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Preservers of totally positive kernels and Pólya frequency functions

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(Recommended by Boris Hasselblatt)

ABSTRACT. Fractional powers and polynomial maps preserving structured totally positive matrices, one-sided Pólya frequency functions, or totally positive kernels are treated from a unifying perspective. Besides the stark rigidity of the polynomial transforms, we unveil an ubiquitous separation between discrete and continuous spectra of such inner fractional powers. Classical works of Schoenberg, Karlin, Hirschman, and Widder are completed by our classification. Concepts of probability theory, multivariate statistics, and group representation theory naturally enter into the picture.

1. Overview

The purpose of this note is to announce some of the results in a sequence of three closely related recent papers [4, 45, 6] which study Pólya frequency functions and the post-composition transforms that preserve them and other classes of totally positive kernels. Specifically, we complete results in total positivity due to Karlin's 1964 *Transactions* [42] and Schoenberg's 1955 *Annals* [71] papers by providing extensions and converses. These concern fractional powers of totally non-negative Toeplitz kernels and of their Laplace transforms. We isolate a spectral-threshold phenomenon in total positivity which is similar to the structure of Berezin–Gindikin–Wallach-type sets, but which was discovered in particular situations prior to the work of these mathematicians.

Next, we focus on powers and other polynomial preservers of more general non-smooth one-sided Pólya frequency (PF) functions. The underlying theme is probabilistic: one-sided PF functions are the density functions of linear combinations $\sum_j \alpha_j X_j$ of independent standard exponential random variables: for example, Karlin's 1964 kernel is the density of $X_1 + X_2$. The densities $\Lambda_{\boldsymbol{\alpha}}$ of finite linear combinations $\sum_{j=1}^m \alpha_j X_j$ were studied by Hirschman and Widder [38], and we prove that, if the coefficients $\boldsymbol{\alpha} = (\alpha_j) \in (0,\infty)^m$ lie outside a null set then the only polynomials p with the property that $p \circ \Lambda_{\boldsymbol{\alpha}}$ is a PF function, are homotheties $p(x) = cx$, where $c > 0$. A previously unexplored connection between recovering $\boldsymbol{\alpha}$ from the moments of $\Lambda_{\boldsymbol{\alpha}}$ and the Jacobi–Trudi identity from symmetric function theory comes as a bonus.

Finally, we study general one-sided PF functions, which are the densities of countable sums of exponential random variables, as well as other classes of totally non-negative kernels, including general PF functions and sequences. For these classes of maps, we

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show there are very few preservers beyond positive homotheties $p(x) = cx$. The culmination of this line of inquiry is the characterization of preservers of total positivity and total non-negativity for kernels on $X \times Y$, where X and Y are arbitrary totally ordered domains.

In addition, we provide a small correction to Schoenberg's classification of discontinuous PF functions [66].

2. The pervasive nature of Pólya frequency functions

2.1. Total positivity and Pólya frequency functions. Given totally ordered sets X and Y, a kernel $K : X \times Y \to \mathbb{R}$ is *totally positive* if, for all integers $n \geq 1$ and all choices of $x_1 < \cdots < x_n$ in X and $y_1 < \cdots < y_n$ in Y, the determinant $\det(K(x_i, y_j))_{i,j=1}^n$ is positive; the kernel K is *totally non-negative* if these determinants are non-negative. We refer to such kernels, which are matrices if the domains X and Y are finite, as TP and TN kernels, respectively. If non-negativity or positivity are only required to hold whenever $n \leq p$ then we speak of TN_p or TP_p kernels, respectively. (The monographs [43, 61] refer to *strict total positivity* and *total positivity* instead of total positivity and total non-negativity.)

Total positivity is a long-studied and evergreen area of mathematics. For almost a century, totally positive and totally non-negative kernels surfaced in the most unexpected circumstances, and this trend continues in full force today. Although this chapter of matrix analysis remains somewhat recondite, it has reached maturity due to the dedicated efforts of several generations of mathematicians. The foundational work by Gantmacher and Krein [27], the survey [2], the early monograph [28], and the more recent publications [43, 29, 61, 19] offer ample references to the fascinating history of total positivity, as well as accounts of its many surprising applications. These include analysis [70, 66, 71], differential equations [55], probability and statistics [13, 18, 43, 48], and interpolation theory [15, 72], to provide a few areas and early references. Total positivity continues to make very recent impacts in areas such as representation theory and cluster algebras [8, 9, 25, 26, 56, 57, 64], Gabor analysis [33, 34], combinatorics [11, 12], as well as integrable systems and positive Grassmannians (the geometric avatar of total positivity) [50, 49, 63].

The origins of total positivity lie in the property of diminishing variation, which can be traced back to Descartes (1600s, via his rule of signs [16]), but concretely at least as far back as Laguerre [52] (1883) and Fekete [23] (1912). Pólya then coined the phrase 'variationsvermindernd', and Schoenberg showed in [68] (1930) that TP and TN matrices have this property of variation diminution. There has been continuing activity for TN matrices [28, 13, 14], and also for TN kernels on bi-infinite domains, which are the main focus of this note.

Definition 2.1. A function $\Lambda : \mathbb{R} \to \mathbb{R}$ is *totally non-negative* if the associated Toeplitz kernel T_Λ is totally non-negative, where

$$T_\Lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \Lambda(x - y). \tag{2.1}$$

If, further, the function Λ is Lebesgue integrable and non-zero at two or more points, then Λ is a *Pólya frequency function*. A function $\Lambda : \mathbb{Z} \to \mathbb{R}$ whose associated Toeplitz kernel is totally non-negative is a *Pólya frequency sequence*.

Following early work by Pólya and Hamburger, Schoenberg has initiated and developed the theory of Pólya frequency functions in his landmark paper [66] dated 1951 (following announcements in 1947 and 1948 in *Proc. Natl. Acad. Sci. USA*). In the companion work [70] (published a year earlier), Schoenberg proved that, when viewed as convolution integral operators, Pólya frequency functions can be characterized in terms of the variation-diminishing property. This study led to an explosion of work in numerical analysis and approximation theory, via *splines*.

2.2. The Laguerre–Pólya class of entire functions. A second notable appearance of Pólya frequency functions and sequences is within the theory of complex function. We henceforth refer to these classes of maps as PF *functions* and PF *sequences*, respectively.

The Fourier–Laplace transform of a PF function is the reciprocal, on a suitable domain of definition, of a *Laguerre–Pólya entire function* [1, 17, 66]. The natural question of characterizing locally uniform limits of sequences of polynomials with only real roots was answered by Laguerre [51], and completed by Pólya [62]. A related wider program of locating the zeros of entire functions was initiated about a century ago by Pólya and Schur [74]. By now, the topic of Laguerre–Pólya entire functions appears in textbooks [54], reflecting a century of accumulated knowledge passing through early contributions such as [35, 78] and continuing up to the present [3]. Since its inception, Riemann's hypothesis was and remains a background theme in studies concerning entire functions of the Laguerre–Pólya class, and so indirectly to Pólya frequency functions and sequences. We cite here only two recent contributions: [32, 44].

We owe to Schoenberg the aforementioned characterization of PF functions.

Theorem 2.2 (Schoenberg [66])**.** *Given a Pólya frequency function Λ, its bilateral Laplace transform*

$$\mathscr{B}\{\Lambda\}(s) := \int_{\mathbb{R}} e^{-xs}\Lambda(x)\,\mathrm{d}x$$

converges for complex s in an open strip containing the imaginary axis, and equals $1/\Psi$ on this strip, for an entire function Ψ in the Laguerre–Pólya class. Conversely, any function Ψ of the above form agrees with the reciprocal of the bilateral Laplace transform of some Pólya frequency function on its strip of convergence.

In [66], Schoenberg then showed that a PF function either vanishes precisely on a semi-axis, which may be open or closed, or is non-vanishing on $\mathbb{R}$. These functions are termed one-sided and two-sided PF functions, respectively. Schoenberg also showed in [66] that, up to linear transformations, the reciprocals of their Laplace transforms are respectively in the first Laguerre–Pólya class of entire functions [51, 62] that are non-vanishing at 0,

$$\Psi(s) = Ce^{\delta s}\prod_{j=1}^{\infty}(1+\alpha_j s), \quad \text{with } C>0,\ \delta,\alpha_j \geq 0,\ \sum_j \alpha_j < \infty, \tag{2.2}$$

and the second Laguerre–Pólya class of functions not vanishing at the origin,

$$\Psi(s) = Ce^{-\gamma s^2+\delta s}\prod_{j=1}^{\infty}(1+\alpha_j s)e^{-\alpha_j s}, \quad \text{with } C>0, \gamma \geq 0, \delta, \alpha_j \in \mathbb{R}, \sum_j \alpha_j^2 < \infty. \tag{2.3}$$

This remarkable dictionary, established by Schoenberg, provides the building blocks of Pólya frequency functions. This class of functions is closed under convolution, and has Gaussian functions, one-sided exponentials and simple fractions with poles on the real line as generators. The exploitation of this basic observation is paramount for interpolation theory, and the expository lectures by Schoenberg [67] are as fresh and informative today as they were fifty years ago.

2.3. Group representations. The string of discoveries of the same class of objects does not stop at entire function theory. The classification of characters of irreducible unitary representations of the infinite symmetric group $S(\infty) = \cup_n S(n)$ and the infinite unitary group $U(\infty) = \cup_n U(n)$ led Thoma [77] and Voiculescu [81] independently to the class of Fourier transforms of Pólya frequency functions. These remarkable findings in the 1960s and 1970s established the foundations of the representation theory of "big groups": see [59, 80] for details.

Not unrelatedly, the computations of spherical functions pointed to orbital-integral formulae for such character functions, as did the more general question of spectral synthesis on homogeneous spaces. The pioneering work of Gelfand and Naimark [30] opened a whole chapter of explicit expressions linking orbital integrals to invariants of finite groups, in the spirit of Weyl's character formula.

A typical orbital integral has the form

$$f_A : H(n) \to \mathbb{C}; \quad B \mapsto \int_\Omega e^{\mathrm{i}\,\mathrm{tr}(BM)} \mu(\mathrm{d}M), \tag{2.4}$$

where $A \in H(n)$ is a positive semi-definite $n \times n$ complex matrix, Ω denotes its orbit under conjugation by the unitary group $U(n)$, and μ is a $U(n)$-invariant measure carried by Ω. In view of the invariance of the trace under cyclic permutations, f_A is invariant under unitary conjugation:

$$f_A(UBU^*) = f_A(B) \qquad \text{for all } U \in U(n) \text{ and } B \in H(n).$$

In particular $f_A(B)$ depends only on the eigenvalues of B and is a symmetric function of these eigenvalues. Furthermore, being a Fourier transform, the function f_A is positive definite.

To bring Pólya frequency functions into view, one considers the inductive limit of such measures and functions defined on the union $H(\infty) = \bigcup_n H(n)$ of Hermitian matrices of arbitrary size. These spherical functions, normalized by the condition that $f(0) = 1$, form a convex set and the extremal points of this set are multiplicative, in the sense that

$$f\big(\mathrm{diag}(b_1, b_2, \dots, b_m)\big) = F(b_1)F(b_2)\cdots F(b_m)$$

for some function of a real variable F. This occurs precisely when the corresponding unitarily invariant measure μ on the union $H(\infty)$ is ergodic. The main classification theorem [59, Theorem 2.9] asserts the existence of a *bijective correspondence between ergodic, unitarily invariant probability measures on $H(\infty)$ and Pólya frequency functions.* To be more precise, F is the Fourier transform of a Pólya frequency function associated with the ergodic measure μ. Moreover, specific invariant measures provide the building blocks of the class of Pólya frequency functions [59, Corollaries 2.5 to 2.7].

Let us consider the case when $A = \mathrm{diag}(a_1, \dots, a_m)$, where $a_1, \dots, a_m$ are positive, and let $B = E_{11} = \mathrm{diag}(1, 0, 0, \dots, 0)$. Passing lightly over the technicalities required to extend f_A, and using the symmetry $f_A(\mathrm{i}xB) = f_B(\mathrm{i}xA)$, which follows from the tracial property, we have that

$$f_A(\mathrm{i}xE_{11}) = \int_{\Omega'} \exp\Big(-x\sum_{j=1}^m a_j|z_j|^2\Big)\sigma(\mathrm{d}z) \qquad (x > 0), \tag{2.5}$$

where $\Omega' = S^{2m-1} \cong U(m)/U(m-1)$ is the unit sphere in $\mathbb{C}^m$ and σ is the normalized rotationally invariant measure on the sphere.

Up to proper normalization, the spherical average (2.4) is equal to the Hirschman–Widder distribution $\Lambda_{\boldsymbol{\alpha}}$ at the point x, where $\boldsymbol{\alpha} = (a_1^{-1}, \dots, a_m^{-1})$. The right-hand side of the identity (2.5) is known as a Harish-Chandra–Itzykson–Zuber integral [36, 39].

We can go further and establish an explicit link between the orbital integral $f_A(B)$ above and Hirschman–Widder densities. Further details, including complete proofs, are contained in [59, Section 5]. Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{C}^m$ have corresponding diagonal matrices $A = \mathrm{diag}\,\mathbf{a} \in H(m)$ and $B = \mathrm{diag}\,\mathbf{b}$, respectively. As in (2.4), we let f_A denote the characteristic function of the invariant probability measure μ with support Ω, where Ω is the $U(m)$-orbit of A under conjugation:

$$f_A(B) := \int_\Omega e^{\mathrm{i}\,\mathrm{tr}(BM)}\mu(\mathrm{d}M) = \int_{U(m)} e^{\mathrm{i}\,\mathrm{tr}(BUAU^*)}\,\mathrm{d}U.$$

Since f_A is entire and symmetric as a function of the coordinates of $\mathbf{b}$, it admits a Taylor-series expansion that is convergent everywhere, so also a convergent expansion in terms of Schur polynomials:

$$f_A(\operatorname{diag}\mathbf{b}) = \sum_\nu c_\nu s_\nu(\mathbf{b}),$$

where the sum runs over Young diagrams with at most m rows. A computation by Olshanski and Vershik, using characters of $U(m)$ and change-of-bases formulas between symmetric power-sum polynomials and Schur polynomials, provides a closed-form expression for the coefficient c_ν: see [59, Theorem 5.1]. This strategy appeared a few decades earlier, in explicit computations of Gel'fand and Naimark [30], and quite remarkably (and independently) in multivariate statistics: see James [41] and the comments in [22]. From here, one derives the following expansions: see [59, Corollaries 5.2 and 5.4].

Proposition 2.3. *If the tuples $\mathbf{a} = (a_1,\dots,a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1,\dots,b_m) \in \mathbb{C}^m$ each have distinct coordinates and $A = \operatorname{diag}\mathbf{a}$ then the orbital integral f_A is given by the Harish-Chandra–Itzykson–Zuber formula:*

$$f_A(-\mathrm{i}\operatorname{diag}\mathbf{b}) = \frac{\prod_{j=0}^{m-1} j!}{V(\mathbf{a})V(\mathbf{b})}\det\begin{pmatrix} e^{b_1a_1} & e^{b_1a_2} & \cdots & e^{b_1a_m} \\ e^{b_2a_1} & e^{b_2a_2} & \cdots & e^{b_2a_m} \\ \vdots & \vdots & \ddots & \vdots \\ e^{b_ma_1} & e^{b_ma_2} & \cdots & e^{b_ma_m}\end{pmatrix}.$$

If instead $B = \operatorname{diag}(1,0,\dots,0) = E_{11}$ then

$$f_A(-\mathrm{i}xB) = (m-1)!\sum_{j=0}^\infty \frac{h_j(a_1,\dots,a_m)}{(j+m-1)!}x^j,$$

where h_j is the j^{th} complete homogeneous symmetric polynomial.

In particular, if $a_1,\dots,a_m$ are positive and distinct, and $x>0$ then, by the second part of Proposition 2.3 and the identity (4.5) below,

$$f_A(\mathrm{i}xE_{11}) = (m-1)!(-x)^{1-m}\sum_{n=m-1}^\infty \frac{h_{n-m+1}(a_1,\dots,a_m)}{n!}(-x)^n = \frac{(m-1)!x^{1-m}}{a_1\cdots a_m}\Lambda_{\boldsymbol{\alpha}}(x),$$

where $\boldsymbol{\alpha} = (a_1^{-1},\dots,a_m^{-1})$.

In conclusion, the Hirschman–Widder density possesses the following integral and determinantal representations: if $x>0$ then

$$\begin{aligned}\Lambda_{\boldsymbol{\alpha}}(x) &= \frac{a_1\cdots a_m}{(m-1)!}x^{m-1}\int_{S^{2m-1}}\exp\Big(-x\sum_{j=1}^m a_j|z_j|^2\Big)\sigma(\mathrm{d}z) \\ &= \frac{a_1\cdots a_m}{V(\mathbf{a})}\det\begin{pmatrix} e^{-a_1x} & e^{-a_2x} & e^{-a_3x} & \cdots & e^{-a_mx} \\ 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{m-2} & a_2^{m-2} & a_3^{m-2} & \cdots & a_m^{m-2}\end{pmatrix}.\end{aligned}$$

The second representation can be obtained from the first identity in Proposition 2.3 by setting $\mathbf{b} = (-x,0,y_1,\dots,y_{m-2})$ and taking successively the j^{th} partial derivative at zero with respect to y_j for $j=1$ to $m-2$.

2.4. Probability theory. The aspects of Pólya frequency functions described above lead to some natural interpretations of a probabilistic flavor. Here we discuss three such perspectives.

First we return to the Hadamard–Weierstrass factorizations of Laplace transforms of Pólya frequency functions. Suppose we are given some continuous random variables, whose associated probability density functions admit bilateral Laplace transforms convergent in some common open strip containing the imaginary axis. Linear combinations of these random variables correspond to convolutions of their densities, which in turn correspond to products at the level of Laplace transforms. From the perspective of (2.2) and (2.3), Schoenberg's Theorem 2.2 can be recast using countably many exponential random variables and a Gaussian random variable.

Theorem 2.4. *Given a Pólya frequency function $\Lambda : \mathbb{R} \to \mathbb{R}$, exactly one of the following holds:*

(1) *Λ is discontinuous at exactly one point x_0, and vanishes on one side of x_0. Moreover, ignoring the value at x_0 and up to a shift of argument and positive rescaling, Λ equals the density of αX, where $\alpha \in \mathbb{R} \setminus \{0\}$ and X is a standard exponential random variable with mean* 1.
(2) *Λ is continuous, vanishes on a semi-axis $\pm(-\infty, x_0)$, and is non-vanishing on the interior of the complement. In this case, up to a shift in argument and positive rescaling, Λ equals the density of a linear combination $\pm \sum_j \alpha_j X_j$, where the α_j are positive and summable, and the X_j are independent standard exponential variables.*
(3) *Λ is non-vanishing on $\mathbb{R}$. In this case, up to a shift in argument and positive rescaling, Λ equals the density of a linear combination $\alpha Y \pm \sum_j \alpha_j X_j$, where (a) the series $\sum_j \alpha_j^2$ converges, (b) if $\alpha = 0$ then there exist at least one positive and one negative α_j, (c) the X_j are independent standard exponential random variables, and (d) Y is a standard Gaussian random variable independent of the X_j.*

As straightforward as it may be, we were unable to find this observation in the literature. We recorded it in a series of results and remarks in [6, Section 2.4]. In particular, the observation about the indeterminateness of a PF function at its point of discontinuity revealed a gap in the literature, which is now clarified in [45]; see Theorem 3.8 and the discussion preceding it.

A second probabilistic interpretation of the density $\Lambda_{\boldsymbol{\alpha}}$ can be derived from random matrix theory. Consider the diagonal matrix $D = \operatorname{diag}(a_1, \ldots, a_m)$ with positive non-zero entries and its orbit Ω under unitary conjugation in the space of $m \times m$ positive semidefinite matrices. If μ is the normalized $U(m)$-invariant measure on Ω then $\Lambda_{\boldsymbol{\alpha}}(x)\, \mathrm{d}x$ is the distribution of any diagonal entry of a random positive semidefinite matrix of arbitrary size distributed according to μ. See Section 3 and [59, Section 8], or [20], for details.

The third occurrence of Pólya frequency functions as characteristic functions of probability distributions arises indirectly from multivariate statistics via orbital integrals. Such integrals over matrix groups go back to Wishart's original work, widely considered to be the origin of modern random matrix theory. The precise calculation of orbital integrals of the type we discussed above produced closed-form expressions for various probability distributions of matricial variables. Of particular interest is the setting where one collects n independent observations $X^{(1)}, \ldots, X^{(n)}$ from a p-dimensional Gaussian vector $X \sim N(\mu, \Sigma)$ with mean $\mu \in \mathbb{R}^p$ and covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$. A fundamental problem of great interest to applied scientists is to detect non-spurious correlations between the components of X. If $\mathbf{X}$ denotes the $n \times p$ matrix whose rows are the vectors $X^{(1)}, \ldots, X^{(n)}$, then detecting dependencies between the variables can be performed by computing the associated *sample covariance matrix*, which, up to rescaling, equals $\mathbf{X}^T\mathbf{X}$. A large entry in $\mathbf{X}^T\mathbf{X}$ indicates a high level of dependence between the corresponding variables and understanding the exact distribution of $\mathbf{X}^T\mathbf{X}$ is critical to assessing whether a large entry arises purely by chance or as the result of a real interaction. Wishart was able to compute the density of such matrices in closed form via changes of variables and the calculation of Jacobians. The resulting distribution is named after him. When the observations $X^{(1)}, \dots, X^{(n)}$ have different means $\mu_1, \dots, \mu_n$ (the *non-central* case), computing the density of $\mathbf{X}^T\mathbf{X}$ involves integrals over the orthogonal group and the result can be written in terms of zonal polynomials [76]. The challenges encountered by statisticians along this path are well recorded in the monograph by Farrell [22]. It seems Karlin himself lectured around 1960 on this subject, providing a versatile Haar measure disintegration formula. No doubt elements of [43] were on his desk at that time.

3. Two atoms: the Berezin–Gindikin–Wallach phenomenon in total positivity

As seen in Theorem 2.4, the 'atoms' that make up a general Pólya frequency function are exponential random variables, together with at most one Gaussian. If we consider a single atom then it is evident that any positive real power of the associated Pólya frequency function is also in the same class. However, as we now explain, the picture is markedly different for more than one exponential variable.

The question of which powers of the density of the sum of two standard exponential variables are totally non-negative is partially answered in the 1964 paper [42] of Karlin. His results show that all integer powers of the associated PF function Ω are again PF functions, but the non-integer powers are only shown to be TN of some finite order.

Definition 3.1. Given totally ordered sets X and Y, and an integer $p \geq 1$, a kernel $K : X \times Y \to \mathbb{R}$ is said to be *totally non-negative of order p*, denoted TN_p, if for all $x_1 < \cdots < x_r$ in X and $y_1 < \cdots < y_r$ in Y, where $1 \leq r \leq p$, the determinant $\det(K(x_i, y_j))_{i,j=1}^r \geq 0$.

A function $\Lambda : \mathbb{R} \to \mathbb{R}$ is said to be TN_p if its associated Toeplitz kernel T_Λ given by (2.1) is TN_p. We say Λ is *totally non-negative* if Λ is TN_p for all $p \geq 1$.

With this notation at hand, the result of Karlin can now be stated. We denote the set of non-negative integers $\{0, 1, 2, \dots\}$ by $\mathbb{Z}_{\geq 0}$.

Theorem 3.2 (Karlin [42])**.** *Let* $\Omega : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \mathbf{1}_{x \geq 0}\, x e^{-x}$ *be the probability density function for the sum of two independent standard exponential random variables. Given an integer* $p \geq 2$ *and a scalar* $\alpha \geq 0$*, the function* $\Omega^\alpha : x \mapsto \Omega(x)^\alpha$ *is* TN_p *if* $\alpha \in \mathbb{Z}_{\geq 0} \cup (p-2, \infty)$.

In particular, if α is a non-negative integer power, then Ω^α is TN. This case was shown previously by Schoenberg in [66] as an immediate consequence of Theorem 2.2.

It is natural to ask if the converse to this 1964 result of Karlin holds. To the best of our knowledge, this question was not answered in the literature, and it is the first result that we announce in this note. In fact, we prove a twofold strengthening.

Theorem 3.3 ([45])**.** *Given* q*,* $r \in (0, \infty)$*, let* $\Omega_{(q,r)}$ *be the probability density function for* $qX_1 + rX_2$*, where* X_1 *and* X_2 *are independent standard exponential random variables. Now fix an integer* $p \geq 2$ *and a real number* $\alpha \geq 0$.

(1) *The function*
$$\Omega_{(q,r)}^\alpha : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \Omega_{(q,r)}(x)^\alpha$$
is TN_p *if* $\alpha \in \mathbb{Z}_{\geq 0} \cup (p-2, \infty)$.

(2) *If* $\alpha \in (0, p-2) \setminus \mathbb{Z}$*, then* $\Omega_{(q,r)}^\alpha$ *is not* TN_p*. More strongly, given arbitrary real numbers* $x_1 < \cdots < x_p$ *and* $y_1 < \cdots < y_p$*, there exists* $a \in \mathbb{R}$ *such that the matrix*
$$(\Omega_{(q,r)}(x_j - y_k - a)^\alpha)_{j,k=1}^p$$

is TP *if* $\alpha > p-2$, TN *if* $\alpha \in \{0,1,\ldots,p-2\}$, *and has a negative principal minor if* $\alpha \in (0,p-2)\setminus\mathbb{Z}$.

The proof of this result is obtained by exploiting a variant of Descartes' rule of signs that was very recently shown and used by Jain [40] in her study of entrywise powers preserving positive semidefiniteness. Note that Theorem 3.3 strengthens Karlin's result: (a) it extends Theorem 3.2 from $q = r = 1$ to all q, $r > 0$; (b) it extends the total non-negativity in Theorem 3.2 to total positivity on a large collection of $p \times p$ matrices, and (c) it shows that the converse to the extended Theorem 3.2 holds for all q, $r > 0$.

This last point means that the set of powers preserving the TN_p property for one-sided Pólya frequency functions built out of two exponential variables is comprised of an arithmetic progression and a semi-infinite axis. In the subsequent decade to Karlin's 1964 work, this phenomenon was observed in numerous different contexts in mathematics.

- In complex analysis, Rossi and Vergne [79] classified the powers α of a Bergman kernel on a fixed tube domain $D \subset \mathbb{C}^n$ which are the reproducing kernels for some Hilbert space of holomorphic functions on D. They termed this set of powers the *Wallach set*, and showed that it equals an arithmetic progression together with a half-line, precisely as above.
- Rossi and Vergne named the above set following Wallach, who was then following Harish-Chandra and studying holomorphic discrete series of connected simply connected Lie groups G. Wallach classified in [82] the twist parameters α of the center of the maximal compact reductive subgroup K of G for which the K-finite highest weight module over the Lie algebra of G has certain unitarizability properties. Once again, α belongs to a similar set.
- In his pioneering work on quantization, Berezin encountered a similar set while classifying admissible values of Planck's constant h in deformations of bounded symmetric domains [10]. Specifically, [10, Theorem 1.1] provides a complete picture of such admissible h, adapted to Cartan's four classes of classical symmetric domains. As might be expected, these tables display a mixture of continuous and discrete values for h. Moreover, [10, Lemma 1.1] is perfectly aligned with the classification of fractional powers which preserve the positivity of a structured kernel: *Given a bounded homogeneous domain with associated Bergman kernel K, for a real number h to be admissible it is necessary and sufficient that the kernel $K^{1/h}$ is positive semidefinite.*
- Gindikin worked in [31] with Riesz distributions R_μ associated to symmetric cones and indexed by a complex parameter μ and showed that R_μ is a positive measure if and only if μ lies in some similar set.
- Finally, FitzGerald and Horn [24] classified the set of entrywise powers preserving positive semidefiniteness on $p \times p$ matrices, and showed that this set also equals $\mathbb{Z}_{\geq 0} \cup (p-2,\infty)$.

It is remarkable that all of the above named authors obtained their results in such diverse areas of mathematics within a few years of each other (all in the 1970s). Each of these works has been followed by tremendous activity. More recently, such a set has been found in the theory of non-central Wishart distributions (see, for example, [21, 53, 58, 60]). Karlin's work predated all of these works and results, thus providing an earlier instance of such a Berezin–Gindikin–Wallach set. We also mention the very recent result of Sra (see [75, Theorem 2]) obtaining such a spectrum from the characterization of fractional powers that preserve the positivity of Hua–Bellman matrices.

In his comprehensive 1968 book [43] on total positivity, Karlin provided a second such set of powers. Let the set of positive integers $\{1,2,3,\dots\}$ be denoted by $\mathbb{Z}_{>0}$.

Theorem 3.4 (Karlin [43]). *If Λ is a one-sided* PF *function and $p \geq 2$ is an integer, then $\mathscr{B}\{\Lambda\}^{\alpha}$ is the Laplace transform of a* TN_p *function for all $\alpha \in \mathbb{Z}_{>0} \cup (p-1,\infty)$.*

Once again, it is natural to ask if the converse holds. We answer this question in the affirmative, and add a third equivalent condition involving a single test function.

Proposition 3.5 ([45]). *Fix an integer $p \geq 2$ and a scalar $\alpha \geq 1$. The following are equivalent:*

(1) *If Λ is a one-sided* PF *function, then $\mathscr{B}\{\Lambda\}^{\alpha}$ is the Laplace transform of a* TN_p *function.*

(2) *For the 'single atom'* PF *function*

$$\lambda_1 : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \mathbf{1}_{x \geq 0}\, e^{-x},$$

the density of a standard exponential variable, the power $x \mapsto \mathscr{B}\{\lambda_1\}(x)^{\alpha}$ is the Laplace transform of a TN_p *function.*

(3) $\alpha \in \mathbb{Z}_{>0} \cup (p-1,\infty)$.

That (3) $\implies$ (1) was Karlin's 1968 result above, and it is a consequence of Schoenberg's representation Theorem 2.2 for PF functions. (The implications (1) $\implies$ (2) $\implies$ (3) are similarly not hard to prove.)

We close this section with a few results related to the topics addressed above. The first is from Schoenberg's 1955 work [71] that initiated the study of TN_p kernels (which Schoenberg termed *multiply positive functions*). In this work, Schoenberg studied the order of total non-negativity of powers of the *Wallis kernel*.

Theorem 3.6 (Schoenberg [71]). *Let*

$$W : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \mathbf{1}_{|x| \leq \pi/2} \cos x.$$

For any integer $p \geq 2$, the power W^{α} is TN_p *if and only if $\alpha \geq p-2$.*

In analogy with Theorem 3.3, we strengthen the total non-negativity to total positivity, and the lack thereof to principal minors, on large subsets of arguments.

Theorem 3.7 ([45]). *Let $p \geq 2$ be an integer. Given arbitrary reals $x_1 < \cdots < x_p$ and $y_1 < \cdots < y_p$, there exists a 'multiplicative shift' $m_0 \in (0,\infty)$ such that the matrices*

$$(W(m(x_j - y_k))^{\alpha})_{j,k=1}^{p}, \qquad 0 < m < m_0$$

are each TP *if $\alpha > p-2$,* TN *if $\alpha \in \{0,1,\dots,p-2\}$, and each have a negative principal minor if $\alpha \in (0,p-2) \setminus \mathbb{Z}$.*

The other result which we address here is the classification of the discontinuous Pólya frequency functions. In his 1951 paper [66], as well as the preceding announcements in *Proc. Natl. Acad. Sci. USA*, Schoenberg asserts that the only discontinuous Pólya frequency function is the standard exponential density $\lambda_1(x) := \mathbf{1}_{x \geq 0}\, e^{-x}$ "up to changes in scale and origin" (which includes reflecting about the y axis). This also implies that the only discontinuous totally non-negative function is the Heaviside function $H_1(x) := \mathbf{1}_{x \geq 0}$, up to changes in scale and origin and multiplying by an exponential factor e^{ax+b}, where $a, b \in \mathbb{R}$.

It turns out that these statements are not quite true, precisely at the point of discontinuity. Our explorations in [4] led us to a family of discontinuous PF functions $\{\lambda_d : d \in [0,1]\}$ that lies outside the above class, and subsequently, to a small correction of Schoenberg's classification.

Theorem 3.8 ([45])**.** *A Pólya frequency function is discontinuous if and only if it equals, up to changes in scale and origin, the following function λ_d for some $d \in [0,1]$:*

$$\lambda_d : \mathbb{R} \to \mathbb{R}; \quad x \mapsto \begin{cases} e^{-x} & \text{if } x > 0, \\ d & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Similarly, a TN *function is discontinuous if and only if (up to changes in scale and origin) it equals $x \mapsto e^{ax+b}\lambda_d(x)$ for some $d \in [0,1]$ and $a, b \in \mathbb{R}$.*

Note that the Laplace transform of λ_d does not depend on d.

4. Three or more atoms: non-smooth Pólya frequency functions as hypoexponential densities

We now turn from Berezin–Gindikin–Wallach type sets – encountered here in the study of TN_p powers of functions – to transforms of broader classes of Pólya frequency functions. The preceding result classifies the discontinuous PF functions, following Schoenberg [66]. In the same paper, Schoenberg classified the non-smooth Pólya frequency functions. In the language of probability theory, they are precisely the *densities of finite linear combinations of independent exponential variables.* Such functions were studied in detail by Hirschman and Widder, first in their 1949 paper [38], and then in the 1955 memoir [37]. In recent work [6], we further investigate these maps, obtaining connections to probability and to symmetric function theory; the goal of this section is to announce some of those results. Here and in [6], we term these frequency functions *Hirschman–Widder densities.*

Before we discuss the power preservers of such densities (following Karlin, Schoenberg, and the results in the preceding section), we present some pleasing properties of Hirschman–Widder densities. The preceding section dealt with single exponential random variables which led to discontinuous PF functions; thus, in this section we consider the densities of variables

$$\alpha_1 X_1 + \cdots + \alpha_m X_m, \qquad \text{where } m \geq 2 \text{ and } \boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}^\times)^m. \tag{4.1}$$

Here $X_1, \ldots, X_m$ are independent standard exponential random variables and $\mathbb{R}^\times$ denotes the set of non-zero real numbers.

We denote the corresponding density function by $\Lambda_{\boldsymbol{\alpha}}$. Such functions are known in probability and statistics as *hypoexponential densities*, or as *Erlang densities* if the coefficients α_j are all equal, and are relevant to several applied fields. However, the connection to the work of Hirschman and Widder seems to not be widely known in the probability literature.

Here are some of the properties enjoyed by Hirschman–Widder densities.

(1) The function $\Lambda_{\boldsymbol{\alpha}} : \mathbb{R} \to [0,\infty)$ is the unique continuous function with bilateral Laplace transform

$$\mathscr{B}\{\Lambda_{\boldsymbol{\alpha}}\}(s) = \prod_{j=1}^{m} \frac{1}{1+\alpha_j s} \tag{4.2}$$

on the open half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\alpha_j^{-1} \text{ for } j = 1, \ldots, m\}$.

(2) In particular, $\Lambda_{\boldsymbol{\alpha}}$ is both the probability density function of the random variable (4.1) and a Pólya frequency function.

(3) $\Lambda_{\boldsymbol{\alpha}}$ has an additive representation via one-sided exponential densities, as well as a multiplicative one via the bilateral Laplace transform, which corresponds to the convolution of these densities.

The additive representation is particularly gratifying when the parameters α_j are pairwise distinct and positive:

$$\Lambda_{\boldsymbol{\alpha}}(x) = \mathbf{1}_{x\geq 0} \sum_{j=1}^{m} a_j e^{-a_j x} \prod_{k\neq j} \frac{a_k}{a_k - a_j}, \qquad \text{where } a_j := \alpha_j^{-1} \text{ for all } j. \tag{4.3}$$

4.1. Taylor coefficients, moments, and symmetric functions. We now turn to some connections between Hirschman–Widder densities and symmetric function theory. As discussed above, Schoenberg's results in [66] imply that the densities $\Lambda_{\boldsymbol{\alpha}}$ are the *only* one-sided, non-smooth, continuous Pólya frequency functions that vanish on $(-\infty,0)$ and are non-zero on $(0,\infty)$.

It is natural to take a closer look at the Taylor expansion of $\Lambda_{\boldsymbol{\alpha}}$ at 0^+ and also at the moments of this density function. The descriptions of both of these quantities involve a well-known family of symmetric functions, the *complete homogeneous symmetric polynomials*:

$$h_p(a_1,\ldots,a_m) := \sum_{1\leq j_1\leq j_2\leq\cdots\leq j_p\leq m} a_{j_1} a_{j_2} \cdots a_{j_p}, \qquad \text{for all } p \in \mathbb{Z}_{>0}. \tag{4.4}$$

(We set $h_0 \equiv 1$.) These polynomials are clearly symmetric in their arguments, and equal Schur polynomials corresponding to tableaux for a single row with p cells and an alphabet of size m. In particular, they carry representation-theoretic content: they correspond to the characters of certain polynomial representations of the symmetric group, or of the special Lie algebra $\mathfrak{sl}_m(\mathbb{C})$.

With these polynomials at hand, we provide closed-form expressions for the Maclaurin coefficients at 0^+, as well as the moments, of Hirschman–Widder densities.

Theorem 4.1 ([6])**.** *Fix an integer $m \geq 2$ and parameters $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m) \in (0,\infty)^m$.*

(1) *The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ has the expansion*

$$\Lambda_{\boldsymbol{\alpha}}(x) = \alpha_1 \cdots \alpha_m \sum_{n=m-1}^{\infty} \frac{(-1)^{n-m+1} h_{n-m+1}(\alpha_1^{-1},\ldots,\alpha_m^{-1})}{n!} x^n \tag{4.5}$$

as its Maclaurin series, convergent for all $x \in [0,\infty)$. In particular, $\Lambda_{\boldsymbol{\alpha}}$ is smooth except at the origin, where it is of continuity class C^{m-2} but not C^{m-1}.

(2) *The Hirschman–Widder density $\Lambda_{\boldsymbol{\alpha}}$ has p^{th} moment*

$$\mu_p := \int_{\mathbb{R}} x^p \Lambda_{\boldsymbol{\alpha}}(x)\,\mathrm{d}x = p!\, h_p(\alpha_1,\ldots,\alpha_m) \qquad \textit{for all } p \geq 0.$$

While both the Maclaurin coefficients and the moments are expressed by the same family of symmetric functions, the arguments are different, involving $\{a_j = \alpha_j^{-1}\}$ and $\{\alpha_j\}$, respectively. Corresponding to these are two familiar identities which appear as byproducts of the proofs in [6]. The first is the 'local' version of the well-known generating function for the symmetric polynomials h_p:

$$\sum_{p=0}^{\infty} h_p(\alpha_1^{-1},\ldots,\alpha_m^{-1}) z^p = \prod_{j=1}^{m} \frac{1}{1-\alpha_j^{-1} z} \quad \text{whenever } |z| < \min\{\alpha_j : j = 1,\ldots,m\}.$$

The second is the moment-generating function for the density $\Lambda_{\boldsymbol{\alpha}}$ of $X := \sum_{j=1}^{m} \alpha_j X_j$:

$$\sum_{p=0}^{\infty} \frac{\mu_p}{p!} z^p = \mathbb{E}[e^{zX}] = \mathscr{B}\{\Lambda_{\boldsymbol{\alpha}}\}(-z) = \prod_{j=1}^{m} \frac{1}{1-\alpha_j z}.$$

As a final result in this vein, we show in [6] that for each $m \geq 2$, the parameters $\boldsymbol{\alpha}$ can be recovered from finitely many moments or Maclaurin coefficients.

Theorem 4.2 ([6]). *Given $m \geq 2$ and $\boldsymbol{\alpha} \in (0,\infty)^m$, the parameters $\boldsymbol{\alpha}$ may be recovered, up to permuting its entries, from the first m moments $\mu_1, \dots, \mu_m$ of the density $\Lambda_{\boldsymbol{\alpha}}$, and also from the lowest $m+1$ non-trivial Maclaurin coefficients of $\Lambda_{\boldsymbol{\alpha}}$, that is, $\Lambda_{\boldsymbol{\alpha}}^{(k)}(0^+)$ with $m-1 \leq k \leq 2m-1$.*

The proof crucially relies upon the Jacobi–Trudi identity from symmetric function theory. As we mention below, this is one of several hitherto unexplored connections between symmetric functions and positivity that have emerged in recent works.

4.2. Power and polynomial preservers of Hirschman–Widder densities. We now return to the theme of understanding which transformations preserve the class of Hirschman–Widder densities. As Karlin and Schoenberg have shown, the density of $X_1 + X_2$ (with X_1 and X_2 independent standard exponential random variables) is a PF function, every positive integer power of which is also a PF function. As we asserted in Theorem 3.3, the situation is identical for $\alpha_1 X_1 + \alpha_2 X_2$ whenever $\alpha_1, \alpha_2 > 0$. More generally, we have the following result; note that, since $\Lambda_{\boldsymbol{\alpha}}$ is symmetric in $\boldsymbol{\alpha}$, the entries of $\boldsymbol{\alpha}$ may be assumed to be monotonic.

Theorem 4.3 ([6]). *Consider the class of probability densities*

$$\{\Lambda_{\boldsymbol{\alpha}} : \alpha_1^{-1}, \dots, \alpha_m^{-1} \text{ are positive and form an arithmetic progression}\}$$

of finite linear combinations $\sum_{j=1}^m \alpha_j X_j$ of independent standard exponential random variables. This class of densities is closed under taking positive integer powers.

In particular, if $m = 2$, then all positive integer powers of $\Lambda_{\boldsymbol{\alpha}}$ are Hirschman–Widder densities, so Pólya frequency functions, so totally non-negative functions.

However, the preceding result is somewhat misleading as far as general parameters $\boldsymbol{\alpha}$ go. For $m \geq 3$ and almost all $\boldsymbol{\alpha}$, we show that not only do integer powers not remain PF functions, or even TN functions, but in fact *no* polynomial transform enjoys this property, save for the obvious ones.

Theorem 4.4. *Fix an integer $m \geq 3$. There exists a set $\mathcal{N} \subset (0,\infty)^m$ with zero Lebesgue measure such that*

$$p \circ \Lambda_{\boldsymbol{\alpha}} : \mathbb{R} \to \mathbb{R}; \quad x \mapsto p(\Lambda_{\boldsymbol{\alpha}}(x))$$

is not a TN *function, and so not a Pólya frequency function, for any $\boldsymbol{\alpha} \in (0,\infty)^m \setminus \mathcal{N}$ and any non-homothetic real polynomial p, that is, any real polynomial not of the form $p(x) = cx$ for some $c > 0$.*

5. One and two-sided Pólya frequency functions and sequences

Having studied finite linear combinations of exponential random variables and the transformations that preserve total non-negativity of their densities, we now tackle the same preserver problem for the general case: one-sided and two-sided Pólya frequency functions. This is the focus of the third paper [4] that is being announced in the present note.

In studying these preservers, we were naturally motivated by the celebrated paper of Schoenberg [69], which classified the preservers of a related notion: positive semidefiniteness. Coming from considerations of distance geometry (specifically, positive definite functions of the form $F \circ \cos$ on Euclidean spheres) Schoenberg classified the continuous functions $F : [-1,1] \to \mathbb{R}$ which preserve positivity when applied entrywise to positive semidefinite matrices of all dimensions.

Theorem 5.1 (Schoenberg [69]). *Suppose $F : [-1,1] \to \mathbb{R}$ is continuous. The following are equivalent:*

(1) *The* entrywise transform $F[A] := (F(a_{ij}))_{i,j=1}^n$ *is positive semidefinite for every positive semidefinite matrix $A = (a_{ij})_{i,j=1}^n$ with entries in $[-1,1]$ and every $n \geq 1$.*

(2) *F admits a power-series representation $\sum_{k=0}^\infty c_k x^k$ on $[-1,1]$ with non-negative Maclaurin coefficients.*

As observed by Pólya and Szegő in 1925, the implication (2) $\implies$ (1) essentially follows from the Schur product theorem [73], and functions which such a representation are called *absolutely monotonic*. Schoenberg's theorem [69] provides the highly non-trivial converse result, that continuous preservers are precisely the absolutely monotonic functions. The continuity assumption in Theorem 5.1 was subsequently removed by Rudin, who showed moreover that the test set can be greatly reduced in every dimension, to low-rank Toeplitz matrices.

Theorem 5.2 (Rudin [65]). *Suppose $I = (-1,1)$ and $F : I \to \mathbb{R}$. The following are equivalent:*

(1) *$F[A]$ is positive semidefinite for every positive semidefinite $A \in I^{n \times n}$ and every $n \geq 1$.*

(2) *$F[A]$ is positive semidefinite for every positive semidefinite $A \in I^{n \times n}$ which is Toeplitz of rank at most* 3 *and every $n \geq 1$.*

(3) *F is represented by a power series $\sum_{k=0}^\infty c_k x^k$ on I, with all $c_k \geq 0$.*

Inspired by Schoenberg's theorem, and with applications to moment problems in mind, we recently strengthened Theorem 5.1 in a parallel direction to Rudin's result.

Theorem 5.3 ([7]). *The three conditions in Theorem 5.2 are also equivalent to the following:*

(4) *$F[A]$ is positive semidefinite for every positive semidefinite $A \in I^{n \times n}$ which is Hankel of rank at most* 3 *and every $n \geq 1$. Equivalently, $F[-]$ preserves the class of moment sequences of positive measures on I having moments of all orders.*

These results can be recast in the language of composition maps C_F. A real $n \times n$ matrix A can be identified with a function $A : [n] \times [n] \to \mathbb{R}$, where $[n] := \{1, \ldots, n\}$. Now, the entrywise transform $F[A]$ is simply the composition $C_F(A) := F \circ A$. Thus, the above results are equivalent to classifying the composition maps C_F that preserve the class of *positive semidefinite kernels* on an infinite domain X. In particular, Theorem 5.3 classifies the composition maps that preserve positivity on the family of real Hankel kernels on $\mathbb{Z}_{\geq 0}^2$.

It is natural to move from studying the preservers of structured Hankel kernels to those of structured Toeplitz kernels. In this section, we consider endomorphisms of several different classes of functions: Pólya frequency functions, totally non-negative functions, and Pólya frequency sequences. We also discuss one-sided variants of these. We begin by asserting that the classes of Pólya frequency functions and sequences are as rigid as the class of Hirschman–Widder densities, not just for polynomials, but for arbitrary composition transforms.

Theorem 5.4 ([4]). *Let $F : [0,\infty) \to [0,\infty)$ be a function. Consider the composition transform C_F, taking any function of the form $\Lambda : X \to [0,\infty)$ to the map $C_F(\Lambda) := F \circ \Lambda : X \to [0,\infty)$.*

(1) *The composition transform C_F preserves the class of Pólya frequency functions if and only if F is a positive homothety: there exists $c > 0$ such that $F(x) = cx$ for all x.*

(2) *The transform C_F preserves the class of Pólya frequency sequences if and only if F is a positive homothety or a non-negative constant: either $F \equiv 0$ or there exists $c > 0$ such that $F \equiv c$ or $F(x) = cx$ for all x.*
(3) *The transform C_F preserves the class of totally non-negative functions if and only if $F \equiv 0$ or there exists $c > 0$ such that $F \equiv c$ or $F(x) = cx$ for all x or $F(x) = c\mathbf{1}_{x>0}$ for all x.*

We next present a one-sided version of Theorem 5.4. Given a domain $X \subset \mathbb{R}$, we say that a function $\Lambda : X \to \mathbb{R}$ is *one sided* if there exists $x_0 \in \mathbb{R}$ such that $\Lambda \equiv 0$ on $X \cap (-\infty, x_0)$ or $\Lambda \equiv 0$ on $X \cap (x_0, \infty)$.

Theorem 5.5 ([4]). *Suppose $F : [0,\infty) \to [0,\infty)$.*

(1) *The composition transform C_F preserves the class of one-sided Pólya frequency functions if and only if there exists $c > 0$ such that $F(x) = cx$ for all x.*
(2) *The transform C_F preserves the class of one-sided Pólya frequency sequences if and only if $F \equiv 0$ or there exists $c > 0$ such that $F(x) = cx$ for all x.*
(3) *The transform C_F preserves the class of one-sided totally non-negative functions if and only if $F \equiv 0$ or there exists $c > 0$ such that either $F(x) = cx$ for all x or $F(x) = c\mathbf{1}_{x>0}$ for all x.*

These results reveal the rigidity of the endomorphisms of several classes of totally non-negative functions. A similar phenomenon is uncovered when understanding the endomorphisms of totally positive functions, or of TP Pólya frequency functions or sequences. Recall that these are the subsets of the corresponding classes of maps for which all determinants in the defining conditions are positive instead of just non-negative.

Theorem 5.6 ([4]). *Suppose $F : (0,\infty) \to (0,\infty)$. The composition transform C_F preserves any of the following classes (and so all of them) if and only if there exists $c > 0$ such that $F(x) = cx$ for all x: (a) totally positive* PF *functions, (b) totally positive* PF *sequences, and (c) totally positive functions.*

The proof of Theorem 5.6 involves two interesting ingredients. The first is a density phenomenon for totally positive Pólya frequency functions.

Proposition 5.7 ([4]). *The class of totally positive Pólya frequency functions is dense in the class of Pólya frequency functions Λ that are* regular, *that is, such that $\Lambda(x) = (\Lambda(x^+) + \Lambda(x^-))/2$ for all x, where $\Lambda(x^+)$ and $\Lambda(x^-)$ are the right-hand and left-hand limits of the function Λ at the point x.*

The reader may be reminded of the density result of Anne M. Whitney [83], which asserts that the set of $m \times n$ TP_p matrices is dense in that of $m \times n$ TN_p matrices, for all positive integers m, n and p. The proposition above is the analogous result for Toeplitz kernels arising from Pólya frequency functions.

The second ingredient in its proof is the discontinuous test function $\lambda_{1/2}$ from Theorem 3.8, which is indeed a regular Pólya frequency function. In fact, it was in this context that we first encountered $\lambda_{1/2}$ and were led to the more general family of functions λ_d and to the classification Theorem 3.8.

6. Preservers of total positivity on arbitrary domains

In this final section, we go from working with structured kernels to arbitrary ones. Just as the preservers of all positive semidefinite matrices are classified by Schoenberg's theorem 5.1 and its strengthenings in [65] and [7], a natural and parallel question is to understand the preservers of total positivity. Here, we present the complete solution to the following problems.

Given non-empty totally ordered sets X and Y and a positive integer p, classify the composition transforms C_F which preserve the classes of TN_p, TP_p, TN, *and* TP *kernels on* $X \times Y$.

Remark 6.1. Two initial observations are in order. First, from the definitions, it is clear that we can assume X and Y have cardinality at least p, for the TN_p and TP_p problems, and are infinite otherwise, to ensure the relevant class is not empty. Second, the kernel $K \equiv 0$ is TN and $K' \equiv 1$ is TP_1 on any non-empty domain $X \times Y$, so there are no further restrictions for the TN_p and TP_1 classification problems.

As in [4], we examine the totally non-negative and totally positive cases separately.

6.1. Preservers of total non-negativity. It is immediate that the functions F such that C_F preserves the TN_1 or TP_1 property for kernels are precisely the self maps $F : [0,\infty) \to [0,\infty)$ or $F : (0,\infty) \to (0,\infty)$ respectively. Thus, in the sequel we will assume $p \geq 2$.

To write down systematically all total non-negativity preservers, we first introduce the following compact notation.

Definition 6.2. Given totally ordered sets X and Y, and a positive integer p, let

$$\mathscr{F}^{\mathrm{TN}}_{X,Y}(p) := \{F : [0,\infty) \to \mathbb{R} \mid F \circ K \text{ is } \mathrm{TN}_p \text{ for any } \mathrm{TN}_p \text{ kernel } K \text{ on } X \times Y\},$$
$$\mathscr{F}^{\mathrm{TP}}_{X,Y}(p) := \{F : [0,\infty) \to \mathbb{R} \mid F \circ K \text{ is } \mathrm{TP}_p \text{ for any } \mathrm{TP}_p \text{ kernel } K \text{ on } X \times Y\}.$$

We also let $\mathscr{F}^{\mathrm{TN}}_{X,Y}(\infty)$ and $\mathscr{F}^{\mathrm{TP}}_{X,Y}(\infty)$ denote the analogous classes of functions for TN and TP kernels, respectively, and use TN_∞ and TP_∞ synonymously with TN and TP.

To classify the total non-negativity preservers, we first note that if X and Y have cardinality at least p then every preserver of the set of TN_p matrices in $\mathbb{R}^{p\times p}$ is automatically a preserver of the set of TN_p kernels on $X \times Y$, and vice versa. Indeed, one inclusion of preservers is immediate, and the reverse inclusion follows by padding a $p \times p$ TN_p matrix by zeros to yield a TN_p kernel on $X \times Y$. With this observation at hand, we present the solution to the easier of the two questions posed above.

Theorem 6.3 ([4])**.** *Let X and Y be totally ordered sets with cardinality at least p, where $2 \leq p \leq \infty$. Then*

(1) $\mathscr{F}^{\mathrm{TN}}_{X,Y}(2) = \{cx^\alpha : c > 0, \alpha > 0\} \cup \{c : c \geq 0\} \cup \{c\mathbf{1}_{x>0} : c > 0\}$,
(2) $\mathscr{F}^{\mathrm{TN}}_{X,Y}(3) = \{cx^\alpha : c > 0, \alpha \geq 1\} \cup \{c : c \geq 0\}$, *and*
(3) $\mathscr{F}^{\mathrm{TN}}_{X,Y}(p) = \{cx : c > 0\} \cup \{c : c \geq 0\}$ *if* $4 \leq p \leq \infty$.

We briefly remark that proving Theorem 6.3 involves showing that every preserver of TN_p preserves TN_2, whence is measurable. Now the problem reduces to solving the multiplicative Cauchy functional equation in this setting, to deduce that the preserver is, up to rescaling, a power function or $\mathbf{1}_{x>0}$.

Remark 6.4. Interestingly, the preserver $c\mathbf{1}_{x>0}$ also features in Theorem 5.4, and the preservers of PF sequences, which may be thought of as bi-infinite TN Toeplitz matrices, coincide with those of TN_p kernels for $4 \leq p \leq \infty$.

Remark 6.5. As Theorem 6.3 shows, there are very few preservers of total non-negativity of order at least 4. The situation was the same when we considered the preservers of TN Toeplitz kernels on infinite domains such as $\mathbb{R}$ or $\mathbb{Z}$. However, this changes dramatically upon going from the Toeplitz to the Hankel setting: there are large classes of preservers of Hankel TN matrices of a fixed dimension or even of all dimensions. We illustrate this with a few concrete examples.

(1) The map $F : [0,\infty) \to [0,\infty)$ is such that the composition transform C_F preserves all TN Hankel matrices of all dimensions precisely when it equals a convergent power series $\sum_{k=0}^{\infty} c_k x^k$ on $(0,\infty)$ with all $c_k \geq 0$ and is such that $0 \leq F(0) \leq F(0^+)$. This is similar to the class of preservers in Schoenberg's Theorem 5.1, and far larger than the collection of constants and homotheties in Theorem 6.3.
(2) The set of entrywise powers that preserve the class of Hankel TN_p matrices coincides [7] with the Berezin–Gindikin–Wallach-type set $\mathbb{Z}_{\geq 0} \cup (p-2,\infty)$ obtained by Karlin: see Theorems 3.2 and 3.3. The same set of powers also appears as the set of power preservers of the class of continuous Hankel TN_p kernels on arbitrary sub-intervals of $\mathbb{R}$ with positive measure, as shown in [45]. Once again, this set differs vastly from the one in Theorem 6.3.
(3) In a fixed dimension $p \geq 2$, there are polynomial preservers with negative coefficients, as obtained in [5, 47]. This line of investigation uncovered unexpected connections between positivity and Schur polynomials, which preceded the findings in Section 4.1: see [6, 46].

6.2. Preservers of finite-order total positivity. We now come to the preservers of totally positive kernels. There are two distinct cases, when p is finite and when p is infinite, and each has its own subtleties.

The first observation is that if X and Y each have cardinality at least 2 and there exists a TP_2 kernel on $X \times Y$, then X and Y necessarily embed into the real line.

Proposition 6.6 ([4])**.** *Let X and Y be non-empty totally ordered sets. The following are equivalent:*

(1) *There exists a totally positive kernel on $X \times Y$.*
(2) *There exists a TP_2 kernel on $X \times Y$.*
(3) *At least one of X or Y is a singleton set, or there exist order-preserving embeddings from X into $(0,\infty)$ and from Y into $(0,\infty)$.*

Thus, the general problem of classifying total positivity preservers reduces to considering domains embedded in the real line. A key difference with the corresponding problem for total non-negativity preservers is that one can no longer extend kernels on smaller domains to the whole of $X \times Y$ by padding with zeros. Thus, TP_p preservers of $p \times p$ matrices will preserve TP_p kernels on $X \times Y$ whenever X and Y have cardinality at least p, but to show the converse we need to develop additional tools.

The following result is the complete classification in the finite-order case.

Theorem 6.7 ([4])**.** *Suppose X and Y are totally ordered sets such that there exists a TP_p kernel on $X \times Y$ for some integer $p \geq 2$. Then*

(1) $\mathscr{F}_{X,Y}^{\mathrm{TP}}(2) = \{cx^\alpha : c > 0, \alpha > 0\}$,
(2) $\mathscr{F}_{X,Y}^{\mathrm{TP}}(3) = \{cx^\alpha : c > 0, \alpha \geq 1\}$, *and*
(3) $\mathscr{F}_{X,Y}^{\mathrm{TP}}(p) = \{cx : c > 0\}$ *if* $p \geq 4$.

This result should be compared with Theorem 6.3. The proof involves first showing that every preserver is continuous; we elaborate on this below. Once this is done, when X and Y are finite, we may invoke Whitney's density theorem [83], which reduces the classification of preservers of TP_p kernels on $X \times Y$ to the case of TN_p kernels, which was done in Theorem 6.3. Now some matrix analysis completes the proof.

For finite X, Y and p, we still need to show that every preserver is continuous. In fact, a stronger result holds.

Proposition 6.8 ([4]). *Let X and Y be totally ordered sets such that there exists a* TP_2 *kernel on $X \times Y$.*

(1) *Every* 2×2 TP *matrix can be embedded in arbitrary position into a* TP *kernel on $X \times Y$.*
(2) *If $F : (0,\infty) \to (0,\infty)$ is such that the composition map C_F preserves the class of* TP_p *kernels on $X \times Y$ for some $p \geq 2$ (including $p = \infty$) then F is continuous and there exist $c > 0$ and $\alpha > 0$ such that $F(x) = cx^\alpha$ for all x.*

The proof of part (1) involves showing that every 2×2 TP matrix is, up to scaling and taking the transpose, a generalized Vandermonde matrix, and then invoking Proposition 6.6(3). Moreover, a generalized Vandermonde matrix can easily be embedded in the TP kernel

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad (x, y) \mapsto \exp(xy),$$

from which part (2) follows easily.

This completes the outline of the proof of Theorem 6.7 when X and Y are finite. However, the proof techniques used for the results above no longer suffice if at least one of X and Y is infinite, even after we know that $F(x) = cx^\alpha$ as above. The missing ingredient that is required is an analogue of Whitney's density theorem for infinite domains. Thus, we extend that result.

Theorem 6.9 ([4]). *Let X and Y be subsets of the real line which each contain at least two distinct points. If $K : X \times Y \to \mathbb{R}$ is a bounded* TN_p *kernel for some integer $p \geq 2$ then there exists a sequence of* TP_p *kernels on $X \times Y$ which converges to K locally uniformly on the set of points in $X \times Y$ where K is continuous.*

6.3. Total-positivity preservers on bi-infinite domains. If X or Y is finite, then the TP kernels on $X \times Y$ are precisely the TP_p kernels, where p is the minimum of the cardinalities of X and Y. The preservers in this setting are classified by Theorem 6.7. Thus, the only remaining case requires X, Y and p to be infinite.

Theorem 6.10 ([4]). *If X and Y are infinite totally ordered sets such that there exists a* TP_2 *kernel on $X \times Y$ then the only preservers of* TP *kernels on $X \times Y$ are the positive homotheties: functions of the form $F(x) = cx$ for some $c > 0$.*

In conjunction with Theorem 6.7, this result completely resolves the problem of classifying total positivity preservers for kernels on an arbitrary domain $X \times Y$.

We briefly sketch here the arguments in the proof. The idea is to use the classification of the preservers of Pólya frequency functions and sequences from a previous section. To do so, one requires a novel ingredient: order-preserving embeddings of sets containing arbitrarily long arithmetic progressions. Thus, we introduce the following definition.

Definition 6.11. Two sets X and Y of real numbers are said to form an *admissible pair* if for each integer $n \geq 2$ there exist n-step arithmetic progressions

$$x_1 < \cdots < x_n \text{ in } X \quad \text{and} \quad y_1 < \cdots < y_n \text{ in } Y$$

that are equally spaced: $x_{j+1} - x_j = y_{j+1} - y_j$ for $j = 1, \ldots, n-1$.

Also, let the Minkowski difference $X - Y := \{x - y : x \in X, y \in Y\}$, and call a kernel $K : X \times Y \to \mathbb{R}$ *Toeplitz* if there exists a function $f : X - Y \to \mathbb{R}$ such that $K(x, y) = f(x - y)$ for all $x \in X$ and $y \in Y$.

Now the first step in proving Theorem 6.10 is the following common extension of parts (a) and (b) of Theorem 5.6.

Proposition 6.12 ([4]). *Suppose X, $Y \subset \mathbb{R}$ form an admissible pair, and let $F : (0,\infty) \to (0,\infty)$. The composition map C_F preserves total positivity for all Toeplitz kernels on $X \times Y$ if and only if there exists $c > 0$ such that $F(x) = cx$ for all x.*

With this result at hand, the proof of Theorem 6.10 goes as follows. The set X contains an infinite ascending chain or an infinite descending chain. We construct an order-preserving bijection $\varphi_X : X \to X'$, where $X' \subset \mathbb{R}$ is a subset that contains an arithmetic progression of length 2^n and step-size 4^{-n} for each $n \geq 2$, and similarly for $\varphi_Y : Y \to Y'$. Since X' and Y' form an admissible pair, Proposition 6.12 says that if F is not a homothety, then F does not preserve some TP Toeplitz kernel K' on $X' \times Y'$. Transferring the kernel K' back to a totally positive kernel on $X \times Y$ via (φ_X, φ_Y) shows the contrapositive of the non-trivial implication in Theorem 6.10.

6.4. Preservers of TP and TN on symmetric kernels. We conclude by mentioning that there exist symmetrical analogues of all of the results in this section, characterizing the preservers of symmetric TN_p or TP_p kernels on $X \times X$ for an arbitrary totally ordered set X. While the results have similar statements, we are unable to employ Pólya frequency functions or sequences as test kernels in our proofs, and so are forced to look elsewhere.

Here we discuss only one result: the symmetric version of the final theorem above.

Theorem 6.13 ([4]). *Given an infinite totally ordered set X such that there exists a symmetric TP_2 kernel on $X \times X$, the only preservers of symmetric TP kernels on $X \times X$ are the positive homotheties.*

The proof of this result employs all of the above tools and techniques. We conclude this note with a sketch of it.

First, by the symmetric variant of Proposition 6.8, every preserver is a power function of the form $F(x) = cx^\alpha$ with $c > 0$ and $\alpha > 0$. Moreover, as F is continuous, it also preserves the class of symmetric TN kernels on $X \times X$ that are limits of symmetric TP kernels.

Using the order-preserving bijection $\varphi_X : X \to X'$ from Proposition 6.6(3), we transfer a family of kernels on $X' \times X'$, which are Hankel on $Z_n \times Z_n$ for each n-step arithmetic progression $Z_n \subset X'$, to symmetric TN kernels on $X \times X$. As these Hankel kernels on $X' \times X'$ are limits of TP Hankel kernels, the preceding paragraph and a variant of the stronger Schoenberg-type Theorem 5.3 from [7] imply that F is absolutely monotonic. Hence $\alpha \in \mathbb{Z}_{>0}$.

We now fix a real number $\beta > 0$ and consider the test function

$$M_\beta : \mathbb{R} \to (0,\infty); \quad x \mapsto (\beta+1)\exp(-\beta|x|) - \beta\exp(-(\beta+1)|x|).$$

It follows from Schoenberg's representation theorem, Theorem 2.2, that M_β is a PF function, but that that M_β^k is not a PF function for any integer $k \geq 2$. Using convolution with a family of Gaussian densities, it is readily seen that the Toeplitz kernel associated with M_β is the limit of symmetric TP kernels, and so $F \circ M_\beta = M_\beta^\alpha$ is TN on $X \times X$ by our assumption on F.

Finally, we suppose $\alpha \geq 2$. Employing a discretization technique and using the continuity of M_β, we produce an arithmetic progression $Z' = (z'_1 < \cdots < z'_{n_\alpha})$ in $\mathbb{R}$ such that $(M_\beta(z'_i, z'_j)^\alpha)_{i,j=1}^{n_\alpha}$ is not TN. We transfer this kernel on $Z' \times Z'$ first to $X' \times X'$, via a change of scale and origin, then to $X \times X$ using φ_X. This shows that M_β^α is not TN on $X \times X$, a contradiction, and so the power α cannot be 2 or more, forcing $\alpha = 1$.

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