Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 28, n° 3 (2016), p. 811-828.

<http://jtnb.cedram.org/item?id=JTNB_2016__28_3_811_0>

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Height and Weber's Class Number Problem

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Résumé. Nous étudions la non divisibilité par un nombre premier ℓ du nombre de classes h_n du n-ième étage $\mathbb{B}_n$ de la $\mathbb{Z}_p$-extension cyclotomique de $\mathbb{Q}$, où p est un nombre premier fixé. Posons $q = 4$ si $p = 2$ et $q = p$ si $p \geq 3$ et notons $D(p, s, f)$ l'ensemble des nombres premiers ℓ dont l'ordre modulo q vaut f et dont p^s est la plus grande puissance de p divisant $\ell^f - 1$. Dans cet article nous définissons une constante explicite $G(p, s, f)$ ayant la propriété que chaque h_n est non divisible par les ℓ dans $D(p, s, f)$ tels que $\ell > G(p, s, f)$.

Abstract. We discuss indivisibility by prime numbers ℓ of the class number of the n-th layer $\mathbb{B}_n$ of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ where p is an arbitrary fixed prime number.

We denote by h_n the class number of $\mathbb{B}_n$. Put $q = 4$ if $p = 2$ or $q = p$ if $p \geq 3$. For positive integers f and s, let $D(p, s, f)$ be the set of prime numbers ℓ satisfying the following two conditions: (1) the order of ℓ modulo q is f and (2) p^s is the exact power of p dividing $\ell^f - 1$. In this paper, we define an explicit function $G(p, s, f)$ which depends only on p, s and f. We show that h_n is indivisible by every prime number ℓ in $D(p, s, f)$ with $\ell > G(p, s, f)$ for every non-negative integer n.

1. Introduction

Let p be a prime number and μ_m the group of all m-th roots of unity. We put $q = 4$ if $p = 2$ or $q = p$ if $p \geq 3$. We denote by $\mathbb{B}_n$ the unique real subfield of $\mathbb{Q}(\mu_{qp^n})$ which is the cyclic extension of the rational number field $\mathbb{Q}$ with degree p^n. Note that the Galois group of $\mathbb{B}_\infty = \bigcup_{n \geq 0} \mathbb{B}_n$ over $\mathbb{Q}$ is isomorphic to the p-adic integer ring $\mathbb{Z}_p$ as additive group. The fields $\mathbb{B}_\infty$ and $\mathbb{B}_n$ are called the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and its n-th layer, respectively. We denote by h_n the class number of $\mathbb{B}_n$. We consider the following problem.

Weber's class number problem. *Is the class number h_n equal to* 1 *for every non-negative integer n?*

Manuscrit reçu le 6 novembre 2014, révisé le 27 juin 2015, accepté le 7 juillet 2015.
Mathematics Subject Classification. 11R06, 11R18, 11R29.
Mots-clefs. Class number, $\mathbb{Z}_p$-extension, Height of algebraic number.

In the case $p = 2$, H. Weber proved $h_1 = h_2 = h_3 = 1$. Later, several authors showed $h_n = 1$ for $(p, n) = (2, 4)$, $(2, 5)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, $(5, 1)$ and $(7, 1)$ (see [1], [3], [18] and [19]). And recently, J. C. Miller obtained striking results determining $h_n = 1$ for $(p, n) = (2, 6)$, $(5, 2)$, $(11, 1)$, $(13, 1)$, $(17, 1)$ and $(19, 1)$ (see [20] and [21]). However, calculating one class number by one gives information on the class numbers for only finitely many layers. Thus, we are lead to a problem of different aspect.

Problem 1.1. *Fix a prime number ℓ. Is the class number h_n indivisible by ℓ for every non-negative integer n?*

In the case $\ell = p$, H. Weber [28] and K. Iwasawa [17] showed that h_n is indivisible by p for every non-negative integer n.

In the case $\ell \neq p$, by studying generalized Bernoulli numbers, L. C. Washington [27] proved that the quotient h_n/h_{n-1} is indivisible by ℓ for sufficiently large n.

However, this result does not immediately imply the ℓ-indivisibility of h_n. On the indivisibility problem, Washington [26] also showed that the minus part of the class number of $\mathbb{Q}(\mu_{5^{n+1}})$ is indivisible by every prime number ℓ with $\ell^8 \not\equiv 1 \pmod{100}$ for every non-negative integer n. Later, K. Horie [7, 8, 9, 10] and K. Horie–M. Horie [11, 12, 13, 14, 15] made a breakthrough. Indeed, they succeeded in controlling cyclotomic units which relate to our class numbers.

We introduce notation before presenting a summary of their results. Let $f_p(\ell)$ be the order of ℓ modulo q and $p^{s_p(\ell)}$ the exact power of p dividing $\ell^{f_p(\ell)} - 1$. And we define the set $D(p, s, f)$ of prime numbers to be

$$D(p, s, f) = \{\ell \neq p \mid f_p(\ell) = f, s_p(\ell) = s\}.$$

If f divides $\varphi(q)$, where φ is the Euler function, and (p, s, f) is not in $\{(2, 1, 1), (2, 1, 2), (2, 2, 2)\}$, then $D(p, s, f)$ contains infinitely many prime numbers. Moreover, $D(p, s, f)$ can be written as a union of congruence classes of prime numbers modulo p-power. For example, $D(2, 6, 2) = \{\ell \equiv 31 \bmod 64\}$ and $D(3, 1, 2) = \{\ell \equiv 2 \bmod 9\} \cup \{\ell \equiv 5 \bmod 9\}$.

Theorem 1.2 (K. Horie–M. Horie)**.** *Let p be a prime number.*

(1) *Let s be a positive integer and f a positive divisor of $\varphi(q)$. There exists an explicit positive constant $H(p, s, f)$ such that the class number h_n is indivisible by every prime number ℓ in $D(p, s, f)$ with $\ell > H(p, s, f)$ for every non-negative integer n.*
(2) *If $p = 2$, then the class number h_n is indivisible by every prime number ℓ such that $\ell \not\equiv \pm 1 \pmod 8$ for every non-negative integer n.*
(3) *If $3 \leq p \leq 23$, then the class number h_n, for every non-negative integer n, is indivisible by every prime number ℓ such that ℓ is a primitive root modulo p^2.*

Remark 1.3. They wrote $H(p,s,f)$ explicitly. We give a few of its numerical values at the end of this section.

T. Fukuda–K. Komatsu [5] proved the following theorem on the basis of the works of K. Horie.

Theorem 1.4 (T. Fukuda–K. Komatsu, [5]). *Let $p = 2$ and ℓ a prime number. Assume that $\ell < 10^9$ or $\ell \not\equiv \pm 1 \pmod{32}$. Then the class number h_n is indivisible by ℓ for every non-negative integer n.*

In our previous papers [22, 24, 25], we proposed new methods for controlling cyclotomic units, which enabled us to prove the following theorem.

Theorem 1.5. *Let p, f and s be the same as in Theorem 1.2 and $c = (p-1)p^{s-1}$. We put*

$$G_1(p,s,f) = \begin{cases} (c!)^{1/f} & \text{if } p = 2, \\ (2^{c/2} \cdot c!)^{1/f} & \text{if } p = 3, \\ \left(\left(\dfrac{\sqrt{6}p}{2}\right)^c \cdot c!\right)^{1/f} & \text{if } p > 3. \end{cases}$$

Assume that ℓ is greater than $G_1(p,s,f)$. Then the class number h_n is indivisible by ℓ for every non-negative integer n.

The technical condition of Theorem 1.5 on the magnitude of ℓ is weaker than that of Theorem 1.2.

In this paper, we show the ℓ-indivisibility under even weaker technical condition on the magnitude.

Theorem A. *Let p, f, s and c be the same as in Theorem 1.5. We put*

$$G(p,s,f) = \begin{cases} \left(2\left(\dfrac{\sqrt{\pi}}{\sqrt{2}\log\left(2+\sqrt{5}\right)}\right)^c \dfrac{c+2}{2}!\right)^{1/f} & \text{if } p = 2, \\ \left(\left(\dfrac{\sqrt{2\pi}}{3^{3/4}\log((3^{40/81}+\sqrt{3^{80/81}+4})/2)}\right)^c \dfrac{c+2}{2}!\right)^{1/f} & \text{if } p = 3, \\ \left(\left(\dfrac{2\sqrt{\pi}}{5^{3/8}\log((5^{31/125}+\sqrt{5^{62/125}+4})/2)}\right)^c \dfrac{c+2}{2}!\right)^{1/f} & \text{if } p = 5, \\ \left(\left(\dfrac{\sqrt{\pi}(p-1)\sqrt{p+1}}{2\sqrt{6}p^{(p-2)/2(p-1)}\log((p^{(p+1)/p^2}+\sqrt{p^{2(p+1)/p^2}+4})/2)}\right)^c \dfrac{c+2}{2}!\right)^{1/f} & \text{if } 7 \le p \le 19, \\ \left(\left(\dfrac{\sqrt{\pi}(p-1)\sqrt{p+1}}{2\sqrt{6}p^{(p-2)/2(p-1)}\log((p^{1/p}+\sqrt{p^{2/p}+4})/2)}\right)^c \dfrac{c+2}{2}!\right)^{1/f} & \text{if } p \ge 23. \end{cases}$$

Then the class number h_n is indivisible by every prime number ℓ in $D(p,s,f)$ with $\ell > G(p,s,f)$ for every non-negative integer n.

For example, $G(2,6,1)$ and $G(2,6,2)$ are smaller than 7.8×10^{12} and 2.8×10^6, respectively.

Recalling Theorem 1.4, we see Theorem A implies the following corollary.

Corollary B. *Let $p = 2$. If ℓ is not congruent to ± 1 modulo 64, then h_n is indivisible by ℓ for every non-negative integer n.*

Example 1.6. We can compute H, G_1 and G. In the following table, we give a few examples of their values rounded up to two significant figures.

(p,s,f)	$D(p,s,f)$	$H(p,s,f)$	$G_1(p,s,f)$	$G(p,s,f)$
$(2,5,1)$	$\ell \equiv 33 \pmod{64}$	6.2×10^{66}	2.1×10^{13}	7.6×10^4
$(3,3,2)$	$\ell \equiv 26, 53 \pmod{81}$	5.5×10^{32}	1.9×10^9	4.3×10^4
$(5,1,1)$	$\ell \equiv 6, 11, 16, 21 \pmod{25}$	2.0×10^{13}	3.4×10^4	3.8×10^2

Remark 1.7. In this paper, we don't study small primes. For small primes ℓ, the reader should consult papers of H. Ichimura–S. Nakajima [16] or K. Horie–M. Horie [15]. In the case $\ell = 2$, H. Ichimura–S. Nakajima showed that if $p \leq 509$, then h_n is odd for every non-negative integer n. For small odd primes, there are several results proven by K. Horie–M. Horie. For example, they showed that if $3 \leq \ell \leq 13$ and $p \leq 101$, then h_n is indivisible by ℓ for every non-negative integer n.

2. Lemmas

2.1. Horie unit. Let p be a prime number. We put $\zeta_n = \exp(2\pi\sqrt{-1}/p^n)$, $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{n=1}^{\infty} \mathbb{Q}(\mu_{p^n})$, σ the topological generator of the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_q))$ with $\zeta_{n+2}^\sigma = \zeta_{n+2}^5$ if $p = 2$ or $\zeta_{n+1}^\sigma = \zeta_{n+1}^{1+p}$ if $p > 2$. Set $\tau_n = \sigma^{p^{n-1}}$. Then the restriction of σ and τ_n to $\mathbb{B}_n$ generate the Galois groups $\mathrm{Gal}(\mathbb{B}_n/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{B}_n/\mathbb{B}_{n-1})$, respectively. Thus, we use the same symbols σ and τ_n for their restriction to $\mathbb{B}_n$.

Let E_n and C_n be the group of units and of cyclotomic units of $\mathbb{B}_n$, respectively. Since

$$[E_n^{1-\tau_n} : C_n^{1-\tau_n}] = h_n/h_{n-1},$$

we study $E_n^{1-\tau_n}$ and $C_n^{1-\tau_n}$ (see [8]).

Since $(1-\tau_n)(1+\tau_n+\cdots+\tau_n^{p-1}) = 0$, the ring $\mathbb{Z}[\zeta_n]$ acts on $(\mathbb{B}_n^\times)^{1-\tau_n}$, $E_n^{1-\tau_n}$ and $C_n^{1-\tau_n}$ via the isomorphism:

$$\begin{aligned} \mathbb{Z}[\mathrm{Gal}(\mathbb{B}_n/\mathbb{Q})]/(1+\tau_n+\cdots+\tau_n^{p-1}) \quad &\cong \quad \mathbb{Z}[\zeta_n], \\ \sigma \bmod (1+\tau_n+\cdots+\tau_n^{p-1}) \quad &\longmapsto \quad \zeta_n. \end{aligned}$$

Hence we regard $(\mathbb{B}_n^\times)^{1-\tau_n}$, $E_n^{1-\tau_n}$ and $C_n^{1-\tau_n}$ as $\mathbb{Z}[\zeta_n]$-modules.

We define the n-th Horie unit η_n by

$$(2.1)\qquad \eta_n = \begin{cases} \dfrac{\zeta_{n+3} - \zeta_{n+3}^{-1}}{\sqrt{-1}(\zeta_{n+3} + \zeta_{n+3}^{-1})} & \text{if } p = 2, \\ Nr_{\mathbb{Q}(\zeta_{n+1}+\zeta_{n+1}^{-1})/\mathbb{B}_n} \left(\dfrac{\zeta_{n+1} - \zeta_{n+1}^{-1}}{\zeta_1\zeta_{n+1} - \zeta_1^{-1}\zeta_{n+1}^{-1}} \right) & \text{if } p > 2. \end{cases}$$

In the case $p = 2$, η_n is also called Weber's normal unit (see e.g. [4] and [29]). The n-th Horie unit is essential for controlling our unit groups since $\eta_n^{1-\sigma}$ generates $C_n^{1-\tau_n}$ as $\mathbb{Z}[\zeta_n]$-module. Indeed, K. Horie showed the following lemma.

Lemma 2.1 (K. Horie, [8])**.** *Let ℓ be a prime number different from p and F an intermediate field of $\mathbb{Q}(\zeta_n)$ and the decomposition field of ℓ for $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. Then ℓ divides the integer h_n/h_{n-1} if and only if there exists a prime ideal $\mathfrak{L}$ of F dividing ℓ such that η_n^α is an ℓ-th power in E_n for every element α of the integral ideal $\ell\mathfrak{L}^{-1}$ of F.*

In the case where p is an odd prime number, we consider another cyclotomic unit δ_n defined by

$$\delta_n = Nr_{\mathbb{Q}(\zeta_{n+1}+\zeta_{n+1}^{-1})/\mathbb{B}_n} \left(\frac{(\zeta_{n+1} - \zeta_{n+1}^{-1})^p}{\zeta_n - \zeta_n^{-1}} \right)$$

which enables us to obtain precise information on the Horie unit η_n through the relation

$$(2.2)\qquad \delta_n^{1-\tau_n} = \eta_n^p.$$

For simplicity, we also put $\delta_n = \eta_n$ if $p = 2$.

2.2. Height of unit. Let ε be a totally real unit of degree N with conjugates $\varepsilon_1 = \varepsilon$, ε_2, $\cdots$, ε_N. We define the height of unit ε.

Definition 2.2 (Height of unit)**.** We define the L_2-height of the Dirichlet embedding of ε by

$$ht(\varepsilon) = \sqrt{\sum_{i=1}^{N} (\log |\varepsilon_i|)^2}.$$

For simplicity, we call $ht(\varepsilon)$ the height of ε.

The height of totally real units allows quantitative control as described below.

Lemma 2.3. *Let ε be a totally real unit of degree $N > 1$. We put $C = |Nr_{\mathbb{Q}(\varepsilon)/\mathbb{Q}}(\varepsilon^2 - 1)|$. Then we have*

$$ht(\varepsilon) \geq \sqrt{N} \log \left(\frac{C^{1/N} + \sqrt{C^{2/N} + 4}}{2} \right).$$

In particular, we have

$$ht(\varepsilon) \geq \sqrt{N} \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Proof. Let

$$M(\varepsilon) = \prod_{i=1}^{N} \max\{1, |\varepsilon_i|\}$$

be the Mahler measure of ε. Then we have

$$M(\varepsilon) \geq \left(\frac{C^{1/N} + \sqrt{C^{2/N}+4}}{2}\right)^{N/2} \tag{2.3}$$

(see e.g. [24, Theorem 2.2]).

On the other hand, we know that

$$\log M(\varepsilon) = \frac{1}{2}\sum_{i=1}^{N} |\log|\varepsilon_i||.$$

Hence we obtain

$$\frac{\sqrt{N}}{2} ht(\varepsilon) \geq \log M(\varepsilon) \tag{2.4}$$

from the Cauchy–Schwarz inequality. The assertions follow after (2.3) and (2.4). □

Up to this point, ε is just an irrational totally real unit. To apply Lemma 2.3 to our case, we have the following lemma.

Lemma 2.4. *Let ε be a unit in $E_n \setminus E_{n-1}$ with $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$.*

(1) ([23, Lemma 5.2]) *If $p = 2$, then ε is congruent to* 1 *modulo* 2. *In particular, we have*

$$|Nr_{\mathbb{B}_n/\mathbb{Q}}(\varepsilon^2 - 1)| \geq 4^{2^n}.$$

(2) ([24, Lemma 9.1]) *If p is odd, then we know*

$$|Nr_{\mathbb{B}_n/\mathbb{Q}}(\varepsilon^2 - 1)| \geq p^{(p^n-1)/(p-1)}.$$

Proof. For the convenience of the reader, we give an alternative proof of (1).

Let ε be a unit in $E_n \setminus E_{n-1}$ with $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$. Then there exist integers $b_0, \cdots, b_{N-1}$ such that

$$\varepsilon = b_0 + \sum_{i=1}^{N-1} b_i(\zeta_{n+2}^{i} + \zeta_{n+2}^{-i}).$$

We put

$$\gamma_+(\varepsilon) = b_0 + b_2(\zeta_{n+2}^{2} + \zeta_{n+2}^{-2}) + \cdots + b_{N-2}(\zeta_{n+2}^{N-2} + \zeta_{n+2}^{-N+2}),$$
$$\gamma_-(\varepsilon) = b_1(\zeta_{n+2} + \zeta_{n+2}^{-1}) + \cdots + b_{N-1}(\zeta_{n+2}^{N-1} + \zeta_{n+2}^{-N+1}).$$

Then we have

$$\varepsilon = \gamma_+(\varepsilon) + \gamma_-(\varepsilon), \quad \gamma_+(\varepsilon)^{\tau_n} = \gamma_+(\varepsilon), \quad \gamma_-(\varepsilon)^{\tau_n} = -\gamma_-(\varepsilon).$$

Since $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$, we obtain

$$\gamma_+(\varepsilon)^2 - \gamma_-(\varepsilon)^2 = 1.$$

This implies

$$(\gamma_+(\varepsilon) - 1)(\gamma_+(\varepsilon) + 1) = \gamma_-(\varepsilon)^2. \tag{2.5}$$

Let $\mathfrak{p}_n$ be the prime ideal of $\mathbb{B}_n$ lying above 2. We denote by ν_2 the additive $\mathfrak{p}_n$-adic valuation nomalized by $\nu_2(\zeta_{n+2} + \zeta_{n+2}^{-1}) = 1$. Assume that $\nu_2(\gamma_+(\varepsilon) - 1) < N$. Then we have

$$\nu_2(\gamma_+(\varepsilon) - 1) = \nu_2(\gamma_+(\varepsilon) + 1) = \nu_2(\gamma_-(\varepsilon))$$

from (2.5). However, $\nu_2(\gamma_+(\varepsilon)-1)$ and $\nu_2(\gamma_+(\varepsilon)+1)$ are even and $\nu_2(\gamma_-(\varepsilon))$ is odd or ∞. This is a contradiction.

Thus, we have $\nu_2(\gamma_+(\varepsilon) - 1) \geq N$. Therefore, we obtain $\gamma_+(\varepsilon) \equiv 1 \pmod 2$ and $\gamma_-(\varepsilon) \equiv 0 \pmod 2$, that is, $\varepsilon \equiv 1 \pmod 2$. □

By combining Lemma 2.3 and Lemma 2.4, we obtain the following lemma.

Lemma 2.5. *Let ε be a unit in $E_n \setminus E_{n-1}$ with $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$.*

(1) *If $p = 2$, then we have*

$$ht(\varepsilon) \geq \sqrt{2^n} \log\left(2 + \sqrt{5}\right).$$

(2) *If p is an odd prime number, then we have*

$$ht(\varepsilon) \geq \sqrt{p^n} \log\left(\frac{p^{(p^n-1)/p^n(p-1)} + \sqrt{p^{2(p^n-1)/p^n(p-1)} + 4}}{2}\right).$$

2.3. Geometry of numbers. For applying Lemma 2.1, it is desirable to choose an α in an ideal of F so that $ht(\delta_n^\alpha)$ is small.

Let r and n be positive integers with $r \leq n$ and $p^r \neq 2$. We put $d = \varphi(p^r)$ where φ is the Euler function. We start with geometry of numbers for an arbitrary ideal $\mathfrak{a}$ of $\mathbb{Z}[\zeta_r]$ ($F = \mathbb{Q}(\zeta_r)$).

Definition 2.6. We define a map λ_n from E_n to $\mathbb{R}^{p^n}$ by

$$\lambda_n(\varepsilon) = \left(\log|\varepsilon|, \log|\varepsilon^\sigma|, \cdots, \log|\varepsilon^{\sigma^{p^n-1}}|\right).$$

We define d-dimensional $\mathbb{R}$-vector space

$$V_n = \mathbb{R}\lambda_n(\delta_n) \oplus \mathbb{R}\lambda_n(\delta_n^{\zeta_r}) \oplus \cdots \oplus \mathbb{R}\lambda_n(\delta_n^{\zeta_r^{d-1}})$$

with standard metric. Let $\mathfrak{a}$ be an integral ideal of $\mathbb{Q}(\zeta_r)$. We associate the lattice

$$\Lambda = \lambda_n(\mathfrak{a}) = \{\lambda_n(\delta_n^\alpha)\ ;\ \alpha \in \mathfrak{a}\}$$

in V_n with $\mathfrak{a}$.

Let $\beta_1, \cdots, \beta_d$ be a $\mathbb{Z}$-basis of $\mathfrak{a}$. The metric of V_n induces the quadratic form

$$F_{\mathfrak{a}}(x_1, \cdots, x_d) = ht(\delta_n^{\sum_{i=1}^d x_i\beta_i})^2.$$

Then

$$F_{\mathfrak{a}}(x_1, \cdots, x_d) = \sum_{j=0}^{p^n-1} \left(\sum_{i=1}^{d} x_i \log |\delta_n^{\beta_i \sigma^j}| \right)^2$$

is a positive definite quadratic form in d variables and of determinant $\mathrm{vol}^{(d)}(\Lambda)^2$ where $\mathrm{vol}^{(d)}$ is the d-dimensional volume on V_n.

We will detect a non-zero lattice point of Λ by the following theorem in geometry of numbers.

Theorem 2.7 (Blichfeldt, [2, Theorem II])**.** *Let Λ be a lattice in the metric vector space of dimension d. Then there exists a non-zero vector v of Λ such that*

$$|v|^2 \leq \frac{2}{\pi} \left(\Gamma \left(1 + \frac{d+2}{2} \right) \right)^{2/d} \mathrm{vol}^{(d)}(\Lambda)^{2/d}.$$

In our setting of Λ, this implies the following lemma.

Lemma 2.8. *Let $\mathfrak{a}$ be an integral ideal of $\mathbb{Z}[\zeta_r]$. Then there exists a non-zero element α in $\mathfrak{a}$ such that*

$$ht(\delta_n^{\alpha}) \leq \sqrt{\frac{2}{\pi}} \left(\frac{d+2}{2}! \, [\mathbb{Z}(\zeta_r) : \mathfrak{a}] \, \mathrm{vol}^{(d)}(\lambda_n(\mathbb{Z}[\zeta_r])) \right)^{1/d}.$$

3. Proof in the case $p = 2$

In this section, we prove Theorem A for $p = 2$.

Let $p = 2$, ℓ a prime number in $D(2, s, f)$ and n a positive integer. We put $r = \min\{n, s\}$ and $d = 2^{r-1}$.

3.1. Height of Horie unit for $p = 2$.

Lemma 3.1. *Assume $p = 2$. We have*

$$ht(\eta_n) \leq \frac{\pi}{2}\sqrt{2^n}.$$

Proof. We rewrite (2.1) as

$$\eta_n = \frac{\zeta_{n+3} - \zeta_{n+3}^{-1}}{\zeta_{n+3}^{1+2^{n+1}} - \zeta_{n+3}^{-1-2^{n+1}}}.$$

Then, we see

$$\eta_n^{\sigma^i} = \tan \frac{5^i \pi}{2^{n+2}}.$$

Hence we obtain

$$ht(\eta_n)^2 = \sum_{i=1}^{2^n} \left(\log \left| \tan \frac{(2i-1)\pi}{2^{n+2}} \right| \right)^2$$
$$= 2 \sum_{i=1}^{2^{n-1}} \left(\log \tan \frac{(2i-1)\pi}{2^{n+2}} \right)^2 .$$

Since

(3.1)
$$\frac{d}{d\theta}(\log \tan\theta)^2 = \frac{2\log\tan\theta}{\sin\theta\cos\theta} < 0$$

and

(3.2)
$$\frac{d^2}{d\theta^2}(\log\tan\theta)^2 = \frac{8(1-\cos 2\theta \log\tan\theta)}{(\sin 2\theta)^2} > 0$$

for $0 < \theta < \pi/4$, we have

$$2 \sum_{i=1}^{2^{n-1}} \left(\log \left| \tan \frac{(2i-1)\pi}{2^{n+2}} \right| \right)^2 \leq \frac{2^{n+2}}{\pi} \int_0^{\pi/4} (\log\tan\theta)^2 d\theta$$
$$= \frac{\pi^2}{4} 2^n. \qquad \square$$

3.2. Volume of lattice for $p = 2$. We assume $n \geq 2$. Then we have the following lemma.

Lemma 3.2. *Assume $p = 2$. Let $\mathfrak{L}$ be a prime ideal of $\mathbb{Q}(\zeta_r)$ lying above ℓ. We have*

$$\mathrm{vol}^{(d)}(\lambda_n((1-\zeta_r)\ell\mathfrak{L}^{-1})) \leq 2\ell^{d-f} \left(\frac{\pi}{2}\sqrt{2^n} \right)^d .$$

Proof. Note that

$$\mathrm{vol}^{(d)}(\lambda_n((1-\zeta_r)\ell\mathfrak{L}^{-1})) = [\mathbb{Z}[\zeta_r] : (1-\zeta_r)\ell\mathfrak{L}^{-1}]\mathrm{vol}^{(d)}(\lambda_n(\mathbb{Z}[\zeta_r])).$$

From $r \leq s$, we have

$$[\mathbb{Z}[\zeta_r] : (1-\zeta_r)\ell\mathfrak{L}^{-1}] = 2\ell^{d-f}.$$

Since $ht(\eta_n) = ht(\eta_n^{\zeta_r}) = \cdots = ht(\eta_n^{\zeta_r^{d-1}})$, we obtain

$$\mathrm{vol}^{(d)}(\lambda_n(\mathbb{Z}[\zeta_r])) \leq ht(\eta_n)^d$$
$$\leq \left(\frac{\pi}{2}\sqrt{2^n} \right)^d . \qquad \square$$

3.3. Concluding the proof of Theorem A for $p = 2$. We prove the contrapositive. Suppose that ℓ divides h_n/h_{n-1}. It is sufficient to show that $\ell \leq G(2, s, f)$.

Since $h_1 = 1$, we may assume that $n \geq 2$. From Lemma 2.1, there exists a prime ideal $\mathfrak{L}$ in $\mathbb{Q}(\zeta_r)$ lying above ℓ such that η_n^{α} is an ℓ-th power in E_n for every element α of $\ell\mathfrak{L}^{-1}$.

We put $\mathfrak{a} = (1 - \zeta_r)\ell\mathfrak{L}^{-1}$. From Lemmas 2.8 and 3.2, there exists a non-zero element α in $\ell\mathfrak{L}^{-1}$ such that

$$ht\left(\eta_n^{(1-\zeta_r)\alpha}\right) \leq \sqrt{\frac{2}{\pi}}\left(\frac{d+2}{2}!\ 2\ell^{d-f}\left(\frac{\pi}{2}\sqrt{2^n}\right)^d\right)^{1/d}. \tag{3.3}$$

From Lemma 2.1, there exists a unit ε in E_n such that $\eta_n^{\alpha} = \varepsilon^{\ell}$. Therefore, we have

$$\eta_n^{(1-\zeta_r)\alpha} = \left(\varepsilon^{1-\zeta_r}\right)^{\ell}. \tag{3.4}$$

Since $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\eta_n) = -1$ and $(1 - \zeta_r)\alpha$ is non-zero, the degree of $\varepsilon^{1-\zeta_r}$ is 2^n and $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}\left(\varepsilon^{1-\zeta_r}\right) = 1$. Hence we have

$$ht\left(\varepsilon^{1-\zeta_r}\right) \geq \sqrt{2^n}\log\left(2+\sqrt{5}\right). \tag{3.5}$$

from Lemma 2.5 (1).

From (3.3), (3.4) and (3.5), we obtain

$$\ell\sqrt{2^n}\log\left(2+\sqrt{5}\right) \leq \sqrt{\frac{2}{\pi}}\left(\frac{d+2}{2}!2\ell^{d-f}\left(\frac{\pi}{2}\sqrt{2^n}\right)^d\right)^{1/d}.$$

This implies

$$\ell \leq \left(2\left(\frac{\sqrt{\pi}}{\sqrt{2}\log\left(2+\sqrt{5}\right)}\right)^d \frac{d+2}{2}!\right)^{1/f}.$$

Since $c = 2^{s-1}$ and $s \geq r$, we have $c \geq d$. Hence we can replace d with c. Therefore, we have

$$\ell \leq \left(2\left(\frac{\sqrt{\pi}}{\sqrt{2}\log\left(2+\sqrt{5}\right)}\right)^c \frac{c+2}{2}!\right)^{1/f} = G(2, s, f).$$

4. Proof in the case $p \geq 3$

In this section, we prove Theorem A for $p \geq 3$.

Let p be a prime number with $p \geq 3$, ℓ a prime number in $D(p, s, f)$ and n a positive integer. We put $r = \min\{n, s\}$ and $d = (p - 1)p^{r-1}$.

4.1. Height of δ_n. From the definition of δ_n and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} ht(\delta_n)^2 &= \sum_{i=0}^{p^n-1} \left(\log\left|\delta_n^{\sigma^i}\right|\right)^2 \\ &\leq \frac{p-1}{4} \sum_{i=1, p\nmid i}^{p^{n+1}-1} \left(\log\left|\frac{(\zeta_{n+1}^i - \zeta_{n+1}^{-i})^p}{\zeta_n^i - \zeta_n^{-i}}\right|\right)^2 \\ &= \frac{p-1}{4} \sum_{i=1, p\nmid i}^{p^{n+1}-1} \left(p\log\left|2\sin\left(\frac{2i\pi}{p^{n+1}}\right)\right| - \log\left|2\sin\left(\frac{2i\pi}{p^n}\right)\right|\right)^2. \end{aligned}$$

Since 2 acts on $(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$, we have

$$\begin{aligned} &\frac{p-1}{4} \sum_{i=1, p\nmid i}^{p^{n+1}-1} \left(p\log\left|2\sin\left(\frac{2i\pi}{p^{n+1}}\right)\right| - \log\left|2\sin\left(\frac{2i\pi}{p^n}\right)\right|\right)^2 \\ &\quad = \frac{p-1}{4} \sum_{i=1, p\nmid i}^{p^{n+1}-1} \left(p\log\left|2\sin\left(\frac{i\pi}{p^{n+1}}\right)\right| - \log\left|2\sin\left(\frac{i\pi}{p^n}\right)\right|\right)^2 \\ &\quad = \frac{p-1}{4} \sum_{i=1, p\nmid i}^{p^n-1} \sum_{j=0}^{p-1} \left(p\log\left|2\sin\left(\frac{i\pi}{p^{n+1}} + \frac{j\pi}{p}\right)\right| - \log\left|2\sin\left(\frac{i\pi}{p^n}\right)\right|\right)^2 \\ &\quad = \frac{p(p-1)}{4} \sum_{i=1,\ p\nmid i}^{p^n-1} g\left(\frac{i\pi}{p^{n+1}}\right) \\ &\quad = \frac{p(p-1)}{4} \sum_{k=0}^{p^{n-1}-1} \sum_{i=1}^{p-1} g\left(\frac{(i+pk)\pi}{p^{n+1}}\right) \end{aligned}$$

where

$$g(\theta) = \frac{1}{p} \sum_{j=0}^{p-1} \left(p\log\left|2\sin\left(\theta + \frac{j\pi}{p}\right)\right| - \log|2\sin(p\theta)|\right)^2.$$

From the equality

$$\prod_{j=0}^{p-1} (\zeta_1^j x - \zeta_1^{-j} x^{-1}) = x^p - x^{-p},$$

we get

$$\prod_{j=0}^{p-1} \left|2\sin\left(\theta + \frac{j\pi}{p}\right)\right| = |2\sin(p\theta)|.$$

Hence we obtain

$$g(\theta) = p\sum_{j=0}^{p-1}\left(\log\left|2\sin\left(\theta+\frac{j\pi}{p}\right)\right|\right)^2 - (\log|2\sin(p\theta)|)^2. \tag{4.1}$$

In order to get an upper bound on the height of δ_n, we give another description of $g(\theta)$. We put $C(\theta,\alpha) = (\log|2\sin\theta| - \log|2\sin(\theta+\alpha)|)^2$. Then we get

$$\begin{aligned}
&\frac{1}{2}\sum_{i'=0}^{p-1}\sum_{j'=0}^{p-1} C\left(\theta+\frac{i'\pi}{p}, \frac{j'\pi}{p}\right)\\
&\quad= \frac{1}{2}\sum_{i'=0}^{p-1}\sum_{j'=0}^{p-1}\left(\log\left|2\sin\left(\theta+\frac{i'\pi}{p}\right)\right| - \log\left|2\sin\left(\theta+\frac{i'\pi}{p}+\frac{j'\pi}{p}\right)\right|\right)^2\\
&\quad= p\sum_{j=0}^{p-1}\left(\log\left|2\sin\left(\theta+\frac{j\pi}{p}\right)\right|\right)^2\\
&\qquad - \sum_{i'=0}^{p-1}\log\left|2\sin\left(\theta+\frac{i\pi}{p}\right)\right|\sum_{j'=0}^{p-1}\log\left|2\sin\left(\theta+\frac{i'\pi}{p}+\frac{j'\pi}{p}\right)\right|\\
&\quad= p\sum_{j=0}^{p-1}\left(\log\left|2\sin\left(\theta+\frac{j\pi}{p}\right)\right|\right)^2 - (\log|2\sin(p\theta)|)^2.
\end{aligned}$$

This implies

$$g(\theta) = \frac{1}{2}\sum_{i'=0}^{p-1}\sum_{j'=0}^{p-1} C\left(\theta+\frac{i'\pi}{p}, \frac{j'\pi}{p}\right). \tag{4.2}$$

Note that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{d}{d\theta}\right)^2 C(\theta,\alpha)\\
&\quad= \left(\frac{\cos\theta}{\sin\theta} - \frac{\cos(\theta+\alpha)}{\sin(\theta+\alpha)}\right)^2 - \left(\frac{1}{(\sin\theta)^2} - \frac{1}{(\sin(\theta+\alpha))^2}\right)\log\left|\frac{\sin\theta}{\sin(\theta+\alpha)}\right|.
\end{aligned}$$

We see

$$\left(\frac{d}{d\theta}\right)^2 C(\theta,\alpha) \geq 0$$

for θ and $\theta+\alpha$ not equal to multiples of π. Therefore, $g(\theta)$ is a convex function on $]0,\pi/p[$ from (4.2).

Now we need the following proposition.

Proposition 4.1. *Let M be a positive integer and $F(\theta)$ a convex function on an interval $]a,b[$. Assume that $\int_a^b F(\theta)d\theta$ is convergent. Then we have*

$$\sum_{i=1}^{M} F\left(a+\frac{b-a}{M+1}i\right) \le \frac{M}{b-a}\int_a^b F(\theta)d\theta.$$

Note that $g(\theta)$ is a convex function on the interval $]0,\pi/p[$. By applying Proposition 4.1 for $M=p-1$, $a=k\pi/p^n$ and $b=(k+1)\pi/p^n$, we see

$$\sum_{i=1}^{p-1} g\left(\frac{(i+pk)\pi}{p^{n+1}}\right) \le \frac{(p-1)p^n}{\pi}\int_{k\pi/p^n}^{(k+1)\pi/p^n} g(\theta)d\theta.$$

for $0\le k\le p^{n-1}-1$. Therefore, by taking sum, we obtain

$$ht(\delta_n)^2 \le \frac{(p-1)^2p^{n+1}}{4\pi}\int_0^{\pi/p} g(\theta)d\theta.$$

From the equality (4.1), we have

$$\begin{aligned}
&\int_0^{\pi/p} g(\theta)d\theta \\
&\quad = p\sum_{j=0}^{p-1}\int_0^{\pi/p}\left(\log\left|2\sin\left(\theta+\frac{j\pi}{p}\right)\right|\right)^2 d\theta - \int_0^{\pi/p}(\log|2\sin(p\theta)|)^2 d\theta \\
&\quad = p\sum_{j=0}^{p-1}\int_{j\pi/p}^{(j+1)\pi/p}(\log|2\sin\theta|)^2\, d\theta - \frac{1}{p}\int_0^{\pi}(\log|2\sin\theta|)^2 d\theta \\
&\quad = \frac{p^2-1}{p}\int_0^{\pi}(\log|2\sin\theta|)^2 d\theta \\
&\quad = \frac{(p^2-1)\pi^3}{12p}.
\end{aligned}$$

Hence we obtain

$$ht(\delta_n)^2 \le \frac{p^n(p-1)^3(p+1)\pi^2}{48}.$$

Therefore, we get the following lemma.

Lemma 4.2. *Assume $p\ge 3$. We have*

$$ht(\delta_n) \le \frac{(p-1)\pi\sqrt{3(p^2-1)}}{12}\sqrt{p^n}.$$

4.2. Volume of lattice for $p\ge 3$. Let m and d be positive integers with $m\le d$ and V a d-dimensional $\mathbb{R}$-vector space. For v_0, v_1, $\cdots$, v_m in V, we define the parallelotope $S(v_0,v_1,\cdots,v_m)$ by

$$S(v_0,v_1,\cdots,v_m) = \left\{\sum_{i=0}^{m} t_iv_i\ ;\ 0\le t_i\le 1, \sum_{i=0}^{m} t_i = 1\right\}.$$

We quote the following estimate ([6, Theorem 2.2]) of its volume.

Proposition 4.3. *If* $||v_0|| = ||v_1|| = \cdots = ||v_m|| = h$*, then we have*

$$\mathrm{vol}^{(m)}(S(v_0, v_1, \cdots, v_m)) \leq \frac{(m+1)^{(m+1)/2}}{m! m^{m/2}} h^m.$$

We put

$$Q(v_1, \cdots, v_m) = \left\{ \sum_{i=1}^{m} t_i v_i \ ; \ 0 \leq t_i \leq 1 \right\}$$

and

$$Q_{j,k} = Q\left(\lambda_n(\delta_n^{\zeta_r^j}), \lambda_n(\delta_n^{\zeta_r^j \zeta_1}), \cdots, \lambda_n(\delta_n^{\zeta_r^j \zeta_1^{k-1}}), \lambda_n(\delta_n^{\zeta_r^j \zeta_1^{k+1}}), \cdots, \lambda_n(\delta_n^{\zeta_r^j \zeta_1^{p-1}})\right).$$

Then we have the following proposition.

Proposition 4.4. *For* $0 \leq j \leq p^{r-1} - 1$*, we have*

$$\frac{p}{(p-1)!}\mathrm{vol}^{(p-1)}(Q_{j,p-1}) = \mathrm{vol}^{(p-1)}\left(S\left(\lambda_n(\delta_n^{\zeta_r^j}), \lambda_n(\delta_n^{\zeta_r^j \zeta_1}), \cdots, \lambda_n(\delta_n^{\zeta_r^j \zeta_1^{p-1}})\right)\right).$$

Proof. Note that, since $\sum_{k=0}^{p-1} \lambda_n(\delta_n^{\zeta_r^i \zeta_1^k}) = 0$, we have

$$\mathrm{vol}^{(p-1)}\left(S\left(\lambda_n(\delta_n^{\zeta_r^j}), \lambda_n(\delta_n^{\zeta_r^j \zeta_1}), \cdots, \lambda_n(\delta_n^{\zeta_r^j \zeta_1^{p-1}})\right)\right) = \frac{1}{(p-1)!}\sum_{k=0}^{p-1}\mathrm{vol}^{(p-1)}(Q_{j,k}).$$

and $\mathrm{vol}^{(p-1)}(Q_{j,k}) = \mathrm{vol}^{(p-1)}(Q_{j,k'})$ for $0 \leq k, k' \leq p-1$. Therefore, we obtain the assertion. □

Then we obtain the following lemma.

Lemma 4.5. *Assume* $p \geq 3$*. Let* $\mathfrak{L}$ *be a prime ideal of* $\mathbb{Q}(\zeta_r)$ *lying above* ℓ*. Then we have*

$$\mathrm{vol}^{(d)}(\lambda_n((1-\zeta_1)\ell\mathfrak{L}^{-1})) \leq \ell^{d-f}\left(\frac{\pi(p-1)p^{p/2(p-1)}\sqrt{p+1}}{4\sqrt{3}}\sqrt{p^n}\right)^d.$$

Proof. From Propositions 4.3 and 4.4, we have

$$
\begin{aligned}
&[\mathbb{Z}[\zeta_r] : (1-\zeta_1)\ell\mathfrak{L}^{-1}]\mathrm{vol}^{(d)}(\lambda_n(\mathbb{Z}[\zeta_r])) \\
&\qquad = p^{p^{r-1}}\ell^{d-f}\mathrm{vol}^{(d)}(\lambda_n(\mathbb{Z}[\zeta_r])) \\
&\qquad \leq p^{p^{r-1}}\ell^{d-f}\prod_{j=0}^{p^{r-1}-1}\mathrm{vol}^{(p-1)}(Q_{j,p-1}) \\
&\qquad \leq p^{p^{r-1}}\ell^{d-f}\prod_{j=0}^{p^{r-1}-1}\frac{(p-1)!}{p}\mathrm{vol}^{(p-1)} \\
&\qquad\qquad \times \left(S\left(\lambda_n(\delta_n^{\zeta_r^j}), \lambda_n(\delta_n^{\zeta_r^j\zeta_1}), \cdots, \lambda_n(\delta_n^{\zeta_r^j\zeta_1^{p-1}})\right)\right) \\
&\qquad \leq \ell^{d-f}\frac{p^{p^r/2}}{(p-1)^{d/2}}ht(\delta_n)^d.
\end{aligned}
$$

From Lemma 4.2, we obtain the assertion. □

4.3. Concluding the proof of Theorem A for odd p. We prove the contrapositive. Suppose that ℓ divides h_n/h_{n-1}. It is sufficient to show that $\ell \leq G(p,s,f)$. Since $h_n = 1$ for $(p,n) = (3,1)$, $(3,2)$, $(3,3)$, $(5,1)$, $(5,2)$, $(7,1)$, $(11,1)$, $(13,1)$, $(17,1)$ and $(19,1)$, we may assume that $n \geq 4$ if $p=3$, $n \geq 3$ if $p=5$ and $n \geq 2$ if $7 \leq p \leq 17$.

From Lemma 2.1, there exist a prime ideal $\mathfrak{L}$ in $\mathbb{Q}(\zeta_r)$ lying above ℓ such that η_n^α is an ℓ-th power in E_n for every element α of $\ell\mathfrak{L}^{-1}$.

We put $\mathfrak{a} = (1-\zeta_1)\ell\mathfrak{L}^{-1}$. From Lemmas 2.8 and 4.5, there exists a non-zero element α in $\ell\mathfrak{L}^{-1}$ such that

(4.3)

$$ht\left(\delta_n^{(1-\zeta_1)\alpha}\right) \leq \sqrt{\frac{2}{\pi}}\left(\frac{d+2}{2}!\,\ell^{d-f}\left(\frac{\pi(p-1)p^{p/2(p-1)}\sqrt{p+1}}{4\sqrt{3}}\sqrt{p^n}\right)^d\right)^{1/d}.$$

From (2.2), we have $\delta_n^{1-\zeta_1} = \eta_n^p$. Moreover, from Lemma 2.1, there exist a unit ε in E_n such that $\eta_n^\alpha = \varepsilon^\ell$. These two assertions imply that

$$\delta_n^{(1-\zeta_1)\alpha} = \varepsilon^{p\ell}. \tag{4.4}$$

Since $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\delta_n) = 1$ and $(1-\zeta_1)\alpha$ is non-zero, the degree of ε is p^n and $Nr_{\mathbb{B}_n/\mathbb{B}_{n-1}}(\varepsilon) = 1$. Hence we have

$$ht(\varepsilon) \geq \sqrt{p^n}\log\left(\frac{p^{(p^n-1)/p^n(p-1)} + \sqrt{p^{2(p^n-1)/p^n(p-1)}+4}}{2}\right). \tag{4.5}$$

from Lemma 2.5 (2).

From (4.3), (4.4) and (4.5), we obtain

$$p\ell\sqrt{p^n}\log\left(\frac{p^{(p^n-1)/p^n(p-1)}+\sqrt{p^{2(p^n-1)/p^n(p-1)}+4}}{2}\right)$$
$$\leq \sqrt{\frac{2}{\pi}}\left(\frac{d+2}{2}!\,\ell^{d-f}\left(\frac{\pi(p-1)p^{p/2(p-1)}\sqrt{p+1}}{4\sqrt{3}}\sqrt{p^n}\right)^d\right)^{1/d}.$$

This implies

$$\ell \leq \left(\left(\frac{\sqrt{\pi}(p-1)\sqrt{p+1}}{2\sqrt{6}p^{(p-2)/2(p-1)}\log\left((p^{(p^n-1)/p^n(p-1)}+\sqrt{p^{2(p^n-1)/p^n(p-1)}+4})/2\right)}\right)^d\frac{d+2}{2}!\right)^{1/f}.$$

Since $c \geq d$, we can replace d by c. Therefore, we have

$$\ell \leq \left(\left(\frac{\sqrt{\pi}(p-1)\sqrt{p+1}}{2\sqrt{6}p^{(p-2)/2(p-1)}\log\left((p^{(p^n-1)/p^n(p-1)}+\sqrt{p^{2(p^n-1)/p^n(p-1)}+4})/2\right)}\right)^c\frac{c+2}{2}!\right)^{1/f}.$$

From the assumption on n, we obtain

$$\frac{p^n-1}{p^n(p-1)} \geq \begin{cases} 40/81, & \text{if } p=3, \\ 31/125, & \text{if } p=5, \\ (p+1)/p^2, & \text{if } 7\leq p\leq 19, \\ 1/p, & \text{if } p\geq 23. \end{cases}$$

This implies $\ell \leq G(p,s,f)$.

5. Corollary B

In this section, we show the ℓ-indivisibility of the class number h_n for $p=2$ and $\ell \not\equiv \pm 1 \pmod{64}$.

From Theorem 1.4, we study the cases $\ell \equiv 31 \pmod{64}$ and $\ell \equiv 33 \pmod{64}$.

5.1. $\boldsymbol{\ell \equiv 31 \pmod{64}}$. Let ℓ be a prime number with $\ell \equiv 31 \pmod{64}$. Then $f=2$, $s=6$ and $c=32$. Hence we have

$$G(2,6,2)=\sqrt{2\left(\frac{\sqrt{\pi}}{\sqrt{2}\log\left(2+\sqrt{5}\right)}\right)^{32}17!} < 2777715 < 10^9.$$

From Theorem 1.4 and Theorem A, h_n is indivisible by ℓ for every non-negative integer n if $\ell \equiv 31 \pmod{64}$.

5.2. $\ell \equiv 33 \pmod{64}$. Let ℓ be a prime number with $\ell \equiv 33 \pmod{64}$. Then $f = 1$, $s = 5$ and $c = 16$. Hence we have

$$G(2,5,1) = 2\left(\frac{\sqrt{\pi}}{\sqrt{2}\log\left(2+\sqrt{5}\right)}\right)^{16} 9! < 75585 < 10^9.$$

From Theorem 1.4 and Theorem A, h_n is indivisible by ℓ for every non-negative integer n if $\ell \equiv 33 \pmod{64}$.

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