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Uniform L^p Resolvent Estimates on the Torus

Jonathan Hickman

(Recommended by Christopher Sogge)

Abstract. A new range of uniform L^p resolvent estimates is obtained in the setting of the flat torus, improving previous results of Bourgain, Shao, Sogge and Yao. The arguments rely on the ℓ^2-decoupling theorem and multidimensional Weyl sum estimates.

1. Introduction

This article continues a line of investigation pursued by Dos Santos Ferreira, Kenig and Salo [DSFKS14] and Bourgain, Shao, Sogge and Yao [BSSY15] concerning uniform L^p estimates for resolvents of Laplace–Beltrami operators on compact manifolds. Here new bounds are obtained only in the special case of the flat n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n \setminus \mathbb{Z}^n$ but, in order to contextualise the results, it is useful to recall the general setup from [DSFKS14, BSSY15]. To this end, let (M, g) be a smooth, compact manifold of dimension $n \geq 3$ without boundary and Δ_g be the associated Laplace–Beltrami operator. In [DSFKS14] the following problem was introduced: determine the regions $\mathscr{R} \subseteq \mathbb{C}$ for which there is a uniform bound

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C_{\mathscr{R}} \|(\Delta_g + z)u\|_{L^{\frac{2n}{n+2}}(M)} \qquad \text{for all } z \in \mathscr{R}. \tag{1.1}$$

Interest in inequalities of the form (1.1) was partly inspired by earlier work on the standard Laplacian on n-dimensional euclidean space. In the euclidean setting, it was shown by Kenig, Ruiz and Sogge [KRS87] that, *inter alia*, the euclidean analogue of (1.1) holds for $\mathscr{R} = \mathbb{C}$; scaling considerations imply that $(\frac{2n}{n+2}, \frac{2n}{n-2})$ is the only exponent pair lying on the line of duality for which such uniformity is possible.[1] This observation partially motivates the choice of Lebesgue exponents featured above. By contrast, uniformity in (1.1) over the whole of $\mathbb{C}$ patently fails for compact manifolds (M, g): in this case $-\Delta_g$ has a discrete spectrum and therefore (1.1) cannot hold whenever z is an eigenvalue of $-\Delta_g$. Therefore, it is natural when working in the compact manifold setting to consider regions $\mathscr{R}$ which are bounded away from the nonnegative real line, and thereby avoid the spectrum.

As in [BSSY15], it is convenient to write $z = (\lambda + i\mu)^2$ for some $\lambda, \mu \in \mathbb{R}$ and express the results in terms of these real parameters. For $\lambda \leq 1$ the situation is relatively easy to understand and is treated in [BSSY15, §2]. Henceforth, it is assumed that $\lambda \geq 1$. The problem is to determine how small $|\mu|$ can be (in terms of λ) whilst retaining uniformity in (1.1).

Theorem 1. *Let $n \geq 3$ and $\Delta_{\mathbb{T}^n}$ be the Laplacian on the flat torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. For all $\varepsilon > 0$ the uniform L^p resolvent bound*

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \leq C_\varepsilon \|(\Delta_{\mathbb{T}^n} + z)u\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}$$

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[1] For certain proper subsets $\mathscr{R}$ of $\mathbb{C}$ many other (p, q) exponent pairs are possible: see [KL19].

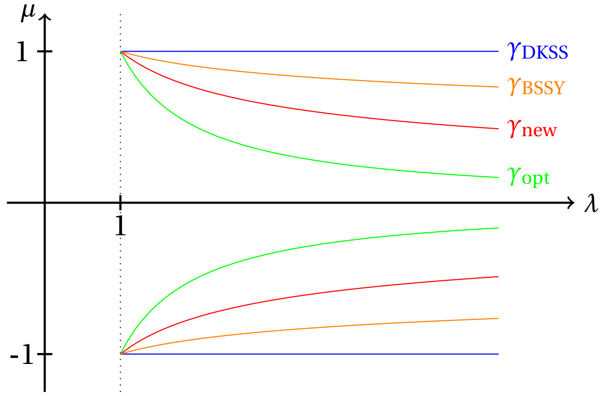

FIGURE 1. Successive results and the optimal region. Each curve γ_{DKSS}, γ_{BSSY}, γ_{new} and γ_{opt} corresponds to the interesting part of the boundary of $\mathscr{R}_{\mathrm{DKSS}}$, $\mathscr{R}_{\mathrm{BSSY}}$, $\mathscr{R}_{\mathrm{new}}$ and $\mathscr{R}_{\mathrm{opt}}$, respectively, in the coordinates (λ,μ).

holds whenever $z \in \mathbb{C}$ belongs to the region

$$\mathscr{R}_{\mathrm{new}} := \{z = (\lambda + i\mu)^2 \in \mathbb{C} : \lambda, \mu \in \mathbb{R}, \lambda \geq 1, |\mu| \geq \lambda^{-1/3+\varepsilon}\}.$$

It is useful to compare the theorem with existent results. Shen [She01] previously showed that Theorem 1 holds in the more restrictive region

$$\mathscr{R}_{\mathrm{DKSS}} := \{z = (\lambda + i\mu)^2 \in \mathbb{C} : \lambda, \mu \in \mathbb{R}, \lambda \geq 1, |\mu| \geq 1\}.$$

This was later generalised to arbitrary compact manifolds by Dos Santos Ferreira, Kenig and Salo [DSFKS14]. In [DSFKS14] it was also asked whether it is possible to extend the uniform bounds beyond $\mathscr{R}_{\mathrm{DKSS}}$ for general manifolds. Interestingly, Bourgain, Shao, Sogge and Yao [BSSY15] showed that the region $\mathscr{R}_{\mathrm{DKSS}}$ is, in fact, optimal in the case of Zoll manifolds (one example being the standard euclidean sphere S^n), in the sense that here it is not possible to relax $|\mu| \geq 1$ to $|\mu| \geq \lambda^{-\alpha}$ for any $\alpha > 0$ in $\mathscr{R}_{\mathrm{DKSS}}$. Underpinning such behaviour in the Zoll case is the tight spectral clustering exhibited by $-\Delta_g$. Clustering does not occur for the torus and, consequently, improvements may be obtained for $\mathbb{T}^n$. Indeed, in [BSSY15] it was shown that for all $\varepsilon > 0$ Theorem 1 holds for the region

$$\mathscr{R}_{\mathrm{BSSY}} := \{z = (\lambda + i\mu)^2 \in \mathbb{C} : \lambda, \mu \in \mathbb{R}, \lambda \geq 1, |\mu| \geq \lambda^{-\varepsilon_n+\varepsilon}\},$$

where $\varepsilon_n > 0$ is given by

$$\varepsilon_n := \frac{2(n-1)}{n(n+1)} \text{ if } n \geq 3 \text{ is odd}, \quad \varepsilon_n := \frac{2(n-1)}{n^2+2n-2} \text{ if } n \geq 4 \text{ is even};$$

furthermore, by using additional number-theoretic input, it was also shown in [BSSY15] that for $n = 3$ the slightly relaxed condition $\varepsilon_3 := \frac{85}{252}$ is sufficient.

Theorem 1 provides a further improvement over the ranges $\mathscr{R}_{\mathrm{DKSS}}$ and $\mathscr{R}_{\mathrm{BSSY}}$ (at least for $n > 3$); see Figure 1. Note for $n = 3$ the numerology of the new result coincides with the $\frac{2(n-1)}{n(n+1)}$ exponent from [BSSY15]. A pleasant feature of Theorem 1 is that $\mathscr{R}_{\mathrm{new}}$ provides a "uniform" strengthening over $\mathscr{R}_{\mathrm{DKSS}}$ in all dimensions.

It is remarked that $\mathscr{R}_{\mathrm{new}}$ is certainly not sharp and a natural conjecture would be the following.

Conjecture 2. *Let $n \geq 3$ and $\Delta_{\mathbb{T}^n}$ be the Laplacian on the flat torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. For all $\varepsilon > 0$ the uniform L^p resolvent bound*

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \leq C_\varepsilon \|(\Delta_{\mathbb{T}^n} + z)u\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}$$

holds whenever $z \in \mathbb{C}$ belongs to the region

$$\mathscr{R}_{\mathrm{opt}} := \big\{z = (\lambda + i\mu)^2 \in \mathbb{C} : \lambda, \mu \in \mathbb{R}, \lambda \geq 1, |\mu| \geq \lambda^{-1+\varepsilon}\big\}.$$

A slightly larger region, given by taking $\varepsilon = 0$ in the definition of $\mathscr{R}_{\mathrm{opt}}$, featured in the original question posed in [DSFKS14]. Conjecture 2 is closely related to the so-called *discrete restriction conjecture* for the sphere studied in [Bou93], which partially motivates the above definition of $\mathscr{R}_{\mathrm{opt}}$; this connection is discussed in more detail in §2 below.

The proof of Theorem 1 follows the strategy of [BSSY15] but takes advantage of new estimates available due to the Bourgain–Demeter ℓ^2-decoupling theorem [BD15b]. In [BSSY15] uniform resolvent estimates were shown to be equivalent to $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ bounds for certain spectral projectors with thin bandwidths; the precise details of this equivalence are recalled in §2. The desired spectral projection bounds are then proved using the ℓ^2-decoupling inequality. It is not surprising that decoupling should play a rôle here since it has already had numerous applications to the spectral theory of $\Delta_{\mathbb{T}^n}$ [Bou13, BD15b, BSSY15].

The Bourgain–Demeter theorem yields an $L^{\frac{2(n+1)}{n+3}} \to L^{\frac{2(n+1)}{n-1}}$ bound for the projector; see Corollary 9 below. Roughly speaking, to obtain the desired $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ inequality, one interpolates the $L^{\frac{2(n+1)}{n+3}} \to L^{\frac{2(n+1)}{n-1}}$ estimate with an $L^1 \to L^\infty$ estimate. The L^∞ bound for the projector follows from a pointwise estimate for the kernel which, as in [BSSY15], is established using the classical lattice point counting method of Hlawka [Hla50] (see also [Sog17, Chapter 1]).

Hlawka's original argument [Hla50] has been refined by numerous authors (see, for instance, [KN91, KN92, Mül99, Guo12]). In [BSSY15] exponential sum bounds from [KN92] were applied to yield the slightly improved exponent $\varepsilon_3 = \frac{85}{252}$ mentioned above. Similarly, by applying a more refined analysis involving the multidimensional Weyl sum estimates from [Mül99], it is possible to slightly extend $\mathscr{R}_{\mathrm{new}}$ in all dimensions.

Theorem 3. *For $n \geq 3$ and all $\varepsilon > 0$ the result of Theorem 1 holds for*

$$\mathscr{R}'_{\mathrm{new}} := \big\{z = (\lambda + i\mu)^2 \in \mathbb{C} : \lambda, \mu \in \mathbb{R}, \lambda \geq 1, |\mu| \geq \lambda^{-\beta_n+\varepsilon}\big\},$$

where

$$\beta_n := \frac{1}{3} + \frac{n}{3} \cdot \frac{1}{21n^2 - n - 24}.$$

Taking $n = 3$, the exponent becomes $\beta_3 = \frac{55}{162}$ which is slightly larger than the previous best exponent $\varepsilon_3 = \frac{85}{252}$ from [BSSY15]. This improvement for $n = 3$ is due in part to the use of stronger multidimensional Weyl sum estimates from [Mül99] (as opposed to the estimates of [KN92] used in [BSSY15]) and also due in part to the use of the ℓ^2-decoupling inequality, which allows for greater leverage of the exponential sum bounds.

This article is structured as follows:

- In §2 preliminary results from [BSSY15] and, in particular, the details of the equivalence between resolvent and spectral projection estimates, are reviewed.
- In §3 spectral projection bounds are proven, following the scheme described above. Using the equivalence discussed in §2, this provides the proof of Theorem 1.
- In §4 exponential sum estimates from [Mül99] are applied to refine the argument from §3, yielding Theorem 3.

Notation. Given positive numbers $A, B \geq 0$ and a list of objects L, the notation $A \lesssim_L B$, $B \gtrsim_L A$ or $A = O_L(B)$ signifies that $A \leq C_L B$ where C_L is a constant which depends only on the objects in the list and the dimension n. Furthermore, $A \sim_L B$ signifies that $A \lesssim_L B$ and $B \lesssim_L A$.

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2. Spectral projections

An equivalent formulation. It was shown in [BSSY15] that the desired resolvent estimates are equivalent to certain spectral projection bounds. Given $\lambda \geq 1$ and $\rho > 0$, define

$$A(\lambda, \rho) := \{\xi \in \hat{\mathbb{R}}^n : \big||\xi| - \lambda\big| < \rho\};$$

here $\hat{\mathbb{R}}^n$ denotes the frequency space. In the case of the torus, [BSSY15, Theorem 1.3] implies the following.

Theorem 4 ([BSSY15]). *Given $n \geq 3$ and $0 < \alpha \leq 1$, the following are equivalent:*

i) For all $\lambda \geq 1$ there is a uniform spectral projection estimate

$$\Big\| \sum_{k \in \mathbb{Z}^n \cap A(\lambda, \lambda^{-\alpha})} \hat{f}(k) e^{2\pi i x \cdot k} \Big\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\alpha \lambda^{1-\alpha} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}. \tag{2.1}$$

ii) There is a uniform resolvent estimate

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\alpha \|(\Delta_{\mathbb{T}^n} + z)u\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}$$

for all $z = (\lambda + i\mu)^2 \in \mathbb{C}$ such that $\lambda, \mu \in \mathbb{R}$ satisfy $\lambda \geq 1$, $|\mu| \geq \lambda^{-\alpha}$.

Remark 5. In [BSSY15] a more general statement is proven for compact manifolds.

The remaining sections of this paper will focus on proving spectral projection bounds of the type featured above.

Relationship with discrete Fourier restriction. Although it will not play any rôle in later arguments, it is nevertheless instructive to remark that Theorem 4 relates the resolvent and discrete restriction conjectures.

Conjecture 6 (Discrete restriction conjecture [Bou93]). *For $n \geq 3$, $\lambda \geq 1$ and $\varepsilon > 0$,*

$$\Big\| \sum_{k \in \mathbb{Z}^n \cap \lambda S^{n-1}} \hat{f}(k) e^{2\pi i x \cdot k} \Big\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon \|f\|_{L^2(\mathbb{T}^n)}. \tag{2.2}$$

In particular, if e_λ is an L^2-normalised eigenfunction for $-\Delta_{\mathbb{T}^n}$ with eigenvalue λ^2, then Conjecture 6 implies that $\|e_\lambda\|_{L^{2n/(n-2)}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon$. Various partial results on this problem are known, establishing weaker versions of (2.2) with larger values of p on the left-hand side: see [Bou93, Bou97, BD13, BD15a, BD15b].

By elementary separation properties of concentric lattice spheres, (2.2) is equivalent to

$$\Big\| \sum_{k \in \mathbb{Z}^n \cap A(\lambda, \lambda^{-1})} \hat{f}(k) e^{2\pi i x \cdot k} \Big\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon \|f\|_{L^2(\mathbb{T}^n)}.$$

using a T^*T argument, it is not difficult to see, that the above estimate would follow from (2.1) with $\alpha = 1 - \varepsilon$. Thus, by Theorem 4, the resolvent conjecture (Conjecture 2) implies the discrete restriction conjecture (Conjecture 6).

3. The proof of Theorem 1

By Theorem 4, the uniform resolvent estimates in Theorem 1 are equivalent to the following spectral projection bounds.

Proposition 7. *Let $n \geq 3$, $\lambda \geq 1$ and $\varepsilon > 0$. If $\rho := \lambda^{-1/3+\varepsilon}$, then*

$$\Big\| \sum_{k \in \mathbb{Z}^n \cap A(\lambda,\rho)} \hat{f}(k) e^{2\pi i x \cdot k} \Big\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \rho\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}. \tag{3.1}$$

Given $m \in \ell^\infty(\mathbb{Z}^n)$ let $m(D)$ denote the associated Fourier multiplier operator, defined initially on $C^\infty(\mathbb{T}^n)$ by

$$m(D)f(x) := \sum_{k \in \mathbb{Z}^n} m(k)\hat{f}(k) e^{2\pi i x \cdot k}.$$

If $m \in L^\infty(\hat{\mathbb{R}}^n)$, then $m(D) := m|_{\mathbb{Z}^n}(D)$ where $m|_{\mathbb{Z}^n}$ denotes the restriction of m to the integer lattice. Thus, with this notation, one may write (3.1) as

$$\|\chi_{A(\lambda,\rho)}(D)f\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \rho\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}. \tag{3.2}$$

The remainder of this section deals with the proof of Proposition 7.

Smooth multipliers. In proving Proposition 7, one may replace the rough cutoff function $\chi_{A(\lambda,\rho)}$ with a smoothed out version. Indeed, by T^*T, (3.2) is equivalent to

$$\|\chi_{A(\lambda,\rho)}(D)f\|_{L^2(\mathbb{T}^n)} \lesssim_\varepsilon (\rho\lambda)^{1/2} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}. \tag{3.3}$$

Fix $\beta \in C_c^\infty(\mathbb{R})$ nonnegative with $\beta(r) = 1$ for $|r| \leq 1$ and $\beta(r) = 0$ for $|r| \geq 2$ and define the multiplier

$$m^{\lambda,\rho}(\xi) := \beta\big(\rho^{-1}(|\xi| - \lambda)\big). \tag{3.4}$$

By L^2-orthogonality, (3.3) would follow from the bound

$$\big\| m^{\lambda,\rho}(D)^{1/2} f \big\|_{L^2(\mathbb{T}^n)} \lesssim_\varepsilon (\rho\lambda)^{1/2} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}$$

and, by a second application of T^*T, this would further follow from

$$\|m^{\lambda,\rho}(D)f\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \rho\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}. \tag{3.5}$$

Consequences of ℓ^2-decoupling. The proof of Proposition 7 relies on the ℓ^2-decoupling theorem proved in [BD15b]. It is convenient to work with a rescaled version of the decoupling theorem, in the special case of the euclidean sphere. For $\lambda \geq 1$ and $g \in L^1(\lambda S^{n-1})$ let

$$E_\lambda g(x) := \int_{\lambda S^{n-1}} g(\omega) e^{2\pi i x \cdot \omega} \, \mathrm{d}\sigma_{\lambda S^{n-1}}(\omega), \qquad x \in \mathbb{R}^n,$$

where the integration is with respect to the normalised (to have unit mass) surface measure on λS^{n-1}.

Theorem 8 (Bourgain–Demeter [BD15b])**.** *Let $\lambda \gtrsim 1$, $1 \gtrsim \rho \geq \lambda^{-1}$ and $\Theta(\lambda,\rho)$ be a finitely-overlapping covering of λS^{n-1} by $(\rho\lambda)^{1/2}$-caps. Given $g \in L^1(\lambda S^{n-1})$ write $g_\theta := g \cdot \chi_\theta$. For all $\varepsilon > 0$,*

$$\|E_\lambda g\|_{L^{\frac{2(n+1)}{n-1}}(B_{\rho^{-1}})} \lesssim_\varepsilon \lambda^\varepsilon \Big(\sum_{\theta \in \Theta(\lambda,\rho)} \|E_\lambda g_\theta\|^2_{L^{\frac{2(n+1)}{n-1}}(w_{B_{\rho^{-1}}})} \Big)^{\frac{1}{2}}.$$

Here B_r is used to denote an *r-ball*: that is, B_r is a ball in $\mathbb{R}^n$ with (arbitrary) centre $c(B_r)$ and radius $r > 0$. The weight w_{B_r} is the function concentrated on B_r given by

$$w_{B_r}(x) := \big(1 + r^{-1}|x - c(B_r)|\big)^{-6N} \tag{3.6}$$

where $N := 100n$. Finally, an *r-cap* on the sphere λS^{n-1} is the intersection of λS^{n-1} with an r-ball centred at a point on λS^{n-1}.

Using Theorem 8, one may prove an $L^{\frac{2(n+1)}{n+3}} \to L^{\frac{2(n+1)}{n-1}}$ bound for the projector in (3.1).

Corollary 9. *Let $n \geq 3$, $\lambda \gtrsim 1$ and $1 \gtrsim r \geq \lambda^{-1}$. For all $\varepsilon > 0$,*

$$\big\|\chi_{A(\lambda,r)}(D)f\big\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon (r\lambda)^{\frac{n-1}{n+1}} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}. \tag{3.7}$$

By duality and T^*T, (3.7) is equivalent to either of the following inequalities:

$$\|\chi_{A(\lambda,r)}(D)f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon (r\lambda)^{\frac{n-1}{2(n+1)}} \|f\|_{L^2(\mathbb{T}^n)}, \tag{3.8}$$

$$\|\chi_{A(\lambda,r)}(D)f\|_{L^2(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon (r\lambda)^{\frac{n-1}{2(n+1)}} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}. \tag{3.9}$$

Remark 10. If $r = \lambda^{-1}$, then Corollary 9 corresponds to a special case of the discrete Fourier restriction theorem of Bourgain–Demeter [BD15b, Theorem 2.2]. On the other hand, if $r \sim 1$, then (3.7) holds with no ε-loss as a simple consequence of the Stein–Tomas restriction theorem for the sphere, as discussed below.

Proof (of Corollary 9). As remarked earlier, it suffices to prove (3.8). It is well known (see, for instance, [BD15b]) that Theorem 8 implies a discrete version of itself. In particular, defining $R := r^{-1}$, given any 1-separated subset $\Omega_\lambda \subseteq \lambda S^{n-1}$ and any sequence $(a_\omega)_{\omega\in\Omega_\lambda}$, it follows that

$$\Big\|\sum_{\omega\in\Omega_\lambda} a_\omega e^{2\pi i x\cdot\omega}\Big\|_{L^{\frac{2(n+1)}{n-1}}(B_R)} \lesssim_\varepsilon \lambda^\varepsilon \Big(\sum_{\theta\in\Theta(\lambda,r)} \Big\|\sum_{\omega\in\Omega_\lambda\cap\theta} a_\omega e^{2\pi i x\cdot\omega}\Big\|^2_{L^{\frac{2(n+1)}{n-1}}(w_{B_R})}\Big)^{\frac{1}{2}}. \tag{3.10}$$

Indeed, this may be deduced by fixing $\psi \in C_c^\infty(\hat{\mathbb{R}}^n)$ with $\psi(0) = 1$, applying Theorem 8 to the functions

$$g_\delta(w) := \sum_{\omega\in\Omega_\lambda} a_\omega \psi(\delta^{-1}(w - \omega))$$

for $\delta > 0$ and applying a simple limiting argument; see [BD15b].

The spatial variable in (3.10) is localized to a ball of radius $R = r^{-1}$, inducing frequency uncertainty at scale r. In particular, one can, at least heuristically, replace the family of points Ω_λ in this inequality with any perturbed family

$$\tilde{\Omega}_\lambda = \{\omega + O(r) : \omega \in \Omega_\lambda\}.$$

For instance, one may take $\tilde{\Omega}_\lambda := \mathbb{Z}^n \cap A(\lambda, r)$, in which case (3.10) implies the heuristic inequality

$$\|\chi_{A(\lambda,r)}(D)f\|_{L^{\frac{2(n+1)}{n-1}}(B_R)} \lesssim_\varepsilon \lambda^\varepsilon \Big(\sum_{\theta\in\Theta(\lambda,r)} \|\chi_{A_\theta(\lambda,r)}(D)f\|^2_{L^{\frac{2(n+1)}{n-1}}(w_{B_R})}\Big)^{1/2} \tag{3.11}$$

where $A_\theta(\lambda, r)$ is the intersection of $A(\lambda, r)$ with the sector generated by θ. Giving a rigorous justification for this uncertainty heuristic is a messy affair and is therefore postponed until the end of the proof.

Since the functions appearing in either side of (3.11) are 1-periodic, it follows that

$$\|\chi_{A(\lambda,r)}(D)f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon \Big(\sum_{\theta\in\Theta(\lambda,r)} \|\chi_{A_\theta(\lambda,r)}(D)f\|^2_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)}\Big)^{1/2}.$$

To bound the right-hand side, observe the elementary estimate

$$\|\chi_{A_\theta(\lambda,r)}(D)f\|_{L^\infty(\mathbb{T}^n)} \leq [\#\mathbb{Z}^n \cap A_\theta(\lambda, r)]^{1/2} \|\chi_{A_\theta(\lambda,r)}(D)f\|_{L^2(\mathbb{T}^n)}$$

holds by a combination of Cauchy–Schwarz and Plancherel's theorem. Thus, given $2 \leq p \leq \infty$, it follows that

$$\|\chi_{A_\theta(\lambda,r)}(D)f\|_{L^p(\mathbb{T}^n)} \leq [\#\mathbb{Z}^n \cap A_\theta(\lambda,r)]^{1/2-1/p}\|\chi_{A_\theta(\lambda,r)}(D)f\|_{L^2(\mathbb{T}^n)}. \tag{3.12}$$

Applying the bound $\#\mathbb{Z}^n \cap A_\theta(\lambda,r) \lesssim (r\lambda)^{\frac{n-1}{2}}$, taking ℓ^2-norms in θ of both sides of the above inequality and using Plancherel's theorem to sum, the desired estimate follows.

It remains to give a rigorous justification of the uncertainty principle heuristic used in the above argument. Given $k \in \mathbb{Z}^n \cap A(\lambda,r)$ let ω_k denote the point on λS^{n-1} closest to k, so that $|\omega_k - k| < r$, and Ω_λ denote the collection of all such ω_k. Suppose $\bar{x} \in \mathbb{R}^n$ is the centre of B_R. Applying the Taylor series expansion for the exponential,

$$\begin{aligned}\chi_{A(\lambda,r)}(D)f(x) &= \sum_{\alpha\in\mathbb{N}_0^n}\frac{(2\pi i)^{|\alpha|}(x-\bar{x})^\alpha}{\alpha!}\sum_{\omega_k\in\Omega_\lambda}(k-\omega_k)^\alpha\hat{f}(k)e^{2\pi i\bar{x}\cdot(k-\omega_k)}e^{2\pi i x\cdot\omega_k}\\ &=: \sum_{\alpha\in\mathbb{N}_0^n}\frac{(2\pi i)^{|\alpha|}(x-\bar{x})^\alpha}{\alpha!}\sum_{\omega\in\Omega_\lambda}a_{\alpha,\omega}e^{2\pi i x\cdot\omega},\end{aligned}$$

where $|\alpha| = \alpha_1+\cdots+\alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$ and $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for $\alpha\in\mathbb{N}_0^n$ and $x\in\mathbb{R}^n$. Thus, by the triangle inequality and (3.10), the left-hand side of (3.11) is dominated by

$$\lambda^\varepsilon\sum_{\alpha\in\mathbb{N}_0^n}\frac{(2\pi R)^{|\alpha|}}{\alpha!}\Big(\sum_{\theta\in\Theta(\lambda,r)}\Big\|\sum_{\omega\in\Omega_\lambda\cap\theta}a_{\alpha,\omega}e^{2\pi i x\cdot\omega}\Big\|^2_{L^{\frac{2(n+1)}{n-1}}(w_{B_R})}\Big)^{\frac{1}{2}}.$$

Given $l\in\mathbb{Z}^n$ write $\bar{x}_l := Rl$ and $B^l := B(\bar{x}_l, \sqrt{n}R)$ so that

$$\Big\|\sum_{\omega\in\Omega_\lambda\cap\theta}a_{\alpha,\omega}e^{2\pi i x\cdot\omega}\Big\|_{L^{\frac{2(n+1)}{n-1}}(w_{B_R})} \lesssim \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-N}\Big\|\sum_{\omega\in\Omega_\lambda\cap\theta}a_{\alpha,\omega}e^{2\pi i x\cdot\omega}\Big\|_{L^{\frac{2(n+1)}{n-1}}(B^l)},$$

where $N := 100n$ is the exponent appearing in the definition of the weight function from (3.6). Indeed, this follows by pointwise dominating w_{B_R} by a weighted sum of characteristic functions thus:

$$w_{B_R}(x) \lesssim \sum_{l\in\mathbb{Z}^n}(1+|l|)^{-6N}\chi_{B^l}(x).$$

As before, one may write

$$\begin{aligned}\sum_{\omega\in\Omega_\lambda\cap\theta}a_{\alpha,\omega}e^{2\pi i x\cdot\omega} &= \sum_{\beta\in\mathbb{N}_0^n}\frac{(2\pi i)^{|\beta|}(x-\bar{x}_l)^\beta}{\beta!}\sum_{\omega_k\in\Omega_\lambda\cap\theta}(\omega_k-k)^\beta a_{\alpha,\omega_k}e^{2\pi i\bar{x}_l\cdot(\omega_k-k)}e^{2\pi i x\cdot k}\\ &= \sum_{\beta\in\mathbb{N}_0^n}\frac{(2\pi i)^{|\beta|}(x-\bar{x}_l)^\beta}{\beta!}\chi_{A_\theta(\lambda,r)}(D)m_{\alpha,\beta,l}(D)f(x)\end{aligned}$$

where $m_{\alpha,\beta,l}$ is supported on $\mathbb{Z}^n\cap A(\lambda,r)$ and is given by

$$m_{\alpha,\beta,l}(k) := (-1)^{|\beta|}(k-\omega_k)^{\alpha+\beta}e^{2\pi i(\bar{x}-\bar{x}_l)\cdot(k-\omega_k)} \quad \text{for } k\in\mathbb{Z}^n\cap A(\lambda,r).$$

In particular,

$$\max_{k\in\mathbb{Z}^n\cap A(\lambda,r)}|m_{\alpha,\beta,l}(k)| \lesssim r^{|\alpha|+|\beta|}.$$

By combining the above observations, applying the triangle inequality and exploiting periodicity, one concludes that $\|\chi_{A(\lambda,r)}(D)f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)}$ is dominated by

$$\lambda^\varepsilon\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ l\in\mathbb{Z}^n}}\frac{(2\pi R)^{|\alpha|+|\beta|}}{\alpha!\beta!}(1+|l|)^{-N}\Big(\sum_{\theta\in\Theta(\lambda,r)}\|\chi_{A_\theta(\lambda,r)}(D)m_{\alpha,\beta,l}(D)f\|^2_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)}\Big)^{1/2}.$$

Finally, a slight modification of the argument used to prove (3.12) shows that, given $2 \le p \le \infty$,

$$\|\chi_{A_\theta(\lambda,r)}(D)m_{\alpha,\beta,l}(D)f\|_{L^p(\mathbb{T}^n)} \lesssim r^{|\alpha|+|\beta|}[\#A_\theta(\lambda,r)]^{1/2-1/p}\|\chi_{A_\theta(\lambda,r)}(D)f\|_{L^2(\mathbb{T}^n)}.$$

The gain in r in the previous inequality compensates for the earlier losses in R and the desired estimate (3.8) now readily follows from Plancherel's theorem. □

It is remarked that discretization arguments similar to those above have frequently appeared elsewhere in the literature: see, for instance, [TV00].

Corollary 11. *Let $n \ge 3$, $\lambda \ge 1$ and $1 \gtrsim r > \lambda^{-1}$ and suppose $m \in \ell^\infty(\mathbb{Z}^n)$ is supported in $A(\lambda, r)$. For all $\varepsilon > 0$,*

$$\|m(D)f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim_\varepsilon \lambda^\varepsilon (r\lambda)^{\frac{n-1}{n+1}}\|m\|_{\ell^\infty(\mathbb{Z}^n)}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}.$$

Proof. The corollary follows easily by writing

$$m = \chi_{A(\lambda,r)} \cdot m \cdot \chi_{A(\lambda,r)}$$

and successively applying (3.8), Plancherel's theorem and (3.9). □

Consequences of the Stein–Tomas theorem. An equivalent formulation of the Stein–Tomas restriction theorem for the sphere is that

$$\Big(\int_{A(\lambda,1)} |\hat{F}(\xi)|^2\,\mathrm{d}\xi\Big)^{1/2} \lesssim \lambda^{\frac{n-1}{2(n+1)}}\|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}; \tag{3.13}$$

see, for instance, [Tao04] or [Sog17, Chapter 5]. This implies a version of Corollary 9 for $r = 1$ with no ε-loss in the exponent.

Corollary 12. *Let $n \ge 3$ and $\lambda \gtrsim 1$. Then*

$$\big\|\chi_{A(\lambda,1)}(D)f\big\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-1}{n+1}}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}.$$

Remark 13. Corollary 12 is also a special case of a more general spectral projection bound for compact Riemann manifolds: see [Sog88] or [Sog17, Chapter 5].

Proof (of Corollary 12). As before, by T^*T the desired estimate is equivalent to

$$\big\|\chi_{A(\lambda,1)}(D)f\big\|_{L^2(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-1}{2(n+1)}}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}. \tag{3.14}$$

Fix $f \in C^\infty(\mathbb{T}^n)$ and let $\psi \in \mathscr{S}(\mathbb{R}^n)$ be nonzero and Fourier supported in a ball of radius $1/2$. Letting $F \in \mathscr{S}(\mathbb{R}^n)$ be defined by

$$F(x) := \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{2\pi i x\cdot k}\psi(x),$$

the estimate (3.14) now follows by applying (3.13) to this function. □

Arguing precisely as in the previous subsection, Corollary 12 implies a version of Corollary 11 for $r = 1$ with no ε-loss.

Corollary 14. *Let $n \ge 3$ and $\lambda \ge 1$ and suppose $m \in \ell^\infty(\mathbb{Z}^n)$ is supported in $A(\lambda, 1)$. Then*

$$\|m(D)f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{T}^n)} \lesssim \lambda^{\frac{n-1}{n+1}}\|m\|_{\ell^\infty(\mathbb{Z}^n)}\|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{T}^n)}.$$

Remark 15. Corollary 14 is also a special instance of the multiplier lemma from [BSSY15, Lemma 2.3], which applies to more general compact Riemannian manifolds.

Proof of the spectral projection bound. The ingredients introduced above may now be combined to prove the desired spectral projection bound.

Proof (of Proposition 7). Fixing $\varepsilon > 0$, recall that it suffices to show (3.5) holds for $\rho = \lambda^{-1/3+\varepsilon}$. In order to justify this choice of ρ, and in view of the proof of Theorem 3 below, it will be convenient to initially let ρ denote some unspecified parameter satisfying $1 \gtrsim \rho \geq \lambda^{-1}$ and only fix the value later in the argument.

Fix a Schwartz function η on $\hat{\mathbb{R}}^n$ satisfying $\check{\eta}(x) = 1$ whenever $|x| \leq 1$. Recalling the definition of the smoothed out multiplier $m^{\lambda,\rho}$ from (3.4), decompose

$$m^{\lambda,\rho} = m_0^{\lambda,\rho} + m_1^{\lambda,\rho}$$

where $m_0^{\lambda,\rho} := m^{\lambda,\rho} * \eta$. Writing $p_1 := \frac{2n}{n-2}$, it follows that

$$\|m^{\lambda,\rho}(D)\|_{p_1'\to p_1} \leq \|m_0^{\lambda,\rho}(D)\|_{p_1'\to p_1} + \|m_1^{\lambda,\rho}(D)\|_{p_1'\to p_1} \tag{3.15}$$

where p' denotes the Hölder conjugate of a Lebesgue exponent p.

Both terms on the right-hand side of (3.15) are estimated via complex interpolation between an $L^{p_0'} \to L^{p_0}$ bound for $p_0 := \frac{2(n+1)}{n-1}$ and an $L^1 \to L^\infty$ bound. In particular, by the Riesz–Thorin theorem,

$$\|m_j^{\lambda,\rho}(D)\|_{p_1'\to p_1} \leq \|m_j^{\lambda,\rho}(D)\|_{p_0'\to p_0}^{\frac{(n-2)(n+1)}{n(n-1)}} \|m_j^{\lambda,\rho}(D)\|_{1\to\infty}^{\frac{2}{n(n-1)}} \quad \text{for } j = 0,1. \tag{3.16}$$

To bound $m_0^{\lambda,\rho}(D)$, apply a partition of unity to decompose

$$\eta = \sum_{\ell\in\mathbb{Z}^n} (1+|\ell|)^{-N}\tilde{\eta}_\ell$$

where $N := 100n$ and each $\tilde{\eta}_\ell$ is supported on the ball of radius $\sqrt{n}$, say, centred at ℓ and satisfies $\|\tilde{\eta}_\ell\|_\infty \lesssim 1$. Note that the latter property holds due to the rapid decay of η. This induces a corresponding decomposition of the multiplier

$$m_0^{\lambda,\rho} = \sum_{\ell\in\mathbb{Z}^n} (1+|\ell|)^{-N}\tilde{m}_\ell^{\lambda,\rho} \tag{3.17}$$

where each $\tilde{m}_\ell^{\lambda,\rho}$ is supported on the Minkowski sum

$$\operatorname{supp} m^{\lambda,\rho} + \operatorname{supp}\tilde{\eta}_\ell \subseteq \ell + A(\lambda, 4\sqrt{n}).$$

Furthermore,

$$\|\tilde{m}_\ell^{\lambda,\rho}\|_{\ell^\infty(\mathbb{Z}^n)} \lesssim \rho \quad \text{and} \quad \|\tilde{m}_\ell^{\lambda,\rho}\|_{\ell^1(\mathbb{Z}^n)} \lesssim \rho\lambda^{n-1}. \tag{3.18}$$

To see this, observe that $|\tilde{m}_\ell^{\lambda,\rho}(\xi)| \lesssim |B(\ell+\xi,1)\cap A(\lambda,\rho)|$, which immediately yields the ℓ^∞ estimate. The ℓ^1 bound then follows from the ℓ^∞ estimate and the fact that $\#\big(\mathbb{Z}^n \cap \ell + A(\lambda, 4\sqrt{n})\big) \lesssim \lambda^{n-1}$. Consequently, and in view of Corollary 14,

$$\|\tilde{m}_\ell^{\lambda,\rho}(D)\|_{p_0'\to p_0} \lesssim \rho\lambda^{\frac{n-1}{n+1}} \quad \text{and} \quad \|\tilde{m}_\ell^{\lambda,\rho}(D)\|_{1\to\infty} \lesssim \rho\lambda^{n-1}. \tag{3.19}$$

More precisely, the first inequality in (3.19) follows from Corollary 14 together with the ℓ^∞ estimate from (3.18). Here it is important to use Corollary 14 rather than Corollary 11 to ensure that there is no ε-loss in the exponent: indeed, otherwise one obtains (3.1) with a $\rho\lambda^{1+\varepsilon}$ factor (as opposed to a $\rho\lambda$ factor) on the right-hand side, which is unsuitable for the desired application of Theorem 4. The second inequality in (3.19) is a direct consequence of the ℓ^1 estimate in (3.18) (which allows one to bound the ℓ^∞ norm of the kernel associated to $m_\ell^{\lambda,\rho}(D)$).

Using the triangle inequality and the decay factor in (3.17) to sum the above estimates,

$$\|m_0^{\lambda,\rho}(D)\|_{p_0'\to p_0} \lesssim \rho\lambda^{\frac{n-1}{n+1}} \quad \text{and} \quad \|m_0^{\lambda,\rho}(D)\|_{1\to\infty} \lesssim \rho\lambda^{n-1}. \tag{3.20}$$

Interpolating the two inequalities in (3.20) via (3.16), one deduces that

$$\|m_0^{\lambda,\rho}(D)\|_{p_1'\to p_1} \lesssim \rho\lambda. \tag{3.21}$$

It remains to bound $m_1^{\lambda,\rho}(D)$. Since the multiplier $m^{\lambda,\rho}$ is supported in $A(\lambda,2\rho)$ and is uniformly bounded, it follows from Corollary 11 that

$$\|m_1^{\lambda,\rho}(D)\|_{p_0'\to p_0} \le \|m_0^{\lambda,\rho}(D)\|_{p_0'\to p_0} + \|m^{\lambda,\rho}(D)\|_{p_0'\to p_0} \lesssim_\varepsilon \lambda^\varepsilon(\rho\lambda)^{\frac{n-1}{n+1}}, \tag{3.22}$$

where the first term on the right-hand side is estimated using (3.20). On the other hand, it is claimed that

$$\|m_1^{\lambda,\rho}(D)\|_{1\to\infty} \lesssim (\lambda/\rho)^{(n-1)/2}. \tag{3.23}$$

Temporarily assuming this bound, interpolating (3.23) against (3.22) via (3.16) yields

$$\|m_1^{\lambda,\rho}(D)\|_{p_1'\to p_1} \lesssim_\varepsilon \lambda^\varepsilon\rho^{1-3/n}\lambda^{1-1/n}. \tag{3.24}$$

Substituting (3.21) and (3.24) into (3.15), one concludes that

$$\|m^{\lambda,\rho}(D)\|_{p_1'\to p_1} \lesssim_\varepsilon \rho\lambda + \lambda^\varepsilon\rho^{1-3/n}\lambda^{1-1/n}. \tag{3.25}$$

Replacing ε with $3\varepsilon/n$ in the above display and choosing $\rho = \lambda^{-1/3+\varepsilon}$ so as to optimize the estimate, one deduces the desired bound. Thus, it remains to verify (3.23).

Computing the kernel of $m_1^{\lambda,\rho}(D)$ and applying the Poisson summation formula,

$$\|m_1^{\lambda,\rho}(D)\|_{1\to\infty} \le \sup_{x\in\mathbb{T}^n}\Big|\sum_{k\in\mathbb{Z}^n} m_1^{\lambda,\rho}(k)e^{2\pi i x\cdot k}\Big| = \sup_{x\in\mathbb{T}^n}\Big|\sum_{k\in\mathbb{Z}^n}\big(m_1^{\lambda,\rho}\big)\check{}\,(x+k)\Big|. \tag{3.26}$$

Note that $\big(m_1^{\lambda,\rho}\big)\check{}\,(x) = \big(m^{\lambda,\rho}\big)\check{}\,(x)\big(1-\check{\eta}(x)\big)$. If σ denotes the surface measure on S^{n-1}, then applying polar coordinates to the definition of the Fourier transform yields

$$\big(m^{\lambda,\rho}\big)\check{}\,(x) = \int_0^\infty \check{\sigma}(rx)\beta\big(\rho^{-1}(r-\lambda)\big)r^{n-1}\,\mathrm{d}r. \tag{3.27}$$

By stationary phase (see, for instance, [Ste93, Chapter VIII] or [Sog17, Chapter 1])

$$\check{\sigma}(x) = \sum_\pm e^{\pm 2\pi i|x|}a_\pm(x),$$

where each $a_\pm \in C^\infty(\mathbb{R}^n)$ is a symbol of order $-(n-1)/2$ in the sense that $|\partial_x^\alpha a_\pm(x)| \lesssim_\alpha (1+|x|)^{-(n-1)/2-|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$. Substituting this identity into (3.27) and applying a change of variables,

$$\big(m^{\lambda,\rho}\big)\check{}\,(x) = \rho\sum_\pm\int_0^\infty e^{\pm 2\pi i r\rho|x|}a_\pm(\rho r x)\beta\big(r-\rho^{-1}\lambda\big)(\rho r)^{n-1}\,\mathrm{d}r. \tag{3.28}$$

Applying repeated integration by parts, it follows that

$$|\big(m^{\lambda,\rho}\big)\check{}\,(x)| \lesssim \rho\lambda^{n-1}(1+\lambda|x|)^{-(n-1)/2}(1+\rho|x|)^{-N} \lesssim \frac{\rho\lambda^{(n-1)/2}}{|x|^{(n-1)/2}}(1+\rho|x|)^{-N}. \tag{3.29}$$

To bound the right-hand side of (3.26) the sum is broken into two pieces. Fix $x \in \mathbb{T}^n$ and write

$$\Big|\sum_{k\in\mathbb{Z}^n}\big(m_1^{\lambda,\rho}\big)\check{}\,(x+k)\Big| \lesssim \big|\big(m_1^{\lambda,\rho}\big)\check{}\,(x)\big| + \Big|\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\big(m_1^{\lambda,\rho}\big)\check{}\,(x+k)\Big|$$

Since $1-\check{\eta}$ vanishes to infinite order at the origin, (3.29) implies that

$$\big|\big(m_1^{\lambda,\rho}\big)\check{}\,(x)\big| = \big|\big(m^{\lambda,\rho}\big)\check{}\,(x)\big(1-\check{\eta}(x)\big)\big| \lesssim \rho\lambda^{(n-1)/2}.$$

The remaining term satisfies the following, more restrictive, bound.

Lemma 16.
$$\Big|\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \big(m_1^{\lambda,\rho}\big)^{\vee}(x+k)\Big| \lesssim (\lambda/\rho)^{(n-1)/2}.$$

Proof. Since $|\big(m_1^{\lambda,\rho}\big)^{\vee}(x)| \lesssim |\big(m^{\lambda,\rho}\big)^{\vee}(x)|$, applying (3.29) yields
$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} |\big(m_1^{\lambda,\rho}\big)^{\vee}(x+k)| \lesssim \rho\lambda^{(n-1)/2} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{-(n-1)/2}(1+\rho|k|)^{-N} \lesssim (\lambda/\rho)^{(n-1)/2}.$$
□

Combining these observations, (3.23) immediately follows, concluding the proof of Proposition 7.

4. Improvements via multidimensional Weyl sum estimates

By Theorem 4 and the reductions in §2, the uniform resolvent estimates in Theorem 3 are equivalent to the following multiplier bound.

Proposition 17. *Let $n \geq 3$, $\lambda \geq 1$ and $\varepsilon > 0$. If $\rho := \lambda^{-\beta_n+\varepsilon}$, then*
$$\big\| m^{\lambda,\rho}(D) f\big\|_{L^{\frac{2n}{n-2}}(\mathbb{T}^n)} \lesssim_\varepsilon \rho\lambda \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{T}^n)}.$$

Proposition 17 follows by combining the argument from §3 with a more delicate estimation of the kernel. The use of the triangle inequality in the first step of the proof of Lemma 16 introduces losses and the idea is to exploit cancellation between the terms of the sum. This is analogous to the refinements of Hlawka's argument found in [KN92, Mül99, Guo12]. In particular, the exponential sum estimates from [Mül99] imply the following strengthened version of Lemma 16.

Lemma 18. *Let $\lambda \geq 1$ and $1 \gtrsim \rho \geq \lambda^{-1}$. For all $q \in \mathbb{N}$ satisfying*
$$\lambda \geq \rho^{-(q-1-2/n+2^{1-q})}, \tag{4.1}$$
the kernel estimate
$$\Big|\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \big(m_1^{\lambda,\rho}\big)^{\vee}(x+k)\Big| \lesssim_{\varepsilon,q} \lambda^{\varepsilon}(\rho^{q+1}\lambda)^{\omega_{n,q}}(\lambda/\rho)^{(n-1)/2}$$
holds for
$$\omega_{n,q} := \frac{n}{2n(2^q-1)+2^{q+1}}.$$

Provided ρ and q are chosen so that ρ^{q+1} is much smaller than λ^{-1}, this provides an improvement over the crude estimate from Lemma 16.

Assuming Lemma 18, it is not difficult to adapt the argument of the previous section to prove the desired spectral projection bounds.

Proof (of Proposition 17). Let $q \in \mathbb{N}$ satisfy the hypotheses of Lemma 18. Arguing as before, Lemma 18 implies that
$$\begin{aligned}\|m_1^{\lambda,\rho}(D)\|_{1\to\infty} &\lesssim_{\varepsilon,q} \rho\lambda^{(n-1)/2} + \lambda^{\varepsilon}(\rho^{q+1}\lambda)^{\omega_{n,q}}(\lambda/\rho)^{(n-1)/2} \\ &\lesssim \lambda^{\varepsilon}(\rho^{q+1}\lambda)^{\omega_{n,q}}(\lambda/\rho)^{(n-1)/2}.\end{aligned}$$

This refined estimate can be used in place of (3.23) in the proof of Proposition 7. In particular, one deduces that
$$\|m^{\lambda,\rho}(D)\|_{p_1'\to p_1} \lesssim_\varepsilon \rho\lambda + \lambda^{\varepsilon}(\rho^{q+1}\lambda)^{2\omega_{n,q}/n(n-1)}\rho^{1-3/n}\lambda^{1-1/n}$$

which provides an improved version of (3.25). In this case, one is led to the choice $\rho = \lambda^{-\beta_{n,q}+\varepsilon}$, where

$$\beta_{n,q} := \frac{1}{3} + \frac{n}{3}\cdot\frac{q-2}{3(n^2-1)2^q - qn - (3n-2)n}.$$

To optimize the estimate, q should be chosen so as to make the exponent as large as possible. Note that $\beta_{n,q} > 1/3$ whenever $q \geq 3$. Fixing n, a simple calculus exercise show that $\beta_{n,q}$ is a decreasing function for $q \geq 4$. Direct comparison between $\beta_{n,3}$ and $\beta_{n,4}$ then shows that $q = 3$ is always the optimal choice of parameter, if no additional constraint is imposed in the form of (4.1). However, it is not difficult to show that $\rho := \lambda^{-\beta_{n,3}+\varepsilon}$ automatically satisfies (4.1), provided ε is sufficiently small. Since $\beta_n = \beta_{n,3}$, Proposition 17 follows. □

It remains to prove Lemma 18. The argument uses two ingredients from [Mül99], the first of which is an elementary exponential sum bound.

Theorem 19 (Müller [Mül99])**.** *Let $n, q \in \mathbb{N}$, $n \geq 2$, and $\lambda, M \geq 1$ satisfy*

$$\lambda \geq M^{q-1-2/n+2^{1-q}}. \tag{4.2}$$

Suppose that $w \in C^\infty(\mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$ is real-valued and that these functions satisfy the following conditions:

- *i)* $\operatorname{supp} w$ *is contained in $B(0,M)$;*
- *ii)* $|\partial_u^\alpha w(u)| \lesssim_\alpha M^{-|\alpha|}$ *and* $|\partial_u^\alpha \phi(u)| \lesssim_\alpha \lambda M^{1-|\alpha|}$ *for all $u \in \operatorname{supp} w$, $\alpha \in \mathbb{N}_0^n$;*
- *iii) There exists some $\alpha(q) \in \mathbb{N}_0^n$ with $|\alpha(q)| = q$ such that*

$$|\mathrm{Hess}\,\partial_u^{\alpha(q)}\phi(u)| \gtrsim (\lambda M^{-(q+1)})^n \qquad \textit{for all } u \in \operatorname{supp} w.$$

Then there is a weighted exponential sum estimate

$$\Big|\sum_{k\in\mathbb{Z}^n} e^{2\pi i\phi(k)} w(k)\Big| \lesssim_\varepsilon \lambda^\varepsilon M^n (M^{-(q+1)}\lambda)^{\omega_{n,q}}.$$

Here Hess is used to denote the Hessian determinant and, as before, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

For the phases and weights arising in the proof of Lemma 18, it is straightforward to verify conditions i) and ii) of Theorem 19. Condition iii), however, only holds locally and after applying a linear coordinate transformation. The existence of such a coordinate transformation is the second ingredient from [Mül99].

Lemma 20 (Müller [Mül99])**.** *For $n, q \in \mathbb{N}$, $n \geq 2$ there exist open regions $S_\ell \subset \mathbb{R}^n \setminus \{0\}$ and integer matrices $Q_\ell \in \mathrm{GL}(n,\mathbb{R})$ for $1 \leq \ell \leq L = L(n,q) \in \mathbb{N}$ with the following properties:*

- *i)* $\mathbb{R}^n \setminus \{0\} \subseteq \bigcup_{\ell=1}^L S_\ell$ *and if $x \in S_\ell$ and $\lambda > 0$, then $\lambda x \in S_\ell$;*
- *ii) The function $\Phi_\ell\colon \mathbb{R}^n \to \mathbb{R}$ given by $\Phi_\ell(u) := |Q_\ell u|$ satisfies*

$$\Big|\mathrm{Hess}\,\frac{\partial^q \Phi_\ell}{\partial u_1 \partial u_n^{q-1}}(u)\Big| \gtrsim |u|^{-(q+1)n} \qquad \textit{for all } u \in Q_\ell^{-1} S_\ell.$$

This follows from [Mül99, Lemma 3]. In particular, it suffices to find an open covering of the unit sphere (rather than the whole of $\mathbb{R}^n \setminus \{0\}$) satisfying property ii), since the full result then follows by homogeneity. The desired cover can then be obtained by combining [Mül99, Lemma 3] with a compactness argument.

Proof (of Lemma 18). The proof is similar to that of Theorem 1 in [Mül99].

By (3.28), one may write

$$\big(m_1^{\lambda,\rho}\big)^{\vee}(x) = \rho \sum_{\pm} e^{\pm 2\pi i\lambda|x|} I_\pm^{\lambda,\rho}(x)$$

where

$$I_{\pm}^{\lambda,\rho}(x) := \int_0^\infty e^{\pm 2\pi i (r-\rho^{-1}\lambda)\rho|x|} a_{\pm}(\rho r x)\beta\big(r-\rho^{-1}\lambda\big)(\rho r)^{n-1}\,\mathrm{d}r\cdot(1-\check{\eta}(x)).$$

Applying integration-by-parts as in (3.29), it follows that

$$|\partial_x^\alpha I_{\pm}^{\lambda,\rho}(x)| \lesssim_\alpha \frac{\lambda^{(n-1)/2}}{|x|^{(n-1)/2+|\alpha|}}(1+\rho|x|)^{-N} \quad \text{for all } \alpha\in\mathbb{N}_0^n \tag{4.3}$$

where $N := \lceil 100 n\varepsilon^{-1}\rceil$. Note that this is a substantially larger (but still admissible) choice of N than that used in the previous arguments. With this choice, it follows, for instance, that $(1+\rho|x|)^{-N}\lesssim_\varepsilon \rho^{100n}$ whenever $|x|>\rho^{-1-\varepsilon}$.

Since the functions $I_{\pm}^{\lambda,\rho}$ decay rapidly when $|x|\geq\rho^{-1}$, it suffices to show that

$$\sup_{x\in[-1/2,1/2]^n}\Big|\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x+k|\leq\rho^{-1-\varepsilon}}} e^{2\pi i\lambda|x+k|}I_{\pm}^{\lambda,\rho}(x+k)\Big| \lesssim_\varepsilon \lambda^\varepsilon\rho^{-1}(\rho^{q+1}\lambda)^{\omega_{n,q}}(\lambda/\rho)^{(n-1)/2} \tag{4.4}$$

holds for all $q\in\mathbb{N}$ satisfying (4.1). The support of the weight functions $I_{\pm}^{\lambda,\rho}$ are decomposed dyadically by writing

$$I_{\pm}^{\lambda,\rho} = \sum_{j\in\mathbb{Z}} I_{\pm,j}^{\lambda,\rho} \qquad\text{where}\qquad I_{\pm,j}^{\lambda,\rho}(x) := I_{\pm}^{\lambda,\rho}(x)\zeta(2^{-j}|x|)$$

for a suitable choice of $\zeta\in C_c^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\zeta\subseteq[1/2,2]$. For any fixed value of $x\in[-1/2,1/2]^n$, there are only $O(\log\rho^{-1})$ values of j for which $I_{\pm,j}^{\lambda,\rho}(x+k)$ is nonzero as k varies over all $k\in\mathbb{Z}^n\setminus\{0\}$ satisfying $|x+k|\leq\rho^{-1-\varepsilon}$. Thus, by dyadic pigeonholing, it suffices to show (4.4) holds with $I_{\pm}^{\lambda,\rho}$ replaced with $I_{\pm,j}^{\lambda,\rho}$ for some fixed choice of j satisfying $1\lesssim 2^j\lesssim\rho^{-1-\varepsilon}$.

Fix $q\in\mathbb{N}$ satisfying (4.1) and a choice of sign $\pm$ and let

$$w^{\lambda,j}(u) := \lambda^{-(n-1)/2}2^{j(n-1)/2}I_{\pm,j}^{\lambda,\rho}(u) \qquad\text{and}\qquad \phi^\lambda(u) := \pm\lambda|u|.$$

Given any $x\in\mathbb{R}^n$, define the translates

$$w_x^{\lambda,j}(u) := w^{\lambda,j}(x+u) \qquad\text{and}\qquad \phi_x^\lambda(u) := \phi^\lambda(x+u),$$

and observe that, by (4.3), if $u\in\operatorname{supp} w_x^{\lambda,j}$, then

$$|\partial_u^\alpha w_x^{\lambda,j}(u)|\lesssim_\alpha 2^{-j|\alpha|} \quad\text{and}\quad |\partial_u^\alpha\phi_x^\lambda(u)|\lesssim_\alpha \lambda 2^{j(1-|\alpha|)} \quad \text{for all } \alpha\in\mathbb{N}_0^n. \tag{4.5}$$

Thus, in view of the above reductions, it suffices to show that

$$\sup_{x\in\mathbb{R}^n}\Big|\sum_{k\in\mathbb{Z}^n} e^{2\pi i\phi_x^\lambda(k)}w_x^{\lambda,j}(k)\Big| \lesssim_{\varepsilon,q} \lambda^\varepsilon 2^{jn}(2^{-j(q+1)}\lambda)^{\omega_{n,q}}. \tag{4.6}$$

Note that the reduction in (4.6) relies upon the (readily checked) fact

$$\frac{n+1}{2}-(q+1)\omega_{n,q}>0 \quad \text{for all } n,q\in\mathbb{N} \text{ with } n\geq 2$$

which, in particular, implies that

$$2^{j(n+1)/2}2^{-j(q+1)\omega_{n,q}} \lesssim \rho^{-O(\varepsilon)}\rho^{-1}\rho^{(q+1)\omega_{n,q}}\rho^{-(n-1)/2}.$$

The estimate (4.6) will follow from Theorem 19, although some preparatory steps are needed to ensure the conditions of the theorem hold in this case.

Let $S_\ell\subset\mathbb{R}^n\setminus\{0\}$ and $Q_\ell\in\mathrm{GL}(n,\mathbb{R})$ for $1\leq\ell\leq L$ be open sets and integer matrices, respectively, satisfying the properties i) and ii) from Lemma 20. By forming a homogeneous partition of unity adapted to the $(S_\ell)_{\ell=1}^L$ and pigeonholing, it suffices to show that

$$\sup_{x\in\mathbb{R}^n}\Big|\sum_{k\in\mathbb{Z}^n} e^{2\pi i\phi_x^\lambda(k)}w_x^{\lambda,j}(k)\psi_x(k)\Big| \lesssim \lambda^\varepsilon 2^{jn}(2^{-j(q+1)}\lambda)^{\omega_{n,q}},$$

where $\psi_x(u) := \psi(x+u)$ for $\psi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ real-valued, homogeneous of degree 0 and supported in $S := S_{\ell_0}$ for some $1 \le \ell_0 \le L$.

Let $Q := Q_{\ell_0}$ and note that the lattice $Q\mathbb{Z}^n$ is a finite index subgroup of $\mathbb{Z}^n$. Thus, there exists some $\mathscr{B} \subseteq \mathbb{Z}^n$ with $\#\mathscr{B} \lesssim_q 1$ such that

$$\mathbb{Z}^n = \bigcup_{b \in \mathscr{B}} (b + Q\mathbb{Z}^n),$$

where the union is disjoint. Fix $b \in \mathscr{B}$ and write

$$\tilde{\phi}_x^\lambda(u) := \phi_x^\lambda(b+Qu) \qquad \text{and} \qquad \tilde{w}_x^{\lambda,j}(u) := w_x^{\lambda,j}(b+Qu)\psi_x(b+Qu).$$

Once again by pigeonholing, the desired estimate would follow from

$$\sup_{x \in \mathbb{R}^n} \Big| \sum_{k \in \mathbb{Z}^n} e^{2\pi i \tilde{\phi}_x^\lambda(k)} \tilde{w}_x^{\lambda,j}(k) \Big| \lesssim \lambda^\varepsilon 2^{jn} (2^{-j(q+1)}\lambda)^{\omega_{n,q}}.$$

To conclude the proof, it suffices to show that for any $x \in \mathbb{R}^n$, the functions $\tilde{\phi}_x^\lambda$ and $\tilde{w}_x^{\lambda,j}$ satisfy the hypotheses of Theorem 19 with $M \sim_q 2^j$ and $\alpha(q) := (1, 0, \dots, q-1)$; since q is chosen so as to satisfy (4.1), one may safely assume (4.2) holds for such a choice of M. Clearly the support condition i) holds. By (4.5) and the homogeneity of ψ, it follows that

$$|\partial_u^\alpha \tilde{w}_x^{\lambda,j}(u)| \lesssim_\alpha 2^{-j|\alpha|} \quad \text{and} \quad |\partial_u^\alpha \tilde{\phi}_x^\lambda(u)| \lesssim_\alpha \lambda 2^{j(1-|\alpha|)} \quad \text{for all } \alpha \in \mathbb{N}_0^n,$$

which is condition ii). Finally, Lemma 20 ensures that

$$\big|\mathrm{Hess}\, \partial_u^{\alpha(q)} \tilde{\phi}_x^\lambda(u)\big| \gtrsim (\lambda 2^{-j(q+1)})^n \qquad \text{for all } u \in \operatorname{supp} \tilde{w}_x^{\lambda,j}.$$

Indeed, if $u \in \operatorname{supp} \tilde{w}_x^{\lambda,j}$, then $x + b + Qu \in S$ and so $\tilde{x} + u \in Q^{-1}S$ for $\tilde{x} := Q^{-1}(x+b)$. If $\Phi(u) := |Qu|$, then $\tilde{\phi}_x^\lambda(u) = \pm\lambda\Phi(\tilde{x}+u)$ and so Lemma 20 implies that

$$|\mathrm{Hess}\, \partial_u^{\alpha(q)} \tilde{\phi}_x^\lambda(u)| = \lambda^n |\mathrm{Hess}\, \partial_u^{\alpha(q)} \Phi(\tilde{x}+u)| \gtrsim \lambda^n 2^{-(q+1)n},$$

as required. □

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