Atsuhira NAGANO

Icosahedral invariants and Shimura curves

Icosahedral invariants and Shimura curves

par Atsuhira NAGANO

Résumé. Une courbe de Shimura est un espace de modules de surfaces abéliennes avec multiplication par une algèbre de quaternions. En utilisant les périodes pour une famille des surfaces $K3$ paramétrées par les invariants icosaédriques qui ont été étudiés par Klein, nous obtenons des modèles explicites de certaines courbes de Shimura.

Abstract. Shimura curves are moduli spaces of abelian surfaces with quaternion multiplication. Models of Shimura curves are very important in number theory. Klein's icosahedral invariants $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ give the Hilbert modular forms for $\sqrt{5}$ via the period mapping for a family of $K3$ surfaces. Using the period mappings for several families of $K3$ surfaces, we obtain explicit models of Shimura curves with small discriminant in the weighted projective space $\mathrm{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}])$.

1. Introduction

This paper gives an application of the moduli theory of $K3$ surfaces to the number theory. We obtain an explicit relation among abelian surfaces with quaternion multiplication, Hilbert modular functions and periods of $K3$ surfaces.

The moduli spaces for principally polarized abelian surfaces determined by the structure of the ring of endomorphisms are very important in number theory ([10, Chap. IX, Prop. 1.2], see also Table 1.1). In this paper, we study the moduli spaces of principally polarized abelian surfaces with quaternion multiplication (for the detailed definition, see Section 2.2). They are called Shimura curves.

Abelian surface A	$\mathrm{End}_0(A)$	Moduli Space
Generic	Rational field	Igusa 3-fold $\mathcal{A}_2$
Real multiplication	Real quadratic field	Humbert surface $\mathcal{H}$
Quaternion multiplication	Quaternion algebra	Shimura curve $\mathcal{S}$
Complex multiplication	CM field	CM points

TABLE 1.1. The moduli spaces of abelian surfaces

To the best of the author's knowledge, to obtain explicit models of Shimura curves is a non trivial problem because Shimura curves have no cusps. In this paper, we shall obtain new models of Shimura curves for quaternion algebras with small discriminant. We consider the weighted projective space $\mathbb{P}(1:3:5) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C}])$, where $\mathfrak{A},\mathfrak{B}$ and $\mathfrak{C}$ are Klein's icosahedral invariants of weight $1, 3$ and 5 respectively. We shall give the explicit defining equations of the Shimura curves for small discriminant in $\mathrm{Proj}(\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C}])$.

Here, let us see the reason why we consider the icosahedral invariants. The moduli space $\mathcal{H}_\Delta$ of principally polarized abelian surfaces with real multiplication by $\mathcal{O}_\Delta$ is called the Humbert surface (for detail, see Section 2.1). The Humbert surface $\mathcal{H}_\Delta$ is uniformized by Hilbert modular functions for Δ. Among Humbert surfaces, the case for $\mathbb{Q}(\sqrt{5})$ is the simplest, since its discriminant is the smallest. In [22], we studied the family $\mathcal{F} = \{S(\mathfrak{A}:\mathfrak{B}:\mathfrak{C})\}$ of elliptic $K3$ surfaces. We can regard $\mathcal{F}$ as a family parametrized over $\mathcal{H}_5$.

By the way, the Igusa 3-fold $\mathcal{A}_2$ is the moduli space of principally polarized abelian surfaces. The family $\mathcal{F}_{CD} = \{S_{CD}(\alpha:\beta:\gamma:\delta)\}$ of $K3$ surfaces for $(\alpha:\beta:\gamma:\delta) \in \mathbb{P}(2:3:5:6)$ is studied by Kumar [18], Clingher and Doran [4] and [25]. This family $\mathcal{F}_{CD}$ is parametrized over $\mathcal{A}_2$. Our family $\mathcal{F}$ can be regarded as a subfamily of $\mathcal{F}_{CD}$. However, it is not apparent to describe the embedding $\mathcal{F} \hookrightarrow \mathcal{F}_{CD}$ explicitly. Our first result of this paper is to obtain the embedding $\Psi_5 : \mathbb{P}(1:3:5) \hookrightarrow \mathbb{P}(2:3:5:6)$ of the parameter spaces (see Theorem 3.16).

A Shimura curve $\mathcal{S}$ is a 1-dimensional subvariety of $\mathcal{A}_2$. In several cases, $\mathcal{S}$ is contained in the image $\Psi_5(\mathbb{P}(1:3:5))$ and the pull-back $\Psi_5^*(\mathcal{S})$ is a curve in $\mathbb{P}(1:3:5)$. In this paper, we obtain models of $\Psi_5^*(\mathcal{S})$ for such cases. We note that $\mathcal{S}$ and $\Psi_5^*(\mathcal{S})$ are isomorphic as varieties. Good modular properties of $\mathrm{Proj}(\mathbb{C}[\mathfrak{A},\mathfrak{B},\mathfrak{C}])$ enable us to study $\Psi_5^*(\mathcal{S})$ effectively. Also, our study is based on the results of quaternion algebras due to Hashimoto [12] and elliptic $K3$ surfaces due to Elkies and Kumar [9].

As a result, we obtain the following explicit defining equations for the Shimura curves $\Psi_5^*(\mathcal{S})$ for discriminant $6, 10, 14$ and 15. Setting $X = \dfrac{\mathfrak{B}}{\mathfrak{A}^3}$,

$Y = \dfrac{\mathfrak{C}}{\mathfrak{A}^5}$, (X, Y) are affine coordinates of $\mathcal{H}_5$. We have the models (see Theorem 5.1, 5.5, 6.1 and 6.2):

$$
\begin{cases}
\Psi_5^*(\mathcal{S}_6) : 3125 - 3375X + 243Y = 0, \\
\Psi_5^*(\mathcal{S}_{10}) : 1 - 5X + Y = 0, \\
\Psi_5^*(\mathcal{S}_{14}) : 30517578125 + 911865234375X + 42529296875X^2 \\
\qquad\qquad - 97897974609375X^3 + 424490000000000X^4 \\
\qquad\qquad - 345600000000000X^5 + 2383486328125Y \\
\qquad\qquad + 32875975781250XY - 147816767984375X^2Y \\
\qquad\qquad + 228155760000000X^3Y + 19189204671875Y^2 \\
\qquad\qquad - 29675018141125XY^2 + 344730881243Y^3 = 0, \\
\Psi_5^*(\mathcal{S}_{15}) : 30517578125 + 911865234375X + 42529296875X^2 \\
\qquad\qquad - 97897974609375X^3 + 424490000000000X^4 \\
\qquad\qquad - 345600000000000X^5 + 2383486328125Y = 0.
\end{cases}
$$

Some researchers obtained models of Shimura curves (for example, Kurihara [19], Hashimoto and Murabayashi [13], Besser [2], Elkies [7], [8], Kohel and Verrill [17], Voight [33], Bonfanti and van Geemen [3]). In comparison with already known models, our new models have the following features.

- They are closely related to the classical invariant theory. Namely, our coordinates $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ of the common ambient space of Shimura curves are coming from Klein's icosahedral invariants. Especially, the Shimura curves for discriminant 6 and 10 have very simple forms. These two curves are just lines touching the locus of Klein's icosahedral equation (see Figure 5.1).
- The moduli of our family $\mathcal{F}$ of $K3$ surfaces were studied in detail. We have an explicit expression of the period mapping ([22],[23]) and the Gauss–Manin connection ([21]) for $\mathcal{F}$. These properties are very useful to study Shimura curves effectively (for example, see the proof of Theorem 5.5).
- Shimura ([31]) studied unramified class fields over CM fields of certain types. In [24], an explicit construction of such class fields over quartic CM fields using the special values of X and Y is given.
- In fact, our $K3$ surface $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ is a toric hypersurface. To study the mirror symmetry for toric $K3$ hypersurfaces is an interesting problem in recent geometry and physics. In [11], our $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ are studied from the viewpoint of mirror symmetry. Especially, our parameters X and Y are directly related to the secondary stack for the toric $K3$ hypersurface.

Thus, our new models of Shimura curves are naturally related to various topics.

2. Moduli of principally polarized abelian surfaces

2.1. Principally polarized abelian surfaces with real multiplication. Let $\mathfrak{S}_2$ be the Siegel upper half plane of rank 2. Let us consider a principally polarized abelian variety (A, Θ) with the theta divisor Θ and the period matrix (Ω, I_2), where $\Omega \in \mathfrak{S}_2$. The symplectic group $Sp(4, \mathbb{Z})$ acts on $\mathfrak{S}_2$. The quotient space $Sp(4, \mathbb{Z}) \backslash \mathfrak{S}_2$ gives the moduli space $\mathcal{A}_2$ of principally polarized abelian varieties. This is called the Igusa 3-fold.

The ring of endomorphisms is given by $\mathrm{End}(A) = \{a \in M_2(\mathbb{C}) \,|\, a(\Omega, I_2) = (\Omega, I_2)M$ for some $M \in M(4, \mathbb{Z})\}$. The principal polarization given by Θ induces the alternating Riemann form $E(z, w)$. Set $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. If A is a simple abelian variety, then $\mathrm{End}_0(A)$ is a division algebra. The Rosati involution $a \mapsto a^\circ$ is an involution on $\mathrm{End}_0(A)$ and gives an adjoint of the alternating Riemann form: $E(az, w) = E(z, a^\circ w)$. Note that the Rosati involution satisfies $\mathrm{Tr}(aa^\circ) > 0$.

A point $\Omega = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{S}_2$ is said to have the singular relation with the invariant Δ if there exist relatively prime integers a, b, c, d and e such that the following equations hold:

$$(2.1) \qquad a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0, \quad \Delta = b^2 - 4ac - 4de.$$

Definition 2.1. Set $N_\Delta = \{\tau \in \mathfrak{S}_2 \,|\, \tau$ has a singular relation with $\Delta\}$. The image of N_Δ under the canonical mapping $\mathfrak{S}_2 \to Sp(4, \mathbb{Z}) \backslash \mathfrak{S}_2$ is called the Humbert surface of invariant Δ.

Let $\mathcal{O}_\Delta$ be the ring of integers of the field $\mathbb{Q}(\sqrt{\Delta})$. The Humbert surface of invariant Δ gives the moduli space of principally polarized abelian surfaces (A, Θ) with $\mathcal{O}_\Delta \subset \mathrm{End}(A)$ and $\mathbb{Q}(\sqrt{\Delta}) \cap \mathrm{End}(A) = \mathcal{O}_\Delta$. Such an abelian surface is said to have real multiplication by $\mathcal{O}_\Delta$ (see [10] or [12]).

2.2. Quaternion multiplication and Shimura curves. In this subsection, we recall the properties of Shimura curves. For detail, see [27], [26] or [32].

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ with $B \not\simeq M_2(\mathbb{Q})$. We have an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$. Let $p_1, \cdots, p_t$ be the distinct primes at which B ramifies. We can show that $t \in 2\mathbb{Z}$. The number $D = p_1 \cdots p_t$ is called the discriminant of B. Two quaternion algebras B and B' are isomorphic as $\mathbb{Q}$-algebras if and only if the discriminant of B coincides with that of B'.

For $\alpha \in B$, let $\alpha \mapsto \alpha'$ be the canonical involution defined by $\alpha' = \mathrm{Tr}_{B/\mathbb{Q}}(\alpha) - \alpha$. An element $\alpha \in B$ is called integral if both $\mathrm{Tr}_{B/\mathbb{Q}}(\alpha)$ and $\mathrm{Nr}_{B/\mathbb{Q}}(\alpha)$ are in $\mathbb{Z}$. If a subring $\mathfrak{O}(\subset B)$ of integral elements is a finitely generated $\mathbb{Z}$-module of B and satisfies $\mathbb{Q}\mathfrak{O} = B$, we call $\mathfrak{O}$ an order of B. A maximal order is an order that is maximal under inclusion. We note that a maximal order in B is unique up to conjugation.

For a maximal order $\mathfrak{O}$ in B with discriminant D, we put $\Gamma^{(1)} = \{\gamma \in \mathfrak{O} \mid \mathrm{Nr}_{B/\mathbb{Q}}(\gamma) = 1\}$. The group $\Gamma^{(1)}$ gives a discrete subgroup of $SL(2,\mathbb{R})$. For $\rho \in \mathfrak{O}$ satisfying $\rho^2 < 0$, we have an involution given by $i_\rho : \alpha \mapsto \rho^{-1}\alpha'\rho$. For $\xi, \eta \in B$, put $E_\rho(\xi, \eta) = \mathrm{Tr}(\rho\xi\eta')$. The pairing E_ρ gives a skew symmetric form on B. Moreover, we can show that for $\gamma \in \Gamma^{(1)}$ we have

$$(2.2) \qquad E_\rho(\xi\gamma, \eta\gamma) = E_\rho(\xi, \eta).$$

Definition 2.2. Take $\rho \in \mathfrak{O}$ such that $\rho^2 = -D$ and $\rho\mathfrak{O} = \mathfrak{O}\rho$. We call (A, Θ, ι) a principally polarized abelian surface with quaternion multiplication by $(\mathfrak{O}, i_\rho)$ if $\iota : \mathfrak{O} \hookrightarrow \mathrm{End}(A)$ and $\iota(\alpha) \mapsto \iota \circ i_\rho(\alpha)$ coincides with the Rosati involution $\iota(\alpha) \mapsto \iota(\alpha)^\circ$.

The quotient space $\Gamma^{(1)} \backslash \mathbb{H}$ for B of discriminant D is already a compact Riemann surface. This is called the Shimura curve $\mathbb{S}_D$ for B_D. Shimura proved that $\mathbb{S}_D$ is isomorphic to the moduli space of (A, Θ, ι) with quaternion multiplication by $(\mathfrak{O}, i_\rho)$. Such Shimura curves were firstly studied in [28], [29] and [30]. There exists a quaternion modular embedding (see the following diagram):

$$(2.3) \qquad \Omega : \quad \mathbb{H} \to \mathfrak{S}_2; \quad w \mapsto \Omega(w).$$

$$
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\ \Omega\ } & \mathfrak{S}_2 \\
{\scriptstyle \Gamma^{(1)}} \downarrow & & \downarrow {\scriptstyle Sp(4,\mathbb{Z})} \\
\mathbb{H} & \xrightarrow[\ \Omega\ ]{} & \mathfrak{S}_2
\end{array}
$$

We note that the Shimura curve $\mathbb{S}_D$ does not be embedded in the Igusa 3-fold $\mathcal{A}_2$. There exist a projection $\varphi_D : \mathbb{S}_D \to \mathcal{S}_D \subset \mathcal{A}_2$ and a group $W_D \subset \mathrm{Aut}(\mathbb{S}_D)$ such that $\mathcal{S}_D$ is birationally equivalent to $\mathbb{S}_D/W_D$. We note that φ_D is either generically 2 to 1 or generically 4 to 1. In this paper, we also call the curve $\mathcal{S}_D$ the Shimura curve. The curve $\mathcal{S}_D$ gives isomorphism classes of (A, Θ). The above projection φ_D is coming from the forgetful mapping $(A, \Theta, \iota) \mapsto (A, \Theta)$. For detail, see [27] and [26].

Remark 2.3. The moduli spaces $\mathcal{S}_D$ are also called Shimura curves in several works (for example, see Hashimoto–Murabayashi [13] or Yang [34]).

Remark 2.4. The choice of the mapping $\varphi_D : \mathbb{S}_D \to \mathcal{A}_2$ is not unique and depends on the principal polarizations on the corresponding abelian surfaces. The number of the choice of φ is calculated by Rotger [27]. Especially, if $D = 6, 10, 15$, then the choice of φ_D is unique. Moreover, if $D = 6, 10, 15$, then φ_D is a 4 to 1 mapping ([26]).

2.3. The result of Hashimoto. In this subsection, we review the result of Hashimoto [12].

Letting B be an indefinite quaternion algebra with discriminant $D = p_1 \cdots p_t$ and $\mathfrak{O}$ be a maximal order of B, take a prime number p such that $p \equiv 5 \mod 8$ and $\left(\dfrac{p}{p_j}\right) = -1$ for $p_j \neq 2$. Here, $\left(\dfrac{a}{b}\right)$ denotes the Legendre symbol. We can assume that B is expressed as $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$ with $i^2 = -D, j^2 = p, ij = -ji$. In this subsection, we consider the quaternion multiplication by $(\mathfrak{O}, i_\rho)$ for $\rho = i^{-1} = \dfrac{-i}{D}$.

Remark 2.5. We note that ρ is not always an element of $\mathfrak{O}$. Nevertheless, the notation in Definition 2.2 is available for $\rho = i^{-1}$. In fact, $i \in \mathfrak{O}$ satisfies $i^2 = -D$, $i\mathfrak{O} = \mathfrak{O}i$ and $\iota_i(\alpha) = i^{-1}\alpha'i = i\alpha'i^{-1} = \iota_{i^{-1}}(\alpha)$ $(\alpha \in B)$.

Take $a, b \in \mathbb{Z}$ such that $a^2 D + 1 = pb$. We have a basis $\eta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ of $\mathfrak{O}$ is given by $\eta_1 = \frac{i+ij}{2} - \frac{p-1}{2}\frac{aDj+ij}{p}, \eta_2 = -aD - \frac{aDj+ij}{p}, \eta_3 = 1, \eta_4 = \frac{1+j}{2}$. We can see that

$$(2.4) \qquad (E_\rho(\eta_j, \eta_k)) = J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

For $\gamma \in \Gamma^{(1)}$, due to (2.2) and (2.4), $\{\eta_1\gamma, \cdots, \eta_4\gamma\}$ gives another symplectic basis of $\mathfrak{O}$ with respect to E_ρ. Hence, there exists $M_\gamma \in Sp(4, \mathbb{Z})$ such that $(\eta_1\gamma, \cdots, \eta_4\gamma) = (\eta_1, \cdots, \eta_4)^t M_\gamma$. for any $\gamma \in \Gamma^{(1)}$. For $w \in \mathbb{H}$, we set an $\mathbb{R}$-linear isomorphism $f_w : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \to \mathbb{C}^2$ given by $\alpha \mapsto \alpha \begin{pmatrix} w \\ 1 \end{pmatrix}$. Put $\omega_j = f_w(\eta_j) \in \mathbb{C}^2$ $(j = 1, \cdots, 4)$. Then, $\Lambda_w = f_w(\mathfrak{O}) = \langle \omega_1, \cdots, \omega_4 \rangle_{\mathbb{Z}}$ gives a lattice in $\mathbb{C}^2$ and $\mathbb{C}^2/\Lambda_w$ gives a complex torus with the period matrix $(\omega_1, \omega_2, \omega_3, \omega_4) = (\Omega_1(w)\Omega_2(w))$. Then, $E_x : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by $E_w(f_w(\xi), f_w(\eta)) = -E_\rho(\xi, \eta)$ induces a non-degenerate skew symmetric pairing $E_w : \Lambda_w \times \Lambda_w \to \mathbb{Z}$. This gives an alternating Riemann form on the complex tours $\mathbb{C}^2/\Lambda_w$. Therefore, we have the holomorphic embedding Ω in (2.3) given by

$$(2.5) \quad w \mapsto \Omega(w) = \Omega_2^{-1}(w)\Omega_1(w)$$

$$= \frac{1}{pw}\begin{pmatrix} \bar{\varepsilon}^2 + \frac{(p-1)aD}{2}w + D\varepsilon^2 w^2 & \bar{\varepsilon} - (p-1)aDw - D\varepsilon w^2 \\ \bar{\varepsilon} - (p-1)aDw - D\varepsilon w^2 & -1 - 2aDw + Dw^2 \end{pmatrix},$$

where $\varepsilon = \frac{1+\sqrt{p}}{2}$ (see [12] Theorem 3.5).

Letting pr be the canonical projection $\mathfrak{S}_2 \to \mathcal{A}_2$, the image of $\mathbb{H}$ under the mapping $pr \circ \Omega$ corresponds to the Shimura curve $\mathcal{S}_D \subset \mathcal{A}_2$.

The matrix $\Omega(w) = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{S}_2$ in (2.5) satisfies the singular relations in (2.1). This is explicitly given by

$$(2.6) \quad m\tau_1 + (m + 2aDn)\tau_2 - \frac{p-1}{4}m\tau_3 + n(\tau_2^2 - \tau_1\tau_3) + (a^2 D - b)Dn = 0$$

with two parameters $m, n \in \mathbb{Z}$ ([12] Theorem 5.1). The invariant Δ in (2.1) for the singular relation (2.6) is given by the quadratic form

$$(2.7) \qquad \Delta(m, n) = pm^2 + 4aDmn + 4bDn^2,$$

where $m, n \in \mathbb{Z}$. Moreover, he showed the following theorem.

Theorem 2.6.

(1) [12, Theorem 5.2] *For a positive non-square integer Δ such that $\Delta \equiv 1, 0 \mod 4$, the following conditions are equivalent:*

    (a) *The number Δ is represented by the quadratic form $\Delta(m, n)$ in (2.7) with relatively prime integers $m, n \in \mathbb{Z}$.*

    (b) *The image $\mathcal{S}_D = \varphi_D(\mathbb{S}_D) \subset \mathcal{A}_2$ of the Shimura curve is contained in the Humbert surface $\mathcal{H}_\Delta$.*

(2) [12, Corollary 5.3] *The Shimura curve $\mathcal{S}_D$ is contained in the intersection $\mathcal{H}_{\Delta_1} \cap \mathcal{H}_{\Delta_2}$ of two Humbert surfaces if and only if Δ_1 and Δ_2 are given by $\Delta(m, n)$ with relatively prime integers m, n.*

Remark 2.7. As we noted in Remark 2.4, the mapping φ_D is not always unique. For a generic D, the mapping φ_D and the model $\mathcal{S}_D$ via the embedding Ω in (2.5) depend on the triple (p, a, b).

Example 2.8. For the case $D = 6$, we can take $(p, a, b) = (5, 2, 5)$. The quadratic form is given by

$$(2.8) \qquad \Delta_6(m, n) = 5m^2 + 48mn + 20n^2.$$

For the case $D = 10$, we can take $(p, a, b) = (13, 3, 7)$. The quadratic form is given by

$$(2.9) \qquad \Delta_{10}(m, n) = 13m^2 + 120mn + 280n^2.$$

For the case $D = 14$, we can take $(p, a, b) = (5, 1, 3)$. The quadratic form is given by

$$(2.10) \qquad \Delta_{14}(m, n) = 5m^2 + 56mn + 168n^2.$$

For the case $D = 15$, we can take $(p, a, b) = (53, 22, 137)$. The quadratic form is given by

$$(2.11) \qquad \Delta_{15}(m, n) = 53m^2 + 1320mn + 8220n^2.$$

Due to Theorem 2.6 (2), we have the following results.

Example 2.9. The image of the Shimura curves $\mathcal{S}_6$ attached to $(p, a, b) = (5, 2, 5)$ and $\mathcal{S}_{10}$ attached to $(p, a, b) = (13, 3, 7)$ are contained in the intersection $\mathcal{H}_5 \cap \mathcal{H}_8$ of the Humbert surfaces because $5 = \Delta_6(5, -1) = \Delta_{10}(5, -1)$ and $8 = \Delta_6(4, -1) = \Delta_{10}(4, -1)$.

Example 2.10. The image of the Shimura curves $\mathcal{S}_6$ attached to $(p,a,b) = (5,2,5)$, $\mathcal{S}_{14}$ attached to $(p,a,b) = (5,1,3)$ and $\mathcal{S}_{15}$ attached to $(p,a,b) = (53,22,137)$ are contained in the intersection $\mathcal{H}_5 \cap \mathcal{H}_{12}$ of the Humbert surfaces because $5 = \Delta_6(5,-1) = \Delta_{15}(25,-2) = \Delta_{14}(1,0)$ and $12 = \Delta_6(6,-1) = \Delta_{15}(12,-1) = \Delta_{14}(6,-1)$.

Example 2.11. The image of the Shimura curves $\mathcal{S}_6$ attached to $(p,a,b) = (5,2,5)$ and $\mathcal{S}_{14}$ attached to $(p,a,b) = (5,1,3)$ are contained in the intersection $\mathcal{H}_5 \cap \mathcal{H}_{21}$ of the Humbert surfaces because $5 = \Delta_6(5,-1) = \Delta_{15}(25,-2) = \Delta_{14}(1,0)$ and $21 = \Delta_6(6,-1) = \Delta_{14}(7,-1)$. However, the Shimura curve $\mathcal{S}_{15}$ is not contained in $\mathcal{H}_5 \cap \mathcal{H}_{21}$ because 21 is not represented by $\Delta_{15}(m,n)$ with $m,n \in \mathbb{Z}$.

3. The embedding $\mathcal{F} \hookrightarrow \mathcal{F}_{CD}$

3.1. The lattice polarized $K3$ surfaces. A $K3$ surface S is a compact complex surface such that the canonical bundle $K_S = 0$ and $H^1(S, \mathcal{O}_S) = 0$.

By the canonical cup product and the Poincaré duality, the homology group $H_2(S, \mathbb{Z})$ has a lattice structure. It is well known that the lattice $H_2(S, \mathbb{Z})$ is isometric to the even unimodular lattice $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$, where $E_8(-1)$ is the negative definite even unimodular lattice of type E_8 and U is the parabolic lattice of rank 2. Let $\mathrm{NS}(S)$ be the Néron–Severi lattice of a $K3$ surface S. This is a sublattice of $H_2(S, \mathbb{Z})$ generated by the divisors on S. The orthogonal complement $\mathrm{Tr}(S) = \mathrm{NS}(S)^\perp$ in $H_2(S, \mathbb{Z})$ is called the transcendental lattice of S.

Let S be a $K3$ surface and M be a lattice. If we have a primitive lattice embedding $\iota : M \to \mathrm{NS}(S)$, the pair (S, ι) is called an M-polarized $K3$ surface. Let (S_1, ι_1) and (S_2, ι_2) be M-polarized $K3$ surfaces. If there exist an isomorphism $S_1 \to S_2$ of $K3$ surfaces such that $\iota_1 = f^* \circ \iota_2$, (S_1, ι_1) and (S_2, ι_2) are isomorphic as M-polarized $K3$ surfaces. From now on, we often omit the primitive lattice embedding ι.

Remark 3.1. When we consider the moduli space and the period mapping of lattice polarized $K3$ surfaces, we should pay attention to the ampleness of lattice polarized $K3$ surfaces (see [6]). However, it is safe to apply the Torelli theorem to our cases. See Remark 3.5 and 3.9.

3.2. The icosahedral invariants. Klein [16] studied the action of the icosahedral group on $\mathbb{P}^2(\mathbb{C}) = \mathrm{Proj}(\mathbb{C}[\zeta_0, \zeta_1, \zeta_2])$. He obtained a system of generators $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in \mathbb{C}[\zeta_0, \zeta_1, \zeta_2]$ of ring of icosahedral invariants. The $\mathfrak{A}$ (resp. $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$) is a homogeneous polynomial of weight 2 (resp. 6, 10, 15) and the ring is given by $\mathbb{C}[\mathfrak{A} : \mathfrak{B} : \mathfrak{C} : \mathfrak{D}]/R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$, where $R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$ is

the Klein's icosahedral relation

$$(3.1) \quad R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}) = 144\mathfrak{D}^2 - (-1728\mathfrak{B}^5 + 720\mathfrak{A}\mathfrak{C}\mathfrak{B}^3 - 80\mathfrak{A}^2\mathfrak{C}^2\mathfrak{B}$$
$$+ 64\mathfrak{A}^3(5\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C})^2 + \mathfrak{C}^3).$$

The Hilbert modular group $PSL(2, \mathcal{O}_5)$ acts on the product $\mathbb{H} \times \mathbb{H}$ of upper half planes. We consider the symmetric Hilbert modular surface $\langle PSL(2, \mathcal{O}_5), \tau\rangle\backslash(\mathbb{H} \times \mathbb{H})$, where τ is the involution given by $(z_1, z_2) \mapsto (z_2, z_1)$.

Hirzebruch obtained the following result.

Proposition 3.2 ([14])**.**

(1) *The ring of symmetric modular forms for $PSL(2, \mathcal{O}_5)$ is isomorphic to the ring $\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}]/R(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D})$. Here, $\mathfrak{A}$ (resp. $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}$) corresponds to a symmetric modular form for $PSL(2, \mathcal{O}_5)$ of weight 2 (resp. $6, 10, 15$).*

(2) *The Hilbert modular surface $\langle PSL(2, \mathcal{O}_5), \tau\rangle\backslash(\mathbb{H} \times \mathbb{H})$ has a compactification by adding one cusp $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0)$. This compactification is the weighted projective plane $\mathbb{P}(1 : 3 : 5) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}])$.*

Set

$$(3.2) \qquad\qquad X = \frac{\mathfrak{B}}{\mathfrak{A}^3}, \qquad\qquad Y = \frac{\mathfrak{C}}{\mathfrak{A}^5}.$$

Then, the pair (X, Y) gives an affine coordinate system of $\{\mathfrak{A} \neq 0\} \subset \mathbb{P}(1 : 3 : 5)$.

Remark 3.3. The symmetric Hilbert modular surface

$$\langle PSL(2, \mathcal{O}_5), \tau\rangle\backslash(\mathbb{H} \times \mathbb{H})$$

coincides with the Humbert surface $\mathcal{H}_5$.

3.3. The family $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$ of $K3$ surfaces. In this subsection, we survey the results of [22].

For $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\}$, we have the elliptic $K3$ surface

$$(3.3) \qquad S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5\mathfrak{A}y^2)x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4.$$

The family $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \,|\, (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\}\}$ is studied in [22]. By a detailed observation, we can prove the following proposition.

Proposition 3.4 ([22, Section 2])**.**

    (1) *For generic* $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\}$, *the Néron–Severi lattice* $\mathrm{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ *is given by the intersection matrix*

$$E_8(-1) \oplus E_8(-1) \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = M_5 \text{ and the transcendental lattice}$$

    $\mathrm{Tr}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ *is given by the intersection matrix* $U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = A_5$.

    (2) *The family* $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$ *gives the isomorphy classes of* M_5-*polarized* $K3$ *surfaces. Especially,* $S(\mathfrak{A}_1 : \mathfrak{B}_1 : \mathfrak{C}_1)$ *and* $S(\mathfrak{A}_2 : \mathfrak{B}_2 : \mathfrak{C}_2)$ *are isomorphic as* M_5-*polarized* $K3$ *surfaces if and only if* $(\mathfrak{A}_1 : \mathfrak{B}_1 : \mathfrak{C}_1) = (\mathfrak{A}_2 : \mathfrak{B}_2 : \mathfrak{C}_2)$ *in* $\mathbb{P}(1 : 3 : 5)$.

The period domain for the family $\mathcal{F}$ is given by the Hermitian symmetric space $\mathcal{D} = \{\xi \in \mathbb{P}^3(\mathbb{C}) \,|\, {}^t\xi A_5 \xi = 0, {}^t\xi A_5 \bar{\xi} > 0\}$ of type IV. The space $\mathcal{D}$ has two connected components $\mathcal{D}_+$ and $\mathcal{D}_-$. We have the multivalued period mapping $\Phi : \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\} \to \mathcal{D}_+$. There exists a biholomorphic mapping $j : \mathbb{H} \times \mathbb{H} \to \mathcal{D}_+$. Then, we have the multivalued mapping $j^{-1} \circ \Phi : \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\} \to \mathbb{H} \times \mathbb{H}$, that is given by
(3.4)
$$(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto (z_1, z_2) = \left(-\frac{\int_{\Gamma_3} \omega + \frac{1-\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega}, -\frac{\int_{\Gamma_3} \omega + \frac{1+\sqrt{5}}{2} \int_{\Gamma_4} \omega}{\int_{\Gamma_2} \omega} \right),$$

where ω is the unique holomorphic 2-form up to a constant factor and $\Gamma_1, \cdots, \Gamma_4$ are certain 2-cycles on $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ (for detail, see [22] and [23]).

Remark 3.5. In [22], the primitive lattice embedding $\iota : M_5 \hookrightarrow \mathrm{NS}(S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$ of the M_5-polarized $K3$ surfaces is given explicitly. Especially, the image $\iota(M_5)$ is given by effective divisors of $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$. In fact, it assures an ampleness of lattice polarized $K3$ surfaces and we can apply the Torelli theorem to our period mapping for $\mathcal{F}$ safely. For detailed argument, see [22, Section 2.2].

Let $\mathfrak{S}_2$ be the Siegel upper half plane consisting of 2×2 complex matrices. The mapping $\mu_5 : \mathbb{H} \times \mathbb{H} \to \mathfrak{S}_2$ given by
(3.5)
$$(z_1, z_2) \mapsto \frac{1}{2\sqrt{5}} \begin{pmatrix} (1 + \sqrt{5})z_1 - (1 - \sqrt{5})z_2 & 2(z_1 - z_2) \\ 2(z_1 - z_2) & (-1 + \sqrt{5})z_1 + (1 + \sqrt{5})z_2 \end{pmatrix}$$

gives a modular embedding (see the following diagram).

$$\begin{array}{ccc}
\mathbb{H} \times \mathbb{H} & \xrightarrow{\;\mu_5\;} & \mathfrak{S}_2 \\
{\scriptstyle \langle PSL(2,\mathcal{O}_5),\tau\rangle} \downarrow & & \downarrow {\scriptstyle Sp(4,\mathbb{Z})} \\
\mathbb{H} \times \mathbb{H} & \xrightarrow[\;\mu_5\;]{} & \mathfrak{S}_2
\end{array}$$

j	0	1	2	3	4	5	6	7	8	9
${}^t a$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$	$(1,0)$	$(0,0)$	$(0,1)$
${}^t b$	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$

TABLE 3.1. The correspondence between j and (a,b).

Moreover, μ_5 in (3.5) gives a parametrization of the surface N_5 in Definition 2.1:

$$(3.6) \qquad N_5 = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{S}_2 \ \middle| \ -\tau_1 + \tau_2 + \tau_3 = 0 \right\}.$$

For $\Omega \in \mathfrak{S}_2$ and $a, b \in \{0,1\}^2$ with ${}^t ab \equiv 0 \mod 2$, set $\vartheta(\Omega; a, b) = \sum_{g \in \mathbb{Z}^2} \exp\left(\pi\sqrt{-1}({}^t(g + \frac{1}{2}a)\Omega(g + \frac{1}{2}a) + {}^t gb)\right)$. For $j \in \{0, 1, \cdots, 9\}$, we set $\theta_j(z_1, z_2) = \vartheta(\mu_5(z_1, z_2); a, b)$, where the correspondence between j and (a, b) is given by Table 3.1.

Let $a \in \mathbb{Z}$ and $j_1, \cdots, j_r \in \{0, \cdots, 9\}$. We set $\theta^a_{j_1, \cdots, j_r} = \theta^a_{j_1} \cdots \theta^a_{j_r}$. The following g_2 (resp. s_6, s_{10}) is a symmetric Hilbert modular form of weight 2 (resp. 6, 10) for $\mathbb{Q}(\sqrt{5})$ (see Müller [20]): $g_2 = \theta_{0145} - \theta_{1279} - \theta_{3478} + \theta_{0268} + \theta_{3569}, s_6 = 2^{-8}(\theta^2_{012478} + \theta^2_{012569} + \theta^2_{034568} + \theta^2_{236789} + \theta^2_{134579}), s_{10} = s_5^2 = 2^{-12}\theta^2_{0123456789}.$

Proposition 3.6 ([22, Theorem 4.1])**.** *Using the coordinates (X, Y) of (3.2), the inverse correspondence $(z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))$ of the period mapping (3.4) for $\mathcal{F}$ has the following theta expression*

$$(3.7) \quad X(z_1, z_2) = 2^5 \cdot 5^2 \cdot \frac{s_6(z_1, z_2)}{g_2^3(z_1, z_2)}, \quad Y(z_1, z_2) = 2^{10} \cdot 5^5 \cdot \frac{s_{10}(z_1, z_2)}{g_2^5(z_1, z_2)}.$$

Moreover, X and Y give a system of generators of the field of symmetric Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$.

We call the divisor $\square = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} \mid z_1 = z_2\}$ the diagonal. On the diagonal $\square$, it holds

$$(3.8) \qquad X(z, z) = \frac{25}{27} \frac{1}{J(z)}, \qquad Y(z, z) = 0.$$

3.4. The family $\mathcal{F}_{CD} = \{S_{CD}(\alpha, \beta, \gamma, \delta)\}$ of $K3$ surfaces. In [4] and [25], the family $\mathcal{F}_{CD} = \{S_{CD}(\alpha, \beta, \gamma, \delta) \mid (\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6) - \{\gamma = \delta = 0\}\}$ of $K3$ surfaces is studied in detail, where

$$(3.9) \quad S_{CD}(\alpha : \beta : \gamma : \delta) : y^2 = x^3 + (-3\alpha t^4 - \gamma t^5)x + (t^5 - 2\beta t^6 + \delta t^7).$$

The above equation gives the structure of an elliptic surface $(x, y, t) \mapsto t$ with the singular fibres $II^* + 5I_1 + III^*$.

Remark 3.7. In this paper, we use the notation $\mathcal{F}_{CD}$ and S_{CD}, since the surface (3.9) appears in the paper of Clingher and Doran [4]. On the other hand, Kumar [18] independently studied elliptic $K3$ surfaces with the same singular fibres. So, we need to recall Kumar's contribution.

Proposition 3.8 ([25, Section 2 and 3])**.**

(1) *If an elliptic $K3$ surface S with the elliptic fibration $(x, y, t) \mapsto t$ has the singular fibres of type II^* at $t = 0$, III^* at $t = \infty$ and other five fibres of type I_1, then S is given by the Weierstrass equation in* (3.9).

(2) *For generic $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6) - \{\gamma = \delta = 0\}$, the Néron–Severi lattice $\mathrm{NS}(S_{CD}(\alpha : \beta : \gamma : \delta)$ is given by the intersection matrix $E_8(-1) \oplus E_7(-1) \oplus U = M_0$ and the transcendental lattice $\mathrm{Tr}(S_{CD}(\alpha : \beta : \gamma : \delta))$ is given by the intersection matrix $U \oplus U \oplus \langle -2 \rangle = A_0$.*

(3) *The family $\mathcal{F}_{CD} = \{S_{CD}(\alpha : \beta : \gamma : \delta)\}$ gives the isomorphism classes of M_0-polarized $K3$ surfaces. Especially, $S_{CD}(\alpha_1 : \beta_1 : \gamma_1 : \delta_1)$ and $S_{CD}(\alpha_2 : \beta_2 : \gamma_2 : \delta_2)$ are isomorphic as M_0-polarized $K3$ surfaces if and only if $(\alpha_1 : \beta_1 : \gamma_1 : \delta_1) = (\alpha_2 : \beta_2 : \gamma_2 : \delta_2)$ in $\mathbb{P}(2 : 3 : 5 : 6)$.*

Let $\mathcal{D}_0 = \{\xi \in \mathbb{P}^4(\mathbb{C}) \,|\, {}^t\xi A_0 \xi = 0, {}^t\xi A_0 \bar{\xi} > 0\}$. The period domain for the family $\mathcal{F}_{CD}$ is given by the quotient space $PO(A_0, \mathbb{Z}) \backslash \mathcal{D}_0$, where $PO(A_0, \mathbb{Z}) = \{M \in GL(4, \mathbb{Z}) \,|\, {}^tMA_0M = A_0\}$. In fact, there exists a holomorphic mapping $\mathcal{D}_0 \to \mathfrak{S}_2$ such that this mapping induces the isomorphism $PO(A_0, \mathbb{Z}) \backslash \mathcal{D}_0 \simeq Sp(4, \mathbb{Z}) \backslash \mathfrak{S}_2 = \mathcal{A}_2$. The transcendental lattice $\mathrm{Tr}(S_{CD}(\alpha : \beta : \gamma : \delta))$ is Hodge isometric to the transcendental lattice $\mathrm{Tr}(A)$ of a generic principally polarized abelian surface and the family $\mathcal{F}_{CD}$ gives the same variations of Hodge structures of weight 2 with the family of principally polarized abelian surfaces (see [25, Section 3]).

Remark 3.9. In [25, Section3], the primitive lattice embedding $\iota : M_0 \hookrightarrow \mathrm{NS}(S_{CD}(\alpha : \beta : \gamma : \delta))$ of M_0-polarized $K3$ surfaces is attained by taking appropriate effective divisors of $S_{CD}(\alpha : \beta : \gamma : \delta)$. This argument guarantees an ampleness of lattice polarized $K3$ surfaces and it is safe to apply the Torelli theorem for lattice polarized $K3$ surface to our family $\mathcal{F}_{CD}$.

Let $\mathcal{M}_2$ be the moduli space of genus 2 curves. Let $\mathbb{P}(1 : 2 : 3 : 5) = \{(\zeta_1 : \zeta_2 : \zeta_3 : \zeta_5)\}$ be the weighted projective space. It is well-known that $\mathcal{M}_2 = \mathbb{P}(1 : 2 : 3 : 5) - \{\zeta_5 = 0\}$. In fact, by the Igusa–Clebsch invariants I_2, I_4, I_6, I_{10} of degree 2, 4, 6, 10 for a genus 2 curve, $(I_2 : I_4 : I_6 : I_{10})$ gives a well-defined point of the moduli space $\mathcal{M}_2$. We note that the moduli space $\mathcal{M}_2$ is a Zariski open set of the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces ($\mathcal{M}_2$ is the complement of the divisor given by the points corresponding to the product of elliptic curves).

By a study of elliptic $K3$ surfaces, we can prove the following proposition.

Proposition 3.10 ([18], [4], see also [25])**.** *The point $(I_2 : I_4 : I_6 : I_{10}) \in \mathcal{M}_2 = \mathbb{P}(1 : 2 : 3 : 5) - \{\zeta_5 = 0\}$ corresponds to the point $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6) - \{\gamma = \delta = 0\}$ of the moduli space of M_0-polarized $K3$ surfaces by the following birational transformation:*

$$(3.10) \qquad \alpha = \frac{1}{9}I_4, \qquad \beta = \frac{1}{27}(-I_2 I_4 + 3I_6), \qquad \gamma = 8I_{10}, \qquad \delta = \frac{2}{3}I_2 I_{10}.$$

The Humbert surface $\mathcal{H}_5$ is a subvariety of the moduli space $\mathcal{A}_2$. Hence, the defining equation of $\mathcal{H}_5$ can be described by the equation in $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6)$. By an observation of the elliptic fibration given by (3.9), we can prove the following theorem. Especially, the equation (3.11) shall give the defining equation of $\mathcal{H}_5$.

Proposition 3.11 ([25, Theorem 4.4])**.**

    (1) *If and only if the equation*

$$(3.11) \qquad\qquad (-\alpha^3 - \beta^2 + \delta)^2 - 4\alpha(\alpha\beta - \gamma)^2 = 0$$

       holds, there exists a non trivial section s_5 of $\{S_{CD}(\alpha : \beta : \gamma : \delta)\}$ as illustrated in Figure 3.1.

    (2) *If the modular equation (3.11) holds, the Néron–Severi lattice of the $K3$ surface $S_{CD}(\alpha : \beta : \gamma : \delta)$ is generically given by the intersection matrix M_5.*

We call the equation (3.11) the modular equation for $\Delta = 5$.
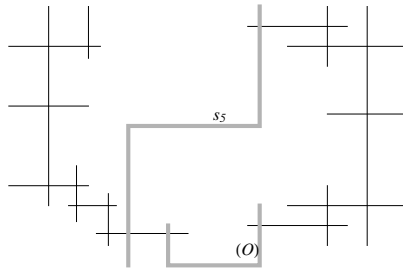

FIGURE 3.1. The section s_5.

Remark 3.12. The modular equation (3.11) for $\Delta = 5$ is much simpler than modular equations for other discriminants. For example, the modular

equation for $\Delta = 8$ is given by

(3.12)

$$1024\alpha^{15} - 5120\alpha^{12}\beta^2 + 10240\alpha^9\beta^4 - 10240\alpha^6\beta^6 + 5120\alpha^3\beta^8 - 1024\beta^{10}$$

$$- 941056\alpha^{11}\beta\gamma + 1053696\alpha^8\beta^3\gamma + 715776\alpha^5\beta^5\gamma - 828416\alpha^2\beta^7\gamma$$

$$- 7556464\alpha^10\gamma^2 + 131492384\alpha^7\beta^2\gamma^2 + 39076880\alpha^4\beta^4\gamma^2 + 13934400\alpha\beta^6\gamma^2$$

$$+ 1491324088\alpha^6\beta\gamma^3 - 918848440\alpha^3\beta^3\gamma^3 - 36968000\beta^5\gamma^3$$

$$+ 13611473901\alpha^5\gamma^4 - 718342500\alpha^2\beta^2\gamma^4 + 9079601250\alpha\beta\gamma^5$$

$$+ 7737809375\gamma^6 - 343808\alpha^12\delta - 647168\alpha^9\beta^2\delta + 2234880\alpha^6\beta^4\delta$$

$$- 1153024\alpha^3\beta^6\delta - 90880\beta^8\delta + 2442144\alpha^8\beta\gamma\delta - 86206272\alpha^5\beta^3\gamma\delta$$

$$+ 12985248\alpha^2\beta^5\gamma\delta - 1669045416\alpha^7\gamma^2\delta - 1449171160\alpha^4\beta^2\gamma^2\delta$$

$$+ 268484800\alpha\beta^4\gamma^2\delta - 157452560\alpha^3\beta\gamma^3\delta - 772939000\beta^3\gamma^3\delta$$

$$- 15745060125\alpha^2\gamma^4\delta + 29370256\alpha^9\delta^2 - 56832480\alpha^6\beta^2\delta^2$$

$$+ 37166352\alpha^3\beta^4\delta^2 - 2626240\beta^6\delta^2 + 1230170496\alpha^5\beta\gamma\delta^2$$

$$- 155485248\alpha^2\beta^3\gamma\delta^2 - 27876720\alpha^4\gamma^2\delta^2 + 2388102200\alpha\beta^2\gamma^2\delta^2$$

$$- 2315093000\beta\gamma^3\delta^2 - 86058160\alpha^6\delta^3 + 3605888\alpha^3\beta^2\delta^3 - 22815760\beta^4\delta^3$$

$$- 1231671584\alpha^2\beta\gamma\delta^3 + 1704478600\alpha\gamma^2\delta^3 + 85375664\alpha^3\delta^4$$

$$+ 53878880\beta^2\delta^4 - 28344976\delta^5 = 0.$$

We can check that for a generic $(\alpha, \beta, \gamma, \delta)$ satisfying (3.12), the transcendental lattice is given by $U \oplus \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$. So, (3.12) gives a counterpart of the equation (3.11).

3.5. The embedding $\Psi_5 : \mathbb{P}(1 : 3 : 5) \hookrightarrow \mathbb{P}(2 : 3 : 5 : 6)$. The family of M_5-polarized $K3$ surfaces is a subfamily of the family of M_0-polarized $K3$ surface. Therefore, Proposition 3.4, 3.8 and 3.11 imply that the family $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$ is a subfamily of the family $\mathcal{F}_{CD} = \{S_{CD}(\alpha : \beta : \gamma : \delta)\}$. Moreover, together with Remark 3.3, the modular equation (3.11) gives the defining equation of the Humbert surface $\mathcal{H}_5$. In this subsection, we realize the embedding $\mathcal{F} \hookrightarrow \mathcal{F}_{CD}$ explicitly. This is given by the embedding $\Psi_5 : \mathbb{P}(1 : 3 : 5) \hookrightarrow \mathbb{P}(2 : 3 : 5 : 6)$ of varieties.

Since the modular equation (3.11) for $\Delta = 5$ is very simple and the coordinates $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ have the explicit theta expressions (3.7) via the period mapping for the family $\mathcal{F}$, we can study the pull-back $\Psi_5^*(V)$ of a variety $V \subset \mathbb{P}(2 : 3 : 5 : 6)$ quite effectively. Especially, in Section 5 and 6, we shall consider the pull-back $\Psi_5^*(\mathcal{S}_D)$ of the Shimura curve $\mathcal{S}_D$ for $D = 6, 10, 14$ and 15 in $\mathbb{P}(2 : 3 : 5 : 6)$.

Lemma 3.13. *The elliptic K3 surface* $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ *is birationally equivalent to the elliptic K3 surface*

$$(3.13) \quad y_0^2 = s^3 + \left(-\frac{25}{12}\mathfrak{A}^2 u^4 - \frac{1}{32}\mathfrak{C}u^5\right) s$$
$$+ \left(u^5 + \left(\frac{125}{108}\mathfrak{A}^3 - \frac{5}{4}\mathfrak{B}\right) u^6 + \left(\frac{25}{64}\mathfrak{B}^2 - \frac{5}{96}\mathfrak{A}\mathfrak{C}\right) u^7\right).$$

The elliptic fibration $\pi : (s, u, y_0) \mapsto u$ *given by* (3.13) *has singular fibres* $\pi^{-1}(0)$ *of type* II^*, $\pi^{-1}(\infty)$ *of type* III^* *and other five singular fibres of type* I_1.

Proof. First, by the correspondence

$$x = \frac{x_1}{16t}, \qquad y = -\frac{x_1}{16t^2}, \qquad z = \frac{x_1 y_1}{256t^4},$$

the surface $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ in (3.3) is transformed to

$$(3.14) \qquad y_1^2 = x_1(x_1^2 + (20\mathfrak{A}t^2 - 20\mathfrak{B}t + \mathfrak{C})x + 16t^5).$$

The elliptic surface given by (3.14) has the singular fibres of type $I_{10} + 5I_1 + III^*$.

Next, by the birational transformation

$$(3.15) \qquad x_1 = \frac{\mathfrak{C}^4 u_0^5}{x_2^4}, \qquad y_1 = \frac{\sqrt{\mathfrak{C}^9} u_0^5}{x_2^6} y_2, \qquad t = \frac{\mathfrak{C}u_0}{x_2},$$

we have

$$(3.16) \qquad y_2^2 = x_2^4 + (16 - 20\mathfrak{B}u_0)x_2^3 + 20\mathfrak{A}\mathfrak{C}u_0^2 x_2 + \mathfrak{C}u_0^5.$$

The equation (3.16) gives a double covering of a polynomial of degree 4 in x_2. According to Section 3.1 of [1], such a polynomial can be transformed to a Weierstrass equation. In our case, putting

$$\begin{cases} x_2 = \dfrac{6s_1(-4 + 5\mathfrak{B}u_0) - 60\mathfrak{A}\mathfrak{C}u_0^2(-4 + 5\mathfrak{B}u_0) + \sqrt{6}y_3}{6(-96 + s_1 + 240\mathfrak{B}u_0 - 150\mathfrak{B}^2 u_0^2 + 20\mathfrak{A}\mathfrak{C}u_0^2)}, \\[4mm] y_2 = \dfrac{48 + s_1 - 120\mathfrak{B}u_0 + 75\mathfrak{B}^2 u_0^2 - 10\mathfrak{A}\mathfrak{C}u_0^2}{3} \\[4mm] \qquad - \dfrac{(27(2(-4 + 5\mathfrak{B}u_0)(16 - 40\mathfrak{B}u_0 + 25\mathfrak{B}^2 u_0^2 - 5\mathfrak{A}\mathfrak{C}u_0^2) + \frac{y_3}{3\sqrt{6}})^2)}{3(-96 + s_1 + 240\mathfrak{B}u_0 - 150\mathfrak{B}^2 u_0^2 + 20\mathfrak{A}\mathfrak{C}u_0^2)^2}, \end{cases}$$

we have the Weierstrass equation

$$(3.17) \qquad y_3^2 = \text{polynomial in } s_1 \text{ of degree 3}.$$

Put

$$s_1 = 3\mathfrak{C}s, \qquad y_3 = \sqrt{27\mathfrak{C}^3}y_0, \qquad u_0 = \frac{u}{2},$$

to (3.17). Then, we have (3.13). $\qquad\square$

Remark 3.14. The transformation in (3.15) gives an example of 2-neighbor step, that is a method to find a new elliptic fibration. By (3.15), we have $u_0 = \dfrac{x}{t^4}$. The new parameter u_0 has a pole of order 4 at $t = 0$. This implies that we have a singular fibre of type III^* at $u_0 = \infty$ (Figure 3.2).
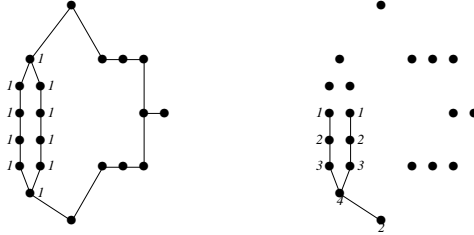

FIGURE 3.2. 2-neiborgh step in (3.15)

Remark 3.15. The section s_5 in Proposition 3.11(1) has the explicit form

(3.18) $$t \mapsto (x(t), y(t), t),$$

where $x(t) = \frac{32}{Y} + \frac{40X}{Y}t + \frac{5(15X^2 - 2Y)}{6Y}t^2$ and $y(t) = \frac{128\sqrt{2}}{Y^{3/2}} + \frac{240\sqrt{2}X}{Y^{3/2}}t - \frac{10\sqrt{2}(-15X^2 + Y)}{Y^{3/2}}t^2 - \frac{25(-5X^3 + XY)}{2\sqrt{2}Y^{3/2}}t^3$.

Theorem 3.16. *The point* $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6)$ *satisfies the modular equation* (3.11) *if and only if the point* $(\alpha : \beta : \gamma : \delta)$ *is in the image of the embedding* $\Psi_5 : \mathbb{P}(1 : 3 : 5) \to \mathbb{P}(2 : 3 : 5 : 6)$ *given by* $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto (\alpha : \beta : \gamma : \delta) = (\alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}))$, *where*

(3.19)
$$\begin{cases} \alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \dfrac{25}{36}\mathfrak{A}^2, \\[2mm] \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \dfrac{1}{2}\Big(-\tfrac{125}{108}\mathfrak{A}^3 + \tfrac{5}{4}\mathfrak{B}\Big), \\[2mm] \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \dfrac{1}{32}\mathfrak{C}, \\[2mm] \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \dfrac{25}{64}\mathfrak{B}^2 - \tfrac{5}{96}\mathfrak{A}\mathfrak{C}. \end{cases}$$

Proof. According to Proposition 3.11, if $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6)$ satisfies the modular equation (3.11), then the Néron–Severi lattice $\mathrm{NS}(S(\alpha : \beta : \gamma : \delta))$ is generically given by the intersection matrix M_5.

On the other hand, a family of the isomorphism classes of M_5-marked $K3$ surfaces is given by $\mathcal{F}$. By Lemma 3.13, $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ is birationally equivalent to the surface given by (3.13) with the section (3.18). Therefore,

the Weierstrass equation (3.13) induces an embedding $\mathcal{F} \hookrightarrow \mathcal{F}_{CD}$ of the family of elliptic $K3$ surfaces with the singular fibers of type $II^*+5I_1+III^*$. Together with Proposition 3.8, we have (3.19) by comparing the coefficients of (3.9) and (3.13). □

Remark 3.17. We can check that any point of the image $\Psi_5(\mathbb{P}(1:3:5))$ satisfies the modular equation (3.11).

4. The family $\mathcal{F}^{\Delta}_{EK}$ of $K3$ surfaces

Elkies and Kumar [9] obtained rational models of Hilbert modular surfaces. Especially, in their argument, they used parametrizations of the Humbert surfaces $\mathcal{H}_{\Delta}$ for fundamental discriminants Δ such that $1 < \Delta < 100$. Their method was the following. They consider a family, that is called $\mathcal{F}^{\Delta}_{EK}$ in this paper, of elliptic $K3$ surfaces with two complex parameters. A generic member of $\mathcal{F}^{\Delta}_{EK}$ has a suitable transcendental lattice and the moduli space of $\mathcal{F}^{\Delta}_{EK}$ is birationally equivalent to the Humbert surface $\mathcal{H}_{\Delta}$. Moreover, the family $\mathcal{F}^{\Delta}_{EK}$ can be regarded as a subfamily of $\mathcal{F}_{CD}$. It follows that the two complex parameters of $\mathcal{F}_{CD}$ give a parametrization χ_{Δ} of $\mathcal{H}_{\Delta}$.

In this paper, we shall use the parametrization χ_8 (resp. χ_{12}, χ_{21}) for the Humbert surface $\mathcal{H}_8$ (resp. $\mathcal{H}_{12}, \mathcal{H}_{21}$). We survey their results in this section.

However, we remark that the explicit forms of the parametrization χ_{Δ} appeared in the paper [9] only for the case $\Delta = 5$ and 8. Then, we need to calculate the explicit forms of χ_{12} and χ_{21} from the families $\mathcal{F}^{12}_{EK}$ and $\mathcal{F}^{21}_{EK}$ (see Section 4.3 and 4.4).

Remark 4.1. The choice of the parametrization of the Humbert surface $\mathcal{H}_{\Delta}$ is not unique. In fact, the parametrization χ_{Δ} due to Elkies and Kumar depends on the choice of an elliptic fibration of a generic member of $\mathcal{F}^{\Delta}_{EK}$. To the best of the author's knowledge, it is not easy to study modular properties of the parametrization χ_{Δ}. For example, it seems highly non trivial problem to obtain an explicit expression of the parametrization of χ_{Δ} via the Hilbert modular forms for $\mathbb{Q}(\sqrt{\Delta})$. See Remark 4.2 also.

4.1. The case of discriminant 5. Before we consider the cases $\Delta = 8, 12, 21$, let us see the case $\Delta = 5$.

In [9, Section 6], the family $\mathcal{F}^5_{EK}$ of $K3$ surfaces is studied. A generic member of $\mathcal{F}^5_{EK}$ is given by the defining equation

$$y^2 = x^3 + \frac{1}{4}t^3(-3g^2t + 4)x - \frac{1}{4}t^5(4h^2t^2 + (4h + g^3)t + (4g + 1)),$$

where g and h are two complex parameters.

Using this family, Elkies and Kumar obtained a parametrization of the Humbert surface $\mathcal{H}_5$. In [9], $\mathcal{H}_5$ is realized as a surface in the moduli space $\mathcal{M}_2 = \mathrm{Proj}(\mathbb{C}[I_2, I_4, I_6, I_{10}])$ by the parametrization

$$I_2 = 6(4g+1), \quad I_4 = 9g^2, \quad I_6 = 9(4h + 9g^3 + 2g^2), \quad I_{10} = 4h^2.$$

Together with (3.10), we obtain the mapping $\chi_5 : \mathbb{P}(1:2:5) \to \mathbb{P}(2:3:5:6)$ given by $(k:g:h) \mapsto (\alpha_5'(k:g:h) : \beta_5'(k:g:h) : \gamma_5'(k:g:h) : \delta_5'(k:g:h))$, where $\alpha_5'(k:g:h) = g^2, \beta_5'(k:g:h) = g^3 + 4hk, \gamma_5'(k:g:h) = 32h^2$ and $\delta_5'(k:g:h) = 16h^2(4g+k^2)$. The mapping χ_5 gives a parametrization of the Humbert surface $\mathcal{H}_5$. Any point of the image of χ_5 satisfies the modular equation (3.11).

Remark 4.2. The above χ_5 is a parametrization different from $\Psi_5 : \mathbb{P}(1 : 3 : 5) \hookrightarrow \mathbb{P}(2:3:5:6)$ in Theorem 3.16. In Section 5 and 6, we shall use only Ψ_5. The parametrization Ψ_5 has good modular properties and is more convenient than χ_5 for our purpose. For example, the weighted projective space $\mathbb{P}(1:3:5) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}])$ is a canonical compactification of the Hilbert modular surface (see Section 3.2) and the coordinates $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ have an expression by Hilbert modular forms (see Proposition 3.6).

4.2. The case of discriminant 8. In [9, Section 7], the family $\mathcal{F}_{EK}^8$ of $K3$ surfaces is studied. A generic member of $\mathcal{F}_{EK}^8$ is given by the defining equation

$$y^2 = x^3 + t((2r+1)t + r)x^2 + 2rst^4(t+1)x + rst^7$$

where r and s are two complex parameters. This Weierstrass equation gives the elliptic fibration $(x, y, t) \mapsto t$ with the singular fibres of type III^* and I_5^*. A generic member of $\mathcal{F}_{EK}^8$ admits another elliptic fibration with the singular fibres of type $II^* + 5I_1 + III^*$ and we regard $\mathcal{F}_{EK}^8$ as a subfamily of $\mathcal{F}_{CD}$ (see Proposition 3.8). Thus, Elkies and Kumar gave the correspondence

$$\begin{cases} I_2 = -4(3s + 8r - 2), \ I_4 = 4(9rs + 4r^2 + 4r + 1), \ I_{10} = -8s^2 r^3, \\ I_6 = -4(36rs^2 + 94r^2 s - 35rs + 4s + 48r^3 + 40r^2 + 4r - 2). \end{cases}$$

Together with (3.10), we have the following correspondence $\chi_8 : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}(2 : 3 : 5 : 6)$ given by $(q : r : s) \mapsto (\alpha_8(r : s : q) : \beta_8(r : s : q) : \gamma_8(r : s : q) : \delta_8(r : s : q))$ where
(4.1)
$$\begin{cases} \alpha_8(r : s : q) = \frac{4}{9}(q^2 + 4qr + 4r^2 + 9rs), \\ \beta_8(r : s : q) = \frac{1}{27}\Big(16(-2q + 8r + 3s)(q^2 + 4qr + 4r^2 + 9rs) \\ \quad - 12(-2q^3 + 4q^2 r + 40qr^2 + 48r^3 + 4q^2 s - 35qrs + 94r^2 s + 36rs^2)\Big), \\ \gamma_8(r : s : q) = -64r^3 s^2, \quad \delta_8(r : s : q) = \frac{64}{3}r^3 s^2(-2q + 8r + 3s). \end{cases}$$

The correspondence χ_8 gives a parametrization of the Humbert surface $\mathcal{H}_8$. We shall use χ_8 in Section 5.

4.3. The case of discriminant 12. In [9, Section 8], Elkies and Kumar considered the elliptic $K3$ surfaces given by the following equation

$$(4.2) \quad y^2 = x^3 + ((1 - f^2)(1 - t) + t)tx^2 + 2et^3(t - 1)x + e^2(t - 1)^2t^5.$$

Here, e and f are two complex parameters. The Weierstrass equation (4.2) defines the elliptic fibration $(x, y, t) \mapsto t$. For a generic point (e, f), the equation (4.2) gives an elliptic surface with singular fibres I_2^*, I_3 and II^* at $t = 0, 1$ and ∞, respectively.

Proposition 4.3. *The $K3$ surface given by (4.2) is birationally equivalent to the elliptic $K3$ surface given by the Weierstrass equation*

$$(4.3) \quad y_1^2 = x_1^3 + \left(\frac{1}{3}(-9e + 15ef - f^4)u^4 + (-e^3(1 + f))u^5 \right)x$$

$$+ \left(u^5 + \frac{1}{27}(-54e^2 - 81ef^2 + 63ef^3 + 2f^6)u^6 \right.$$

$$\left. + \frac{1}{3}e^3(3 + 3e + 3f - 2f^2 - 2f^3)u^7 \right),$$

with singular fibres of type $II^* + 5I_1 + III^*$.

Proof. Putting

$$(4.4) \qquad x = (t - 1)t^3 \left(\frac{e}{(f - 1)t} + u_1 \right), \qquad\qquad y = t^3(t - 1)y_0$$

to (4.2), we obtain an equation in the form

$$(4.5) \qquad\qquad y_0^2 = \text{a polynomial in } t \text{ of degree} 4.$$

Applying the canonical method of Section 3.1 in [1] to (4.5), we have an equation in the form

$$(4.6) \qquad\qquad y_0^2 = 4x_0^3 - I_0(u_1)x_0 - J_0(u_1).$$

By the birational transformation $x_0 = \frac{e^6 x_1}{4(-1+f)^2}, u_1 = -\frac{e^3 u}{-1+f}, y_0 = \frac{e^9 y_1}{4(-1+f)^3}$ to (4.6), we have (4.3). □

Remark 4.4. The birational transformation (4.4) gives an example of 2-neighbor step to obtain the singular fibre of type III^*. See Figure 4.1.
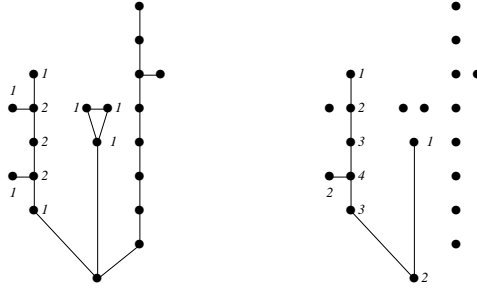
FIGURE 4.1. 2-neiborgh step in (4.4)

From (3.9) and (4.3), we obtain the following proposition.

Proposition 4.5. *The mapping* $\chi_{12} : \mathbb{P}(2 : 1 : 1) \to \mathbb{P}(2 : 3 : 5 : 6) = \mathbb{P}(4 : 6 : 10 : 12)$ *given by* $(e : f : g) \mapsto (\alpha_{12}(e : f : g) : \beta_{12}(e : f : g) : \gamma_{12}(e : f : g) : \delta_{12}(e : f : g))$ *where*

(4.7)
$$\begin{cases} \alpha_{12}(e : f : g) = \frac{1}{9}(f^4 - 15efg + 9eg^2), \\ \beta_{12}(e : f : g) = \frac{1}{54}f(-2f^6 - 63ef^3g + 54e^2g^2 + 81ef^2g^2), \\ \gamma_{12}(e : f : g) = e^3g^3(f + g), \\ \delta_{12}(e : f : g) = \frac{1}{3}e^3g^3(-2f^3 + 3eg - 2f^2g + 3fg^2 + 3g^3), \end{cases}$$

gives a parametrization of the Humbert surface $\mathcal{H}_{12}$.

4.4. The case of discriminant 21. In [9, Section 11], they studied the elliptic $K3$ surface given by the Weierstrass equation

$$(4.8) \quad y^2 = x^3 + (a_0 + a_1t + a_2t^2)x^2 \\ + 2t^2(t - 1)(b_0 + b_1t)x + t^4(t - 1)^2(c_0 + c_1t),$$

where

(4.9)
$$\begin{cases} a_0 = 1, \quad a_1 = -r^2 + 2rs - 1, \quad a_2 = (r - s)^2, \\ b_0 = (r^2 - 1)(s - r)^2, \quad b_1 = (r^2 - 1)(s - r)^2(rs - 1), \\ c_0 = (r^2 - 1)^2(s - r)^4, \quad c_1 = (r^2 - 1)^3(s - r)^4. \end{cases}$$

Here, r and s are two complex parameters. For generic (r, s), an elliptic surface given by (4.8) has singular fibers I_7, I_3 and II^* at $t = 0, 1$ and ∞ respectively.

Proposition 4.6. *The K3 surface given by* (4.8) *is birationally equivalent to the elliptic K3 surface given by the Weierstrass equation*

$$(4.10) \quad y_1^2 = x_1^3 + \Big(-(-1+r)^6(1+r)^4(r-s)u^5$$
$$+ \tfrac{1}{3}(-9 - 30r - 7r^2 + 30r^3 + 15r^4$$
$$+ 30s + 26rs - 30r^2s - 24r^3s - s^2)u^4\Big)x_1$$
$$+ u^5 - \tfrac{2}{27}(r-s)(189 + 252r - 280r^2 - 441r^3$$
$$+ 63r^4 + 189r^5 + 27r^6 - 63s$$
$$- 70rs + 63r^2s + 72r^3s - s^2)u^6$$
$$+ \tfrac{1}{3}(-1+r)^6(1+r)^4(3 - 5r^2 + 3r^4 + 6s + 10rs$$
$$- 6r^3s - 2s^2 + 3r^2s^2)u^7.$$

with singular fibres of type $II^* + 5I_1 + III^*$.

Proof. By the birational transformation given by
$$(4.11)$$
$$\begin{cases} x = (-1+r^2)(r-s)^2(-1+t)t^2x_0, \\ y = \dfrac{1}{2}(-1+r^2)(r-s)^2(-1+t)t^2 \\ \quad \times \Big(2t^3u_0 + 2(1+x_0) - t(1+r^2(-1+x_0) + x_0 - 2rsx_0) \\ \quad - t^2\big(1 + 2u_0 - r^2(-1+x_0) + x_0 - 2sx_0 + 2r(-1+sx_0)\big)\Big), \end{cases}$$

the equation (4.8) is changed to an equation in the form

$$(4.12) \quad a_0 + a_1x_0 + a_2t + a_3x_0^2 + a_4x_0t + a_5t^2$$
$$+ a_6x^3 + a_7x^2t + a_8xt^2 + a_9t^3 = 0.$$

Using the method of Section 3.2 in [1], the equation (4.12) is transformed to the Weierstrass equation in the form

$$(4.13) \qquad\qquad y_0^2 = 4x_0^3 - 108S(u_0)x_0 - 27T(u_0).$$

Then, $x_0 = \tfrac{1}{4}(-1+r)^{10}(1+r)^6(r-s)^3x_1$, $y_0 = \tfrac{1}{4}(-1+r)^{15}(1+r)^9\sqrt{(r-s)^9}y_1$ and $u_0 = \tfrac{1}{2}(-1+r)^5(1+r)^3(r-s)u_1$ to (4.13), we have (4.10). $\qquad\square$

Remark 4.7. The birational transformation (4.11) gives an example of 3-neighbor step to obtain the singular fibre of type III^*. See Figure 4.2.

Setting $s = 1 + r_1 + \frac{r_1(2+r_1)}{s_1}$, $r = 1 + r_1$ and comparing the coefficients of (3.9) and (4.10), we can prove the following proposition.
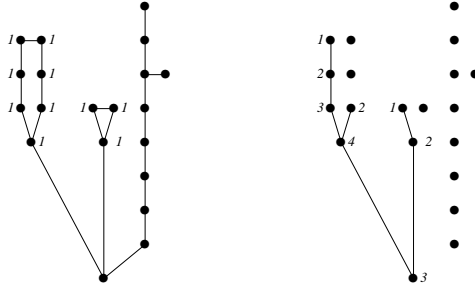
FIGURE 4.2. 3-neiborgh step for (4.11)

Proposition 4.8. *The mapping* $\chi_{21} : \mathbb{P}(1 : 1 : 2) \to \mathbb{P}(2 : 3 : 5 : 6) = \mathbb{P}(4 : 6 : 10 : 12)$ *given by* $(q_1 : r_1 : s_1) \mapsto (\alpha_{21}(q_1 : r_1 : s_1) : \beta_{21}(q_1 : r_1 : s_1) : \gamma_{21}(q_1 : r_1 : s_1) : \delta_{21}(q_1 : r_1 : s_1))$, *where*

(4.14)
$$
\begin{cases}
\alpha_{21}(q_1 : r_1 : s_1) = \frac{1}{9}(q_1^4 + 54q_1^2 s_1 + 24q_1 r_1 s_1 + 9s_1^2), \\
\beta_{21}(q_1 : r_1 : s_1) = \frac{1}{27}(q_1^6 - 135q_1^4 s_1 - 72q_1^3 r_1 s_1 - 405q_1^2 s_1^2 \\
\qquad\qquad\qquad\qquad\qquad - 243q_1 r_1 s_1^2 - 27r_1^2 s_1^2), \\
\gamma_{21}(q_1 : r_1 : s_1) = -r_1^2 s_1^4, \\
\delta_{21}(q_1 : r_1 : s_1) = \frac{1}{3}s_1^4(q_1^2 r_1^2 + 6q_1 r_1^3 + 3r_1^4 + 6q_1 r_1 s_1 + 3s_1^2).
\end{cases}
$$

gives a parametrization of the Humbert surface $\mathcal{H}_{21}$.

5. The Shimura curves of discriminant 6 and 10 in $\mathbb{P}(1 : 3 : 5)$

5.1. The Shimura curves $\Psi_5^*(\mathcal{S}_6)$ and $\Psi_5^*(\mathcal{S}_{10})$. Let us recall that the Humbert surface $\mathcal{H}_\Delta$ of discriminant Δ is a surface in the Igusa 3-fold $\mathcal{A}_2$. Moreover, $\mathcal{A}_2$ is a Zariski open set of the weighted projective space $\mathbb{P}(2 : 3 : 5 : 6) = \text{Proj}(\mathbb{C}[\alpha : \beta : \gamma : \delta])$. Hence, we can regard the Humbert surface $\mathcal{H}_\Delta$ as a divisor in $\mathbb{P}(2 : 3 : 5 : 6)$. Especially, Theorem 3.16 says that the Humbert surface $\mathcal{H}_5 \subset \mathcal{A}_2$ is parametrized by $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) \simeq \overline{\langle PSL(2, \mathcal{O}_5), \tau \rangle \backslash (\mathbb{H} \times \mathbb{H})}$ via the embedding $\Psi_5 : \mathbb{P}(1 : 3 : 5) \hookrightarrow \mathbb{P}(2 : 3 : 5 : 6)$. On the other hand, the Humbert surface $\mathcal{H}_8 \subset \mathcal{A}_2$ is parametrized by χ_8 (see Section 4.2). The intersection $\mathcal{H}_5 \cap \mathcal{H}_8$ of two Humbert surfaces is an analytic subset in $\mathcal{A}_2$. Let us consider the pull-back $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_8)$, that is a curve in $\mathbb{P}(1 : 3 : 5)$.

Theorem 5.1. *The divisor* $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_8)$ *in the weighted projective space* $\mathbb{P}(1:3:5) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A}:\mathfrak{B}:\mathfrak{C}])$ *is given by the defining equation*

$$(5.1) \qquad (\mathfrak{A}^5 - 5\mathfrak{A}^2\mathfrak{B} + \mathfrak{C})$$

$$\times (3125\mathfrak{A}^5 - 3375\mathfrak{A}^2\mathfrak{B} + 243\mathfrak{C})$$

$$\times (384000000000000\mathfrak{A}^9\mathfrak{B}^2 + 1205280000000000\mathfrak{A}^6\mathfrak{B}^3\mathfrak{B}^4$$

$$- 4100625000000\mathfrak{A}^3 - 1845281250000000\mathfrak{B}^5$$

$$+ 25600000000000\mathfrak{A}^10\mathfrak{C} + 69984000000000\mathfrak{A}^7\mathfrak{B}\mathfrak{C}$$

$$+ 34698942000000\mathfrak{A}^4\mathfrak{B}^2\mathfrak{C} + 4253988375000\mathfrak{A}\mathfrak{B}^3\mathfrak{C}$$

$$+ 2576431800000\mathfrak{A}^5\mathfrak{C}^2 + 2714325066000\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2$$

$$+ 146211169851\mathfrak{C}^3) = 0.$$

Proof. We have the parametrization Ψ_5 in (3.19) (resp. χ_8 in (4.1)) of the Humbert surface $\mathcal{H}_5$ (resp. $\mathcal{H}_8$).

For a generic $(\alpha : \beta : \gamma : \delta) \in \mathcal{H}_5 \cap \mathcal{H}_8 \subset \mathbb{P}(2:3:5:6)$, there exist $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in \mathbb{P}(1:3:5)$ and $(r : s : q) \in \mathbb{P}^2(\mathbb{C})$ such that

$$(5.2) \qquad \begin{cases} \alpha = \alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \alpha_8(r, s, q), \\ \beta = \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \beta_8(r, s, q), \\ \gamma = \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \gamma_8(r, s, q), \\ \delta = \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = \delta_8(r, s, q). \end{cases}$$

We have the polynomial $F_1^{(5,8)}$ (resp. $F_2^{(5,8)}, F_3^{(5,8)}, F_4^{(5,8)}$) in $\mathbb{C}[\mathfrak{A} : \mathfrak{B} : \mathfrak{C}, r, s, q]$ of weight 2 (resp. 3, 5, 6) :

$$(5.3) \qquad \begin{cases} F_1^{(5,8)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q) = \alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) - \alpha_8(r, s, q), \\ F_2^{(5,8)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q) = \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) - \beta_8(r, s, q), \\ F_3^{(5,8)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q) = \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) - \gamma_8(r, s, q), \\ F_4^{(5,8)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q) = \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) - \delta_8(r, s, q). \end{cases}$$

We have the weighted homogeneous ideal $I = \langle F_1^{(5,8)}, F_2^{(5,8)}, F_3^{(5,8)}, F_4^{(5,8)} \rangle \subset \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q]$. Let $V(I)$ the zero set of the ideal I. This is an analytic subset of $\mathbb{P}(1:3:5:1:1:1) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, r, s, q])$. From (5.2), the point $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C} : r : s : q) \in V(I)$ gives $(\alpha : \beta : \gamma : \delta) \in \mathcal{H}_5 \cap \mathcal{H}_8$. Let $v : \mathbb{P}(1:3:5:1:1:1) \to \mathbb{P}(1:3:5)$ be the canonical projection given by $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C} : r : s : q) \mapsto (\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$. The Zariski closure of the image $v(V(I))$ corresponds to the zero set $V(I_S)$ of the elimination ideal $I_S = I \cap \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$. From (5.2) again, it follows that $(\alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})) \in \mathcal{H}_5 \cap \mathcal{H}_8$ if and only if $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \in V(I_S)$.

By the theory of Gröbner basis (see [5]) and a computer aided calculation powered by Mathematica, we can show that the ideal I_S is a principal ideal generated by the polynomial in (5.1). Thus, the theorem is proved. $\square$

Using the explicit expression of the parameters of $\mathcal{F}$ in (3.7), let us study the divisor $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_8) \subset \mathbb{P}(1:3:5)$ in detail. According to Example 2.9, $\mathcal{H}_5 \cap \mathcal{H}_8$ contains the Shimura curves $\mathcal{S}_6$ and $\mathcal{S}_{10}$ as irreducible components. We shall give explicit forms of the pull-backs of these two Shimura curves as divisors in $\mathbb{P}(1:3:5)$. We note that the pull-back $\Psi_5^*(\mathcal{S}_6)$ (resp. $\Psi_5^*(\mathcal{S}_{10})$) is isomorphic to $\mathcal{S}_6$ (resp. $\mathcal{S}_{10}$) as varieties because Ψ_5 is an embedding of varieties and $\mathcal{S}_6$ and $\mathcal{S}_{10}$ is contained in the image $\mathrm{Im}(\Psi_5)$. Set

$$(5.4) \quad \begin{cases} R_1 : \mathfrak{A}^5 - 5\mathfrak{A}^2\mathfrak{B} + \mathfrak{C} = 0, \\[4pt] R_2 : 3125\mathfrak{A}^5 - 3375\mathfrak{A}^2\mathfrak{B} + 243\mathfrak{C} = 0, \\[4pt] L_1 : 384000000000000\mathfrak{A}^9\mathfrak{B}^2 + 120528000000000\mathfrak{A}^6\mathfrak{B}^3 \\[4pt] \qquad - 4100625000000\mathfrak{A}^3\mathfrak{B}^4 - 1845281250000000\mathfrak{B}^5 \\[4pt] \qquad + 2560000000000\mathfrak{A}^{10}\mathfrak{C} + 69984000000000\mathfrak{A}^7\mathfrak{B}\mathfrak{C} \\[4pt] \qquad + 346989420000000\mathfrak{A}^4\mathfrak{B}^2\mathfrak{C} + 42539883750000\mathfrak{A}\mathfrak{B}^3\mathfrak{C} \\[4pt] \qquad + 25764318000000\mathfrak{A}^5\mathfrak{C}^2 + 2714325066000\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 \\[4pt] \qquad \qquad \qquad \qquad \qquad + 146211169851\mathfrak{C}^3 = 0. \end{cases}$$

Lemma 5.2. *The curve L_1 in (5.4) is neither $\Psi_5^*(\mathcal{S}_6)$ nor $\Psi_5^*(\mathcal{S}_{10})$.*

Proof. By a direct calculation, the divisor L_1 intersects the divisor $\{\mathfrak{C} = 0\}$ at the three points

$$(5.5) \qquad (\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0), \left(1 : \frac{25}{27} : 0\right), \left(1 : -\frac{64}{135}\right).$$

On the other hand, the Shimura curves $\mathcal{S}_6, \mathcal{S}_{10}$ are already compact. Then, $\Psi_5^*(\mathcal{S}_6)$ and $\Psi_5^*(\mathcal{S}_{10})$ never touch the cusp of $\langle PSL(2, \mathcal{O}_5), \tau \rangle \backslash (\mathbb{H} \times \mathbb{H})$. According to Proposition 3.2(2), the cusp is given by $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0)$. So, from (5.5), the curve L_1 is neither $\Psi_5^*(\mathcal{S}_6)$ nor $\Psi_5^*(\mathcal{S}_{10})$. $\square$

According to [13], we have the quaternion modular embedding $\check{\Omega}_6 : \mathbb{H} \to \mathfrak{S}_2$ for $D = 6$ given by

$$(5.6) \qquad w \mapsto \check{\Omega}_6(w) = \begin{pmatrix} \dfrac{3w}{2} - \dfrac{1}{4w} & -\dfrac{3\sqrt{2}w}{4} - \dfrac{1}{2} - \dfrac{\sqrt{2}}{8w} \\[10pt] -\dfrac{3\sqrt{2}w}{4} - \dfrac{1}{2} - \dfrac{\sqrt{2}}{8w} & \dfrac{3w}{4} - \dfrac{1}{2} - \dfrac{1}{8w} \end{pmatrix}$$

for $w \in \mathbb{H}$.

Remark 5.3. The modular embedding $\check{\Omega}_6$ is different from the embedding Ω in Section 2.3. However, as we noted in Remark 2.4, we have the unique choice of the Shimura curve $\mathcal{S}_6$. So, our argument for discriminant 6 is not dependent on the choice of a quaternion modular embeddings.

For

$$M = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z}).$$

and $w \mapsto \check{\Omega}_6(w)$ in (5.6), we have another modular embedding $\mathbb{H} \to \mathfrak{S}_2$ given by $w \mapsto \tilde{\Omega}_6(w)$ where

$$(5.7) \qquad \tilde{\Omega}_6(w) = (A\check{\Omega}_6(w) + B)(C\check{\Omega}_6(w) + D)^{-1}$$

$$= \begin{pmatrix} \dfrac{-2+\sqrt{2}+4w+6(2+\sqrt{2})w^2}{-1+\sqrt{2}+8w+6(1+\sqrt{2})w^2} & \dfrac{\sqrt{2}+4w+6\sqrt{2}w^2}{-1+\sqrt{2}+8w+6(1+\sqrt{2})w^2} \\ \dfrac{\sqrt{2}+4w+6\sqrt{2}w^2}{-1+\sqrt{2}+8w+6(1+\sqrt{2})w^2} & \dfrac{-2+12w^2}{-1+\sqrt{2}+8w+6(1+\sqrt{2})w^2} \end{pmatrix}.$$

Setting $\tilde{\Omega}_6(w) = \begin{pmatrix} \tilde{\tau}_1(w) & \tilde{\tau}_2(w) \\ \tilde{\tau}_2(w) & \tilde{\tau}_3(w) \end{pmatrix}$, it holds

$$(5.8) \qquad -\tilde{\tau}_1(w) + \tilde{\tau}_2(w) + \tilde{\tau}_3(w) = 0.$$

Especially, $\tilde{\Omega}_6$ embeds $\mathbb{H}$ to N_5 in (3.6). Recall that the surface N_5 is parametrized by the Hilbert modular embedding μ_5 in (3.5).

Lemma 5.4. *Let $\square$ be the diagonal, μ_5 be the Hilbert modular embedding given by (3.5) and pr be the canonical projection $\mathfrak{S}_2 \to Sp(4, \mathbb{Z})\backslash\mathfrak{S}_2 = \mathcal{A}_2$. Set $M_6 = pr \circ \mu_5(\square)(\subset \mathcal{A}_2)$. Then, $\mathcal{S}_6$ intersects M_6 at only one point $pr \circ \mu_5(1 + \sqrt{-1}, 1 + \sqrt{-1})$.*

Proof. The embedding $w \mapsto \tilde{\Omega}_6(w)$ parametrizes a curve in the surface N_5. The surface N_5 is parametrized by $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ via μ_5. Hence, the set $\mu_5(\square) \cap \text{Im}(\tilde{\Omega}_6)$ is given by the condition $\tilde{\tau}_2(w) = 6\sqrt{2}w^2 + 4w + \sqrt{2} = 0$. The solution in the upper half plane $\mathbb{H}$ of the equation $6\sqrt{2}w^2 + 4w + \sqrt{2} = 0$ is only

$$w = \frac{1}{3}\left(\sqrt{-1} - \frac{1}{\sqrt{2}}\right).$$

By a direct calculation, we can check that

$$\tilde{\Omega}_6\left(\frac{1}{3}\left(\sqrt{-1} - \frac{1}{\sqrt{2}}\right)\right) = \mu_5(1 + \sqrt{-1}, 1 + \sqrt{-1}). \qquad \square$$

We note that a point in $N_5 = \text{Im}(\mu_5)$ has an expression by the period mapping for the family $\mathcal{F}$ of M_5-marked $K3$ surfaces (see Section 3.3). The explicit theta expression (3.7) of the inverse of the period mapping for $\mathcal{F}$ enables us to study the quaternion embedding $\tilde{\Omega}_6$ given by (5.7) in detail.

Theorem 5.5. *The Shimura curve* $\Psi_5^*(\mathcal{S}_6)$ *(resp.* $\Psi_5^*(\mathcal{S}_{10})$*) is given by the divisor* R_2 *(resp.* R_1*) in* (5.4).

Proof. According to Theorem 5.1, the Shimura curves $\Psi_5^*(\mathcal{S}_6)$ and $\Psi_5^*(\mathcal{S}_{10})$ are irreducible components of the union of the curves $R_1 \cup R_2 \cup L_1$. However, from Lemma 5.2, the curve L_1 never give any Shimura curve. According to (3.2), the (X, Y)-plane gives an affine plane of $\mathbb{P}(1 : 3 : 5)$. Due to Lemma 5.4, the Shimura curve $\Psi_5^*(\mathcal{S}_6)$ passes the point

$$P_0 = (X(1 + \sqrt{-1}, 1 + \sqrt{-1}), Y(1 + \sqrt{-1}, 1 + \sqrt{-1})).$$

Since we have the formula (3.8),

$$P_0 = \Big(\frac{25}{27} \frac{1}{J(1 + \sqrt{-1})}, 0 \Big) = \Big(\frac{25}{27}, 0 \Big) \in (X, Y) - \text{plane}.$$

On the other hand, by a direct observation, the curve R_1 does not touch the point P_0 and the curve R_2 passes the point P_0.

Therefore, the Shimura curve $\Psi_5^*(\mathcal{S}_6)$ is given by the curve R_2. The other curve R_1 corresponds to the Shimura curve $\Psi_5^*(\mathcal{S}_{10})$. $\qquad\square$

In Figure 5.1, the Shimura curves $R_2 = \Psi_5^*(\mathcal{S}_6)$, $R_1 = \Psi_5^*(\mathcal{S}_{10})$ and the curve coming from Klein's icosahedral equation

$$-1728\mathfrak{B}^5 + 720\mathfrak{A}\mathfrak{C}\mathfrak{B}^3 - 80\mathfrak{A}^2\mathfrak{C}^2\mathfrak{B} + 64\mathfrak{A}^3(5\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C})^2 + \mathfrak{C}^3 = 0$$
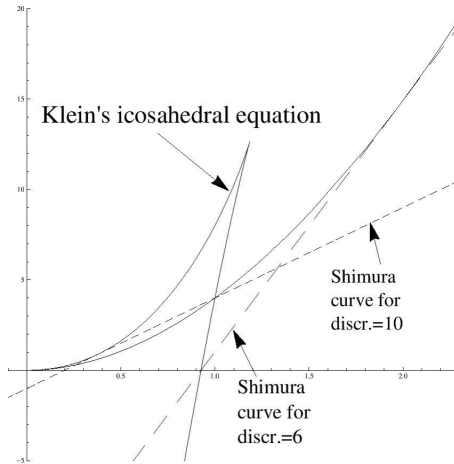
(see (3.1)).

FIGURE 5.1. The Shimura curves of discriminant 6 and 10 and the Klein's icosahedral equation

5.2. The genus 2 curves of Hashimoto and Murabayashi and the family $\mathcal{K}_j$ of Kummer surfaces. Hashimoto and Murabayashi [13] studied the moduli space of genus 2 curves and the Shimura curves for discriminant 6 and 10. In this subsection, let us see the relation between the results of [13] and Theorem 5.5.

Let C be a Riemann surface of genus 2. The Jacobian variety $\mathrm{Jac}(C)$ is a principally polarized abelian surface. Let T be the involution on $\mathrm{Jac}(C)$ induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$ on the universal covering $\mathbb{C}^2$. The minimal resolution $\overline{\mathrm{Jac}(C)/\langle id, T \rangle}$ is called the Kummer surface and denoted by $\mathrm{Kum}(C)$.

For a Riemann surface of genus 2 given by

$$(5.9) \qquad C(\lambda_1, \lambda_2, \lambda_3) : Y^2 = X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3),$$

Humbert [15] obtained explicit conditions when the corresponding Jacobian variety $\mathrm{Jac}(C(\lambda_1, \lambda_2, \lambda_3))$ has real multiplication for $\Delta = 5$ and $\Delta = 8$. These conditions are given by equations, called Humbert's modular equations, in λ_1, λ_2 and λ_3 (see [13, Theorem 2.9 and 2.11]). For example, Humbert's module equation for $\Delta = 5$ is given by

$$
\begin{aligned}
(5.10) \quad &4(\lambda_1^2\lambda_3 - \lambda_2^2 + \lambda_3^2(1-\lambda_1) + \lambda_2^2\lambda_3)(\lambda_1^2\lambda_2\lambda_3 - \lambda_1\lambda_2^2\lambda_3) \\
&= (\lambda_1^2(\lambda_2+1)\lambda_3 - \lambda_2^2(\lambda_1+\lambda_3) + (1-\lambda_1)\lambda_2\lambda_3^2 + \lambda_1(\lambda_2-\lambda_3))^2.
\end{aligned}
$$

It is well known that $\mathrm{Kum}(C(\lambda_1, \lambda_2, \lambda_3))$ is given by the double cover of the projective plane $\mathbb{P}^2(\mathbb{C}) = \mathrm{Proj}(\mathbb{C}[\zeta_0, \zeta_1, \zeta_2])$ branched along 6 lines $\zeta_2 = 0, \zeta_0 = 0, \zeta_2 + 2\zeta_1 + \zeta_0 = 0$ and $\zeta_2 + 2\lambda_j\zeta_1 + \lambda_j^2\zeta_0 = 0$ $(j = 1, 2, 3)$. Humbert's modular equations for $\Delta = 5$ and $\Delta = 8$ are obtained by a study of $\mathrm{Kum}(C(\lambda_1, \lambda_2, \lambda_3))$.

Remark 5.6. Let $\mathcal{M}_{2,2}$ be the moduli space of genus 2 curves with level 2 structure and $\mathcal{Q} : \mathcal{M}_{2,2} \to \mathcal{A}_2$ be the canonical projection. Humbert's modular equation for Δ is not a defining equation of the Humbert surface $\mathcal{H}_\Delta \subset \mathcal{A}_2$, but defines a component of $\mathcal{Q}^{-1}(\mathcal{H}_\Delta)$. To the best of the author's knowledge, to study Humbert's modular equations is not easy, for they have complicated forms in λ_1, λ_2 and λ_3. However, Humbert's modular equation (5.10) for $\Delta = 5$ and our simple modular equation (3.11) for $\mathcal{H}_5$ are explicitly related by the formula of [25] Theorem 8.7.

Hashimoto and Murabayashi [13] studied the genus 2 curve $C(\lambda_1, \lambda_2, \lambda_3)$, where $(\lambda_1, \lambda_2, \lambda_3)$ satisfies Humbert's modular equations $\Delta = 5$ and $\Delta = 8$. Applying Theorem 2.6 (2), they obtained the following results.

Proposition 5.7 ([13, Theorem 1.3, 1.7])**.**

(1) *Set a genus 2 curve $C_6(s,t)$ given by*

$$C_6(s,t) : Y^2 = X(X^4 - PX^3 + QX^2 - RX + 1)$$

with $P = -2(s+t)$, $Q = \frac{(1+2t^2)(11-28t^2+8t^4)}{3(1-t^2)(1-4t^2)}$ and $R = -2(s-t)$. Here, (s,t) satisfies

(5.11) $4s^2t^2 - s^2 + t^2 + 2 = 0.$

Then, $\text{Jac}(C_6(s,t))$ is a principally polarized abelian surface with quaternion multiplication by $\mathfrak{O}_6$.

(2) *Set a genus 2 curve $C_{10}(s,t)$ given by*

$$C_{10}(s,t) : Y^2 = X(X^4 - PX^3 + QX^2 - RX + 1)$$

with $P = \frac{4(2t+1)(t^2-t-1)}{(t-1)^2}$, $Q = \frac{(1+t^2)(t^4+8t^3-10t^2-8t+1)}{t(t-1)^2(t+1)^2}$ and $R = \frac{(t-1)s}{t(t+1)(2t+1)}$. Here, (s,t) satisfies

(5.12) $s^2 - t(t-2)(2t+1) = 0.$

Then, $\text{Jac}(C_{10}(s,t))$ is a principally polarized abelian surface with quaternion multiplication by $\mathfrak{O}_{10}$.

Remark 5.8. The equation (5.11) (resp. (5.12)) does not give the exact defining equation of the Shimura curve $\mathcal{S}_6$ (resp. $\mathcal{S}_{10}$) but defines a covering of $\mathcal{S}_6$ (resp. $\mathcal{S}_{10}$). In fact, each curve defined by (5.11) and (5.12) is of genus 1. On the other hand, $\mathcal{S}_6$ and $\mathcal{S}_{10}$ are genus 0 curves.

Let $j \in \{6, 10\}$. Set $\mathcal{C}_j = \{C_j(s,t)\}$ in Proposition 5.7. For two members $C_j(s_1,t_1)$ and $C_j(s_2,t_2)$ of $\mathcal{C}_j$, if $\text{Jac}(C_j(s_1,t_1))$ and $\text{Jac}(C_j(s_2,t_2))$ are isomorphic as principally polarized abelian surfaces, we call two members are equivalent. Let $[C_j(s,t)]$ be the equivalence class of $C_j(s,t) \in \mathcal{C}_j$. Let $\tilde{\mathcal{C}}_j$ denote the equivalent class of $\mathcal{C}_j$. We have the family $\text{Kum}(\tilde{\mathcal{C}}_j) = \{\text{Kum}(C_j(s,t)) \mid [C_j(s,t)] \in \tilde{\mathcal{C}}_j\}$ of Kummer surfaces.

On the other hand, our $K3$ surface $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ in (3.3) has the Shioda–Inose structure. Namely, there exists an involution σ on $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ such that the minimal resolution $K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ of $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})/\langle id, \sigma \rangle$ is a Kummer surface. The Kummer surface $K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$ is given by the following equation (see [23] Theorem 2.13),

(5.13) $v^2 = (u^2 - 2t^5)(u - (5\mathfrak{A}t^2 - 10\mathfrak{B}t + \mathfrak{C})).$

Remark 5.9. The period mapping for the family $\mathcal{F} = \{S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$ coincides with that of the family $\mathcal{K} = \{K(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})\}$ of Kummer surfaces. (see [23, Section 2.4]).

From Theorem 5.5 and the defining equation (5.13), we have two families of Kummer surfaces $\mathcal{K}_j = \{K_j(\mathfrak{A} : \mathfrak{B})\}$ $(j \in \{6, 10\})$ given by

$$(5.14) \begin{cases} K_6(\mathfrak{A} : \mathfrak{B}) : v^2 = (u^2 - 2t^5)(u - (5\mathfrak{A}t^2 - 10\mathfrak{B}t - \frac{3125\mathfrak{A}^5 - 3375\mathfrak{A}^2\mathfrak{B}}{243})), \\ K_{10}(\mathfrak{A} : \mathfrak{B}) : v^2 = (u^2 - 2t^5)(u - (5\mathfrak{A}t^2 - 10\mathfrak{B}t - (\mathfrak{A}^5 - 5\mathfrak{A}^2\mathfrak{B})). \end{cases}$$

Considering the properties of the $K3$ surface $S(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})$, the above procedure of the families $\mathcal{K}_j$ for $j = 6, 10$, Remark 2.4, Remark 5.6 and Remark 5.9, we have the following proposition.

Proposition 5.10. *For $j \in \{6, 10\}$, the family* $\mathrm{Kum}(\tilde{\mathcal{C}}_j)$ *coincides with the family* $\mathcal{K}_j$.

6. The Shimura curves of discriminant 14 and 15 in $\mathbb{P}(1 : 3 : 5)$

In this section, we obtain the explicit forms of the Shimura curves for discriminants 14 and 15 in the weighted projective space $\mathrm{Proj}(\mathbb{C}[\mathfrak{A} : \mathfrak{B} : \mathfrak{C}])$.

However, as in Remark 2.4, the Shimura curve $\mathcal{S}_D = \varphi_D(\mathbb{S}_D) \subset \mathcal{A}_2$ is not unique for $D = 14$ because there exist two choices of φ_D. So, the image $\mathcal{S}_{14}$ of the Shimura curve depends on the triples (p, a, b) in the argument of Section 2.3.

In this section, as in Example 2.10 and 2.11, we only consider the Shimura curve $\mathcal{S}_{14}$ in $\mathcal{A}_2$ coming form the triple $(p, a, b) = (5, 1, 3)$.

Theorem 6.1. *The pull-back* $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{12})$ *is given by the union of four devisors* R_2, R_3, R_4, L_2, *where* R_2 *is given by (5.4) and* R_3, R_4 *and* L_2 *are curves in* $\mathbb{P}(1 : 3 : 5) = \mathrm{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}])$ *in the following:*

$R_3 : 10485760\mathfrak{A}^{10} - 309657600\mathfrak{A}^7\mathfrak{B} - 1446336000\mathfrak{A}^4\mathfrak{B}^2$
$\qquad - 1574640000\mathfrak{A}\mathfrak{B}^3 + 72721152\mathfrak{A}^5\mathfrak{C} - 272937600\mathfrak{A}^2\mathfrak{B}\mathfrak{C} - 59049\mathfrak{C}^2 = 0,$

$R_4 : 30517578125\mathfrak{A}^{15} + 91186523437500\mathfrak{A}^{12}\mathfrak{B} + 42529296875\mathfrak{A}^9\mathfrak{B}^2$
$\qquad - 97897974609375\mathfrak{A}^6\mathfrak{B}^3 + 424490000000000\mathfrak{A}^3\mathfrak{B}^4$
$\qquad - 34560000000000000\mathfrak{B}^5 + 2383486328125\mathfrak{A}^{10}\mathfrak{C}$
$\qquad + 32875975781250\mathfrak{A}^7\mathfrak{B}\mathfrak{C} - 147816767984375\mathfrak{A}^4\mathfrak{B}^2\mathfrak{C}$
$\qquad + 228155760000000\mathfrak{A}\mathfrak{B}^3\mathfrak{C} + 191892046718750\mathfrak{A}^5\mathfrak{C}^2$
$\qquad - 29675018141125\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + 344730881243\mathfrak{C}^3 = 0,$

$L_2 : -640000000000000000000\mathfrak{A}^{12}\mathfrak{B}^6 + 3700000000000000000000\mathfrak{A}^9\mathfrak{B}^7$

$\qquad + 8152343750000000000000\mathfrak{A}^6\mathfrak{B}^8 - 3902343750000000000000\mathfrak{A}^3\mathfrak{B}^9$

$\qquad - 7119140625000000000000\mathfrak{B}^10 + 3840000000000000000000\mathfrak{A}^{13}\mathfrak{B}^4\mathfrak{C}$

$\qquad - 2239200000000000000000\mathfrak{A}^{10}\mathfrak{B}^5\mathfrak{C} - 10759675000000000000000\mathfrak{A}^7\mathfrak{B}^6\mathfrak{C}$

$\qquad + 3969323437500000000000\mathfrak{A}^4\mathfrak{B}^7\mathfrak{C} + 4702429687500000000000\mathfrak{A}\mathfrak{B}^8\mathfrak{C}$

$\qquad - 7680000000000000000\mathfrak{A}^{14}\mathfrak{B}^2\mathfrak{C}^2 + 4557760000000000000000\mathfrak{A}^{11}\mathfrak{B}^3\mathfrak{C}^2$

$\qquad + 4497306980000000000000\mathfrak{A}^8\mathfrak{B}^4\mathfrak{C}^2 - 1463038602500000000000\mathfrak{A}^5\mathfrak{B}^5\mathfrak{C}^2$

$\qquad - 1122863301562500000000\mathfrak{A}^2\mathfrak{B}^6\mathfrak{C}^2 + 5120000000000000000\mathfrak{A}^{15}\mathfrak{C}^3$

$\qquad - 3077760000000000000\mathfrak{A}^{12}\mathfrak{B}\mathfrak{C}^3 - 7756101040000000000\mathfrak{A}^9\mathfrak{B}^2\mathfrak{C}^3$

$\qquad + 2359593227400000000000\mathfrak{A}^6\mathfrak{B}^3\mathfrak{C}^3 + 1213513231187500000000\mathfrak{A}^3\mathfrak{B}^4\mathfrak{C}^3$

$\qquad + 135237021187500000000\mathfrak{B}^5\mathfrak{C}^3 + 4779900760000000000000\mathfrak{A}^{10}\mathfrak{C}^4$

$\qquad - 13908191752800000000000\mathfrak{A}^7\mathfrak{B}\mathfrak{C}^4 - 8326918293212000000000\mathfrak{A}^4\mathfrak{B}^2\mathfrak{C}^4$

$\qquad - 25308770872275000000\mathfrak{A}\mathfrak{B}^3\mathfrak{C}^4 + 4494155396468000000\mathfrak{A}^5\mathfrak{C}^5$

$\qquad + 1039220333140600000\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^5 + 50787635527751\mathfrak{C}^6 = 0.$

Proof. Recalling (3.19) and (4.7), set

$$\begin{cases} F_1^{(5,12)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, e, f, g) = \alpha_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \alpha_{12}(e, f, g), \\ F_2^{(5,12)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, e, f, g) = \beta_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \beta_{12}(e, f, g), \\ F_3^{(5,12)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, e, f, g) = \gamma_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \gamma_{12}(e, f, g), \\ F_4^{(5,12)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, e, f, g) = \delta_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \delta_{12}(e, f, g). \end{cases}$$

Take the weighted homogeneous ideal $I_{12} = \langle F_1^{(5,12)}, F_2^{(5,12)}, F_3^{(5,12)}, F_4^{(5,12)} \rangle$ of the ring $\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, e, f, g]$. As in the proof of Theorem 5.1, the zero set of the ideal $I_{12,S} = I_{12} \cap \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$ corresponds to the pull-back $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{12})$. By a computer aided calculation, we can show that the zero set of $I_{12,S}$ is the union of the curves R_2, R_3, R_4 and L_2. $\qquad\square$

Theorem 6.2. *The Shimura curve $\Psi_5^*(\mathcal{S}_{15})$ corresponds to R_3 and the Shimura curve $\Psi_5^*(\mathcal{S}_{14})$ corresponds to R_4.*

Proof. First, according to Theorem 6.1, the divisor $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{12})$ consists of only four irreducible components R_2, R_3, R_4 and L_2. However, from Theorem 5.5, the curve R_2 is the Shimura curve $\Psi_5^*(\mathcal{S}_6)$. Moreover, since the curve L_2 passes the cusp $(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) = (1 : 0 : 0)$, the curve L_2 does not corresponds to any Shimura curves.

Then, we shall identify the curves R_3 and R_4. Because we have Example 2.10, the divisor $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{12})$ contains the Shimura curves $\Psi_5^*(\mathcal{S}_{15})$ and $\Psi_5^*(\mathcal{S}_{14})$. So, of the two curves R_3 and R_4, one corresponds to $\Psi_5^*(\mathcal{S}_{15})$ and the other corresponds to $\Psi_5^*(\mathcal{S}_{14})$. By the way, according to Example 2.11, only the Shimura curve $\Psi_5^*(\mathcal{S}_{14})$ is contained in the divisor $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{21})$. Moreover, due to the next lemma, the curve R_4 is an irreducible component of $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{21})$. Therefore, we conclude that the curve R_3 (resp. R_4) gives the explicit model of the Shimura curve $\Psi_5^*(\mathcal{S}_{15})$, (resp. $\Psi_5^*(\mathcal{S}_{14})$). $\qquad \square$

Lemma 6.3. *The curve R_4 is an irreducible component of the divisor $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{21})$.*

Proof. Using the notation in (3.19) and (4.14), we set

$$
\begin{cases}
F_1^{(5,21)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, q_1, r_1, s_1) = \alpha_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \alpha_{21}(q_1, r_1, s_1), \\
F_2^{(5,21)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, q_1, r_1, s_1) = \beta_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \beta_{21}(q_1, r_1, s_1), \\
F_3^{(5,21)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, q_1, r_1, s_1) = \gamma_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \gamma_{21}(q_1, r_1, s_1), \\
F_4^{(5,21)}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, q_1, r_1, s_1) = \delta_5(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) - \delta_{21}(q_1, r_1, s_1).
\end{cases}
$$

Take the weighted homogeneous ideal $I_{21} = \langle F_1^{(5,21)}, F_2^{(5,21)}, F_3^{(5,21)}, F_4^{(5,21)} \rangle$ in the ring $\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, q_1, r_1, s_1]$. The zero set of the ideal $I_{21,S} = I_{21} \cap \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$ gives the curve $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{21})$. However, because of a huge amount of calculations of the Gröbner basis, it is very difficult to obtain a system of generators of the ideal $I_{21,S}$ directly.

Instead, we shall consider the ideal $I_{21,T} = I_{21} \cap \mathbb{C}[q_1, r_1, s_1]$. By a computer aided calculation, we can show that the ideal $I_{21,T}$ is a principal ideal generated by

$$
\begin{aligned}
(6.1) \quad & q_1(2q_1 + r_1)s_1^2(q_1^3 - q_1 s_1 - r_1 s_1)(2q_1^4 - 27q_1^2 s_1 - 27q_1 r_1 s_1 + 81s_1^2) \\
& \times (2q_1^4 + q_1^3 r_1 - 11q_1^2 s_1 - 22q_1 r_1 s_1 - 8r_1^2 s_1 + 9s_1^2) \\
& \times (q_1^6 - 20q_1^4 s_1 - 9q_1^3 r_1 s_1 + 98q_1^2 s_1^2 + 62q_1 r_1 s_1^2 + 8r_1^2 s_1^2 + 9s_1^3).
\end{aligned}
$$

Next, taking a factor $q_1^3 - q_1 s_1 - r_1 s_1$ of (6.1), we consider the new ideal $J_{21} = \langle F_1^{(5,21)}, F_2^{(5,21)}, F_3^{(5,21)}, F_4^{(5,21)}, q_1^3 - q_1 s_1 - r_1 s_1 \rangle$. The zero set $V(J_{21,S})$ of the elimination ideal $J_{21,S} = J_{21} \cap \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$ corresponds to certain components of $\Psi_5^*(\mathcal{H}_5 \cap \mathcal{H}_{21})$. By a computer aided calculation, we can compute the Gröbner basis of the ideal $J_{21,S}$. So, we can check that the curve R_4 is contained in the zero set of the ideal $J_{21,S}$. $\qquad \square$

The author believes that the method using icosahedral invariants and the theta expression (3.7) is effective for the Shimura curves $\mathcal{S}_D$ for $D > 15$ with some skillful computations of Gröbner basis.

References

[1] S. Y. AN, S. Y. KIM, D. C. MARSHALL, S. H. MARSHALL, W. G. MCCALLUM & A. R. PERLIS, "Jacobians of Genus One Curves", *J. Number Theory* **90** (2001), no. 2, p. 304-315.

[2] A. BESSER, "Elliptic fibrations of $K3$ surfaces and QM Kummer surfaces", *Math. Z.* **288** (1998), no. 2, p. 283-308.

[3] M. A. BONFANTI & B. VAN GEEMEN, "Abelian surfaces with an automorphism and quaternionic multiplication", *Can. J. Math.* **68** (2016), no. 1, p. 24-43.

[4] A. CLINGHER & C. F. DORAN, "Lattice polarized $K3$ surfaces and Siegel modular forms", *Adv. Math.* **231** (2012), no. 1, p. 172-212.

[5] D. COX, J. LITTLE & D. O'SHEA, *Using algebraic geometry*, Graduate Texts in Mathematics, vol. 185, Springer, 1998, xii+499 pages.

[6] I. V. DOLGACHEV, "Mirror symmetry for lattice polarized $K3$ surfaces", *J. Math. Sci., New York* **81** (1996), no. 3, p. 2599-2630.

[7] N. ELKIES, "Shimura curve computations", in *Algorithmic number theory. 3rd international symposium*, Lecture Notes in Computer Science, vol. 1423, Springer, 1998, p. 1-47.

[8] ———, "Shimura curve computations via $K3$ surfaces of Néron-Severi rank at least 19", in *Algorithmic number theory. 8th international symposium*, Lecture Notes in Computer Science, vol. 5011, Springer, 2008, p. 196-211.

[9] N. ELKIES & A. KUMAR, "$K3$ Surfaces and equations for Hilbert modular surfaces", *Algebra Number Theory* **8** (2014), no. 10, p. 2297-2411.

[10] G. VAN DER GEER, *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 16, Springer, 1988, ix+291 pages.

[11] K. HASHIMOTO, A. NAGANO & K. UEDA, "Modular surfaces associated with toric $K3$ surfaces", `https://arxiv.org/abs/1403.5818`, 2014.

[12] K.-I. HASHIMOTO, "Explicit form of quaternion modular embeddings", *Osaka J. Math.* **32** (1995), no. 3, p. 533-546.

[13] K.-I. HASHIMOTO & N. MURABAYASHI, "Shimura curves as intersections of Humbert surfaces and defining equations of QM-curves of genus two", *Tôhoku Math. J.* **47** (1995), no. 2, p. 271-296.

[14] F. HIRZEBRUCH, "The ring of Hilbert modular forms for real quadratic fields of small discriminant", in *Modular Funct. of one Var. VI, Proc. int. Conf., Bonn 1976*, Lecture Notes in Mathematics, vol. 627, Springer, 1977, p. 287-323.

[15] G. HUMBERT, "Sur les fonctions abéliennes singulières", in *Oeuvres de G. Humbert 2, pub. par les soins de Pierre Humbert et de Gaston Julia*, Gauthier-Villars, 1936, p. 297-401.

[16] F. KLEIN, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Tauber, 1884.

[17] D. R. KOHEL & H. A. VERRILL, "Fundamental Domains for Shimura Curves", *J. Théor. Nombres Bordx* **15** (2003), no. 1, p. 205-222.

[18] A. KUMAR, "$K3$ surfaces associated to curves of genus two", *Int. Math. Res. Not.* **16** (2008), ArticleID: rnm165.

[19] A. KURIHARA, "On some examples of equations defining Shimura curves and the Mumford uniformization", *J. Fac. Sci., Univ. Tokyo* **25** (1978), p. 277-300.

[20] R. MÜLLER, "Hilbertsche Modulformen und Modulfunktionen zu $\mathbb{Q}(\sqrt{5})$", *Arch. Math.* **45** (1985), p. 239-251.

[21] A. NAGANO, "Period differential equations for the families of $K3$ surfaces with two parameters derived from the reflexive polytopes", *Kyushu J. Math.* **66** (2012), no. 1, p. 193-244.

[22] ———, "A theta expression of the Hilbert modular functions for $\sqrt{5}$ via period of $K3$ surfaces", *Kyoto J. Math.* **53** (2013), no. 4, p. 815-843.

[23] ———, "Double integrals on a weighted projective plane and Hilbert modular functions for $\mathbb{Q}(\sqrt{5})$", *Acta Arith.* **167** (2015), no. 4, p. 327-345.

[24] ———, "Icosahedral invariants and a construction of class fields via periods of $K3$ surfaces", `https://arxiv.org/abs/1504.07500v7`, 2017.

[25] A. NAGANO & H. SHIGA, "Modular map for the family of abelian surfaces via elliptic $K3$ surfaces", *Math. Nachr.* **288** (2015), no. 1, p. 89-114.

[26] V. ROTGER, "Modular Shimura varieties and forgetful maps", *Trans. Am. Math. Soc.* **356** (2004), no. 4, p. 1535-1550.

[27] ———, "Shimura curves embedded in Igusa's threefold", in *Modular curves and Abelian varieties. Based on lectures of the conference, Bellaterra, Barcelona, July 2002*, Prog. Math., vol. 224, Birkhäuser, 2004, p. 263-276.

[28] G. SHIMURA, "Construction of class fields and zeta functions of algebraic curves", *Ann. Math.* **85** (1967), p. 58-159.

[29] ———, "On canonical models of arithmetic quotients of bounded symmetric domains", *Ann. Math* **91** (1970), p. 144-222.

[30] ———, "On the real points of an arithmetic quotient of a bounded symmetric domain", *Math. Ann.* **215** (1975), p. 135-164.

[31] ———, *Abelian Varieties with Complex Multiplication and Modular Functions*, Princeton Mathematical Series, vol. 46, Princeton University Press, 1998, ix+217 pages.

[32] M.-F. VIGNÉRAS, *Arithmétiques des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 80, Springer, 1980, vii+169 pages.

[33] J. VOIGHT, "Shimura Curve Computations", in *Arithmetic geometry*, Clay Mathematics Proceedings, vol. 8, Clay Mathematics Institute, 2009, p. 103-113.

[34] Y. YANG, "Quaternionic loci in Siegel's modular threefolds", 2015, `http//www.tims.ntu.edu.tw/download.talk.Summary.pdf`.

Atsuhira NAGANO
Department of Mathematics
King's College London
Strand, London, WC2R 2LS, UK
E-mail: `atsuhira.nagano@gmail.com`