elements of the chart. The elements of the chart* $m_i \in \mathcal{M}$ *such that* $\Lambda(m_i) \geqslant \tau$ *are called members of the chart.*

So, we distinguish between elements and members of the chart. Namely, we count as members of the chart only big enough occurrences of monomials from $\mathcal{M}$. Now we define a multi-turn that is a ring-theoretic analogue of a group turn.

In the case of groups we have the following situation. Let G be a small cancellation group, $R_i = M_1 M_2^{-1}$ be a relator of its small cancellation presentation. Assume LM_1R and LM_2R are two words, then the transition from LM_1R to LM_2R



is called a *turn* of an occurrence of the subrelation M_1 (to its complement M_2). Analogously, in our case we define a multi-turn.

Definition 3.2. *Let $p = \sum_{j=1}^{n} \alpha_j a_j \in \mathscr{R}$. For every $h = 1, \dots, n$ we call the transition*

$$a_h \longmapsto \sum_{j=1, j\neq h}^{n} (-\alpha_h^{-1} \alpha_j a_j),$$

an elementary multi-turn of a_h with respect to p.

Let $p = \sum_{j=1}^{n} \alpha_j a_j \in \mathscr{R}$. Let a_h be a maximal occurrence in U, $U = La_hR$. The transformation

$$U = La_hR \longmapsto \sum_{j=0, j\neq h}^{n} (-\alpha_h^{-1} \alpha_j La_jR)$$

with the further cancellations if there are any, is called a multi-turn of the occurrence a_h in U that comes from an elementary multi-turn $a_h \mapsto \sum_{j=1, j\neq h}^{n} (-\alpha_h^{-1} \alpha_j a_j)$. Obviously,

$$U - \sum_{j=0, j\neq h}^{n} (-\alpha_h^{-1} \alpha_j La_jR) = \alpha_h^{-1} LpR \in \mathscr{I}.$$

In this case the polynomial $LpR = \sum_{j=1}^{n} \alpha_j La_jR$ (after the cancellations) is called a layout of the multi-turn.

In what follows we undertake a very detailed study of the influence of multi-turns on charts of the monomials. We will trace transformation of a chart under the given multi-turn or set of multi-turns. We also take care of transformations of individual monomials $U_h = La_hR \mapsto U_j = La_jR$ called *replacements*.

Applying the multi-turns of a_h in $U_h = La_hR$ we arrive at monomials $U_j = La_jR$. We describe precisely *how the corresponding maximal occurrences in U_j look compared to maximal occurrences in U_h*.

We consider three variants for the resulting monomial $U_j = La_jR$: a_j is not a small piece; a_j is a small piece; a_j is 1. We show that in the first case the structure of the chart remains almost stable after a multi-turn, in the second case the replacement a_h by a_j can cause merging and restructuring of the chart, and in the third case strong cancellations resulting in complete modification of the chart are possible.

We produce the full list of all appearing arrangements of maximal occurrences. The calculations are based on thorough analysis of all combinatorial possibilities. This list is in fact a theorem that provides ground to further considerations towards a filtration on $k\mathscr{F}$.

Our goal is constructing a special ordering on monomials. This ordering is far from being usual $DegLex$-order. In more precise terms our objective is to build numerical characteristic of a chart that allows to define a filtration on monomials which behaves well with respect to replacements of the monomials caused by multi-turns.

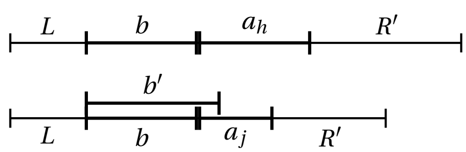
On the way we have to treat several caveats. When we define members of a chart in the terms of their Λ-measure, such definition is not stable enough under multi-turns.

So, we define a quite delicate notion of a *virtual member* of a chart. Virtual members of the chart of a monomial U are those maximal occurrences b which originally are not necessarily members of the chart but they are rather big with $\Lambda(b) \geqslant \tau-2$, and after a series of *admissible replacements* of maximal occurrences by incident monomials become members of the chart (for precise definitions see [3], Definition 6.2 of an admissible replacement and Definition 6.5 of a virtual member of the chart).

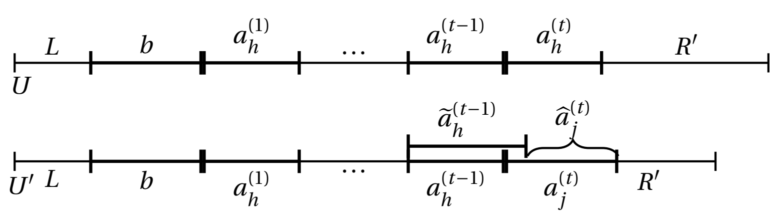
Definitions of admissible replacements and virtual members of the chart takes much of preparatory work. So, in order to give the reader a conceptual understanding of these notions we prefer to give here a number of illustrative examples instead of precise definitions.

First of all we give an example which illuminates the notion of an admissible replacement. Let U_h be a monomial, a_h be a maximal occurrence in U_h, $U_h = La_hR$, and a_h and a_j be incident monomials. We consider the replacement of a_h to a_j in U_h. Then the resulting monomial is La_jR. The important particular case of an admissible replacement in U_h that illustrates the whole idea is the replacement of a_h by a_j such that $\Lambda(a_h) \geqslant \tau-2$ and $\Lambda(a_j) \geqslant 3$ (that is, a_h and a_j are big enough, and a_j can be either of bigger, or smaller, or of the same Λ-measure as a_h). Roughly speaking, all admissible replacements are of such form.

Now we are in a position to illustrate the notion of a virtual member of the chart. Let a monomial $U = LbR$ be given, where b is a maximal occurrence in U. Assume $\Lambda(b) = \tau-1$, so b is not a member of the chart of U. Let $U = Lba_hR'$ where a_h is a maximal occurrence in U. Assume a_h and a_j are incident monomials, $\Lambda(a_h) \geqslant \tau-2$, $\Lambda(a_j) \geqslant 3$. That is, the replacement of a_h by a_j in U is an admissible replacement. Let b' be a maximal occurrence that contains b in the resulting monomial Lba_jR'. It can happen that b' prolongs b to the right. Because of our definition of a small piece, b' can prolong b to the right only by a small piece. So, it is possible that $\Lambda(b') = \Lambda(b)+1 = \tau$. Then b' is a member of the chart of Lba_jR'. In this case b is a virtual member of the chart of U. Graphically this example looks as follows.

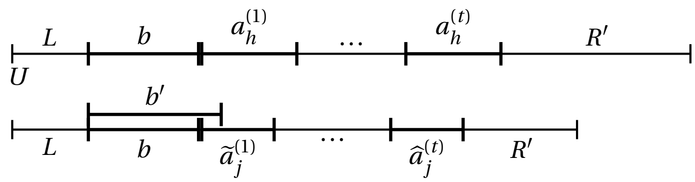


The same effect can take place not only locally. Namely, let $U = Lba_h^{(1)}a_h^{(2)}\dots a_h^{(t)}R'$, where $\Lambda(b) = \tau-1$, $\Lambda(a_h^{(i)}) = \tau-3$ for $1 \leqslant i \leqslant t-1$ and $\Lambda(a_h^{(t)}) \geqslant \tau-2$. Let $a_h^{(t)}$ and $a_j^{(t)}$ be incident monomials, and $\Lambda(a_j^{(t)}) \geqslant 3$. We make the replacement of $a_h^{(t)}$ to $a_j^{(t)}$ in U, let U' be the resulting monomial. Assume the maximal occurrence $\widetilde{a}_h^{(t-1)}$ that prolongs $a_h^{(t-1)}$ in the resulting monomial $U' = Lba_h^{(1)}a_h^{(2)}\dots a_j^{(t)}R'$ from the right becomes of Λ-measure equal to $\tau-2$.



Assume it is possible to find an admissible replacement of $\widetilde{a}_h^{(t-1)}$ to $\widetilde{a}_j^{(t-1)}$ in U' such that $\Lambda(\widetilde{a}_h^{(t-2)}) = \tau-2$, where $\widetilde{a}_h^{(t-2)}$ is the maximal occurrence that prolongs $a_h^{(t-2)}$ in the resulting monomial of this replacement. Move from right to left closer and closer to b

keeping doing admissible replacements of such kind. Suppose that as a result of the last admissible replacement of $\widetilde{a}_h^{(1)}$ to $\widetilde{a}_j^{(1)}$ the maximal occurrence b' that contains b in the last resulting monomial is of Λ-measure equal to τ. That is, b' is a member of the chart of the last resulting monomial in the sequence. Then b is also a virtual member of the chart of U.



Roughly speaking, all virtual members of the chart of U are defined with the use of a process similar to the above done either from the right side of b, or from the left side of b, or both from the left and the right of b.

Let U be a monomial. Consider subsets of $\mathscr{M}(U)$ that cover the same letters in U as the whole $\mathscr{M}(U)$. A covering of such type consisting of the smallest number of elements is called a *minimal covering*. Of course, such covering is not, necessarily, unique.

Given a monomial U, we define $\mathsf{MinCov}(U)$ to be the number of elements in a minimal covering of U. Denote the number of virtual members of the chart of U by $\mathsf{NVirt}(U)$. It is clear that $\mathsf{NVirt}(U) \leqslant \mathsf{MinCov}(U)$.

The next proposition aggregates all calculations beforehand.

Proposition 3.3. *Assume U_h is a monomial, a_h is a virtual member of the chart of U_h. Let a_h and a_j be incident monomials. Consider the replacement $a_h \mapsto a_j$ in U_h. Let U_j be the resulting monomial. If a_j is a virtual member of the chart of U_j, then $\mathsf{MinCov}(U_h) = \mathsf{MinCov}(U_j)$ and $\mathsf{NVirt}(U_h) = \mathsf{NVirt}(U_j)$. If a_j is not a virtual member of the chart of U_j, then either $\mathsf{MinCov}(U_j) < \mathsf{MinCov}(U_h)$, or $\mathsf{MinCov}(U_j) = \mathsf{MinCov}(U_h)$ but $\mathsf{NVirt}(U_j) < \mathsf{NVirt}(U_h)$.*

Definition 3.4. *Let U be a monomial. We introduce f-characteristic U by the rule:*

$$f(U) = (\mathsf{MinCov}(U), \mathsf{NVirt}(U))).$$

If U_1 and U_2 are monomials, we say that $f(U_1) < f(U_2)$ if and only if either $\mathsf{MinCov}(U_1) < \mathsf{MinCov}(U_2)$, or $\mathsf{MinCov}(U_1) = \mathsf{MinCov}(U_2)$ but $\mathsf{NVirt}(U_1) < \mathsf{NVirt}(U_2)$.

We define *derived monomials* of U as the result of applying of a sequence of replacements of virtual members of the chart by incident monomials, starting from U.

Lemma 3.5. *Assume U and Z are monomials, Z is a derived monomial of U. Then $f(Z) \leqslant f(U)$. Moreover, $f(Z) < f(U)$ if and only if in the corresponding sequence of replacements there exists at least one replacement of the form $La_hR \mapsto La_jR$ such that a_h is a virtual member of the chart of La_hR and a_j is not a virtual member of the chart of La_jR.*

The introduced f-characteristic gives rise to a certain function t on natural numbers defined as follows. We put $t(0) = (0,0)$. Assume $t(n) = (r,s)$, then we put

$$t(n+1) = \begin{cases} (r, \quad s+1) & \text{if } r > s, \\ (r+1, 0) & \text{if } r = s. \end{cases}$$

Definition 3.6. *We define an increasing filtration on $k\mathscr{F}$ by the rule*

$$\mathrm{F}_n(k\mathscr{F}) = \langle Z \mid Z \in \mathscr{F}, f(Z) \leqslant t(n) \rangle.$$

That is, the space $\mathrm{F}_n(k\mathscr{F})$ is generated by all monomials with f-characteristics not greater than $t(n)$.

3.2. Derived monomials and dependencies. We need a set of new notions. Let U be a monomial. By $\langle U\rangle_d$ we denote a linear subspace of $\mathrm{F}_n(k\mathscr{F})$ generated by all derived monomials of U. By $\mathrm{L}\langle U\rangle_d$ we denote the subspace generated by all derived monomials of U with f-characteristic smaller than $f(U)$. The next principal object is the set of *dependencies*, defined as follows. Suppose Y is a subspace of $k\mathscr{F}$ linearly generated by a set of monomials and closed under taking derived monomials. We take the set of all layouts of multi-turns of virtual members of the chart of monomials of Y and look at its linear envelope $\mathrm{Dp}(Y)$, which is our set of dependencies related to Y. We prove that $\mathrm{Dp}(k\mathscr{F}) = \mathscr{I}$.

The key statement is the following proposition which describes nice interaction between dependencies and filtration:

Proposition 3.7. $\mathrm{Dp}(\mathrm{F}_n(k\mathscr{F})) \cap \mathrm{F}_{n-1}(k\mathscr{F}) = \mathrm{Dp}(\mathrm{F}_{n-1}(k\mathscr{F}))$.

This proposition yields

Proposition 3.8. *Suppose X, Y are subspaces of $k\mathscr{F}$ generated by monomials and closed under taking derived monomials, $Y \subseteq X$. Then* $\mathrm{Dp}(X) \cap Y = \mathrm{Dp}(Y)$.

The proof of Proposition 3.7 is based on the following lemma:

Lemma 3.9 (**Main Lemma**)**.** *Let U be an arbitrary monomial, $U \in \mathrm{F}_n(k\mathscr{F}) \setminus \mathrm{F}_{n-1}(k\mathscr{F})$. Then*

$$\mathrm{Dp}\langle U\rangle_d \cap \mathrm{L}\langle U\rangle_d \subseteq \mathrm{Dp}(\mathrm{F}_{n-1}(k\mathscr{F})).$$

Here is the place to make some comments. Main Lemma says that there is a natural interaction between dependencies and reduction of f-characteristic, and this interaction causes descending in the filtration. This yields, in essence, that in the quotient algebra $k\mathscr{F}/\mathscr{I}$ *there are no unexpected linear dependencies.* But, first, one has to explain what are the *expected* linear dependencies.

Consider the filtration $\mathrm{F}_n(k\mathscr{F})$, $n \geqslant 0$, on $k\mathscr{F}$ defined as above. Let $U \in \mathrm{F}_n(k\mathscr{F})$ be a monomial such that its chart has m virtual members $u^{(i)}$, $U = L^{(i)}u^{(i)}R^{(i)}$, $i = 1, 2, \ldots, m$. For any $p \in \mathscr{R}$ of the form $p = \alpha u^{(i)} + \sum_{j=1}^{k} \alpha_j a_j$, $\alpha \neq 0$, we consider the polynomial $L^{(i)}pR^{(i)} \in k\mathscr{F}$. All such polynomials obviously belong to $\mathrm{F}_n(k\mathscr{F}) \cap \mathscr{I}$ and regarded as expected dependencies. We shall emphasize that in case the relations $\mathscr{R}$ do not satisfy special conditions, the term $\mathrm{F}_n(k\mathscr{F}) \cap \mathscr{I}$ may contain also arbitrary unexpected dependencies.

In fact, Proposition 3.8 claims that the opposite is also true. In more detail, Proposition 3.8 implies that $\mathrm{F}_n(k\mathscr{F}) \cap \mathscr{I} = \mathrm{F}_n(k\mathscr{F}) \cap \mathrm{Dp}(k\mathscr{F}) = \mathrm{Dp}(\mathrm{F}_n(k\mathscr{F}))$. That is, $\mathrm{F}_n(k\mathscr{F}) \cap \mathscr{I}$ is linearly generated by expected linear dependencies related to $\mathrm{F}_n(k\mathscr{F})$. This can be restated as follows.

Theorem 3.10. $\mathrm{F}_n(k\mathscr{F}) \cap \mathscr{I}$ *is linearly spanned by all polynomials of the form $L^{(i)}pR^{(i)}$, $i = 1, \ldots, m$, for all monomials $U \in \mathrm{F}_n(k\mathscr{F})$ and polynomials $p \in \mathscr{R}$ as above, $n \geqslant 0$.*

3.3. Grading on small cancellation ring. First of all, it can be seen that $\mathrm{Dp}(k\mathscr{F}) = \mathscr{I}$. The quotient space $k\mathscr{F}/\mathscr{I}$ naturally inherits the filtration from $k\mathscr{F}$, namely,

$$\mathrm{F}_n(k\mathscr{F}/\mathscr{I}) = (\mathrm{F}_n(k\mathscr{F}) + \mathrm{Dp}(k\mathscr{F}))/\mathrm{Dp}(k\mathscr{F}) = (\mathrm{F}_n(k\mathscr{F}) + \mathscr{I})/\mathscr{I}.$$

We define a grading on $k\mathscr{F}/\mathscr{I}$ by the rule:

$$\mathrm{Gr}(k\mathscr{F}/\mathscr{I}) = \bigoplus_{n=0}^{\infty} \mathrm{Gr}_n(k\mathscr{F}/\mathscr{I}) = \bigoplus_{n=0}^{\infty} \mathrm{F}_n(k\mathscr{F}/\mathscr{I})/\mathrm{F}_{n-1}(k\mathscr{F}/\mathscr{I}).$$

The next theorem establishes the compatibility of the filtration and the corresponding grading on $k\mathscr{F}$ with the space of dependencies $\mathrm{Dp}(k\mathscr{F})$.

Theorem 3.11. $\mathrm{Gr}_n(k\mathscr{F}/\mathscr{I}) \cong \mathrm{F}_n(k\mathscr{F})/(\mathrm{Dp}(\mathrm{F}_n(k\mathscr{F})) + \mathrm{F}_{n-1}(k\mathscr{F}))$.

3.4. Non-triviality of $k\mathscr{F}/\mathscr{I}$. Construction of a basis of $k\mathscr{F}/\mathscr{I}$.

Lemma 3.12. *Let $\{V_i\}_{i\in I}$ be all different spaces $\{\langle Z\rangle_d \mid Z \in \mathscr{F}\}$. Then not all spaces $V_i/(\mathrm{Dp}(V_i)+\mathrm{L}(V_i))$, $i \in I$, are trivial. Namely, the space $\langle X\rangle_d/(\mathrm{Dp}\langle X\rangle_d + \mathrm{L}\langle X\rangle_d)$, where X is a monomial with no virtual members of the chart, is always non-trivial, and of dimension 1. In particular, $\langle 1\rangle_d/(\mathrm{Dp}\langle 1\rangle_d + \mathrm{L}\langle 1\rangle_d) \neq 0$, where 1 is the empty word.*

Proof. Let X be a monomial with no virtual members of the chart. Then there are no derived monomials of X except X itself, and there are no multi-turns of virtual members of the chart of X. So, by definition, $\langle X\rangle_d$ is linearly generated by X and, therefore, is of dimension 1; $\mathrm{Dp}\langle X\rangle_d = 0$; $\mathrm{L}\langle X\rangle_d = 0$. Therefore,

$$\langle X\rangle_d/(\mathrm{Dp}\langle X\rangle_d + \mathrm{L}\langle X\rangle_d) = \langle X\rangle_d = \langle X\rangle \neq 0,$$

and $\langle X\rangle_d/(\mathrm{Dp}\langle X\rangle_d + \mathrm{L}\langle X\rangle_d)$ is of dimension 1.

By definition, the empty word 1 is a small piece. Therefore, 1 has no virtual members of the chart. So, it follows from the above that $\langle 1\rangle_d/(\mathrm{Dp}\langle 1\rangle_d + \mathrm{L}\langle 1\rangle_d) \neq 0$. □

Now we can prove that the quotient ring $k\mathscr{F}/\mathscr{I}$ is non-trivial.

Corollary 3.13. *The quotient ring $k\mathscr{F}/\mathscr{I}$ is non-trivial.*

Proof. Let U be a monomial. Consider the space $\langle U\rangle_d$ and the corresponding subspace in $k\mathscr{F}/\mathscr{I}$, namely, $(\langle U\rangle_d + \mathscr{I})/\mathscr{I}$. From the isomorphism theorem it follows that

$$(\langle U\rangle_d + \mathscr{I})/\mathscr{I} \cong \langle U\rangle_d/(\langle U\rangle_d \cap \mathscr{I}).$$

Recall that $\mathscr{I} = \mathrm{Dp}(k\mathscr{F})$. From Proposition 3.8 it follows that $\langle U\rangle_d \cap \mathrm{Dp}(k\mathscr{F}) = \mathrm{Dp}\langle U\rangle_d$. Hence,

$$(\langle U\rangle_d + \mathscr{I})/\mathscr{I} \cong \langle U\rangle_d/\mathrm{Dp}\langle U\rangle_d.$$

By Lemma 3.12, there exists a space $\langle U_0\rangle_d$, $U_0 \in \mathscr{F}$, such that $\langle U_0\rangle_d/(\mathrm{Dp}\langle U_0\rangle_d + \mathrm{L}\langle U_0\rangle_d) \neq 0$. Hence, we see that $\langle U_0\rangle_d/\mathrm{Dp}\langle U_0\rangle_d \neq 0$ and $(\langle U_0\rangle_d + \mathscr{I})/\mathscr{I} \neq 0$. So, there exists a non-trivial subspace of $k\mathscr{F}/\mathscr{I}$. Thus, $k\mathscr{F}/\mathscr{I}$ itself is non-trivial. □

Now we are able, at last, to describe a basis of $k\mathscr{F}/\mathscr{I}$. This is done in two steps. First, we construct a basis for non-trivial graded components of our filtration on $k\mathscr{F}/\mathscr{I}$:

$$\mathrm{Gr}_n(k\mathscr{F}/\mathscr{I}) = \mathrm{F}_n(k\mathscr{F}/\mathscr{I})/\mathrm{F}_{n-1}(k\mathscr{F}/\mathscr{I}).$$

Given n we consider the set of spaces $\{\langle Z\rangle_d \mid Z \in \mathscr{F}, Z \in \mathrm{F}_n(k\mathscr{F}) \setminus \mathrm{F}_{n-1}(k\mathscr{F})\}$, such that $\langle Z\rangle_d/(\mathrm{Dp}\langle Z\rangle_d + \mathrm{L}\langle Z\rangle_d) \neq 0$. Let $\{V_i^{(n)}\}_{i\in I^{(n)}}$ be all different spaces from this set. Then,

$$\mathrm{Gr}_n(k\mathscr{F}/\mathscr{I}) \cong \bigoplus_{i\in I^{(n)}} V_i^{(n)}/(\mathrm{Dp}(V_i^{(n)}) + \mathrm{L}(V_i^{(n)})).$$

Assume $\{\overline{W}_j^{(i,n)}\}_j$ is a basis of $V_i^{(n)}/(\mathrm{Dp}(V_i^{(n)}) + \mathrm{L}(V_i^{(n)}))$, $i \in I^{(n)}$. Let $W_j^{(i,n)} \in V_i^{(n)}$ be an arbitrary representative of the coset $\overline{W}_j^{(i,n)}$. Then

$$\bigcup_{i\in I^{(n)}} \left\{W_j^{(i,n)} + \mathscr{I} + \mathrm{F}_{n-1}(k\mathscr{F}/\mathscr{I})\right\}_j$$

is a basis of $\mathrm{Gr}_n(k\mathscr{F}/\mathscr{I})$.

Finally, the next theorem describes a basis of $k\mathscr{F}/\mathscr{I}$. We have

Theorem 3.14. *Let $\{V_i\}_{i\in I}$ be all different spaces $\{\langle Z\rangle_d \mid Z \in \mathscr{F}\}$. Then*

$$k\mathscr{F}/\mathscr{I} \cong \bigoplus_{i\in I} V_i/(\mathrm{Dp}(V_i) + \mathrm{L}(V_i)),$$

as vector spaces, and the right-hand side is explicitly described via a tensor product of subspaces.

Assume $\{\overline{W}_j^{(i)}\}_j$ *is a basis of* $V_i/(\mathrm{Dp}(V_i)+\mathrm{L}(V_i))$, $i \in I$. *Let* $W_j^{(i)} \in V_i$ *be an arbitrary representative of the coset* $\overline{W}_j^{(i)}$. *Then*

$$\bigcup_{i\in I}\left\{W_j^{(i)}+\mathscr{I}\right\}_j$$

is a basis of $k\mathscr{F}/\mathscr{I}$.

Remark 3.15. *We shall informally explain the essence of Isolation axioms. Given a monomial* U, *consider the set of its non-degenerate derived monomials (see Subsection 1.5). Every derived monomial can be imagined as a result of a sequence of replacements of virtual members of a chart by incident monomials. If two essentially different sequences of replacements result in one and the same derived monomial, the exotic dependencies appear in the ideal* $\mathscr{I}$. *Isolation Axiom guarantees that essentially different sequences of replacements result in different monomials. Hence, exotic dependencies are not present in* $\mathscr{I}$.

4. Algorithmic properties

We study algorithmic properties of the constructed small cancellation ring. We show that they are as expected to be for small cancellation objects and similar in a sense to the ones valid for small cancellation groups. However, in the ring case the essential peculiarities arise in many places. Recall that small cancellation groups enjoy Dehn's algorithm [13]. In this section we define and study a corresponding greedy algorithm for rings.

Let a ring $\mathscr{A} = k\mathscr{F}/\mathscr{I}$ with small cancellation condition be given. We extend our set of relations $\mathscr{R}$ to a certain additive closure $\mathrm{Add}(\mathscr{R})$. It is important that $\mathscr{R} = \mathrm{Add}(\mathscr{R})$ for natural examples of the ring $\mathscr{A}$ considered below. We define a linear order on all monomials, based on f-characteristic and other considerations, and denote it by $<_f$. Then, given the order $<_f$ and the set $\mathrm{Add}(\mathscr{R})$, we define a special greedy algorithm (with external source of knowledge) for small cancellation rings. This algorithm has a similar meaning as *Dehn's algorithm* does for the case of groups. Denote it by $\mathrm{GreedyAlg}(<_f, \mathrm{Add}(\mathscr{R}))$.

This algorithm works as follows. Let $\sum_{i=1}^k \gamma_i W_i \in k\mathscr{F}$, $\gamma_i \neq 0$, and let W_{i_0} be its highest monomial with respect to the order $<_f$. Then we try to make the highest monomial smaller using a multi-turn of a virtual member of the chart of W_{i_0}. Namely, we take a polynomial $p = \sum_{j=1}^n \alpha_j L a_j R$ such that $\sum_{j=1}^n \alpha_j a_j \in \mathrm{Add}(\mathscr{R})$, $W_{i_0} = La_hR$ is its highest monomial and a_h is a virtual member of the chart of W_{i_0}, and make a transformation

$$\sum_{i=1}^k \gamma_i W_i \longmapsto \sum_{i=1}^k \gamma_i W_i - \gamma\alpha_h^{-1} p.$$

Then the highest monomial of $\sum_{i=1}^k \gamma_i W_i - \gamma\alpha_h^{-1} p$ is strictly smaller than W_{i_0} because W_{i_0} is cancelled out. If there is no suitable polynomial in $\mathrm{Add}(\mathscr{R})$, then the algorithm terminates. The external source of knowledge answers if there exist appropriate polynomials in $\mathrm{Add}(\mathscr{R})$ and in case they exist it gives such a polynomial.

Recall that given a small cancellation group $G = \langle \mathscr{X} \mid \mathscr{R}_G\rangle$, a word W from a free group is equal to 1 in G if and only if Dehn's algorithm, starting from W, terminates at 1, [13]. Our Theorem 4.1 establishes the similar properties in much more complicated situation of rings.

Namely, assume $W_1, \dots, W_k$ are different monomials. We take an element $\sum_{i=1}^k \gamma_i W_i \in k\mathscr{F}$, $\gamma_i \neq 0$.

Theorem 4.1. *The following statements are equivalent:*

- *some branch of the algorithm* GreedyAlg($<_f$,Add($\mathscr{R}$)), *starting from* $\sum_{i=1}^{k}\gamma_i W_i$, *terminates at* 0;
- $\sum_{i=1}^{k}\gamma_i W_i \in \mathscr{I}$;
- *every branch of the algorithm* GreedyAlg($<_f$,Add($\mathscr{R}$)), *starting from* $\sum_{i=1}^{k}\gamma_i W_i$, *terminates at* 0.

Corollary 4.2. *We have*

- GreedyAlg($<_f$,Add($\mathscr{R}$)) *solves the Ideal Membership Problem for* $\mathscr{I}$,
- Add($\mathscr{R}$) *is a Gröbner basis of the ideal* $\mathscr{I}$ *with respect to monomial ordering* $<_f$.

5. Examples

Example A. First of all, let us notice that the group algebra of a small cancellation group satisfying a small cancellation condition with $C(m)$ for $m \geqslant 22$ (see [13]) is a small cancellation ring (see [3], Section 11.1, for details).

Example B. Let us consider another example of a small cancellation ring, which is studied in detail in [2] and in [3], Section 11.2. See also [2] regarding the motivation for studying such rings.

Let w be a primitive (not a proper power) cyclically reduced word. Let x and y be letters from the set of free generators of $\mathscr{F}$ such that the initial and the final letter of w and w^{-1} differ from $x^{\pm1}$ and $y^{\pm1}$. So, we need a free group $\mathscr{F}$ with at least 4 free generators. Take the word

$$v = x^{n_1} y x^{n_1+1} y \cdots x^{n_2} y,\ n_1, n_2 \in \mathbb{N},$$

such that $n_1 - |w| > 0$ and $n_2 - n_1 \geqslant 21$. We consider $k\mathscr{F}/\mathscr{I}$ such that $\mathscr{I} = \langle v^{-1} - 1 - w\rangle_i$.

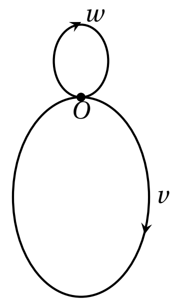
The word v exhibits small cancellation properties, because a subword of v^m, $m \in \mathbb{Z}$, containing at least two letters $y^{\pm1}$, appears in v^m uniquely modulo a shift by multiple of $|v|$. So, it seems natural to have the same non-small pieces in a sense of Definition 2.1.

Notice that $\mathscr{R} = \{v^{-1} - 1 - w\}$ itself does not satisfy Axioms 1—3. However, it is possible to extend the set $\mathscr{R}$ to $\mathscr{R}_1$ such that $\mathscr{R}_1$ generates the same ideal and satisfies Axioms 1—3. Namely, we consider non-commutative Laurent polynomials $P(x_1, x_2)$ such that $P((1+t)^{-1}, t) = 0$. One can show that $P(v, w) \in \mathscr{I}$. Then $\mathscr{R}_1 = \{\gamma A \cdot P(v, w) \cdot B \mid \gamma \in k\}$, where P runs through Laurent polynomials with the above property, a monomial A runs through the set

$$\left\{ v_f, v_i^{-1}, w_f, w_i^{-1} \,\middle|\, \begin{array}{l} v_i \text{ is a prefix of } v,\ v_f \text{ is a suffix of } v, \\ w_i \text{ is a prefix of } w,\ w_f \text{ is a suffix of } w \end{array} \right\},$$

and a monomial B runs through the similar set $\{v_i, v_f^{-1}, w_i, w_f^{-1}\}$. Then one can prove that every subword of v^m, $m \in \mathbb{Z}$, containing at least two letters $y^{\pm1}$, is a non-small piece with respect to $\mathscr{R}_1$ (see [3], Section 11.2). Notice that v can appear in two different polynomials from $\mathscr{R}_1$ but naturally v is not a small piece. So, straightforward intuition, which we mention in Section 2, does not work for our needs.

In groups we represent defining relations as a cyclic words and they correspond to closed paths in a Cayley graph. We do not have Cayley graphs for rings, however, we can produce a similar illustration for our case. Namely, we consider a graph of the form



and call it v-*figure*. Then every relation from $\mathscr{R}_1$ corresponds to a collection of paths in this graph with a fixed initial point and a fixed final point. We show that a monomial is not a small piece with respect to $\mathscr{R}_1$ if and only if it corresponds to a unique path in the above graph (see [3], Section 11.2, Proposition 11.7). This is similar to the group case, where a non small piece appears uniquely in the cyclic path that corresponds to a relation (modulo a period of the relation).

Now we give some comment regarding Isolation Axiom. Roughly speaking, Isolation Axiom says that incident monomials and monomials that are connected by a sequence of incident monomials have a significant difference. Namely, they can not differ only by small pieces at their ends. Notice that for groups such a property is rather trivial (see [3], Section 11.1, Part 3). In the current case monomials that are considered in Isolation Axiom correspond to paths in v-figure with the same initial points and the same final points. So, we check it only for such pairs of monomials. It could be verified via direct calculation that uses the explicit form of our relations (see [3], Section 11.2, Part 3). Let us also note that for more complicated defining relations such a verification can cause significant difficulties.

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