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Equivariant epsilon conjecture for 1-dimensional Lubin-Tate groups

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Résumé. Dans cet article, nous énonçons une conjecture sur les relations entre les constantes équivariantes ϵ (associées à des représentations locales p-adiques V et des extensions finies de corps locaux L/K) et les groupes de cohomologie galoisienne locale de T, un $\mathbb{Z}_p$-réseau stable de V. Nous prouvons cette conjecture pour une extension L/K au plus modérément ramifiée et pour T un module de Tate p-adique d'un groupe de Lubin-Tate de dimension un défini sur $\mathbb{Z}_p$. Cette preuve généralise les idées dévelopées dans [4] dans le cas du groupe multiplicatif $\mathbb{G}_m$ au cas d'un groupe de Lubin-Tate de dimension un quelconque. Pour les relations avec les différentes formulations des conjectures ϵ de [1], [18], [4], [2] et [9], voir [19].

Abstract. In this paper we formulate a conjecture on the relationship between the equivariant ϵ-constants (associated to a local p-adic representation V and a finite extension of local fields L/K) and local Galois cohomology groups of a Galois stable $\mathbb{Z}_p$-lattice T of V. We prove the conjecture for L/K being at most tamely ramified and T being a p-adic Tate module of a one-dimensional Lubin-Tate group defined over $\mathbb{Z}_p$ by extending the ideas of [4] from the case of the multiplicative group $\mathbb{G}_m$ to arbitrary one-dimensional Lubin-Tate groups. For the connection to the different formulations of the ϵ-conjecture in [1], [18], [4], [2] and [9], see [19].

1. Introduction

In order to illustrate the importance and occurrence of ϵ-factors we would like to recall the following basic example. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a (primitive) Dirichlet character modulo $N \in \mathbb{N}$ (i.e., N is minimal). As

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usual we extend χ to $\mathbb{Z}$ by setting $\chi(n) := \chi(n \bmod N)$ for $(n, N) = 1$ and 0 otherwise. Then the Dirichlet L-function of χ is defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s \in \mathbb{C},\ Re(s) > 1.$$

The completed L-function

$$\Lambda(\chi, s) := L_\infty(\chi, s) L(\chi, s), \quad L_\infty(\chi, s) = \Big(\frac{N}{\pi}\Big)^{s/2} \Gamma(\chi, s),$$

where $\Gamma(\chi, s) := \Gamma(\frac{s+k}{2})$ in terms of the usual Γ-function and the exponent $k \in \{0, 1\}$ given by $\chi(-1) = (-1)^k \chi(1)$, admits a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi, s) = \frac{\tau(\chi)}{i^k \sqrt{N}} \Lambda(\overline{\chi}, 1 - s)$$

with the Gauß sum

$$\tau(\chi) = \sum_{\nu=0}^{N-1} \chi(\nu) e^{2\pi i \nu / N},$$

which is the basic example of an ϵ-factor. For the trivial character χ we get the Riemann ζ-function, whose Euler product reflects the prime factor decomposition of natural numbers. Let μ_N be the group of primitive N-th roots of unity and $\mathbb{Q}(\mu_N)$ the corresponding cyclotomic field. In general, for Galois field extensions L/K we shall write $G(L/K)$ for the associated Galois group. Using the isomorphism $G(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ we can view χ as a character of $G(\mathbb{Q}(\mu_N)/\mathbb{Q})$. For $N = 4$ the analytic class number formula specialises to

$$h_{\mathbb{Q}(i)} = \frac{\sharp\mu(\mathbb{Q}(i))\sqrt{N}}{2\pi} L(\chi_1, 1), \quad \chi_1(\bar{1}) = 1,\ \chi_1(\bar{3}) = -1$$

and shows the deep and beautiful connection between special L-values and (algebraic) arithmetic objects like the ideal class group or more generally Galois or étale cohomology groups. The next prominent example is given by the Birch and Swinnerton-Dyer conjecture, which apart from rationality questions relates the order of vanishing at $s = 1$ of the Hasse-Weil L-function attached to an elliptic curve E over a number field to the rank of the Mordell-Weil group of E. Finally, the corresponding leading coefficient again is related to arithmetic invariants of E.

To arbitrary representations coming from motives M the formulae of this kind have been extended conjecturally by Bloch and Kato and they form part of the so called Tamagawa Number Conjecture (TNC). Again we have the complex L-function $L(M, s)$ attached to a motive M which is believed to satisfy the following functional equation relating $L(M, s)$ to the L-function $L(M^*(1), s)$ of the (Kummer) dual motive $M^*(1)$ of M:

$$L(M, s) = \epsilon(M, s)\frac{L_\infty(M^*(1), -s)}{L_\infty(M, s)}L(M^*(1), -s).$$

Here L_∞ is the so called Euler-factor at infinity (attached to M and $M^*(1)$, respectively), which is built up by certain Γ-factors and certain powers of 2 and π (depending on the Hodge structure of M), while the so called ϵ-factor decomposes into local factors

$$\epsilon(M, s) = \prod_v \epsilon_v(M, s).$$

Then the TNC states a relation between the leading term $L^*(0)$ (of the Taylor expansion of $L(M, s)$ at zero s = 0) and certain global Galois cohomology groups of $R\Gamma(G_K, M_p)$ of M (up to the period and regulator). In the following we assume the validity of the functional equation. Then it is by no means evident that the validity of the TNC for M is equivalent to the validity of the TNC for $M^*(1)$ under this functional equation on the complex analytic side and under Artin/Verdier or Poitou/Tate-duality on the p-adic Galois cohomology side. To the contrary, they are only compatible if and only if the local constants $\epsilon_v(M, 0)$ are in a specific way related to certain local Galois cohomology groups of $R\Gamma(G_{K_v}, M_p)$ for all places v. This is - roughly speaking - the content of the absolute ϵ-conjecture. From a more technical point of view,i.e., if one spells out this property in detail, this boils down to an integral p-adic interpolation of the corresponding Bloch-Kato exponential maps.

The *equivariant* ϵ-conjecture is formulated in a similar way by tensoring the coefficients ($\mathbb{Z}_p$-modules) with the group algebra $\mathbb{Z}_p[G]$, where G is the Galois group of some local Galois extension L_v/K_v, i.e., by deforming the presentation along an extension of local fields. This version not only states the relation mentioned above for all twists of M_p by Artin representations of the Galois group G, simultaneously, but is finer: it states (implicitly) congruences between the corresponding invariants attached to each twist which are related to the difference between the (K-goups of the) integral group algebra $\mathbb{Z}_p[G]$ and a maximal order in $\mathbb{Q}_p[G]$. As before it now describes the compatibility of the Equivariant Tamagawa Number Conjecture (ETNC) [9] for a motive and its Kummer dual with respect to the functional equation and duality in Galois cohomology. S. Yasuda proved in [28] the cases in which the residue characteristics of v and p differ, thus here we consider only p-representations of the absolute Galois groups of local fields. For more details about this general background we refer the reader to [18] or the survey article [26] including the references given there.

In [1] Benois and Berger have proved the equivariant conjecture with respect to the extension L/K for arbitrary crystalline representations V of G_K, where K is an unramified extension of $\mathbb{Q}_p$ and L a finite subextension of $K(\mu_{p^\infty})$ over K; here we write μ_{p^∞} for the p-primary part of all roots of unity μ. In the special case $V = \mathbb{Q}_p(r)$, $r \in \mathbb{Z}$, Burns and Flach [10] prove the local ETNC using global ingredients in a semi-local setting. Extending work [2] of Bley and Burns Breuning [4] proves the equivariant conjecture for $V = \mathbb{Q}_p(1)$ with respect to all tamely ramified extensions.

The $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$ can be considered as the p-adic Tate module of the multiplicative formal group $\hat{\mathbb{G}}_m$ defined over $\mathbb{Z}_p$. Following closely the approach of Breuning we extend his ideas to arbitrary one-dimensional Lubin-Tate groups and prove in this paper the equivariant ϵ-conjecture for tamely ramified extensions and p-adic Tate modules of such one-dimensional Lubin-Tate groups defined over $\mathbb{Z}_p$, see Theorem 6.6. For (possibly wildly ramified, p-adic Lie) subextensions of the maximal abelian extension of $\mathbb{Q}_p$ the corresponding result has been proved in [27] based on Kato's ideas. One main aspect of this paper is that unramified characters can be dealt with in the context of local constants by unramified base change (and descending afterwards). This philosophy is also immanent in [21] where a two-variable Perrin-Riou regulator map is used, in which the second variable - in contrast to the first one, that already shows up in [1] - stems from an unramified extension.

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Notation and conventions. Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$, $\chi^{ur} : G_K \to \mathbb{Z}_p^\times$ be a continuous character which is the restriction of a continuous unramified character $\chi^{ur}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ and let $\chi^{cyc} : G_K \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. We consider a continuous representation $T = \mathbb{Z}_p(\chi^{ur})(\chi^{cyc}) =: \mathbb{Z}_p(\chi^{ur})(1)$ of G_K, which appears naturally as a restriction to G_K of the p-adic Tate module $\mathrm{T}_p\mathcal{F}$ of a one-dimensional Lubin-Tate group $\mathcal{F}$ defined over $\mathbb{Z}_p$ (see [19, Exm. 5.20]). We set $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$. Let L be a finite Galois extension of K and let $G = G(L/K)$ denote its Galois group with inertia subgroup I; by Fr_K we denote the arithmetic Frobenius homomorphism of K. Let $\widehat{\mathbb{Q}_p^{ur}}$ denote the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$ and $\widehat{\mathbb{Z}_p^{ur}}$ its ring of integers. By σ we denote the absolute arithmetic Frobenius automorphism acting on the latter rings. Finally, we set $\Lambda := \mathbb{Z}_p[G]$ and $\Omega := \mathbb{Q}_p[G]$, $\tilde{\Lambda} := \widehat{\mathbb{Z}_p^{ur}}[G]$ and $\tilde{\Omega} := \widehat{\mathbb{Q}_p^{ur}}[G]$.

We omit the case, in which χ^{ur} factors over L, as in this case the ϵ-conjecture can easily be derived from [4] as we sketch in the Appendix. Hence we assume from now on that $\chi^{ur}(G_L) \neq 1$.

2. Preliminaries

2.1. *K*-theory, determinants and refined Euler characteristics.

2.1.1. *Review of K_0 and K_1.* Let R be a ring. By an R-module we mean a left R-module.

By definition $K_0(R)$ is an abelian group, whose group law we denote additively, presented by the following generators and relations.

- Generators: $[P]$, where P is a finitely generated projective R-module.
- Relations:
  (i) If $P \cong Q$, then $[P] = [Q]$.
  (ii) $[P \oplus Q] = [P] + [Q]$.

$K_1(R)$ is an abelian group, whose group law we denote multiplicatively, defined by the following generators and relations.

- Generators: $[P, \alpha]$, where P is a finitely generated projective R-module and α is an automorphism of P.
- Relations:
  (i) If there is an isomorphism $P \cong Q$ via which α corresponds to β, then $[P, \alpha] = [Q, \beta]$.
  (ii) $[P \oplus Q, \alpha \oplus \beta] = [P, \alpha] \cdot [Q, \beta]$.
  (iii) $[P, \alpha \circ \beta] = [P, \alpha] \cdot [P, \beta]$ for $\alpha, \beta \in \mathrm{Aut}(P)$.

Then the canonical homomorphisms $GL_n(R) \to K_1(R)$ sending an element $\alpha \in GL_n(R)$ to $[R^n, \alpha]$ induces an isomorphism

$$GL_\infty(R)/[GL_\infty(R), GL_\infty(R)] \xrightarrow{\cong} K_1(R), \tag{2.1}$$

where $[GL_\infty(R), GL_\infty(R)]$ denotes the commutator subgroup, which equals the group of elementary matrices $E_\infty(R)$.

If R is commutative, the determinant map $\det : GL_n(R) \to R^\times$ induces the determinant map

$$\det : K_1(R) \to R^\times$$

via the isomorphism (2.1).

Next we recall Swan's construction of relative K-groups. For any ring R we denote by $\mathcal{P}(R)$ the category of finitely generated projective R-modules. For any homomorphism of rings $\phi : R \to R'$, the relative K-group $K_0(R, R')$ is defined by generators and relations, as follows. Consider triples (M, f, N) with $M, N \in \mathcal{P}(R)$, $f : R' \otimes_R M \cong R' \otimes N$. For brevity, let $M' = R' \otimes_R M$, etc. A morphism

$$(\mu, \nu) : (M_1, f_1, N_1) \to (M_2, f_2, N_2)$$

consists of a pair of maps $\mu \in \mathrm{Hom}_R(M_1, M_2)$, $\nu \in \mathrm{Hom}_R(N_1, N_2)$, such that

$$\nu' \circ f_1 = f_2 \circ \mu' : M_1' \to N_2'.$$

We write $(M_1, f_1, N_1) \cong (M_2, f_2, N_2)$ if both μ and ν are isomorphisms. A short exact sequence of triples is a sequence

$$0 \longrightarrow (M_1, f_1, N_1) \xrightarrow{(\mu_1,\nu_1)} (M_2, f_2, N_2) \xrightarrow{(\mu_2,\nu_2)} (M_3, f_3, N_3) \longrightarrow 0$$

such that each pair (μ_i, ν_i) is a morphism, and where the sequences of R-modules

$$0 \longrightarrow M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \longrightarrow 0$$

and similarly for N_i with ν_i are exact. Now $K_0(R, R')$ is defined as the free abelian group generated by all isomorphism classes of triples, modulo the relations

$$(L, gf, N) = (L, f, M) + (M, g, N)$$

and for each short exact sequence as above

$$(M_2, f_2, N_2) = (M_1, f_1, N_1) + (M_3, f_3, N_3).$$

This relative K-group fits into the following exact sequence of groups

$$K_1(R) \longrightarrow K_1(R') \xrightarrow{\partial} K_0(R, R') \xrightarrow{\iota} K_0(R) \longrightarrow K_0(R'), \tag{2.2}$$

where the map $\partial := \partial_{R,R'}$ (usually we shall omit the decorations) is defined by $\partial(f) = [R^n, f, R^n]$ for $f \in GL_n(R')$, while the map ι is given by $\iota([M, f, N]) = [M]-[N]$, and where the brackets denote classes in $K_0(R, R')$ and $K_0(R)$, respectively. The above exact sequence behaves functorially with respect to change of rings. E.g. we obtain a group homomorphism

$$K_0(\Lambda, \mathbb{Q}_p) \longrightarrow K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}}), \tag{2.3}$$

where we abbreviate $K_0(R, B) := K_0(R, R \otimes_C B)$ for any C-algebra R, where $C \subseteq B$ is any extension of commutative rings. By Taylor's fixed point theorem [17, thm. 10A] it is injective.

2.1.2. *Non-commutative determinants.* As before let $\mathcal{P}(R)$ denote the category of finitely generated projective R-modules and $(\mathcal{P}(R), is)$ its subcategory of isomorphisms, i.e., with the same objects, but whose morphisms are precisely the isomorphisms. Then there exists a category $\mathcal{C}_R$ and a functor

$$\mathbf{d}_R : (\mathcal{P}(R), is) \to \mathcal{C}_R,$$

which satisfies the following properties:

(i) There is an associative and commutative product structure on $\mathcal{C}_R$ $(M, N) \to M \cdot N$ or written just MN with unit object $\mathbf{d}_R(0)$ and inverses. All objects are of the form $\mathbf{d}_R(P)\mathbf{d}_R(Q)^{-1}$ for some $P, Q \in \mathcal{P}(R)$.

(ii) all morphisms of $\mathcal{C}_R$ are isomorphisms, $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic if and only if the classes of P and Q in $K_0(R)$ coincide. There is an identification of groups $\mathrm{Aut}(\mathbf{d}_R(0)) = K_1(R)$ and $\mathrm{Mor}(M,N)$ is either empty or an $K_1(R)$-torsor, where $\alpha : \mathbf{d}_R(0) \to \mathbf{d}_R(0) \in K_1(R)$ acts on $\phi : M \to N$ as $\alpha \cdot \phi : M = \mathbf{d}_R(0) \cdot M \to \mathbf{d}_R(0) \cdot N = N$.

(iii) $\mathbf{d}_R$ preserves the "product" structures: $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

We define the category $\mathcal{C}_R$ as follows. An object of $\mathcal{C}_R$ is a pair (P,Q) of finitely generated projective R-modules and morphisms of $\mathcal{C}_R$ are as follows.

There exists a morphism $(P,Q) \to (P',Q')$ if and only if $[P]-[Q] = [P']-[Q']$ in $K_0(R)$. If $[P]-[Q] = [P']-[Q']$, there is a finitely generated projective R-module T such that $P \oplus Q' \oplus T \cong P' \oplus Q \oplus T$. Let

$$G_T = \mathrm{Aut}(P' \oplus Q \oplus T), \quad I_T = \mathrm{Isom}(P \oplus Q' \oplus T, P' \oplus Q \oplus T).$$

Then I_T is a G_T-torsor (that is, I_T is a non-empty set endowed with an action of G_T and for each $x, y \in I_T$, there exists a unique $g \in G_T$ such that $y = gx$). We define the set $\mathrm{Mor}((P,Q),(P',Q'))$ of morphisms $(P,Q) \to (P',Q')$ in $\mathcal{C}_R$ by

$$\mathrm{Mor}((P,Q),(P',Q')) = K_1(R) \times^{G_T} I_T. \tag{2.4}$$

Here $K_1(R) \times^{G_T} I_T$ denotes the quotient of $K_1(R) \times I_T$ by the action of G_T given by $(x,y) \mapsto (x\bar{g}, g^{-1}y)$ ($x \in K_1(R), y \in I_T, g \in G_T$ and $\bar{g}$ denotes the image of g in $K_1(R)$). It is the $K_1(R)$-torsor obtained from the G_T-torsor I_T by the canonical homomorphism $G_T \to K_1(R)$. This set of morphisms does not depend on the choice of T (see [18]). By definition any morphism in $\mathcal{C}_R$ is an isomorphism and

- For an object (P,Q) of $\mathcal{C}_R$ we denote the object (Q,P) of $\mathcal{C}_R$ by $(P,Q)^{-1}$ and call it the inverse of (P,Q) (with respect to the product structure).
- For objects (P,Q) and (P',Q') of $\mathcal{C}_R$ we denote the object $(P \oplus P', Q \oplus Q')$ of $\mathcal{C}_R$ by $(P,Q) \cdot (P',Q')$ and call it the product of (P,Q) and (P',Q').
- For a finitely generated projective R-module P we denote the object $(P,0)$ of $\mathcal{C}_R$ by $\mathbf{d}_R(P)$. Hence an object (P,Q) of $\mathcal{C}_R$ is described as
$$(P,Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)^{-1}.$$

Remark 2.1. In particular, if $\alpha : P \to Q$ denotes an isomorphism of finitely generated projective R-modules, then this gives rise to a morphism $\mathbf{d}_R(\alpha) = [(1,\alpha)] \in K_1(R) \times^{G_0} \mathrm{Isom}(P,Q) = Mor(\mathbf{d}_R(P), \mathbf{d}_R(Q))$.

Let R' be another ring and let Y be a f.g projective R'-module endowed with a structure of a right R-module such that the actions of R and R' on

Y commute. Then we have a functor

$$Y\otimes_R : \mathcal{C}_R \to \mathcal{C}_{R'}, \quad (P,Q) \mapsto (Y \otimes_R P, Y \otimes_R Q).$$

For example, for a ring homomorphism $R \to R'$ we have a functor $R'\otimes_R : \mathcal{C}_R \to \mathcal{C}_{R'}$, by taking R' as Y.

Also, for example, for a ring homomorphism $R' \to R$ such that R is finitely generated and projective as a (left) R'-module, we have a functor $R\otimes_R$ by taking R as Y, which is the functor to regard a R-module as a R'-module. The induced homomorphism $\mathrm{Aut}(\mathbf{d}_R(0)) \to \mathrm{Aut}(\mathbf{d}_{R'}(0))$ coincides with the norm homomorphism $K_1(R) \to K_1(R')$.

Remark 2.2. In particular, if $\alpha : P \to Q$ denotes an isomorphism of finitely generated projective R-modules, then this gives rise to a morphism $\mathbf{d}_R(\alpha) = [(1,\alpha)] \in K_1(R) \times^{G_0} \mathrm{Isom}(P,Q) = \mathrm{Mor}(\mathbf{d}_R(P), \mathbf{d}_R(Q))$ with $T = 0$ the trivial module. Now consider ring homomorphism $R \to R'$ and assume that the natural map $K_0(R) \to K_0(R')$ is injective. Then, if P, Q are finitely generated projective R-modules, by the definition above the set $\mathrm{Mor}(R' \otimes_R \mathbf{d}_R(P), R' \otimes_R \mathbf{d}_R(P))$ is a $K_1(R')$-torsor (respectively empty) if and only if $\mathrm{Mor}(\mathbf{d}_R(P), \mathbf{d}_R(P))$ is a $K_1(R)$-torsor (respectively empty). Moreover, it follows from (2.4) that

$$\mathrm{Mor}(R' \otimes_R \mathbf{d}_R(P), R' \otimes_R \mathbf{d}_R(P)) \cong \mathrm{Mor}(\mathbf{d}_R(P), \mathbf{d}_R(P)) \times^{K_1(R)} K_1(R').$$

The functor $\mathbf{d}_R$ can be naturally extended to complexes. Let $\mathrm{C}^p(R)$ be the category of bounded complexes in $\mathcal{P}(R)$ and $(\mathrm{C}^p(R), quasi)$ its subcategory of quasi-isomorphisms. For $C \in \mathrm{C}^p(R)$ we set $C^+ = \oplus_{i \text{ even}} C^i$ and $C^- = \oplus_{i \text{ odd}} C^i$ and define $\mathbf{d}_R(C) := \mathbf{d}_R(C^+)\mathbf{d}_R(C^-)^{-1}$ and thus we obtain a functor

$$\mathbf{d}_R : (\mathrm{C}^p(R), quasi) \to \mathcal{C}_R$$

with the following properties ($C, C', C'' \in \mathrm{C}^p(R)$)

(iv) If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of complexes, then there is a canonical isomorphism

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C''),$$

which we take as an identification.

(v) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism

$$\mathbf{d}_R(0) \to \mathbf{d}_R(C).$$

(vi) $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$, where $C[r]$ denotes the r^{th} translate of C.

(vii) The functor $\mathbf{d}_R$ factorizes over the image of $\mathrm{C}^p(R)$ in $\mathrm{D}^p(R)$, the category of perfect complexes (as full triangulated subcategory of the derived category $\mathrm{C}^b(R)$ of the homotopy category of bounded complexes of R-modules), and extends to $(\mathrm{D}^p(R), is)$ (uniquely up to unique isomorphism).

(viii) If $C \in \mathrm{D}^p(R)$ has the property, that all cohomology groups $\mathrm{H}^i(C)$ belong again to $\mathrm{D}^p(R)$, then there is a canonical isomorphism

$$\mathbf{d}_R(C) = \prod_i \mathbf{d}_R(\mathrm{H}^i(C))^{(-1)^i}.$$

(ix) Let R' be another ring and let Y be a f.g projective R'-module endowed with a structure of a right R-module such that the actions of R and R' on Y commute. Then we have a commutative diagram

$$\begin{tikzcd}
(\mathrm{D}^p(R), is) \arrow[r, "\mathbf{d}_R"] \arrow[d, "Y\otimes^{\mathbb{L}}_R -"'] & \mathcal{C}_R \arrow[d, "Y\otimes_R -"] \\
(\mathrm{D}^p(R'), is) \arrow[r, "\mathbf{d}_{R'}"] & \mathcal{C}_{R'}.
\end{tikzcd}$$

For the handling of the determinant functor in practice the following considerations are quite important:

Remark 2.3. The determinant of the complex $C = [P_0 \xrightarrow{\phi} P_1]$ (in degrees 0 and 1) with $P_0 = P_1 = P$ is by definition $\mathbf{d}_R(C) = \mathbf{d}_R(0)$ and is defined even if ϕ is not an isomorphism (in contrast to $\mathbf{d}_R(\phi)$). But if ϕ happens to be an isomorphism, i.e. if C is acyclic, then by the property (v) there is also a canonical map $\mathbf{d}_R(0) \xrightarrow{acyc} \mathbf{d}_R(C)$, which is in fact nothing else then

$$\mathbf{d}_R(0) = \mathbf{d}_R(P_1)\cdot\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}_R(\phi)^{-1}\cdot \mathrm{id}_{\mathbf{d}_R(P_1)^{-1}}} \mathbf{d}_R(P_0)\cdot\mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

(and which depends in contrast to the first identification on ϕ). Hence, the composite $\mathbf{d}_R(0) \xrightarrow{acyc} \mathbf{d}_R(C) = \mathbf{d}_R(0)$ corresponds to $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$ according to the previous remark. In order to distinguish the above identifications between $\mathbf{d}_R(0)$ and $\mathbf{d}_R(C)$ we also say, that C is *trivialized by the identity* when we refer to $\mathbf{d}_R(C) = \mathbf{d}_R(0)$ (or its inverse with respect to composition). For $\phi = \mathrm{id}_P$ both identifications agree obviously.

2.1.3. ***Refined Euler characteristics.*** The precise definition and properties of refined Euler characteristics can be found in [7, 8]. Consider a ring homomorphism $R \to R'$ such that R' becomes a flat R-module and assume that R' is noetherian and regular (in our applications R' is a semi-simple ring). Let C be in $\mathrm{D}^p(R)$ and

$$t : \mathrm{H}^{od}(R'\otimes_R C) := \bigoplus_{i \text{ odd}} \mathrm{H}^i(R'\otimes_R C) \to \mathrm{H}^{ev}(R'\otimes_R C) := \bigoplus_{i \text{ even}} \mathrm{H}^i(R'\otimes_R C)$$

a trivialisation, i.e., an isomorphisms of R'-modules (note that in [8] is normalised in the opposite way, i.e., goes from even to odd). By [6, def. 5.4] one can attach a class $[C] \in K_0(R)$ to C, which equals $[C^+] - [C^-]$ if

C belongs to $C^p(R)$. Then there is canonical class $\chi(C,t) \in K_0(R,R')$ satisfying that $\iota(\chi(C,t)) = [C]$ and, if t' is another trivialisation, then

$$\chi(C,t') = \chi(C,t) + \partial([\mathrm{H}^{od}(R' \otimes_R C), (t')^{-1} \circ t]).$$

For the additivity of the refined Euler characteristic $\chi(C,t)$ see thm. 5.7, rem. 6.1 and thm. 6.2 in [6].

As mentioned above the fundamental groups of the Picard category $\mathcal{C}_R$ ($\pi_0(\mathcal{C}_R)$ is the group of isomorphism classes of objects of $\mathcal{C}_R$ while $\pi_1(\mathcal{C}_R) = \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{d}_R(0))$) are canonically isomorphic to the K-groups $K_0(R)$ and $K_1(R)$ of R. Also the relative K-group $K_0(R,R')$ for a ring homomorphism $R \to R'$ can be realized as fundamental group of a Picard category: Let $\mathcal{P}$ be the Picard category with unique object $\mathbf{1}_{\mathcal{P}}$ and $\mathrm{Aut}_{\mathcal{P}}(\mathbf{1}_{\mathcal{P}}) = 0$. Following [9, (20)] we define $\mathcal{C}(R,R')$ to be the fibre product category $\mathcal{C}_R \times_{\mathcal{C}_{R'}} \mathcal{P}$. Thus objects of $\mathcal{C}(R,R')$ consists of pairs (M,λ) with $M \in \mathcal{C}_R$ and $\lambda : R' \otimes_R M \to \mathbf{d}_{R'}(0)$ an isomorphism in $\mathcal{C}_{R'}$. One has an isomorphism

$$\Psi_{R,R'} : K_0(R,R') \cong \pi_0(\mathcal{C}(R,R')) \tag{2.5}$$

where $[M,f,N]$ is mapped to $[\mathbf{d}_R(M)\mathbf{d}_R(N)^{-1}, \mathbf{d}_{R'}(f)\cdot \mathrm{id}_{\mathbf{d}_{R'}(R' \otimes_R N)^{-1}}]$ and which is compatible with the exact sequence (2.2) and the Mayer-Vietoris exact sequence attached to the fibre product category, see [6, lem. 5.1]. There it is also explained that the refined Euler characteristic (at least in the semi-simple case) can be defined using relative K-groups or fibre-product categories and determinants. In the comparison ([6, thm. 6.2]) signs may arise, which - due to the remark 6.4 in [6] do not show up in our calculations.

2.2. Fontaine's theory of period rings. In this subsection we recall some of Fontaine's conctructions (see [15]) in p-adic Hodge theory. Let K_0 denote the maximal unramified subextension of $K/\mathbb{Q}_p$ and K_0^{ur} its maximal unramified extension.

Let B_{dR}, B_{st} and B_{cris} denote Fontaine's period rings. The field $B_{dR} = B_{dR}^+[\frac{1}{t}]$ is a $\mathbb{Q}_p$-algebra containing $\overline{\mathbb{Q}}_p$, which has an action by G_K and an exhaustive decreasing and separated filtration $\mathrm{Fil}^i B_{dR} = t^i B_{dR}^+$, where $t := log[\xi] \in B_{dR}$ denotes the p-adic period analogous to $2\pi i$. The latter depends on the choice $\xi = (\xi_n)$ of a compatible system of p^nth roots of unity, which we once and for all fix here for the whole paper. Then $g(t) = \chi^{cyc}(g)\cdot t$ for all $g \in G_K$.

The ring B_{st} is a $\mathbb{Q}_p$-algebra containing $\widehat{\mathbb{Q}_p^{ur}}$; it is endowed with an action by G_K, with a (Frobenius) endomorphism ϕ, which is σ-semi-linear and which commutes with the Galois action, as well as with a monodromie operator $N : B_{st} \to B_{st}$, which commutes with the Galois action and satifies $N \circ \phi = p\phi \circ N$. Finally, $B_{cris} = B_{st}^{N=0} \subseteq B_{st}$.

For all p-adic representations W of G_K we set $\mathrm{D}_{dR}^K(W) := (B_{dR} \otimes_{\mathbb{Q}_p} W)^{G_K}$, which is a finite-dimensional filtered K-vector space. Then the tangent space of W is defined as $t_W(K) := \mathrm{D}_{dR}^K(W)/\mathrm{Fil}^0\mathrm{D}_{dR}^K(W)$. Furthermore, we define

$$\mathrm{D}_{cris}^K(W) := (B_{cris} \otimes_{\mathbb{Q}_p} W)^{G_K} \text{ and } \mathrm{D}_{pst}^K(W) := \varinjlim_{L/K} (B_{st} \otimes_{\mathbb{Q}_p} W)^{G_L},$$

where L runs through all finite extensions of K. While $\mathrm{D}_{cris}^K(W)$ is a K_0-vector space endowed with a σ-semi-linear action ϕ, the K_0^{ur}-vector space $\mathrm{D}_{pst}^K(W)$ possesses the operator ϕ and N satisfying $N \circ \phi = p\phi \circ N$. If K is clear from the context we often omit the superscript writing just $\mathrm{D}_{dR}(W)$, $\mathrm{D}_{cris}(W)$ or $\mathrm{D}_{pst}(W)$. In general one has

$$\dim_{K_0} D_{cris}(W) \leq \dim_{K_0^{ur}} D_{pst}(W) \leq \dim_K D_{dR}(W) \leq \dim_{Qp} W$$

A representation W is called crystalline, potentially semi-stable or de Rham, if $\dim_{Qp} W$ is equal to $\dim_{K_0} D_{cris}(W)$, $\dim_{K_0^{ur}} D_{pst}(W)$ or $\dim_K D_{dR}(W)$, respectively.

Following Bloch and Kato we define

$$\mathrm{H}^1_e(K, W) := \ker\left(\mathrm{H}^1(K, W) \to \mathrm{H}^1(K, B_{cris}^{\phi=1} \otimes_{\mathbb{Q}_p} W)\right),$$
$$\mathrm{H}^1_f(K, W) := \ker\left(\mathrm{H}^1(K, W) \to \mathrm{H}^1(K, B_{cris} \otimes_{\mathbb{Q}_p} W)\right)$$

and

$$\mathrm{H}^1_g(K, W) := \ker\left(\mathrm{H}^1(K, W) \to \mathrm{H}^1(K, B_{dR} \otimes_{\mathbb{Q}_p} W)\right).$$

We remind the reader that under the local Tate duality pairing $\mathrm{H}^1_f(K, W)$ is the orthogonal complement of $\mathrm{H}^1_f(K, W^*(1))$. Here W^* denotes the dual of W.

2.3. Epsilon-factors. We recall the definition of ϵ-factors associated to representations of the Weil group of $\mathbb{Q}_p$ (or more generally K), for which the canonical reference is [13], see also Tate [25], [1, §2.3/4] or [18, 3.3.3]. In particular, our convention is that under the local reciprocity law uniformizers π_K correspond to *geometric* Frobenius automorphisms. These are constants

$$\epsilon_E(\mathbb{Q}_p, D, \psi, \mathrm{d}x) \in E^\times$$

where E is a field of characteristic 0 containing μ_{p^∞}, ψ is a locally constant E-valued character of $\mathbb{Q}_p$, $\mathrm{d}x$ is a Haar measure on $\mathbb{Q}_p$, and D is a finite-dimensional E-linear representation of the Weil group $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ which is locally constant (i.e. the image of the inertia group $I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is finite).

Following [18, §3.2], we shall restrict to the case when $\mathrm{d}x$ is the usual Haar measure giving $\mathbb{Z}_p$ measure 1, and ψ has kernel equal to $\mathbb{Z}_p$,i.e., conductor[1]

[1] $n(\psi)$ is defined to be the largest integer n such that $\psi(p^{-n}\mathbb{Z}_p) = 1$.

$n(\psi) = 0$; the data of such a character ψ is equivalent to the data of a compatible system of p-power roots of unity $\xi = (\xi_n)_{n\geq 1}$, via the map sending ψ to $\left(\psi(p^{-n})\right)_{n\geq 1}$. Since $\mathrm{d}x$ and $\mathbb{Q}_p$ are fixed, and ψ is determined by ξ, which we have fixed above, we shall drop them usually from the notation and write the ϵ-factor sometimes as $\epsilon_E(D)$.

We are interested in the case when $D = D_{pst}(W)$ for a de Rham representation W of $G_{\mathbb{Q}_p}$, with the linearized action of the Weil group given as in [14]. If W is an F-linear representation of dimension d, for F a finite extension of $\mathbb{Q}_p$, then $D_{pst}(W)$ is naturally a free module of rank d over $\mathbb{Q}_p^{\mathrm{ur}} \otimes_{\mathbb{Q}_p} F$, and we may obtain the necessary roots of unity by extending scalars to $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} F$; but this is, of course, not a field but rather a finite product of fields indexed by embeddings $f : F \hookrightarrow \overline{\mathbb{Q}}_p$. While [18, §3.3.4] consider

$$\left(\epsilon_{\overline{\mathbb{Q}}_p}\left(\overline{\mathbb{Q}}_p \otimes_{(F\otimes\mathbb{Q}_p^{\mathrm{ur}},f)} D_{pst}(W)\right)\right)_f \in \left(\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} F\right)^{\times} = \prod_f \overline{\mathbb{Q}}_p^{\times}.$$

in this paper we fix an embedding $\iota : F \hookrightarrow \overline{\mathbb{Q}}_p$ and use the corresponding Weil-Deligne representation $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}^{ur}\otimes_{\mathbb{Q}_p}F} \mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V\otimes\rho_\chi^*))$ to define by abuse of notation

$$\begin{aligned}\epsilon(D_{pst}(W)) &:= \epsilon_{F,\iota}(D_{pst}(W))\\ &:= \epsilon_{\overline{\mathbb{Q}}_p}(\mathbb{Q}_p, \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} F\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V\otimes\rho_\chi^*)))\end{aligned} \tag{2.6}$$

to be the corresponding component. Here ρ_χ denotes the representation attached to the character χ and ρ_χ^* its contragredient one. If necessary we shall add the dependence of ψ as $\epsilon(D_{pst}(W), \psi)$, similarly with the Haar measure. In the case that W is an Artin representation we also allow ourselves to omit the D_{pst} from the notation.

2.4. Galois cohomology. First we compute the continuous Galois cohomology groups $\mathrm{H}^i(L, T)$ as Λ-modules. We point out, that doing this we also determine the Galois cohomology groups of the Kummer dual representation $T^*(1)$ in view of the local duality theorems in the Galois cohomology theory. We start with the following crucial although well-known

Remark 2.4. T is the Tate module $T_p\mathcal{F}_\pi$ of a Lubin-Tate formal group $\mathcal{F} = \mathcal{F}_\pi$ with $\pi = \chi^{ur}_{\mathbb{Q}_p}(Fr_{\mathbb{Q}_p})p$. For more details see [19, Exm. 5.20].

Proposition 2.5. *With the notation as above we have:*

(i) $\mathrm{H}^i(L, T) = 0$ *for* $i \neq 1, 2$.

(ii) $\mathrm{H}^1(L, T) \cong \left(\widehat{(L^{ur})^{\times}}^{p}(\chi^{ur})\right)^{G(L^{ur}/L)}$, *where* $\widehat{-}^p$ *denotes the* p*-completion of a group.*

(iii) *There is an isomorphism of* Λ*-modules*

$$\mathcal{F}(\mathfrak{p}_L) \xrightarrow{\cong} \mathrm{H}^1(L,T).$$

(iv) $\mathrm{H}^2(L,T) \cong \mathrm{H}^0(L,V^*/T^*(1))^\vee \cong \mathbb{Z}_p/p^\omega(\chi^{ur})$ *is finite, where* $\omega = v_p(1-\chi^{ur}(Fr_L)) \neq \infty$*. There is an exact sequence of* Λ*-modules*

$$0 \longrightarrow \mathbb{Z}_p[G/I] \xrightarrow{1-\chi^{ur}(Fr_K)Fr_K} \mathbb{Z}_p[G/I] \longrightarrow \mathrm{H}^2(L,T) \longrightarrow 0\ .$$

In particular, $\mathrm{H}^i(L,T)$ *lies in the category of perfect complexes of* Λ*-modules for all* i *if the extension* L/K *is at most tamely ramified.*

Proof. $\mathrm{H}^i(L,T)=0$ for $i\neq 0,1,2$ because the cohomological dimension of G_K is 2. Further, $\mathrm{H}^0(L,T)=T^{G_L}=0$, as the character $\chi^{ur}\otimes\chi^{cyc}: G_L \to \mathbb{Z}_p^\times$ is not trivial. Using the local duality theorem [23, Thm. (7.2.6)] we get $\mathrm{H}^2(L,T)\cong \mathrm{H}^0(L,V^*(1)/T^*(1))^\vee$. Since $V^*(1)/T^*(1) = \mathbb{Q}_p/\mathbb{Z}_p\big((\chi^{ur})^{-1}\big)$ is unramified, we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p\big((\chi^{ur})^{-1}\big) \xrightarrow{1-Fr_L} \mathbb{Q}_p/\mathbb{Z}_p\big((\chi^{ur})^{-1}\big) \longrightarrow 0,$$

whose Pontryagin dual becomes

$$0 \longrightarrow \mathbb{Z}_p(\chi^{ur}) \xrightarrow{1-Fr_L^{-1}} \mathbb{Z}_p(\chi^{ur}) \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1))^\vee \longrightarrow 0,$$

whence $\mathrm{H}^2(L,T)$ is a finite cyclic group as by assumption $(\chi^{ur})^{-1}(G_L)\neq 1$. On the other hand it is easy to calculate the $\mathbb{Z}_p$-elementary divisors of the matrix representing the operator $1-\chi^{ur}(Fr_K)Fr_K$ on $\mathbb{Z}_p[G/I]$:

$$1,\dots,1,1-\chi^{ur}(Fr_L) = -\chi^{ur}(Fr_L)\left(1-\chi^{ur}(Fr_L^{-1})\right),$$

whence the exactness in (iv) is clear because $-\chi^{ur}(Fr_L)$ is a unit. Since $R\Gamma(G_L,T)$ is a perfect complex, the remaining assertions follow immediately by standard homological algebra.

To compute the group $\mathrm{H}^1(L,T)$ we use the Hochschild-Serre spectral sequence for the closed subgroup $G_{L^{ur}}$ of G_L, which exists a priori only for finite discrete modules T/p^n, but with [23, Thm. 2.7.5] also for the compact module T. Note that the character χ^{ur} factors over L^{ur}, such that $G_{L^{ur}}$ acts via the cyclotomic character on T. The five-term exact sequence takes the form

$$0 \to \mathrm{H}^1(G(L^{ur}/L),T^{G_{L^{ur}}}) \to \mathrm{H}^1(L,T) \to \mathrm{H}^1(G_{L^{ur}},T)^{G(L^{ur}/L)} \to \mathrm{H}^2(G(L^{ur}/L),T^{G_{L^{ur}}}) \to \mathrm{H}^2(L,T).$$

The module of invariants $T^{G_{L^{ur}}}$ is a zero-module, because the cyclotomic character is not trivial, thus the first and the fourth term in the exact sequence above vanish and we get a canonical isomorphism

$$\mathrm{H}^1(L,T) \cong \mathrm{H}^1(G_{L^{ur}},T)^{G(L^{ur}/L)} = \mathrm{H}^1(G_{L^{ur}},\mathbb{Z}_p(\chi^{ur})(1))^{G(L^{ur}/L)}.$$

From Kummer theory and the isomorphism

$$\mathrm{H}^1(G_{L^{ur}},\mathbb{Z}_p(\chi^{ur})(1))^{G(L^{ur}/L)} = \Big(\mathrm{H}^1(G_{L^{ur}},\mathbb{Z}_p(1))(\chi^{ur})\Big)^{G(L^{ur}/L)}$$

we obtain $H^1(L,T) \cong \Big(\widehat{(L^{ur})^\times}^p(\chi^{ur})\Big)^{G(L^{ur}/L)}$.

By taking G_L-invariants of the exact sequence

$$0 \longrightarrow \mathcal{F}(\bar{\mathfrak{p}})[p^n] \longrightarrow \mathcal{F}(\bar{\mathfrak{p}}) \xrightarrow{[p^n]} \mathcal{F}(\bar{\mathfrak{p}}) \longrightarrow 0 \tag{2.7}$$

we get the following exact sequence of Λ-modules

$$0 \longrightarrow {}^{\mathcal{F}(\mathfrak{p}_L)}/_{[p^n](\mathcal{F}(\mathfrak{p}_L))} \longrightarrow \mathrm{H}^1(L,\mathcal{F}(\bar{\mathfrak{p}})[p^n]) \longrightarrow \mathrm{H}^1(L,\mathcal{F}(\bar{\mathfrak{p}}))[p^n] \longrightarrow 0$$

for each $n \geq 1$.

By Remark 2.4 the inverse limit over n of the exact sequences above results in the exact sequence of Λ-modules

$$0 \longrightarrow \mathcal{F}(\mathfrak{p}_L) \longrightarrow \mathrm{H}^1(L,T) \longrightarrow \mathrm{H}^1(L,T)/\mathcal{F}(\mathfrak{p}_L) \longrightarrow 0$$

$\mathcal{F}(\mathfrak{p}_L)$ being a finitely generated $\mathbb{Z}_p$-module (cf. [4, 4.5.1]).

From [19, Exm. 5.20] we deduce that the quotient $\mathrm{H}^1(L,T)/\mathcal{F}(\mathfrak{p}_L)$ is isomorphic to

$$\Big(\frac{\widehat{(L^{ur})^\times}^p}{U^1(\widehat{L^{ur}})}(\chi^{ur})\Big)^{G(L^{ur}/L)}, \tag{2.8}$$

where U^1 denotes 1-units while for an abelian group A we write $\widehat{A}^p := \varprojlim_n A/p^nA$ for its pro-p-completion.

For the extension $L^{ur}/\mathbb{Q}_p$ we have (both algebraically and topologically)

$$(L^{ur})^\times = (\pi_L) \times \mathcal{O}_{L^{ur}}^\times \cong \mathbb{Z} \oplus \mathcal{O}_{L^{ur}}^\times,$$

where π_L is a prime element of $\mathcal{O}_L$. Let κ_L denote the residue class field of L, then we have a split exact sequence

$$1 \longrightarrow U^1(L^{ur}) \longrightarrow \mathcal{O}_{L^{ur}}^\times \longrightarrow \overline{\kappa_L}^\times \longrightarrow 1.$$

The group $\overline{\kappa_L}^\times$ is p-divisible, thus

$$\widehat{\overline{\kappa_L}^\times}^p = \varprojlim_n \overline{\kappa_L}^\times/(\overline{\kappa_L}^\times)^{p^n} = 1$$

and

$$\widehat{(L^{ur})^{\times}}^{p} = \widehat{(\pi_L)}^{p} \times \widehat{U^1(L^{ur})}^{p} = \widehat{(\pi_L)}^{p} \times U^1(\widehat{L^{ur}}),$$

whence the quotient in (2.8) is isomorphic to

$$\left(\widehat{(\pi_L)}^{p}(\chi^{ur})\right)^{G(L^{ur}/L)}.$$

The latter module is zero, since the group $G(L^{ur}/L)$ acts trivially on (π_L) while χ^{ur} is a non-trivial character. $\square$

Next we compute the finite part $\mathrm{H}^1_f(L,T) \subseteq \mathrm{H}^1(L,T)$ defined as a preimage of $\mathrm{H}^1_f(L,V)$ under the map $i : \mathrm{H}^1(L,T) \to \mathrm{H}^1(L,V)$.

Lemma 2.6. $\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V) = [L : \mathbb{Q}_p]$.

Proof. Both, V and $V^*(1)$, are de Rham representations of G_L, thus from [3, pp. 355-356] we have

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) + \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V^*(1)) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V) \tag{2.9}$$

and

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p}(t_V(L)) + \dim_{\mathbb{Q}_p} \mathrm{H}^0(L,V). \tag{2.10}$$

The same is true for $V^*(1)$. But $\mathrm{H}^0(L,V)$ and $\mathrm{H}^0(L,V^*(1))$ are both trivial by the proof of Proposition 2.5, so that

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p} t_V(L)$$

and

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V^*(1)) = \dim_{\mathbb{Q}_p} t_{V^*(1)}(L).$$

For a de Rham representation W by [15, p. 148]

$$t_W(L) = gr^{-1}(W) \hookrightarrow (\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} W)^{G_L}.$$

Moreover, by Corollary 3.57 in [3]

$$(\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} V)^{G_L} = (\mathbb{C}_p(\chi^{ur}))^{G_L} \cong L$$

and

$$(\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} V^*(1))^{G_L} = (\mathbb{C}_p((\chi^{ur})^{-1})(-1))^{G_L} = 0.$$

Thus from equality (2.9) we get

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(L,V) = \dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V).$$

Finally, using the formula for the Euler characteristic (see [24, II. 5.7] or [16])

$$\sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(L,V) = -[L : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} V$$

we see that $\dim_{\mathbb{Q}_p} \mathrm{H}^1(L,V) = [L : \mathbb{Q}_p]$, as $\mathrm{H}^i(L,V) = 0$ for $i \neq 1$ (cf. Proposition 2.5). $\square$

Corollary 2.7. *From the above proposition it follows that*

(i) $\mathrm{H}^1_f(L,T) = \mathrm{H}^1(L,T)$ *is a* $\mathbb{Z}_p$*-module of rank* $[L:\mathbb{Q}_p]$ *and*

(ii) $\mathrm{H}^1_f(L,T^*(1)) = \mathrm{H}^1(L,T^*(1))_{tors} \cong \mathrm{H}^0(L,V^*(1)/T^*(1))$ *is a finite torsion group.*

Proof. The first part is obvious. By the definition of $\mathrm{H}^1_f(L,T^*(1))$ it contains the torsion subgroup of $\mathrm{H}^1(L,T^*(1))$ and, since the image of the group $\mathrm{H}^1_f(L,T^*(1))$ in $\mathrm{H}^1(L,V^*(1))$ is zero, they are equal. Consider the exact sequence of G_L-modules

$$0 \longrightarrow T^*(1) \longrightarrow V^*(1) \longrightarrow V^*(1)/T^*(1) \longrightarrow 0.$$

The associated long exact sequence in cohomology is

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(L,T^*(1)) \longrightarrow \mathrm{H}^0(L,V^*(1)) \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1)) \\ \longrightarrow \mathrm{H}^1(L,T^*(1)) \longrightarrow \mathrm{H}^1(L,V^*(1)) \longrightarrow \mathrm{H}^1(L,V^*(1)/T^*(1)) \longrightarrow \dots \end{aligned}$$

The groups $\mathrm{H}^0(L,T^*(1))$ and $\mathrm{H}^0(L,V^*(1))$ are trivial. Furthermore, the third group $\mathrm{H}^0(L,V^*(1)/T^*(1))$ is a finite torsion group, since $\chi^{ur}(Fr_L) \neq 1$, so that we can replace $\mathrm{H}^1(L,T^*(1))$ by $\mathrm{H}^1_f(L,T^*(1))$ in the exact sequence above getting

$$0 \longrightarrow \mathrm{H}^0(L,V^*(1)/T^*(1)) \xrightarrow{\cong} \mathrm{H}^1_f(L,T^*(1)) \longrightarrow 0. \qquad \square$$

3. Bloch-Kato exponential map

In this section we recall the definition of the Bloch-Kato exponential map (see [3] or [1, p. 611f]). The short exact sequences

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cris}^{\phi=1} \longrightarrow B_{dR}/\mathrm{Fil}^0 B_{dR} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cris} \xrightarrow{(1-\phi,1)} B_{cris} \oplus B_{dR}/\mathrm{Fil}^0 B_{dR} \longrightarrow 0$$

induce after tensoring with W and taking G_L-invariants the following exact sequences

$$0 \longrightarrow \mathrm{H}^0(L,W) \longrightarrow D^L_{cris}(W)^{\phi=1} \longrightarrow t_W(L) \longrightarrow \mathrm{H}^1_e(L,W) \longrightarrow 0 \tag{3.1}$$

and

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(L,W) \longrightarrow D^L_{cris}(W) \\ \longrightarrow D^L_{cris}(W) \oplus t_W(L) \longrightarrow \mathrm{H}^1_f(L,W) \longrightarrow 0. \end{aligned} \tag{3.2}$$

The connecting homomorphism $exp_W : t_W(L) \to \mathrm{H}^1_e(L, W)$ of the first sequence is the Bloch-Kato exponential map for a de Rham representation W. As explained before [1, lem. 1.3] there is also a dual exponential map $exp^*_W : \mathrm{H}^1(L, W) \to \mathrm{Fil}^0 D_{dR}(W)$. Patching the sequence (3.2) with its dual along local Tate duality we obtain the following exact sequence of $\mathbb{Q}_p[G]$-modules (see [1, (2.2)]):

$$(3.3) \quad 0 \longrightarrow \mathrm{H}^0(L, V) \longrightarrow \mathrm{D}^L_{cris}(V) \longrightarrow \mathrm{D}^L_{cris}(V) \oplus t_V(L)$$
$$\xrightarrow{exp_V} \mathrm{H}^1(L, V) \xrightarrow{exp^*_V} \mathrm{D}^L_{cris}(V^*(1))^* \oplus t^*_{V^*(1)}(L)$$
$$\longrightarrow \mathrm{D}^L_{cris}(V^*(1))^* \longrightarrow \mathrm{H}^2(L, V) \longrightarrow 0,$$

in which we have already specialised to our representation V again. By Proposition 2.5 and the proof of Lemma 2.6 we know that the groups $\mathrm{H}^0(L, V)$, $\mathrm{H}^2(L, V)$, $\mathrm{H}^1_f(L, V^*(1))$, $t_{V^*(1)}(L)$ are trivial, whence the exact sequence above degenerates into the two short exact sequences

$$0 \longrightarrow \mathrm{D}^L_{cris}(V^*(1))^* \xrightarrow{1-\phi^*} \mathrm{D}^L_{cris}(V^*(1))^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{D}^L_{cris}(V) \longrightarrow \mathrm{D}^L_{cris}(V) \oplus t_V(L) \xrightarrow{exp_V} \mathrm{H}^1(L, V) \longrightarrow 0,$$

where ϕ^* denotes the dual of ϕ. Furthermore, for weight reasons or using (the proof of) Lemma 6.3 we see that

$$\mathrm{D}^L_{cris}(V) \xrightarrow{1-\phi} \mathrm{D}^L_{cris}(V)$$

is an isomorphism. Using the exact sequence

$$0 \longrightarrow t^*_{V^*(1)}(L) \longrightarrow \mathrm{D}^L_{dR}(V) \longrightarrow t_V(L) \longrightarrow 0$$

$(t^*_{V^*(1)}(L) \cong \mathrm{Fil}^0\mathrm{D}^L_{dR}(V))$ and the isomorphism

$$0 = \frac{\mathrm{D}^L_{cris}(V)}{(1-\phi)\mathrm{D}^L_{cris}(V)} \cong \frac{\mathrm{H}^1_f(L, V)}{\mathrm{H}^1_e(L, V)}$$

of [1, Lem. 1.3] we deduce that

$$\mathrm{D}^L_{dR}(V) \xrightarrow{exp_V} \mathrm{H}^1(L, V)$$

is an isomorphisms, too.

Now let $\mathcal{G}$ be a commutative formal Lie group of finite height over $\mathcal{O}_K$ and W be the p-adic de Rham representation coming from the p-adic Tate module of $\mathcal{G}$. In [3, pp. 359-360] a commutative diagram is described, which connects the Bloch-Kato exponential map with the classical exponential map of $\mathcal{G}$:

$$\begin{CD} \tan(\mathcal{G}_K)(L) @>{exp}>> \mathcal{G}(\mathfrak{p}_L)\otimes\mathbb{Q}_p \\ @V{=}VV @VVV \\ t_W(L) @>{exp}>> \mathrm{H}^1(L,W), \end{CD}$$

where $t_W(L)$ is identified with the tangent space of $\mathcal{G}_K$, the upper (resp. lower) *exp* is the exponential map in the classical sense (resp. Bloch-Kato) and the right vertical map is the boundary map of the Kummer sequence (2.7).

By Proposition 2.5(3) and Lemma 2.6 the representation T being the p-adic Tate module of a formal group $\mathcal{F}$ by Remark 2.4 fits into the commutative diagram

$$\begin{CD} \mathcal{F}(\mathfrak{p}_L) @>{\cong}>> \mathrm{H}^1(L,T) \\ @VVV @VV{i}V \\ t_V(L) @>{exp}>{\cong}> \mathrm{H}^1(L,V), \end{CD} \tag{3.4}$$

where the left vertical arrow is a Λ-homomorphism induced by the classical logarithm $log_{\mathcal{F}}$ of $\mathcal{F}$.

4. Comparison isomorphisms

Let T be a finitely generated (free) $\mathbb{Z}_p$-module with a continuous G_K-action. We assume that $V := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is a de Rham representation. Furthermore we define $\mathbb{Z}_p[G]$-modules

$$\mathbb{T} := \mathbb{Z}_p[G]^\sharp \otimes_{\mathbb{Z}_p} T$$

and

$$\mathbb{V} := \mathbb{Z}_p[G]^\sharp \otimes_{\mathbb{Z}_p} V,$$

where the $\mathbb{Z}_p[G]$-action is given by left-multiplication on the left tensor-factor. The sharp indicates that these modules are endowed with the following G_K-actions: $\sigma(\lambda\otimes t) := \lambda\bar{\sigma}^{-1}\otimes\sigma t$, where $\bar{\sigma}$ denotes the image of σ under the natural projection map. Henceforth we use the following explicit realization for the induction

$$\mathrm{Ind}_{L/\mathbb{Q}_p}T := \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_K]} \mathbb{T} \left(\cong \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_L]} T\right)$$

and similarly for $\mathrm{Ind}_{L/\mathbb{Q}_p}V$. These are $G_{\mathbb{Q}_p}$-modules by the action on the left tensor-factor while they become Λ- and Ω-modules via the corresponding module structures of $\mathbb{T}$ and $\mathbb{V}$, respectively.

The p-adic comparison isomorphism for (the induction of) V

$$\begin{aligned} comp_V := comp_{V,L/\mathbb{Q}_p} : B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V) &\overset{\cong}{\to} B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V, \\ c \otimes x &\mapsto cx \end{aligned} \tag{4.1}$$

is a $B_{dR}[G]$-linear map, which commutes with the action of $G_{\mathbb{Q}_p}$, if $G_{\mathbb{Q}_p}$ acts diagonally.

We apply the determinant functor to $comp_V$ and by Remark 2.1 we obtain a map

$$\begin{aligned} \tilde{\alpha}_{V,L/K} &= (x, y) \\ &\in \mathrm{Isom}\Big(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p} V)\Big) \times^{K_1(\Omega)} K_1(B_{dR}[G]). \end{aligned}$$

Multiplying $\tilde{\alpha}_{V,L/K}$ with t (in general the precise power of t depends on the weights (with multiplicities), see [18] or [1]) we get

$$\begin{aligned} \alpha_{V,L/K} &= (x, ty) \\ &\in \mathrm{Isom}\Big(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p} V)\Big) \times^{K_1(\Omega)} K_1(K'[G]) \end{aligned}$$

for some finite abelian extension K' of $\widehat{\mathbb{Q}_p^{ur}}$.

The maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$ is the composite of the maximal unramified extension $\mathbb{Q}_p^{ur}$ and the cyclotomic extension $\mathbb{Q}_{p,\infty}$ which is obtained by adjoining all p-power roots of 1. For $g \in G_{\mathbb{Q}_p}$ we define $g^{ur} \in G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ by $g^{ur}|_{\mathbb{Q}_p^{ur}} = g|_{\mathbb{Q}_p^{ur}}$ and $g^{ur}|_{\mathbb{Q}_{p,\infty}} = \mathrm{id}$. We also define $g^{ram} \in G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ by $g^{ram}|_{\mathbb{Q}_p^{ur}} = \mathrm{id}$ and $g^{ram}|_{\mathbb{Q}_{p,\infty}} = g|_{\mathbb{Q}_{p,\infty}}$. Thus $g|_{\mathbb{Q}_p^{ab}} = g^{ur} g^{ram}$.

Let $\Gamma(V) := \prod_{\mathbb{Z}} \Gamma^*(j)^{-h(-j)}$, where $h(j) = \dim_{\mathbb{Q}_p} gr^j D_{dR}(V)$ and where $\Gamma^*(-j)$ is defined to be $\Gamma(j) = (j-1)!$ if $j > 0$ and $\lim_{s \to j}(s-j)\Gamma(s) = (-1)^j((-j)!)^{-1}$ otherwise.

We set

$$\beta_{V,L/K} := \Gamma(V)\epsilon_D(L/K, V) \cdot \alpha_{V,L/K} = (x, \Gamma(V)\epsilon_D(L/K, V) \cdot t \cdot y) =: (x, \tilde{y}),$$

with

$$\begin{aligned} \epsilon_D(L/K, V) &:= \Big(\epsilon(\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V \otimes \rho_\chi^*)))\Big)_{\chi \in Irr(G)} \\ &\in \prod_{\chi \in Irr(G)} \overline{\mathbb{Q}}_p^\times \cong K_1(\overline{\mathbb{Q}}_p[G]), \end{aligned}$$

where ρ_χ denotes a representation with character χ and ρ_χ^* its dual. Here we use definition (2.6) with respect to a fixed finite extension F of $\mathbb{Q}_p$ which is large enough so that all characters of G can be realized over it. The above identification for K_1 arises by the Wedderburn decomposition of $\overline{\mathbb{Q}}_p[G]$.

Note also that we have canonical isomorphisms

$$V_\rho \otimes_\Lambda \mathrm{Ind}_{L/\mathbb{Q}_p} T \cong \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_K]} (V_\rho \otimes_\Lambda \mathbb{T}) \cong \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_K]} (V_{\rho^*} \otimes_{\mathbb{Q}_p} V),$$

whence the ρ-component above in the definition of the ϵ-factor is the one which is used by Fukaya and Kato [18] for the module $V_\rho \otimes_\Lambda \mathrm{Ind}_{L/\mathbb{Q}_p} T$, compare also with appendix A.

According to [1, Lem. 2.15] $\beta_{V,L/K}$ is an element of

$$\mathrm{Isom}\Big(\mathbf{d}_\Omega(\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)), \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p} V)\Big) \times^{K_1(\Omega)} K_1(\widehat{\mathbb{Q}_p^{ur}}[G])$$

with

$$g(\tilde{y}) = g^{ur}(\tilde{y}) = [\mathrm{Ind}_{L/\mathbb{Q}_p} V, g^{ur}] \cdot \tilde{y},\ \forall g \in G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p),$$

i.e., $\tilde{y} \in K_1(\tilde{\Omega})_{[\mathrm{Ind}_{L/\mathbb{Q}_p} V, \tau_p]^{-1}} := \{x \in K_1 \tilde{\Omega}) | \sigma(x) = [\mathrm{Ind}_{L/\mathbb{Q}_p} V, \tau_p]\}$ in the terminology of [18], where τ_p in $G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ is the unique element with $\tau_p^{ur} = \sigma$ and $\tau_p^{ram} = 1$.

For a representation W of G_K we denote by $\mathfrak{f}(W) = \pi_K^{a(W)}$ its local Artin conductor (sometimes also understanding the ideal generated by this element) while $f(W) = q_K^{a(W)} = |\mathfrak{f}(W)|_p^{-1}$ denotes its absolute norm, where q_K is the cardinality of the residue class field of K. Setting $\pi_{\mathbb{Q}_p} = p$ we have $f(W) = p^{a(W)} = \mathfrak{f}(W)$ for every $G_{\mathbb{Q}_p}$-representations W.

For $a \in F^\times$ and an additive character ψ of F we denote by $a\psi$ the character sending x to $\psi(ax)$.

Lemma 4.1. *We have the equality*

$$\begin{aligned}\epsilon(\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V \otimes \rho_\chi^*)), \psi_\xi)&\\ &= \epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi), -\psi_\xi)^{-1} \chi_{\mathbb{Q}_p}^{ur}(\mathfrak{f}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi)))\end{aligned}$$

where $\mathbb{Z}_p(\chi^{ur}) = \mathrm{T}_p\mathcal{F}(-1)$ and the character $\chi_{\mathbb{Q}_p}^{ur}$ is viewed as a character $\mathbb{Q}_p^\times \to G_{\mathbb{Q}_p}^{ab} \stackrel{\chi_{\mathbb{Q}_p}^{ur}}{\to} \mathbb{Q}_p^\times$ via the local reciprocity law sending p to the geometric Frobenius.

Proof. Since $\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V \otimes \rho_\chi^*)) \cong \mathrm{D}_{pst}(V) \otimes_{\mathbb{Q}^{ur}} \mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi^*))$ we obtain for the linearized Kummer dual

$$\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(V \otimes \rho_\chi^*))^*(1) \cong \mathrm{D}_{pst}(V)^*(1) \otimes_{\mathbb{Q}^{ur}} \mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi^*))^*.$$

Now the (linearized) action on the unramified $\mathrm{D}_{pst}(V)$ is given by the character $g \mapsto (p\chi_{\mathbb{Q}_p}^{ur}(Fr_{\mathbb{Q}_p}))^{v(g)}$, whence $\mathrm{D}_{pst}(V)^*(1)$ bears the action $g \mapsto \chi_{\mathbb{Q}_p}^{ur}(Fr_{\mathbb{Q}_p})^{-v(g)}$ while the Weil-Deligne representations $\mathrm{D}_{pst}(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi^*))^*$ and $\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi)$ can be identified. Now the claim follows from [18, 3.2.2 (3) and (5)] where τ denotes a geometric Frobenius. ◻

The induction property of local Artin conductors (see [4, Lem. 3.3]) gives

$$f(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi^*)) = N_{K/\mathbb{Q}_p}(\mathfrak{f}(\rho_\chi^*)) \cdot d_{K/\mathbb{Q}_p}^{\chi(1)} \tag{4.2}$$

where $d_{K/\mathbb{Q}_p}$ denotes the discriminant of $K/\mathbb{Q}_p$.

For later purposes we also introduce the local Galois Gauss sum

$$\tau_K(\chi) = \epsilon(\rho_\chi|\cdot|_p^{\frac{1}{2}}, \psi_K, dx_{\psi_K})\sqrt{f(\chi)},$$

i.e., dx_{ψ_K} is the Haar measure of K which is self dual with respect to the standard additive character $\psi_K := \psi_{\mathbb{Q}_p} \circ Tr_{K/\mathbb{Q}_p}$. Note that this definition coincides with that in [4, 17] although in Breuning's thesis $\overline{\rho}$ instead of ρ shows up in the definition of τ: there implicitly the norm rest symbol is normalized by sending π_K to the *arithmetic* Frobenius automorphism, i.e., opposite to the convention used here, whence one has to replace a representation by its contragredient one.

Remark 4.2. An easy calculation shows that for our V the factor $\Gamma(V)$, i.e., $\Gamma_L(V)$ of [18, 3.3.4] or $\Gamma^*(V)$ of [1, 2.4], used for the correction of the comparison isomorphism in the definition of $\beta_{V,L/K}$ is equal to 1.

5. Formulation of the ϵ-conjecture

In this section we follow closely the approach of [1] to construct an isomorphism

$$\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) : \mathbf{d}_{\tilde{\Omega}}(0) \to \tilde{\Omega} \otimes_\Omega \left\{\mathbf{d}_\Omega(R\Gamma(L,V)) \cdot \mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p}V)\right\}$$

satisfying the following condition

$(\star)$: Let $\rho : \Omega \to GL_n(F)$, $n \geq 1$, $[F : \mathbb{Q}_p] < \infty$, be a continuous representation. Then the image of $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ under $F^n \otimes_\Omega -$ is the ϵ-isomorphism of de Rham representations described in [18, Sec. 3.3].

The application of the determinant functor to (3.3) results in the isomorphism

$$\mathbf{d}_\Omega(1-\phi) \cdot \mathbf{d}_\Omega(exp^{-1}) \cdot \mathbf{d}_\Omega((1-\phi^*)^{-1}) \tag{5.1}$$

sending

$$\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V)) \cdot \mathbf{d}_\Omega(R\Gamma(L,V))^{-1} \cdot \mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V^*(1))^*)$$

to

$$\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V)) \cdot \mathbf{d}_\Omega(\mathrm{D}_{dR}^L(V)) \cdot \mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V^*(1))^*),$$

which after base change (abbreviating $\tilde{\Omega} \otimes_\Omega \mathbf{d}_\Omega(-)$ by $\mathbf{d}_\Omega(-)_{\tilde{\Omega}}$) and composition with

$$\mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V))_{\tilde{\Omega}}} \cdot \beta_{V,L/K} \cdot \mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V^*(1))^*)_{\tilde{\Omega}}}$$

and multiplication with

$$\mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V))_{\tilde{\Omega}}^{-1}} \cdot \mathrm{id}_{\mathbf{d}_\Omega(R\Gamma(L,V))_{\tilde{\Omega}}} \cdot \mathrm{id}_{\mathbf{d}_\Omega(\mathrm{D}_{cris}^L(V^*(1))^*)_{\tilde{\Omega}}^{-1}}$$

gives the desired isomorphism $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$. Note that $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ satisfies the condition $(\star)$ automatically by construction (cf. the construction of the ϵ-isomorphisms of de Rham representations in [18, Sec. 3.3] and Remark 4.2).

Above we constructed an isomorphism

$$\begin{aligned}\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V) &:= (x,\tilde{y})\\ &\in \mathrm{Isom}(\mathbf{d}_\Omega(0),\mathbf{d}_\Omega(R\Gamma(L,V))\cdot\mathbf{d}_\Omega(\mathrm{Ind}_{L/\mathbb{Q}_p}V))\times^{K_1(\Omega)}K_1(\tilde{\Omega})\end{aligned}$$

satisfying condition $(\star)$. Here we can and do assume that x arises by base change from an isomorphism $\mathbf{d}_\Lambda(0)\cong\mathbf{d}_\Lambda(R\Gamma(L,T))\cdot\mathbf{d}_\Lambda(\mathrm{Ind}_{L/\mathbb{Q}_p}T)$ by [18, Prop. 3.1.3]. In view of of the localization exact sequence for K-groups

$$1\longrightarrow K_1(\tilde{\Lambda})\longrightarrow K_1(\tilde{\Omega})\xrightarrow{\partial}K_0(\tilde{\Lambda},\widehat{\mathbb{Q}_p^{ur}})\longrightarrow 0$$

($SK_1(\tilde{\Lambda})$ being trivial by [20, Cor. 2.28] and the map $\partial := \partial_{\tilde{\Lambda},\tilde{\Omega}}$ being surjective by [4, Lem. 2.5 and §2.4.4]) we formulate

Conjecture 5.1 ($C^{na}_{ep}(L/K,V)$)**.** *With the notation as above we have the following equation $\partial(\tilde{y})=0$ in $K_0(\tilde{\Lambda},\widehat{\mathbb{Q}_p^{ur}})$. Equivalently, the class*

$$[\mathbf{d}_\Lambda(R\Gamma(L,T)\cdot\mathbf{d}_\Lambda(\mathrm{Ind}_{L/\mathbb{Q}_p}T),\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)^{-1}]\in\pi_0(\mathcal{C}(\Lambda,\tilde{\Omega}))$$

associated with $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ becomes trivial in $\pi_0(\mathcal{C}(\tilde{\Lambda},\tilde{\Omega}))$.

In order to relate our conjecture to the approach of [4] we decided to formulate the ϵ-conjecture in the language of relative K_0-groups, but note that due to the above short exact sequence this definition is also compatible with [18, Conj. 3.4.3], which claims that $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}V)$ arises by change of rings from an integral isomorphism

$$\tilde{\epsilon}_{\Lambda,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p}T)\in\mathrm{Isom}(\mathbf{d}_\Lambda(0),\mathbf{d}_\Lambda(R\Gamma(L,T))\cdot\mathbf{d}_\Lambda(\mathrm{Ind}_{L/\mathbb{Q}_p}T))\times^{K_1(\Lambda)}K_1(\tilde{\Lambda}).$$

Remark 5.2. Although for simplicity we restricted to the representation V in the above conjecture it is obvious that the formulation extends easily to general de Rham representations W.

Now we shall reformulate the conjecture in the language of Breuning: The complex $R\Gamma(L,T)$ is a perfect complex of Λ-modules and $\mathrm{Ind}_{L/\mathbb{Q}_p}T$ is a finitely generated projective Λ-module, thus $M^\bullet := R\Gamma(L,T)\oplus\mathrm{Ind}_{L/\mathbb{Q}_p}T[0]$ is a perfect complex of Λ-modules with $\mathrm{H}^0(M^\bullet)\cong\mathrm{Ind}_{L/\mathbb{Q}_p}T$, $\mathrm{H}^1(M^\bullet)\cong\mathrm{H}^1(L,T)$, $\mathrm{H}^2(M^\bullet)\cong\mathrm{H}^2(L,T)$ and $\mathrm{H}^i(M^\bullet)=0$ for $i\geq 3$. There is an isomorphism

$$comp_V\circ exp^{-1}:\mathrm{H}^1(B_{dR}\otimes M^\bullet)\to\mathrm{H}^0(B_{dR}\otimes M^\bullet),$$

and we define $C_{L/K} := \chi(M^\bullet, comp_V\circ exp^{-1})\in K_0(\Lambda,B_{dR})$ to be the associated refined Euler characteristic.

Set

$$U_{cris} := \partial([\mathrm{D}^L_{cris}(V), 1-\phi]) + \partial([\mathrm{D}^L_{cris}(V^*(1))^*, (1-\phi^*)^{-1}]) \in K_0(\Lambda, \mathbb{Q}_p).$$

Note that

$$\partial([\mathrm{D}^L_{cris}(V^*(1))^*, (1-\phi^*)^{-1}]) = -\partial([\mathrm{D}^L_{cris}(V^*(1)), 1-\phi]),$$

since $\partial([W^*, \psi^*]) = \partial([W, \psi])$, so that

$$U_{cris} = \partial([\mathrm{D}^L_{cris}(V), 1-\phi]) - \partial([\mathrm{D}^L_{cris}(V^*(1)), 1-\phi]).$$

Finally, the multiplication of $\tilde{\alpha}_{V,L/K}$ with t and the equivariant ϵ-factor translates in the language of relative K_0-groups into the summation of their images under ∂.

Consider the class

$$C_{L/K} + \partial_{\Lambda, B_{dR}}(t) + \partial_{\Lambda, B_{dR}}(\epsilon_D(L/K, V)).$$

This belongs to $K_0(\Lambda, \widehat{\mathbb{Q}_p^{ur}})$, because it is invariant under the action of $G_{\mathbb{Q}_p^{ur}}$ and $K_0(\Lambda, \widehat{\mathbb{Q}_p^{ur}}) = K_0(\Lambda, B_{dR})^{G_{\mathbb{Q}_p^{ur}}}$. The conjecture $C^{na}_{ep}(L/K, V)$ takes the form:

(5.2) $C_{L/K} + U_{cris} + \partial(t) + \partial(\epsilon_D(L/K, V))$ becomes trivial in $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}})$.

Indeed, we have to compare the two trivialisations, one given by the definition of $\tilde{\epsilon}_{\Omega,\xi}(\mathrm{Ind}_{L/\mathbb{Q}_p} V)$ and the other one hidden in the above relative K-theory classes (more precisely, the latter expression is mapped under $\Psi_{\tilde{\Lambda},\tilde{\Omega}}$ in (2.5) to the first one): the class $C_{L/K}$ contains the isomorphisms $comp_V$, which corresponds to the constituent $\tilde{\alpha}_{V,L/K}$, and $\exp^{-1}$, which is part of the identifications (5.1); the remaining Euler factors in (5.1) correspond to the class U_{cris}. Finally, as mentioned above already, the difference between $\tilde{\alpha}_{V,L/K}$ and $\alpha_{V,L/K}$ is represented by the classes $\partial(t)$ and $\partial(\epsilon_D(L/K, V))$.

This is similar to a conjecture stated and proved by Breuning for $V = \mathbb{Q}_p(1)$, as explained in the Appendix A.1. But we want to point out that his conjecture is an equation in $K_0(\Lambda, \overline{\mathbb{Q}_p})$ which in some sense is stronger.

6. Tamely ramified extension

Let L/K be a tamely ramified extension, then $\mathcal{O}_L$ is a finitely generated projective Λ-module (see [17, Cor. 1]). Let $\chi^{ur}_{\mathbb{Q}_p}$ be a continuous unramified character of $G_{\mathbb{Q}_p}$ such that χ^{ur} is its restriction to G_K. We fix an element $t^{ur} \in (\widehat{\mathbb{Z}_p^{ur}}^{\times}(\chi^{ur}_{\mathbb{Q}_p}))^{G_{\mathbb{Q}_p}}$. Note that two such elements differ by an element of $\mathbb{Z}_p^{\times}$. Then $(\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}$ and $\mathcal{O}_L$ are isomorphic as Λ-modules by

$$l \in \mathcal{O}_L \longmapsto t^{ur} \cdot l \in (\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}.$$

Thus every element $\tilde{l} \in (\mathcal{O}_{\widehat{L^{ur}}}(\chi^{ur}))^{G(L^{ur}/L)}$ can be written as

$$\tilde{l} = t^{ur} \cdot l = t^{ur} \cdot \sum_{g \in G} a_g g(b), \quad a_g \in \mathcal{O}_K$$

for a normal integral basis b of $\mathcal{O}_L$ over $\mathcal{O}_K$ [17, Cor. 1]. Obviously, the same is true for L and $(\widehat{L^{ur}}(\chi^{ur}))^{G(L^{ur}/L)}$.

Denote by v a basis of $\mathbb{Z}_p(\chi^{ur}_{\mathbb{Q}_p})$, then the element $v \otimes \xi$ constitutes a basis of T. Moreover, $\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V) \cong \mathrm{D}^L_{dR}(V)$ (resp. $\mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)) \cong \mathrm{D}^L_{dR}(V(-1))$ is a one-dimensional L-vector space with the basis $e_{\chi^{ur}_{\mathbb{Q}_p},1} := t^{ur} \cdot t^{-1} \otimes (v \otimes \xi)$ (resp. $e_{\chi^{ur}_{\mathbb{Q}_p},0} := t^{ur} \otimes v$). In particular, they are isomorphic as Ω-modules and we have a commutative diagram of $B_{dR}[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$\begin{tikzcd} B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V) \arrow[r, "comp_V", "\cong"'] \arrow[d] & B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V \arrow[d, "t \cdot \otimes f"] \\ B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V(-1)) \arrow[r, "comp_{V(-1)}", "\cong"'] & B_{dR} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \end{tikzcd} \tag{6.1}$$

where the map $t\cdot$ is the multiplication with t and $f(v \otimes \xi) = v$.

Warning: the left vertical arrow in the above diagram is an isomorphism of Ω-modules induced by $e_{\chi^{ur}_{\mathbb{Q}_p},1} \mapsto e_{\chi^{ur}_{\mathbb{Q}_p},0}$, whereas the right vertical arrow is an isomorphism of $B_{dR}[G]$-modules with an action of $G_{\mathbb{Q}_p}$ and is responsible for the later normalization on K_1-groups.

Set $K^\bullet := R\Gamma(L, T) \oplus \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}[0]$, a perfect complex of Λ-modules with

$$\mathrm{H}^0(K^\bullet) \cong \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, \quad \mathrm{H}^1(K^\bullet) \cong \mathrm{H}^1(L, T), \quad \mathrm{H}^2(K^\bullet) \cong \mathrm{H}^2(L, T)$$

and

$$\mathrm{H}^i(K^\bullet) = 0 \text{ for } i \neq 0, 1, 2.$$

The composition rule for the refined Euler characteristic gives the equality

$$C_{L/K} = \chi(K^\bullet, exp^{-1}) + [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] \tag{6.2}$$

in $K_0(\tilde{\Lambda}, B_{dR})$ (or even in $K_0(\Lambda, B_{dR})$).

Recall that $\mathcal{F}(\mathfrak{p}_L)$ is a cohomologically trivial Λ-module (see [12, Proposition 3.9] or apply Proposition 2.5). Moreover, by [11, Lemma 1.1] $\mathcal{F}(\mathfrak{p}_L)[-1]$ is a perfect complex of Λ-modules. We set

$$E_{L/K}(\mathcal{F}(\mathfrak{p}_L)) := \mathcal{F}(\mathfrak{p}_L)[-1] \oplus \mathcal{O}_L[0],$$

a perfect complex of Λ-modules with

$$\mathrm{H}^0(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) \cong \mathcal{O}_L, \quad \mathrm{H}^1(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) \cong \mathcal{F}(\mathfrak{p}_L)$$

and $\mathrm{H}^i(E_{L/K}(\mathcal{F}(\mathfrak{p}_L))) = 0$ for $i \neq 0, 1$. Using the identification

$$Le_{\chi^{ur}_{\mathbb{Q}_p},1} = \mathrm{D}^L_{dR}(V) = t_V(L) \cong \hat{\mathbb{G}}_a(L) = L,$$

the Λ-module isomorphism $\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1} \cong \mathcal{O}_L$ and the diagram (3.4) we get the equality

$$\begin{aligned}(6.3)\quad \chi((K^\bullet), exp^{-1}) &= \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) \\ &\quad + [\mathbb{Z}_p[G/I], 1 - \chi^{ur}(Fr_K)^{-1} Fr_K, \mathbb{Z}_p[G/I]]\end{aligned}$$

in $K_0(\Lambda, \mathbb{Q}_p)$, the maps exp^{-1} and $log_{\mathcal{F}}$ being Ω-module-isomorphisms, by the additivity [7, prop. 1.2.2]. The last term above represents $\mathrm{H}^2(L, T)$ by Proposition 2.5 (4).

Now let $n_0 \in \mathbb{N}$ be big enough such that

$$log_{\mathcal{F}} : \mathcal{F}(\mathfrak{p}_L^{n_0}) \xrightarrow{\cong} \hat{\mathbb{G}}_a(\mathfrak{p}_L^{n_0}) = \mathfrak{p}_L^{n_0}.$$

Then $\mathcal{F}(\mathfrak{p}_L^{n_0})$ is a projective Λ-submodule of finite index in $\mathcal{F}(\mathfrak{p}_L)$ ($\mathfrak{p}_L^{n_0}$ being a projective Λ-module; indeed, since L/K is tame, all $\mathfrak{p}_L^n$ are Λ-projective), hence we can define $E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0}))$ analogously to the previous consideration. But

$$[\mathcal{F}(\mathfrak{p}_L^{n_0}), log_{\mathcal{F}}, \mathfrak{p}_L^{n_0}] = 0 \text{ in } K_0(\Lambda, \mathbb{Q}_p),$$

so that

$$\begin{aligned}(6.4)\quad \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})), log_{\mathcal{F}}) &= [\mathcal{F}(\mathfrak{p}_L^{n_0}), log_{\mathcal{F}}, \mathfrak{p}_L^{n_0}] + [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] \\ &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L].\end{aligned}$$

The exact sequence

$$0 \longrightarrow \mathcal{F}(\mathfrak{p}_L^{n_0}) \xrightarrow{s} \mathcal{F}(\mathfrak{p}_L) \longrightarrow \mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0}) \longrightarrow 0$$

gives rise to a distinguished triangle of perfect complexes of Λ-modules

$$E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})) \xrightarrow{j} E_{L/K}(\mathcal{F}(\mathfrak{p}_L)) \longrightarrow \mathrm{cone}(j),$$

where $j = s_* \oplus id_{\mathcal{O}_L}$, such that

$$\mathrm{H}^0(\mathrm{cone}(j)) = 0, \quad \mathrm{H}^1(\mathrm{cone}(j)) \cong \mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})$$

and $\mathrm{H}^i(\mathrm{cone}(j)) = 0$ for $i \geq 2$. This triangle together with (6.4) leads to the equalities

$$\begin{aligned}(6.5)\quad \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) &= \chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L^{n_0})), log_{\mathcal{F}}) + \chi(\mathrm{cone}(j), 0) \\ &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] + \chi(\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})[-1], 0).\end{aligned}$$

The quotients $\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})$ and $\mathfrak{p}_L/\mathfrak{p}_L^{n_0}$ are filtered by the images of $\mathcal{F}(\mathfrak{p}_L^i)$ and $\mathfrak{p}_L^i$, respectively, for $i \geq 1$. The associated graded objects considered as complexes are canonically isomorphic perfect complexes of Λ-modules, thus by [2, Prop. 2.1(iii)] we have the equality

$$\begin{aligned}\chi(\mathcal{F}(\mathfrak{p}_L)/\mathcal{F}(\mathfrak{p}_L^{n_0})[-1],0) &= \chi(\mathfrak{p}_L/\mathfrak{p}_L^{n_0}[-1],0)\\ &= [\mathfrak{p}_L, id, \mathfrak{p}_L^{n_0}]\\ &= [\mathcal{O}_L, id, \mathfrak{p}_L^{n_0}] - [\mathcal{O}_L, id, \mathfrak{p}_L].\end{aligned} \tag{6.6}$$

Let $q_K = p^f := [\mathcal{O}_K : \mathfrak{p}_K]$ and $e_I := \frac{1}{|I|}\sum_{i\in I} i$ be the idempotent of Ω associated to the inertia subgroup I of G. Let ${}^\sharp x \in K_1(\Omega) \subset K_1(\tilde{\Omega})$ be defined for every element $x \in Cent(\Omega)$ as follows. If $Cent(\Omega) = \prod F_i$ is the Wedderburn decomposition of $Cent(\Omega)$ into a product of fields and $x = (x_i)$ under this decomposition, then ${}^\sharp x = ({}^\sharp x_i)$ with ${}^\sharp x_i = x_i$ if $x_i \neq 0$ and ${}^\sharp x_i = 1$ if $x_i = 0$.

The normal basis theorem for $\mathcal{O}_L/\mathfrak{p}_L$ over $\mathbb{Z}_p/p\mathbb{Z}_p$ implies that there exists a short exact sequence of G/I-modules

$$0 \longrightarrow p\cdot\mathbb{Z}_p[G/I]^f \longrightarrow \mathbb{Z}_p[G/I]^f \longrightarrow \mathcal{O}_L/\mathfrak{p}_L \longrightarrow 0.$$

Using this sequence we compute that

$$[\mathcal{O}_L, id, \mathfrak{p}_L] = -\partial({}^\sharp(q_K e_I)). \tag{6.7}$$

Observing (6.6) and (6.7) the equality (6.5) becomes

$$\begin{aligned}\chi(E_{L/K}(\mathcal{F}(\mathfrak{p}_L)), log_{\mathcal{F}}) &= [\mathfrak{p}_L^{n_0}, id, \mathcal{O}_L] + [\mathcal{O}_L, id, \mathfrak{p}_L^{n_0}] + \partial({}^\sharp(q_K e_I))\\ &= \partial({}^\sharp(q_K e_I)).\end{aligned} \tag{6.8}$$

Write $\Sigma(L)$ for the set of all embeddings $L \to \overline{\mathbb{Q}_p}$ fixing $\mathbb{Q}_p$. For each $\sigma \in \Sigma(K)$ we fix $\hat{\sigma} \in \Sigma(L)$ such that $\hat{\sigma}|_K = \sigma$. Let $b \in \mathcal{O}_L$ be a $K[G]$-basis of L and let χ be an irreducible $\overline{\mathbb{Q}_p}$-valued character of G. The norm resolvent is defined by

$$\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi) := \prod_{\sigma\in\Sigma(K)} \mathrm{Det}_\chi(\sum_{g\in G} \hat{\sigma}(g(b))g^{-1}) \in \overline{\mathbb{Q}_p}^\times,$$

where Det_χ is the homomorphism $\overline{\mathbb{Q}_p}[G]^\times \to \overline{\mathbb{Q}_p}^\times$ given by

$$\mathrm{Det}_\chi(\sum_{g\in G} a_g g) := \det(\sum_{g\in G} a_g \rho_\chi(g))$$

and $\rho_\chi : G \to GL_{\chi(1)}(\overline{\mathbb{Q}_p})$ is a matrix representation with character χ. Note that the definition of $\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)$ depends on the choice of the $\hat{\sigma}$. We also let $\{a_\sigma : \sigma \in \Sigma(K)\}$ be a fixed $\mathbb{Z}_p$-basis of $\mathcal{O}_K$ and define

$$\delta_K := det((\eta(a_\sigma))_{\eta,\sigma\in\Sigma(K)}) \in \overline{\mathbb{Q}_p}^\times.$$

This is a square root of the discriminant of K and depends on the choice of the a_σ.

Lemma 6.1. *There is an equality*

$$[\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] + \partial(t) = \partial(\theta) \text{ in } K_0(\tilde{\Lambda}, B_{dR}),$$

where $\theta = (\theta_\chi)_{\chi \in Irr(G)} \in K_1(\overline{\mathbb{Q}_p}[G])$ *with* $\theta_\chi = \delta_K^{\chi(1)} \mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)$.

Proof. The unramified representation $V(-1)$ is $\mathbb{C}_p$-admissible (see [15, Prop. 3.56]), thus we may replace the ring B_{dR} by $\mathbb{C}_p$ in the definition of the comparison isomorphism getting

$$\begin{aligned} comp_{V(-1),L/\mathbb{Q}_p} : \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}_{dR}(\mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)) &\overset{\cong}{\to} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \\ c \otimes x &\mapsto cx \end{aligned}$$

a $\mathbb{C}_p$-linear map, which commutes with the action of $G_{\mathbb{Q}_p}$. Taking invariants under $G(\overline{\mathbb{Q}_p}/L^{ur})$ on both sides and using the theorem of Ax-Sen-Tate the isomorphism above becomes

$$comp_{V(-1),L/\mathbb{Q}_p} : \widehat{L^{ur}} \otimes_{\mathbb{Q}_p} L e_{\chi^{ur}_{\mathbb{Q}_p},0} \overset{\cong}{\to} \widehat{L^{ur}} \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1)$$

and is induced (via tensor product $\mathbb{Q}_p \otimes_{\mathbb{Z}_p}$) by

$$\mathcal{O}_{\widehat{L^{ur}}} \otimes_{\mathbb{Z}_p} \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0} \overset{\cong}{\to} \mathcal{O}_{\widehat{L^{ur}}} \otimes_{\mathbb{Z}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1).$$

From diagram (6.1) we deduce that

$$\begin{aligned} (6.9) \quad [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},1}, comp_V, \mathrm{Ind}_{L/\mathbb{Q}_p} T] + \partial(t) \\ = [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}, comp_{V(-1)}, \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1)] \end{aligned}$$

in $K_0(\tilde{\Lambda}, \widehat{L^{ur}}) \subseteq K_0(\tilde{\Lambda}, B_{dR})$, whence to prove the lemma we have to compute the last class.

Let $V_{triv} \cong \mathbb{Q}_p$ denote the trivial representation of G_K and let $V(-1) = \mathbb{Q}_p v$. Fix a set R of representatives of $G_{\mathbb{Q}_p}|G_K$ (inducing the set of $\hat{\sigma}$ as chosen above). There is a commutative diagram of $\mathbb{C}_p[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$\begin{tikzcd} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V_{triv}) \arrow[r, "comp_{V_{triv}}", "\cong"'] \arrow[d, "f_1"] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} \arrow[d, "f_2"] \\ \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V(-1)) \arrow[r, "comp_{V(-1)}", "\cong"'] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1), \end{tikzcd}$$

where we use the following identifications

$$\begin{array}{lcl} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} & \cong & \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_K]} \mathbb{C}_p[G]^\sharp, \\ \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V(-1) & \cong & \mathbb{Z}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}_p[G_K]} (\mathbb{C}_p[G]^\sharp \otimes \mathbb{Q}_p v), \\ \mathrm{D}^L_{dR}(V_{triv}) \cong L, & & \mathrm{D}^L_{dR}(V(-1)) \cong L e_{\chi^{ur}_{\mathbb{Q}_p},0}, \end{array}$$

for which the maps are given by the formulas

$$\begin{aligned} &comp_{V_{triv}}(q \otimes l) = \textstyle\sum_{\tau \in R} \left(\tau \otimes \sum_{g \in G} q\tau g(l) g^{-1}\right), \\ &comp_{V(-1)}(q \otimes l e_{\chi^{ur}_{\mathbb{Q}_p},0}) = \textstyle\sum_{\tau \in R} \left(\tau \otimes \sum_{g \in G} q t^{ur} \tau g(l) g^{-1}\right), \\ &f_1(q \otimes l) = q \otimes l e_{\chi^{ur}_{\mathbb{Q}_p},0}, \\ &f_2(\tau \otimes w) = \tau \otimes (t^{ur} w \otimes v), \quad \tau \in R,\ w \in \mathbb{C}_p[G]^\sharp. \end{aligned}$$

Indeed, the two first formulae arise by combining the comparison isomorphism (4.1) with the general formula $\mathrm{Ind}^H_G(B \otimes V) \cong B \otimes \mathrm{Ind}^H_G V$ from representation theory in the same way as in the proof of [27, lem. A.5]. It follows, that

$$\begin{aligned} (6.10)\quad & [\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] \\ &= [\mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}, comp_{V(-1)}, \mathrm{Ind}_{L/\mathbb{Q}_p} T(-1)] + [\mathcal{O}_L, f_1, \mathcal{O}_L e_{\chi^{ur}_{\mathbb{Q}_p},0}] \\ &\qquad + [\mathrm{Ind}_{L/\mathbb{Q}_p} T(-1), f_2^{-1}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}] \end{aligned}$$

in $K_0(\Lambda, \mathbb{C}_p)$. But the images of the last two classes in $K_0(\tilde{\Lambda}, \mathbb{C}_p)$ are trivial, as f_1 and f_2 are $\tilde{\Lambda}$-module-isomorphisms. Now we are reduced to computing $[\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p} T_{triv}]$. For this we set

$$H_L := \bigoplus_{\eta \in \Sigma(L)} \mathbb{Z}_p,$$

which becomes a free Λ-module under the (left) G-action

$$g((a_\eta)_\eta) = (a_{\eta g})_\eta.$$

We consider the following commutative diagram of $\mathbb{C}_p[G]$-modules (with an action of $G_{\mathbb{Q}_p}$)

$$(6.11)\qquad \begin{tikzcd} \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{D}^L_{dR}(V_{triv}) \arrow[r, "comp_{V_{triv}}", "\cong"'] \arrow[dr, "\rho_L", "\cong"'] & \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{Ind}_{L/\mathbb{Q}_p} V_{triv} \arrow[d, "\varphi_1"] \\ & \mathbb{C}_p \otimes_{\mathbb{Z}_p} H_L, \end{tikzcd}$$

where the maps φ_1 and ρ_L are given by the formulas

$$\rho_L(q \otimes l) = (q\eta(l) \otimes 1)_{\eta \in \Sigma(L)}, \quad q \in \mathbb{C}_p,\ l \in L;$$

$$\varphi_1\left(\textstyle\sum_{\tau \in R} (\tau \otimes \sum\limits_{g \in G} a^\tau_g g)\right) = \left(a^\tau_{g^{-1}} \otimes 1\right)_{\tau g = \eta \in \Sigma(L)}, \quad \forall\ a_g \in \mathbb{C}_p, \tau \in R.$$

From the diagram (6.11) we deduce the equality

$$(6.12)\quad [\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p}T_{triv}] + [\mathrm{Ind}_{L/\mathbb{Q}_p}T_{triv}, \varphi_1, H_L] = [\mathcal{O}_L, \rho_L, H_L]$$

in $K_0(\Lambda, \mathbb{C}_p)$. Further, [4, Lem. 4.16] says that the last class is equal to $\partial(\theta)$, so that we have

$$(6.13)\qquad [\mathcal{O}_L, comp_{V_{triv}}, \mathrm{Ind}_{L/\mathbb{Q}_p}T_{triv}] = \partial(\theta).$$

because the class $[\mathrm{Ind}_{L/\mathbb{Q}_p}T_{triv}, \varphi_1, H_L]$ is obviously zero. □

Lemma 6.2. *Let L/K be (at most) tamely ramified. Then there exists $v' \in \Lambda^\times$, such that $\mathrm{Det}_\chi(v') = \chi^{ur}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\mathfrak{f}(\chi)) \cdot d^{\chi(1)}_{K/\mathbb{Q}_p})$ for all $\chi \in Irr(G)$, whence*

$$\partial(\epsilon_D(L/K, V)) = \partial(\Big(\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi), -\psi_\xi)^{-1}\Big)_{\chi\in Irr(G)}).$$

in $K_0(\Lambda, B_{dR})$ by Lemma 4.1 and (4.2).

Proof. The character $\chi^{ur}_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$ being a homomorphism we have

$$\chi^{ur}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\mathfrak{f}(\chi)) \cdot d^{\chi(1)}_{K/\mathbb{Q}_p}) = \chi^{ur}_{\mathbb{Q}_p}(N_{K/\mathbb{Q}_p}(\mathfrak{f}(\chi))) \cdot \chi^{ur}_{\mathbb{Q}_p}(d^{\chi(1)}_{K/\mathbb{Q}_p}).$$

Let $\chi^{ur}_{\mathbb{Q}_p}(p) =: u' \in \mathbb{Z}_p^\times$ and let $d_{K/\mathbb{Q}_p} = p^m$. Then for $u'^m \in \mathbb{Z}_p^\times \subset \Lambda^\times$, $\chi \in Irr(G)$

$$\chi^{ur}_{\mathbb{Q}_p}(d^{\chi(1)}_{K/\mathbb{Q}_p}) = \chi^{ur}_{\mathbb{Q}_p}(p)^{m\cdot\chi(1)} = u'^{m\cdot\chi(1)} = \det(\rho_\chi(u'^m 1_G)) = \mathrm{Det}_\chi(u'^m).$$

Recall $q_K = p^f$ and $e_I = \frac{1}{|I|}\sum_{i\in I} i \in \Lambda$, as $(|I|, p) = 1$. Let $v'' := \left(\frac{u'}{u'\cdot e_I + (1_G - e_I)}\right)^f$. Then for $\chi \in Irr(G)$

$$\begin{aligned}\mathrm{Det}_\chi(v'') &= \mathrm{Det}_\chi(u')^f \cdot \mathrm{Det}_\chi(u' \cdot e_I + (1_G - e_I))^{-f}\\ &= u'^{f\cdot\chi(1)} \cdot \det(u' \cdot \rho_\chi(e_I) + \rho_\chi(1_G) - \rho_\chi(e_I))^{-f}.\end{aligned}$$

There exists a basis of $V_\rho = V_{\rho_\chi}$, such that

$$\rho_\chi(e_I) \;=\; \begin{pmatrix} 1 & * & \dots & * & & & \\ 0 & 1 & * & \dots & * & & \\ & & \ddots & & & & \\ 0 & \dots & 0 & 1 & * & \dots & * \\ 0 & 0 & 0 & \dots & 0 & * & \\ & & & & & \ddots & \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$\rho_\chi(1_G)-\rho_\chi(e_I) = \begin{pmatrix} 0 & * & \dots & * & & & \\ 0 & 0 & * & \dots & * & & \\ & & \ddots & & & & \\ 0 & \dots & 0 & 0 & * & \dots & * \\ 0 & 0 & \dots & 0 & 1 & * & \\ & & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

as $e_I + 1_G - e_I = 1_G$. It follows, that $\det(u'\cdot\rho_\chi(e_I)+\rho_\chi(1_G)-\rho_\chi(e_I)) = u'^{\operatorname{rank}(\rho_\chi(e_I))}$, hence

$$\mathrm{Det}_\chi(v'') = u'^{f\cdot(\chi(1)-\operatorname{rank}(\rho_\chi(e_I)))}.$$

But

$$\chi(1)=\dim V_\rho,\quad \operatorname{rank}(\rho_\chi(e_I)) = \dim \operatorname{Im}(\rho_\chi(e_I)) = \dim V_\rho^I,$$

so that $\chi(1)-\operatorname{rank}(\rho_\chi(e_I)) = \operatorname{codim}\, V_\rho^I$. Further, since χ is a tamely ramified character, the Artin conductor $\mathfrak{f}(\chi)$ is equal to $\mathfrak{p}_K^{\operatorname{codim}\, V_\rho^I}$ (see [22, p. 22]), whence

$$\mathrm{Det}_\chi(v'') = \chi_{\mathbb{Q}_p}^{ur}(p^{f\cdot\operatorname{codim}\, V_\rho^I}) = \chi_{\mathbb{Q}_p}^{ur}(N_{K/\mathbb{Q}_p}(\mathfrak{p}_K^{\operatorname{codim}\, V_\rho^I})) = \chi_{\mathbb{Q}_p}^{ur}(N_{K/\mathbb{Q}_p}(\mathfrak{f}(\chi))).$$

Now we set $v' := v''\cdot(u')^m$. It remains to prove $v'\in\Lambda^\times$ and for this it is enough to show that $u'\cdot e_I + (1-e_I)\in\Lambda^\times$. But

$$(u'\cdot e_I + (1_G-e_I))\cdot(e_I + u'\cdot(1_G-e_I)) = u'1_G\in\Lambda^\times. \qquad \square$$

Lemma 6.3.

(i) $\partial([\mathrm{D}_{cris}^L(V), 1-\phi]) = -\partial(^\sharp(q_K e_I))$

(ii) $\partial([\mathrm{D}_{cris}^L(V^*(1)), 1-\phi]) = [\mathbb{Z}_p[G/I], 1-\chi^{ur}(Fr_K)Fr_K, \mathbb{Z}_p[G/I]]$

Proof.

(i) Let L_0 and K_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in L and K respectively, $[K_0:\mathbb{Q}_p]=f_K$, $[L_0:K_0]=f_{L/K}$. Denote by Fr_L, Fr_K and $\tau = Fr_{\mathbb{Q}_p}$ the arithmetic Frobenius of L, K and $\mathbb{Q}_p$, respectively. Then $Fr_K = Fr_{\mathbb{Q}_p}^{f_K}$. After choosing a normal basis of $\mathcal{O}_{L_0}$ over $\mathbb{Z}_p$ we have

$$\mathrm{D}_{cris}^L(V) = L_0 e_{\chi_{\mathbb{Q}_p}^{ur},1} \cong \bigoplus_{i=0}^{f_K-1}\mathbb{Q}_p[G/I]\tau^i$$

with $\phi(e_{\chi^{ur}_{\mathbb{Q}_p},1}) = p^{-1}\chi^{ur}_{\mathbb{Q}_p}(Fr^{-1}_{\mathbb{Q}_p})e_{\chi^{ur}_{\mathbb{Q}_p},1}$, i.e., by the semi-linearity of ϕ on the last $\mathbb{Q}_p[G/I]$-module the operator $1-\phi$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -AFr_K \\ -A & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -A & 1 & 0 \\ 0 & 0 & 0 & -A & 1 \end{pmatrix}$$

with respect to the basis $1, \tau, \dots, \tau^{f_K-1}$, where $A = \frac{\chi^{ur}(Fr^{-1}_{\mathbb{Q}_p})}{p}$. Thus

$$\begin{aligned} \partial[\mathrm{D}^L_{cris}(V), 1-\phi]) &= \partial\left(^\sharp((1-p^{-f_K}\chi^{ur}(Fr_K^{-1})Fr_K)e_I)\right) \\ &= \partial\Big(^\sharp\left(\frac{1}{p^{f_K}}(p^{f_K}-\chi^{ur}(Fr_K^{-1})Fr_K)e_I\right)\Big) \\ &= -\partial(^\sharp(q_K e_I)), \end{aligned}$$

since $p^{f_K} - \chi^{ur}(Fr_K^{-1})Fr_K \in \mathbb{Z}_p[G/I]^\times$ as can be seen by a standard argument using the geometric series.

(ii) Analogously,

$$\begin{aligned} \partial([\mathrm{D}^L_{cris}(V^*(1)), 1-\phi]) &= \partial\left(^\sharp((1-\chi^{ur}(Fr_K)Fr_K)e_I)\right) \\ &= [\mathbb{Z}_p[G/I], 1-\chi^{ur}(Fr_K)Fr_K, \mathbb{Z}_p[G/I]]. \qquad \square \end{aligned}$$

Theorem 6.4. *Let L/K be a Galois extension of p-adic fields which is (at most) tamely ramified and let $V = \mathbb{Q}_p(\chi^{ur})(1)$. Then $C^{na}_{ep}(L/K, V)$ is equivalent to the vanishing of*

$$\partial(\theta) + \partial((\epsilon_D(L/K, V))) \tag{6.14}$$

in $K_0(\tilde{\Lambda}, \widehat{\mathbb{Q}_p^{ur}}) \subseteq K_0(\tilde{\Lambda}, B_{dR})$.

Proof. The proof is given by (5.2), (4.2), (6.2), (6.3), (6.8), Lemmata 6.1 and 6.3. $\square$

Now by Lemmata 6.2 and 6.1 the equation (6.14) means that there exists $w \in K_1(\tilde{\Lambda})$ such that

$$\mathrm{Det}_\chi(w) = \frac{\delta_K^{\chi(1)}\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)}{\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi), -\psi_\xi)} \tag{6.15}$$

for all $\chi \in \mathrm{Irr}_{\bar{\mathbb{Q}}_p}(G)$.

Let $\tau' := \tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p} 1_K)$, where 1_K is the trivial character of G.

Lemma 6.5. *There exists a unit $u' \in \mathbb{Z}_p^\times$ and $\sigma' \in G$ such that*

$$\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi), -\psi_\xi, dx) = (u'\tau' d_{K/\mathbb{Q}_p}^{-1})^{\chi(1)}\mathrm{Det}_\chi(\sigma')\epsilon(\rho_\chi, \psi_K, dx)$$

for all χ.

Proof. As Gauss sums are additive and behave inductive on degree 0 characters we have

$$\frac{\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(\rho_\chi), -\psi_\xi, dx)}{\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K), -\psi_\xi, dx)^{\chi(1)}} = \frac{\epsilon(\rho_\chi, -\psi_\xi \circ Tr_{K/\mathbb{Q}_p}, dx)}{\epsilon(1_K, -\psi_\xi \circ Tr_{K/\mathbb{Q}_p}, dx)^{\chi(1)}}.$$

But by [25, (3.2.6.1)] $\epsilon(1_K, -\psi_\xi \circ Tr_{K/\mathbb{Q}_p}, dx) = d_{K/\mathbb{Q}_p}$ as the measure of $\mathcal{O}_K$ is normalized to be 1 and the conductor $n(-\psi_\xi \circ Tr_{K/\mathbb{Q}_p})$ equals the exponent of the different of $K/\mathbb{Q}_p$. Note that there exists $\sigma \in G(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{ur})$ such that $\kappa(\sigma)\psi_{\mathbb{Q}_p} = -\psi_\xi$, where κ denotes the cyclotomic character of $G_{\mathbb{Q}_p}$. Hence, using (2) in [18, §3.2.2] we obtain

$$\begin{aligned}
\epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K), -\psi_\xi, dx) &= \epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K)|\cdot|_p^{\frac{1}{2}}, -\psi_\xi, dx)\sqrt{f(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K))}\\
&= \epsilon(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K)|\cdot|_p^{\frac{1}{2}}, \kappa(\sigma)\psi_\xi, dx)\sqrt{f(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K))}\\
&= \tau_{\mathbb{Q}_p}(\mathrm{Ind}_{K/\mathbb{Q}_p}(1_K))\det(\sigma; \mathrm{Ind}_{K/\mathbb{Q}_p}(1_K))\\
&= \tau'\det(\bar{\sigma}; \mathrm{Ind}_{K/\mathbb{Q}_p}(1_K))
\end{aligned}$$

by [25, (3.4.5)] . Using (3.4.4) in [25] we also have

$$\begin{aligned}
\epsilon(\rho_\chi, -\psi_\xi \circ Tr_{K/\mathbb{Q}_p}, dx) &= \epsilon(\rho_\chi, \kappa(\sigma)\psi_{\mathbb{Q}_p} \circ Tr_{K/\mathbb{Q}_p}, dx)\\
&= \mathrm{Det}_\chi(\sigma')\epsilon(\rho_\chi, \psi_{\mathbb{Q}_p} \circ Tr_{K/\mathbb{Q}_p}, dx)
\end{aligned}$$

for some $\sigma' \in G$. $\square$

Moreover, by [4, Lem. 4.29] or [5, Lem. 3.7] $\delta_K/\iota(\tau') \in (\mathbb{Z}_p^{ur})^\times$, hence (6.15) is equivalent to the existence of $w' \in K_1(\tilde{\Lambda})$ such that

$$\mathrm{Det}_\chi(w') = \frac{d_{K/\mathbb{Q}_p}^{\chi(1)}\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)}{\epsilon(\rho_\chi, \psi_K, dx)} = \frac{\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)}{\tau_K(\rho_\chi)} \tag{6.16}$$

for all $\chi \in \mathrm{Irr}_{\bar{\mathbb{Q}}_p}(G)$ where the last equality holds by the following calculation

$$\begin{aligned}
\tau_K(\rho_\chi) &= \epsilon(\rho_\chi|\ |_p^{\frac{1}{2}}, \psi_K, dx_{\psi_K})f(\chi)^{\frac{1}{2}}\\
&= \epsilon(\rho_\chi, \psi_K, dx_{\psi_K})f(\chi)^{-\frac{1}{2}}q_K^{-\chi(1)\frac{n(\psi_K)}{2}}f(\chi)^{\frac{1}{2}}\\
&= \epsilon(\rho_\chi, \psi_K, dx)q_K^{-\chi(1)n(\psi_K)}\\
&= \epsilon(\rho_\chi, \psi_K, dx)d_{K/\mathbb{Q}_p}^{-\chi(1)}
\end{aligned}$$

using (3.4.5) in [25] for the second equality and the normalization factor $\frac{dx_{\psi_K}}{dx} = q_K^{-\chi(1)\frac{n(\psi_K)}{2}}$ in the third one.

By an old result of M. Taylor in Galois module theory (see [4, (32)], [5, (3.4)] or [17, Thm 31]) the term $\frac{\mathcal{N}_{K/\mathbb{Q}_p}(b|\chi)}{\tau_K(\rho_\chi)}$ stems from an integral unit, because the non-ramified characteristic $\sharp(-Fr_K e_I)$ in [17] is an integral unit itself. More precisely, there exists a finite at most tamely ramified extension F of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and ω'' in $\mathcal{O}_F[G]^\times$ satisfying (6.16). Combining lem. 2.4, (the proof of) lem. 2.8 of [5] as well as [4, (31)] one immediately verifies that each σ in the inertia subgroup $I(F/\mathbb{Q}_p)$ of $G(F/\mathbb{Q}_p)$ acts trivially on $\mathrm{Det}(w'')$. By [20, thm. 2.21] we know that

$$\mathrm{Det}(\mathcal{O}_F[G]^\times)^{I(F/\mathbb{Q}_p)} = \mathrm{Det}((\mathcal{O}_F)^{I(F/\mathbb{Q}_p)}[G]^\times)$$

whence we also find $\omega' \in K_1(\tilde{\Lambda})$ satisfying (6.16). Altogether we now have proven the following theorem:

Theorem 6.6 (Main Theorem). *Let K be a finite extension of $\mathbb{Q}_p$ and L/K be a (at most) tamely ramified Galois extension with $G = G(L/K)$. Let $\chi^{ur} : G_K \to \mathbb{Z}_p^\times$ be a continuous unramified character with $\chi^{ur}(G_L) \neq 1$. Let V be either $\mathbb{Q}_p(\chi^{ur})$ or $\mathbb{Q}_p(\chi^{ur})(1)$ – a p-adic representation of G_K. Then the conjecture $C^{na}_{ep}(L/K, V)$ holds.*

Proof. Only the case $V = \mathbb{Q}_p(\chi^{ur})$ is missing, which follows immediately from the functorial behavior under taking Kummer dual (see [19, Prop. 3.14(1)]). ☐

Appendix A. Compatibilities

A.1. Breunings result for $V = \mathbb{Q}_p(1)$ revisited. We just sketch how our approach has to be modified for $V = \mathbb{Q}_p(1)$ and how it then compares to Breunings setting.

As mentioned earlier our formulation of Conjecture 5.1 also extends to the case $V = \mathbb{Q}_p(1)$. If we want to reformulate it in the style of (5.2) we have to observe that now $\mathrm{H}^1(M^\bullet) \cong \mathrm{H}^1(L, T) \cong \widehat{L^\times}$ and $\mathrm{H}^2(M^\bullet) \cong \mathrm{H}^2(L, T) \cong \mathbb{Z}_p$, thus the trivialisation needs the additional summand $\widehat{v_L}$ induced by the valuation map $v_L : L^\times \to \mathbb{Z}$. Then the corresponding class is $C_{L/K} := \chi(M^\bullet, (comp_V \circ exp^{-1}) \oplus \widehat{v_L}) \in K_0(\Lambda, B_{dR})$ and the class $C_{L/K} + \partial(t)$ corresponds to $C^{Breu}_{L/K}$ in [5] as can be easily checked using (6.9) and (6.12). Since $1 - \phi$ is not longer a bijection on $D^L_{cris}(V^*(1)) = D^L_{cris}(\mathbb{Q}_p)$ also the class U_{cris} has to be adjusted and then corresponds to the 'correction term' $M_{L/K}$, the origin of which seems completely unclear in [5] (as there is no reference to p-adic Hodge theory). Finally, our factor $\partial(\epsilon_D(L/K, \mathbb{Q}_p(1)))$ corresponds to the class $T_{L/K}$ in [5] if one uses Lemma 6.2 and the comment on different conventions before Remark 4.2 - apparently Breuning never discusses the relation of his Gauss sums to the involved representation $\mathbb{Q}_p(1)$ at all. Finally, the term $U_{L/K}$ does not show up in our approach as we consider the classes in $K_0(\tilde{\Lambda}, B_{dR})$ in (5.2), which is in accordance with the vanishing (see [5, prop. 2.12]; although Breuning only states that his element $U_{L/K}$ vanishes in $K_0(\mathcal{O}_p^t[G], \overline{\mathbb{Q}_p})$, where $\mathcal{O}_p^t$ denotes the ring of integers in the maximal tamely ramified extension of $\mathbb{Q}_p$, his proof actually shows that it already vanishes in $K_0(\tilde{\Lambda}, \overline{\mathbb{Q}_p}))$ of that class in the latter K-group, whence the statement (5.2) indeed corresponds to the vanishing $R_{L/K} := T_{L/K} + C_{L/K} + U_{L/K} - M_{L/K}$ in that group according to [5, cor. 3.5].

If now $V = \mathbb{Q}_p(1)$ is twisted by an unramified character $\chi^{ur} : G_K \to G(L/K) \to \mathbb{Z}_p^\times$, then the twisting commutes with forming Galois cohomology etc. because χ^{ur} is trivialised by G_L. Since also the ϵ-constants behave well under unramified twists, it is easy to check that Breuning's arguments above can be adjusted to prove our conjecture also in that situation. Since we do not need the result in this article, we leave this to the interested reader.

A.2. Some remarks concerning the various approaches in the literature. In [19, §3.1] one can find a rather detailed comparison with the approaches of Perrin-Riou, Benois and Berger as well as Fukaya and Kato. Here we only want to give some comments. Since Benois and Berger use Perrin-Riou's big exponential map, their setting generalises and refines that of Perrin-Riou, of course. Since Fukaya and Kato allow also non-abelian extensions their setting is more general. Restricting to abelian extensions L/K one of the main differences between the latter two consists in the fact that Fukaya and Kato use the individual specialisations to the various twists by (Artin) characters instead of working with $\tilde{\Omega} = \widehat{\mathbb{Q}_p}[G(L/K)]$ as Benois and Berger. Also Breuning puts all the characters together, i.e., works with the regular representation rather than the irreducible ones. In his language the evaluation at characters is described via the reduced norm, which we compare in the next subsection with the specialisation in [18]. Moreover, he works within relative K-groups. The translation into the language of determinants we have indicated in the main text (the discussion after Conjecture 5.1, and §2.1.3).

A.3. The reduced norm and specialisation in [18]**.** The reduced norm plays the crucial role in the work of Fröhlich, Taylor, Breuning etc. while in the description of Fukaya and Kato [18] it does not show up at all. In order to check whether both approaches coincide (in settings where both are defined and apply) one has to check certain compatibilities, which we are going to recall in this appendix.

We fix a finite group G. Let F be a finite extension of $\mathbb{Q}_p$ over which all the absolutely irreducible representations $\mathrm{Irr}(G)$ of G are defined and set $A := F[G]$. Every such representation ρ induces a F-algebra homomorphism $T_\rho : A \to \mathrm{End}_K(V_\rho)$ sending the central idempotent e_ρ to id_{V_ρ}. Consider the reduced norm isomorphism

$$nr : K_1(A) \cong \prod_{\rho \in \mathrm{Irr}(G)} F^\times = Z(A)^\times$$

sending a class $[P, a]$ consisting of a projective, finitely generated (left) A-module P together with an A-linear automorphism

$$a \longmapsto \left(\det_F \left(\mathrm{Hom}_A(V_\rho, a)\right)\right)_\rho .$$

Using the following lemma we get an alternative description of this map. Let T be finitely generated $\mathbb{Z}_p$-module with a continuous G_K-action and consider its deformation

$$\mathbb{T} := \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} T,$$

a $(\mathbb{Z}_p[G], G_K)$-bimodule, where $\mathbb{Z}_p[G]$ acts by left multiplication on the left tensor factor while the action of $\sigma \in G_K$ is induced by $\sigma(\lambda \otimes t) = \lambda \bar{\sigma}^{-1} \otimes \sigma t$, where $\bar{\sigma}$ denotes the image of σ in G.

Lemma A.1. *There are canonical isomorphisms*

$$\mathrm{Hom}_A(V_\rho, A \otimes_{\mathbb{Z}_p[G]} \mathbb{T}) \cong V_\rho^* \otimes_{\mathbb{Z}_p} T \cong V_\rho \otimes_{\mathbb{Z}_p[G]} \mathbb{T}$$

of $F[G_K]$-modules, where the G_K-action is induced by the action on $\mathbb{T}$ on the outer terms, while it is diagonally in the middle term ($V_\rho^* = V_{\rho^*}$ is considered as G_K-module via the projection onto G here).

Proof. The inverse of the first isomorphism is induced by sending $\omega \otimes t$ to the map

$$\tilde{\omega} : V_\rho \to A \otimes_{\mathbb{Z}_p[G]} \mathbb{T}, \quad v \mapsto \sum_{g \in G} \omega(g^{-1}v) g \otimes t.$$

Now the statement is easily checked. □

Hence we get immediately that

$$nr([P, a]) = \left(\det_F \left(\mathrm{Hom}_A(V_\rho, a)\right)\right)_\rho = \left(\det_F \left(V_\rho \otimes_A a\right)\right)_\rho .$$

In particular, the ρ-component can also be determined in the way of Fukaya and Kato [18]. Furthermore we have a commutative diagram

$$\begin{tikzcd} A^\times \arrow[rr, "can"] \arrow[dr, "Nr"'] & & K_1(A), \arrow[dl, "nr"] \\ & \prod_{\rho \in \mathrm{Irr}(G)} F^\times & \end{tikzcd}$$

where *can* sends $a \in A$ to $[A, a]$ considering now a as right multiplication by it while

$$Nr : A^{\times} \to \prod_{\rho \in \mathrm{Irr}(G)} F^{\times}$$

sends a to $\left(\det_F(T_\rho(a))\right)_\rho$.

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