JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 28, n° 1 (2016), p. 221-236.

<http://jtnb.cedram.org/item?id=JTNB_2016__28_1_221_0>

Journal de Théorie des Nombres
de Bordeaux **28** (2016), 221–236

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par Adam GAMZON

Résumé. Soit $\rho_{f,\lambda}$ une représentation galoisienne ℓ-adique associée à une forme de Hilbert nouvelle f, et soit $\bar{\rho}_{f,\lambda}$ sa réduction semi-simple modulo ℓ. En généralisant l'approche de Weston, nous démontrons que l'anneau de déformations universel de $\bar{\rho}_{f,\lambda}$ de déterminant donné est non obstrué pour presque tout ℓ. Nous donnons également un exemple explicite pour illustrer comment obtenir une borne inférieure sur les ℓ tels que l'anneau de déformations universel de $\bar{\rho}_{f,\lambda}$ de déterminant donné soit non obstrué pour tout λ divisant ℓ.

Abstract. Let $\rho_{f,\lambda}$ be an ℓ-adic Galois representation associated to a Hilbert newform f. Consider its semisimple mod ℓ reduction $\bar{\rho}_{f,\lambda}$. This paper discusses how, under certain conditions on f, the universal ring for deformations of $\bar{\rho}_{f,\lambda}$ with fixed determinant is unobstructed for almost all primes. We follow the approach of Weston, who carried out a similar program for classical modular forms in 2004. As such, the problem essentially comes down to verifying that various local invariants vanish at all places dividing ℓ or the level of the newform. We conclude with an explicit example illustrating how one can in principle find a lower bound on ℓ such that the universal ring for deformations of $\bar{\rho}_{f,\lambda}$ with fixed determinant is unobstructed for all λ over ℓ.

1. Introduction

Let f be a newform of level N and weight $k \geq 2$. Let K_f be the number field obtained from f by adjoining its Hecke eigenvalues to $\mathbf{Q}$. For each prime λ in K_f, Deligne constructed a semisimple mod ℓ representation $\bar{\rho}_{f,\lambda}$. In [13], Mazur conjectured that the universal deformation ring of this residual representation $\bar{\rho}_{f,\lambda}$ is unobstructed for almost all λ. Weston [16] gave a positive answer to Mazur's question in 2004 assuming that $k \geq 3$. He was also able to obtain some results for weight two modular forms, showing that Mazur's conjecture holds on a set of primes of density one. We show

Manuscrit reçu le 5 janvier 2014, révisé le 7 avril 2014, accepté le 11 avril 2014.
Mathematics Subject Classification. 11F80, 11F70, 11S25, 11R34.
Mots-clefs. Galois representations, Hilbert modular forms, deformation problems.

that Weston's methodology and results essentially carry over to the Hilbert modular form setting with a few minor adjustments.

More specifically, let F be a totally real extension of $\mathbf{Q}$ of degree $d > 1$ and let f be a Hilbert newform on F of level $\mathfrak{n} \subset \mathcal{O}_F$ and weight $k = (k_{\tau_1}, \ldots, k_{\tau_d})$. Here the τ_i denote the embeddings of F into $\mathbf{R}$. We assume that $k_{\tau_i} \geq 2$ for all i and that they satisfy the parity condition $k_{\tau_1} \equiv \cdots \equiv k_{\tau_d} \bmod 2$. As in the previous paragraph, let K_f be the number field generated over $\mathbf{Q}$ by the Hecke eigenvalues of f and let $\mathcal{O}_{K_f}$ its ring of integers. For each prime λ of K_f, let

$$\bar{\rho}_{f,\lambda} \colon G_{F,S} \to \mathrm{GL}_2(k_{f,\lambda})$$

be the semisimple mod ℓ Galois representation attached to f by Carayol and Taylor. Here $k_{f,\lambda} = \mathcal{O}_{K_f}/\lambda$ and $G_{F,S} = \mathrm{Gal}(F_S/F)$, where F_S is the maximal algebraic extension of F, unramified outside of a finite set of places $S = \{v|\mathfrak{n}\ell\} \cup \{v|\infty\}$.

Let $\mathbf{D}^{\det=\delta}_{\bar{\rho}_{f,\lambda}}$ denote the functor that associates to a coefficient ring R the set of all deformations of $\bar{\rho}_{f,\lambda}$ to R with fixed determinant (see Section 2 for precise definitions regarding deformation theory). Note that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for almost all λ [4, Proposition 3.1]. For such λ, the functor $\mathbf{D}^{\det=\delta}_{\bar{\rho}_{f,\lambda}}$ is representable by the *universal deformation ring* $R_{f,\lambda}$ for deformations with fixed determinant. Then our main theorem is the following.

Theorem 1.1. *Suppose that f has no CM, is not a twist of a base change of a Hilbert newform on $E \subsetneq F$, and $k_{\tau_i} \geq 3$ for all i. Then $R_{f,\lambda}$ is unobstructed for almost all λ.*

Remark 1.2. Weston [16] did not have this additional condition of deformations with fixed determinant, but in general there are obstructions that come from lifting the determinant, so there is no way around this. For details about calculating $\dim_{\mathbf{F}_p} H^2(G_{F,S}, \mathbf{Z}/p\mathbf{Z})$, see [14, Theorem 10.7.3].

Remark 1.3. The hypotheses that f has no CM and is not a twist of a base change come from ensuring that certain Selmer groups vanish for almost all λ (see Proposition 2.4, [5, Theorem B(i)] and [6, Theorem 2.1]). It is an open problem as to whether or not these hypotheses can be relaxed.

The strategy for proving Theorem 1.1 is to use a generalization of a criterion for unobstructedness (Proposition 2.4) due to Weston [16]. Using this proposition and results of Dimitrov ([5] and [6]), the proof is reduced to checking that for all $v \in S$, the local cohomology groups $H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0 \bar{\rho}_{f,\lambda}) = 0$ for almost all λ. Here ε is the ℓ-adic cyclotomic character, G_v is a decomposition group a v and $\mathrm{ad}^0 \rho$ denotes the restriction of the adjoint representation of ρ to the trace-zero matrices. Section 3 addresses those $v \in S$ such that $v \nmid \ell$, while section 4 shows that for almost all λ,

this vanishing cohomology condition holds for $v|\ell$. We also give a proof of Theorem 1.1 in section 4. We conclude in Section 5 with an explicit example of determining a lower bound on ℓ such that $R_{f,\lambda}$ is unobstructed for all λ over ℓ. Here f is the unique level one newform on $\mathbf{Q}(\sqrt{5})$ of weight (4,8).

It is with great pleasure that the author thanks Tom Weston for suggesting this problem and for several helpful suggestions along the way. Many thanks are also owed to Mladen Dimitrov for patiently answering every question put to him, especially regarding the vanishing of the previously mentioned Selmer groups. The author also benefited from a number of informative conversations with Ehud de Shalit and for this he is most grateful. Finally, the author acknowledges with gratitude that this work was produced while he was a postdoctoral fellow at the Hebrew University of Jerusalem, jointly supported by the Fulbright Commission in Israel, the United States-Israel Educational Foundation, and by the framework of the ERC grant GLC-247049 entitled *Langlands correspondence and its variants* under David Kazhdan.

Notation. For a field F, denote its absolute Galois group by G_F. As above, we let G_v denote a decomposition group at a place v of F and fix embeddings $G_v \hookrightarrow G_F$. Let F_v denote the v-adic completion of F. We use the phrase "almost all" as a substitute for "all but finitely many."

2. Review of Galois deformation theory

We briefly recall the theory of deformations of mod ℓ Galois representations in the sense of Mazur. For a more thorough introduction see [2] or [11].

Let F be a number field and let S be a finite set of places of F. Let k be a finite field of characteristic ℓ and denote the Witt vectors of k by $W(k)$. Consider an absolutely irreducible continuous representation

$$\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(k).$$

Also consider the category $\mathcal{C}$ of complete local noetherian rings R with residue field k. Morphisms in this category are local homomorphisms that induce the identity on k. A *lift* of $\bar{\rho}$ to R is a continuous representation $\rho : G_{F,S} \to \mathrm{GL}_n(R)$ making the following diagram commute:

$$\begin{array}{ccc} G_{F,S} & \xrightarrow{\rho} & \mathrm{GL}_n(R) \\ & \searrow^{\bar{\rho}} & \downarrow \\ & & \mathrm{GL}_n(k) \end{array}$$

where the homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)$ is the map induced by the reduction homomorphism $R \to k$. We say that two lifts ρ and ρ' of $\bar{\rho}$ to R are *strictly equivalent* if $\gamma\rho\gamma^{-1} = \rho'$ for some $\gamma \in \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k))$.

Definition 2.1. A *deformation* of $\bar{\rho}$ to R is a strict equivalence class of lifts of $\bar{\rho}$ to R.

Consider the functor $D_{\bar{\rho}} : \mathcal{C} \to SETS$ given by

$$D_{\bar{\rho}}(R) = \{\text{deformations of } \bar{\rho} \text{ to } R\}.$$

Call such a functor a *deformation problem.*

Theorem 2.2 (Mazur)**.** *If $\bar{\rho}$ is absolutely irreducible then $D_{\bar{\rho}}$ is representable by a complete local noetherian ring $R_{\bar{\rho}}$ and*

$$R_{\bar{\rho}} \cong W(k)[[x_1, \ldots, x_{d_1}]]/I.$$

Here $d_i = \dim_k H^i(G_{F,S}, \operatorname{ad} \bar{\rho})$ and I is generated by at most d_2 elements.

Definition 2.3. The deformation problem $D_{\bar{\rho}}$ is *unobstructed* if $d_2 = 0$.

We can also consider subfunctors of $D_{\bar{\rho}}$ where we ask our deformations to satisfy certain prescribed properties. For example, we can ask for deformations with fixed determinant. By this we mean that $\det \rho$ is the composition of the canonical homomorphism $W(k) \to R$ (making R a $W(k)$-algebra) with a fixed continuous character $\delta : G_{F,S} \to W(k)$. When this occurs, we say that a deformation ρ has $\det = \delta$. Denote by $D_{\bar{\rho}}^{\det=\delta}$ the subfunctor given by

$$D_{\bar{\rho}}^{\det=\delta} = \{\text{deformations of } \bar{\rho} \text{ to } R \text{ with } \det = \delta\}.$$

Note that for the deformation problem $D_{\bar{\rho}}^{\det=\delta}$, an analogue to Theorem 2.2 holds where we replace $\operatorname{ad} \bar{\rho}$ by $\operatorname{ad}^0 \bar{\rho}$ in the statement of the theorem.

We now specialize to two-dimensional residual representations $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(k)$. Let K be a finite extension of $\mathbf{Q}_\ell$ and let $\mathcal{O}$ be its ring of integers. Assume that we have a (fixed) continuous representation

$$\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$$

lifting $\bar{\rho}$. Set $V_\rho = K^3$ and $A_\rho = (K/\mathcal{O})^3$. Give V_ρ and A_ρ a G_F-action via $\operatorname{ad}^0 \rho$. Let $V_\rho(1)$ denote the Tate-twist of V_ρ. Finally, define the Selmer groups $H_f^1(G_F, V_\rho(1))$ and $H_f^1(G_F, A_\rho)$ in the sense of Bloch-Kato [1]. Then we have the following criterion for unobstructedness.

Proposition 2.4. *Suppose*

(1) $H^0(G_v, \bar{\varepsilon} \otimes \operatorname{ad}^0 \bar{\rho}) = 0$ *for all* $v \in S \setminus \{v|\infty\}$,
(2) $H_f^1(G_F, V_\rho(1)) = 0$,
(3) $H_f^1(G_F, A_\rho) = 0$.

Then $H^2(G_{F,S}, \operatorname{ad}^0 \bar{\rho}) = 0$. That is, $\mathbf{D}_{\bar{\rho}}^{\det=\delta}$ is unobstructed.

Proof. The argument follows mutatis mutandis as in the proof of Proposition 2.2 in [16]. □

Thus the strategy for proving Theorem 1.1 is clear. For $\rho = \rho_{f,\lambda}$, we need to check that the hypotheses of Proposition 2.4 hold for almost all primes λ of K_f.

3. Local invariants for $\ell \neq p$

Let F be a number field and let v be a prime in F over a rational $p \in \mathbf{Z}$. In this section, we show that the local invariants $H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho})$ are zero for almost all λ not dividing p. We separate the proof into two cases based on the local Langlands correspondence for $\mathrm{GL}_2(F_v)$.

Let K be any number field with ring of integers $\mathcal{O}$. For all primes λ of $\mathcal{O}$ not dividing $p = v \cap \mathbf{Z}$, fix an isomorphism $\iota_\lambda \colon \mathbf{C} \to \bar{K}_\lambda$ extending the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_\lambda$.

Let L be a finite extension of $\mathbf{Q}_p$. We say that a continuous character $\chi \colon L \to \mathbf{C}$ is of *Galois-type* with respect to ι_λ if the character $\iota_\lambda \circ \chi$ extends to a continuous character $\chi_\lambda \colon G_L \to \bar{K}_\lambda$ via the dense embedding $L^\times \hookrightarrow G_L^{\mathrm{ab}}$ of local class field theory. Call χ *arithmetic* if $\chi(L^\times) \subset \bar{\mathbf{Q}}^\times$.

Let π be an irreducible admissible complex representation of $\mathrm{GL}_2(F_v)$. Call π arithmetic if it satisfies one of the following conditions:

- π is a subquotient of an induced representation $\pi(\chi_1, \chi_2)$ where the $\chi_i \colon F_v^\times \to \mathbf{C}^\times$ are arithmetic characters (i.e., π is principal series or special, coming from arithmetic characters),
- π is the base change of an arithmetic quadratic character $\chi \colon L^\times \to \mathbf{C}^\times$ where L/F_v is a quadratic extension (i.e., π is supercuspidal and comes from the base change of an arithmetic character),
- π is extraordinary.

Lemma 3.1. *Let π be an arithmetic irreducible admissible complex representation of $\mathrm{GL}_2(F_v)$. Let $\{\rho_\lambda \colon G_v \to \mathrm{GL}_2(\bar{K}_\lambda)\}$ be a family of continuous representations for λ not dividing p such that π and ρ_λ are in Langlands correspondence with respect to ι_λ for all λ. If π is principal series or supercuspidal then*

$$H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda^{\mathrm{ss}}) = 0$$

for almost all λ.

Proof. This follows precisely as in [16, Proposition 3.2], so we do not repeat the argument here. □

Corollary 3.2. *We have $H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda) = 0$ for almost all λ.*

Proof. This is clear from Proposition 3.1 since

$$\dim_{\bar{\mathbf{F}}_\ell} H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda) \leq \dim_{\bar{\mathbf{F}}_\ell} H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda^{\mathrm{ss}}).$$

□

Note that for ρ_λ as in Lemma 3.1,

$$\dim_{\bar{F}_\ell} H^0(G_{\mathfrak{p}}, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda^{\mathrm{ss}}) = 1$$

for almost all λ when π is either one-dimensional or special. Although the stronger vanishing result fails when π is either one-dimensional or special, we can show the sufficient (and desired) vanishing of $H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_\lambda)$ for almost all λ by using a level-lowering argument.

Now suppose that π is special. That is, suppose that it is the infinite dimensional quotient of $\pi(\chi|\cdot|, \chi)$ for some arithmetic character $\chi\colon F_v^\times \to \mathbf{C}^\times$. Assume that $\rho_\lambda\colon G_{F,S} \to \mathrm{GL}_2(K_\lambda)$ is a λ-adic Galois representation such that $\rho_\lambda|_{G_v}$ is in Langlands correspondence with π. Then

$$\rho_\lambda|_{G_v} \cong \begin{pmatrix} \varepsilon\chi_\lambda & * \\ 0 & \chi_\lambda \end{pmatrix}$$

where $*$ is nonzero and ramified. Here we use the fact that the norm character corresponds to the compatible system of G_F-characters $\{\varepsilon_\lambda := \varepsilon\}_\lambda$. Set $k_\lambda = \mathcal{O}_F/v$.

Lemma 3.3. *Suppose that $q^2 \not\equiv 1 \bmod \lambda$ where $q = \#(\mathcal{O}_F/v)$. Then*

$$\bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda \cong \begin{pmatrix} \bar{\varepsilon}\bar{\chi}_\lambda & * \\ 0 & \bar{\chi}_\lambda \end{pmatrix}$$

where $*$ may be zero.

Proof. It is clear that the semi-simplification of $\bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda$ has the form $\bar{\varepsilon}\bar{\chi}_\lambda \oplus \bar{\chi}_\lambda$, so it suffices to show that $\bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda$ is not of the form

$$\begin{pmatrix} \bar{\chi}_\lambda & \nu \\ 0 & \bar{\varepsilon}\bar{\chi}_\lambda \end{pmatrix}$$

where ν is nontrivial. It is straightforward to check that $\bar{\varepsilon}^{-1}\bar{\chi}_\lambda^{-1}\nu$ is a 1-cocycle in $H^1(G_v, \bar{k}_{f,\lambda}(-1))$. Consider the inflation-restriction exact sequence

$$H^1(G_{\mathbf{F}_q}, \bar{k}_\lambda(-1)^{I_v}) \to H^1(G_v, \bar{k}_\lambda(-1)) \to H^1(I_v, \bar{k}_\lambda(-1))^{G_{\mathbf{F}_q}}$$

where $I_v \subset G_v$ is the inertia subgroup. An easy calculation shows that in general $H^1(G_{\mathbf{F}_q}, \bar{k}_\lambda(-1)^{I_v}) = 0$ and that $H^1(I_v, \bar{k}_\lambda(-1))^{G_{\mathbf{F}_q}} = 0$ when $\lambda \nmid q^2 - 1$. □

Lemma 3.4. *Suppose $2(q^2-1) \not\equiv 0 \bmod \lambda$. Then $H^0(G_v, \mathrm{ad}^0\, \bar{\rho}_\lambda) \neq 0$ if and only if $\mathrm{ad}^0\, \bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda$ is semi-simple.*

Proof. This is a straightforward matrix calculation using Lemma 3.3. For example, choose the basis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{End}(V)$ where V is the

3-dimensional $k_{f,\lambda}$-vector space endowed with a $G_{F,S}$ action by $\mathrm{ad}^0\,\bar\rho_{f,\lambda}$. By Lemma 3.3,

$$\bar\rho_\lambda \otimes \bar k_\lambda \cong \begin{pmatrix} \bar\varepsilon\bar\chi_\lambda & \nu \\ 0 & \bar\chi_\lambda \end{pmatrix},$$

so

$$\mathrm{ad}^0\,\bar\rho_\lambda|_{G_v} \otimes \bar k_\lambda \cong \begin{pmatrix} 1 & -2\bar\chi_\lambda^{-1}\nu & 0 \\ 0 & \bar\varepsilon & 0 \\ \bar\varepsilon^{-1}\bar\chi_\lambda^{-1}\nu & -\bar\varepsilon^{-1}\bar\chi_\lambda^{-2}\nu^2 & \bar\varepsilon^{-1} \end{pmatrix}.$$

Thus it is clear that if $\nu = 0$ then $H^0(G_v, \mathrm{ad}^0\,\bar\rho_\lambda) \neq 0$. Conversely, if ν is nonzero then using the fact that it is ramified while $\bar\varepsilon$ is not, one checks that there are no Galois invariants. □

Let $\rho_{f,\lambda}\colon G_{F,S} \to \mathrm{GL}_2(K_{f,\lambda})$ be the Galois representation attached to a Hilbert newform f of level $\mathfrak{n}$, weight k and character ψ by Carayol [3] and Taylor [15]. Write $\psi = \psi_f|\cdot|^{k_0-2}$ where ψ_f is a character of finite order and $|\cdot|$ is the norm character. Note that $\det\rho_{f,\lambda} = \psi_f^{-1}\varepsilon^{1-k_0}$ where here ψ_f also denotes by abuse of notation the corresponding Galois character. (We find it more convenient to work with this cohomological normalization rather than the usual normalization.)

Let π be the automorphic representation corresponding to f. Write $\pi = \otimes'\pi_v$ for the decomposition of π into its irreducible admissible complex representations of π_v into $\mathrm{GL}_2 F_v$. Fixing isomorphisms $\iota_\lambda\colon \mathbf{C} \to \bar K_{f,\lambda}$, Carayol [3, Théorème B] showed that each π_v is arithmetic and is in Langlands correspondence with $\rho_{f,\lambda}|_{G_v}$ for λ not dividing p.

Remark 3.5. The irreducible admissible representation π_v must be infinite dimensional so nothing is lost by assuming that π_v is special (as opposed to one-dimensional) in what follows.

Proposition 3.6. *If π_v is special then*

$$H^0(G_v, \bar\varepsilon \otimes \mathrm{ad}^0\,\bar\rho_{f,\lambda}) = 0$$

for almost all λ.

Proof. Note that π_v has central character $\chi^2|\cdot|$ where χ is an arithmetic character giving rise to π_v. By the local Langlands correspondence, this yields the equality $\chi^2 = \psi_{f,v}^{-1}|\cdot|^{-k_0}$ where $\psi_{f,v}$ is the v-component of ψ_f. Set $\chi_v' = \chi^{-1}|\cdot|^{-k_0/2}$. Note that χ_v' has finite order. Extend χ_v' to a Hecke character χ' and twist f by χ' to get an eigenform $f \otimes \chi'$. Let f' denote the newform in the eigenspace spanned by $f \otimes \chi'$ and let π' denote the corresponding automorphic representation. Then the v-component of π' is a subquotient of $\pi(\chi\chi_v'|\cdot|, \chi\chi_v')$. In particular, $\chi\chi_v'$ is unramified at v, so v divides the level $\mathfrak{n}'$ of f' exactly once.

Suppose λ does not divide $2(q^2-1)$ and suppose that

$$H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_{f,\lambda}) \neq 0. \tag{3.1}$$

Then Lemma 3.4 implies that $\bar{\rho}_{f,\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda} \cong \bar{\varepsilon}\bar{\chi}_\lambda \oplus \bar{\chi}_\lambda$. This means that

$$\bar{\rho}_{f',\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda} \cong (\bar{\rho}_{f,\lambda} \otimes \bar{\chi}'_\lambda)|_{G_v} \otimes \bar{k}_{f,\lambda} \cong \bar{\varepsilon}^{1-k_0/2} \oplus \bar{\varepsilon}^{-k_0/2},$$

so $\bar{\rho}_{f',\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda}$ is unramified at v. Since λ does not divide q^2-1, we have that $N_{F/\mathbf{Q}}(v) \not\equiv 1 \bmod \ell$, so we may apply [12, Theorem 0.1] to get a congruent eigenform f'' of level $\mathfrak{n}'/v$. That is, we get a set of Hecke eigenvalues $\{a(\mathfrak{m}, f'')\}$ such that $a(\mathfrak{q}, f'') \equiv a(\mathfrak{q}, f') \bmod \lambda$ for all $\mathfrak{q}$ not dividing $\mathfrak{n}'\ell$. By strong multiplicity one, there are only finitely many sets of eigenvalues, each one corresponding to a newform of level dividing $\mathfrak{n}'/v$. Therefore, if (3.1) holds for infinitely many λ then for some newform g of level dividing $\mathfrak{n}'/v$ and for all $\mathfrak{q}$ not dividing $\mathfrak{n}'$,

$$a(\mathfrak{q}, g) \equiv a(\mathfrak{q}, f') \mod \lambda$$

for infinitely many λ. We conclude that $a(\mathfrak{q}, g) = a(\mathfrak{q}, f')$ for all $\mathfrak{q}$ not dividing $\mathfrak{n}'$, so applying strong multiplicity one again shows that $g = f'$, a contradiction. □

4. Local invariants for $\ell = p$

We now recall the theory of Fontaine-Laffaille. Let K be a finite unramified extension of $\mathbf{Q}_\ell$ and let $E/\mathbf{Q}_\ell$ be another finite extension containing K. Let σ be the frobenius automorphism on K. Given an E-linear representation V of G_K, define the finite free $E \otimes_{\mathbf{Q}_\ell} K$-module

$$D_{\mathrm{crys}}(V) = (B_{\mathrm{crys}} \otimes_{\mathbf{Q}_\ell} V)^{G_K}$$

where B_{crys} is Fontaine's crystalline period ring. Note that $D_{\mathrm{crys}}(V)$ comes with a decreasing filtration $\{D_{\mathrm{crys}}(V)^i\}_i$ such that

$$\cap_i D_{\mathrm{crys}}(V)^i = 0 \text{ and } \cup_i D_{\mathrm{crys}}(V) = D_{\mathrm{crys}}(V).$$

In addition, $D_{\mathrm{crys}}(V)$ comes with a $1_E \otimes \sigma$-semilinear map $\varphi : D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V)$. Call V is *crystalline* if $\dim_E V$ equals the rank of $D_{\mathrm{crys}}(V)$ as a $E \otimes_{\mathbf{Q}_\ell} K$-module.

Suppose V is an E-linear crystalline G_K-representation with Hodge-Tate filtration in the interval $[-(a+\ell-1), -a]$. Consider the category $\mathrm{MF}^{a,a+\ell}(\mathcal{O}_E)$ of strongly divisible lattices in $D_{\mathrm{crys}}(V)$ whose objects consist of finite free $\mathcal{O} := \mathcal{O}_E \otimes_{\mathbf{Z}_\ell} \mathcal{O}_K$-lattices $L \subset D_{\mathrm{crys}}(V)$ with a filtration $\{L^i := L \cap D_{\mathrm{crys}}(V)^i\}$ and $1_{\mathcal{O}_E} \otimes \sigma$-semilinear maps $\{\varphi_i^L : L^i \to L\}$ such that

(1) $L^i \supset L^{i+1}, L^a = L, L^{a+\ell} = 0$ and each L^i is a direct summand of L,
(2) $\varphi_i^L|_{L^{i+1}} = \ell\varphi_{i+1}^L$ and $L = \sum_i \varphi_i^L(L^i)$.

Then Fontaine-Laffaille [8] gives an equivalence of categories between the category of $\mathcal{O}_E[G_K]$-modules that are finitely generated subquotients of E-linear crystalline G_K-representations V with Hodge-Tate weights in the interval $[-(a+\ell-1), -a]$ and the category $\mathrm{MF}^{a,a+\ell}(\mathcal{O}_K)$.

Remark 4.1. Here we use the definition of the Tate twist of a strongly divisible lattice as in Section 4 of [1] to extend the results of [8] to the case where $a \neq 0$.

Example 4.2. Let $\psi : G_K \to \mathcal{O}_E$ be an unramified character of finite order and let $\mathcal{O}_E(\psi)$ denote the $\mathcal{O}_E[G_K]$-module of rank one with G_K-action given by ψ. Then the strongly divisible lattice D_ψ corresponding to $\mathcal{O}_E(\psi)$ can be described as a free rank one $\mathcal{O}$-module such that $L^0_\psi = L_\psi$, $L^1_\psi = 0$ and $\varphi_0^{L_\psi}$ is multiplication by some $u \in \mathcal{O}^\times = (\mathcal{O}_E^\times)^{[K:\mathbf{Q}_\ell]}$. Denote this u by $\psi(\sigma)$.

We adopt this notation since over some finite extension of E, we have that L_ψ is isomorphic to a strongly divisible lattice L where φ_0^L is multiplication by $(\psi(\sigma), 1, \ldots, 1)$ (see [7]). In any case, the precise value of u will not be important for our intended application.

For the remainder of the section, we assume ℓ is unramified in F (a totally real extension of $\mathbf{Q}$ of degree d) and set $K = F_v$ for a place v of F dividing ℓ.

Example 4.3. Let f be a newform on F in $S_k(\mathfrak{n}, \psi)$. Set $k_0 = \max_i\{k_{\tau_i}\}$. For a prime $v|\ell$ of F, let $E = K_{f,\lambda}F_v$ and consider the Galois representation $\rho_{f,\lambda}|_{G_v} : G_v \to \mathrm{GL}_2(E)$. Let $V_{f,\lambda}$ be a 2-dimensional E vector space on which G_v acts by $\rho_{f,\lambda}|_{G_v}$. The fixed embedding $\iota_\ell : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$ gives a partition of the different embeddings τ_i as a disjoint union of subsets J_v where denotes the subset of τ_i that give the different embeddings of F_v into $\bar{\mathbf{Q}}_\ell$ after composition with ι_ℓ. Note that $V_{f,\lambda}$ is crystalline with labeled Hodge-Tate weights $(-\frac{k_0-2+k_\tau}{2}, -\frac{k_0-k_\tau}{2})_{\tau\in J_v}$ if $\ell > k_0$ is unramified in F and prime to $\mathfrak{n}$. Fix a G_v-stable $\mathcal{O}_E$-lattice $T_{f,\lambda} \subset V_{f,\lambda}$. Thus for v dividing such ℓ, there is a $L_{f,\lambda}$ in $\mathrm{MF}^{0,\ell}(\mathcal{O}_{F_v})$ corresponding to $T_{f,\lambda}$. Then using the Hodge-Tate weights, we have the following description of $L_{f,\lambda}$.

Let $k_0^v = \max\{k_\tau | \tau \in J_v\}$. Set $T_0 = \{\tau \in J_v | k_\tau = k_v\}$. For $i \geq 1$, define k_i^v to be the $\max_{\tau\in J_v \setminus \cup_{j<i} T_j}\{k_\tau\}$ and set $T_i = \{\tau \in J_v | k_\tau = k_{v,i}\}$. Let s be the index such that $k_s^v = \min_{\tau\in J_v}\{k_\tau\}$. Let e_i denote the element of $\mathcal{O} = \mathcal{O}_E^{[F_v:\mathbf{Q}_\ell]}$ with a 1 in its ith component and zeroes everywhere else. Finally, define $d_{\geq i} = \sum_{j\in\cup_{k\geq i}T_i} e_j$ and $d_{<i} = \sum_{j\notin\cup_{k<i}T_i} e_j$. Then there is

an $\mathcal{O}$ basis x, y of L_f such that the filtration satisfies:

$$L^i_{f,\lambda} = \begin{cases} \mathcal{O}x \oplus \mathcal{O}y, & \text{for } i \leq \frac{k_0-k_0^v}{2}, \\ \mathcal{O}x \oplus \mathcal{O}d_{\geq 1}y, & \text{for } \frac{k_0-k_0^v}{2}+1 \leq i \leq \frac{k_0-k_1^v}{2}, \\ \mathcal{O}x \oplus \mathcal{O}d_{\geq 2}y, & \text{for } \frac{k_0-k_1^v}{2}+1 \leq i \leq \frac{k_0-k_2^v}{2}, \\ \vdots & \vdots \\ \mathcal{O}x, & \text{for } \frac{k_0-k_s^v}{2}+1 \leq i \leq \frac{k_0-2+k_s^v}{2}, \\ \mathcal{O}d_{<s}x, & \text{for } \frac{k_0-2+k_s^v}{2}+1 \leq i \leq \frac{k_0-2+k_{s-1}^v}{2} \\ \vdots & \vdots \\ \mathcal{O}d_{<1}x, & \text{for } \frac{k_0-2+k_1^v}{2}+1 \leq i \leq \frac{k_0-2+k_0^v}{2}, \\ 0, & \text{for } i \geq \frac{k_0-2+k_0^v}{2}+1. \end{cases}$$

This is not enough to completely identify $L_{f,\lambda}$ up to isomorphism, but it will be enough for our purposes.

We fix some notation for use in Proposition 4.4. Let

$$\varphi_0^{L_{f,\lambda}}(x) = \alpha x + \beta y$$

for some α, β in $\mathcal{O}$. So writing $\alpha = (\alpha_\tau)_{\tau \in J_v}$ and $\beta = (\beta_\tau)_{\tau \in J_v}$, we conclude that the λ-adic valuations $v_\lambda(\alpha_\tau)$ and $v_\lambda(\beta_\tau)$ are at least $\frac{k_0-2+k_{v,0}}{2}$ for all τ by condition 2 of the definition of the objects of $\mathrm{MF}^{0,\ell}(\mathcal{O}_{F_v})$. Here we normalized v_λ so that $v_\lambda(\ell) = 1$.

Proposition 4.4. *Suppose f is a Hilbert newform on F of weight $k = (k_{\tau_1}, \ldots, k_{\tau_d})$, level $\mathfrak{n}$, and character ψ. Assume that $k_0 > 2$. Then for $\ell \geq 2k_0^v - 1$, unramified in F and prime to $\mathfrak{n}$,*

$$H^0(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_{f,\lambda}) = 0.$$

Proof. Since $\det \rho_{f,\lambda} = \psi_f^{-1}\varepsilon^{1-k_0}$, we have the Galois-stable lattice

$$\mathrm{ad}^0\, T_{f,\lambda}(1) \cong (\mathrm{Sym}^2(T_{f,\lambda}) \otimes_{\mathcal{O}_E} \mathcal{O}_E(\psi_f))(k_0).$$

Since $\ell \geq 2k_0^v - 1$, we can apply the Fontaine-Laffaille functor to get a corresponding L in $\mathrm{MF}^{-k_0^v, k_0^v-1}(\mathcal{O}_{F_v})$. By [9, Proposition 1.7],

$$L \cong (\mathrm{Sym}^2(L_{f,\lambda}) \otimes_{\mathcal{O}} L_{\psi_f})(k_0).$$

Note that by [1, Lemma 4.5],

$$H^1(G_v, \bar{\varepsilon} \otimes \mathrm{ad}^0\, \bar{\rho}_{f,\lambda}) \cong \ker(1 - \varphi_0^L : L^0/\lambda L^0 \to L/\lambda L). \tag{4.1}$$

Furthermore, by the definition of Tate twists for strongly divisible lattices,

$$\begin{aligned} L^0 &= (\mathrm{Sym}^2(L_{f,\lambda}) \otimes_{\mathcal{O}} L_{\psi_f})^{k_0} \\ &= \left\{ v \otimes w \,\middle|\, v = \sum_i a_i(u \otimes u') \in \mathrm{Sym}^2(L_{f,\lambda}), u \in L^i_{f,\lambda}, u' \in L^{i'}_{f,\lambda}, i + i' = k_0 \right\} \\ &= L_{f,\lambda}^{\frac{k_0 - k_{v,0}}{2}+1} \otimes_{\mathcal{O}} L_{f,\lambda}^{\frac{k_0-2+k_0^v}{2}} \otimes_{\mathcal{O}} L^0_{\psi_f} \\ &= \mathcal{O} d_{<1}(x \otimes x \otimes w) \end{aligned}$$

where w is a generator of the rank one $\mathcal{O}$-module L_{ψ_f}. Set $v = x \otimes x \otimes w$. Then we have

$$\begin{aligned} \varphi_0^L(ad_{<1}v) &= a^\sigma d^\sigma_{<1} \varphi_{\frac{k_0-k_{v,0}}{2}+1}^{L_{f,\lambda}}(x) \otimes \varphi_{\frac{k_0-2+k_0^v}{2}}^{L_{f,\lambda}}(x) \otimes \varphi_0^{L_{\psi_f}}(w) \\ &= \frac{(ad_{<1})^\sigma \psi_f(\sigma)\alpha^2}{\ell^{\frac{k_0-2+k_0^v}{2}+1}} x \otimes x \otimes w + \cdots \end{aligned} \tag{4.2}$$

where the superscript σ denotes the action of $1_{\mathcal{O}_E} \otimes \sigma$ on the given element of $\mathcal{O}$. Suppose that (4.1) is nonzero. Thus if $ad_{<1}(x \otimes x \otimes w)$ is a nonzero element of the kernel of $1 - \varphi_0^L$ then (4.2) implies that there is some $\tau \in J_v$ such that

$$\frac{(ad_{<1})^\sigma_\tau \psi_f(\sigma)\alpha^2_\tau}{\ell^{\frac{k_0-2+k_0^v}{2}+1}} \equiv (ad_{<1})_\tau \not\equiv 0 \bmod \lambda.$$

This implies that the λ-adic valuation of the numerator is $\frac{k_0-2+k_0^v}{2} + 1$. Since $v_\lambda(a_\tau \psi_f(\sigma)) = 0$, this means that

$$v_\lambda(\alpha_\tau^2) = \frac{k_0 - 2 + k_0^v}{2} + 1.$$

But we also know that $v_\lambda(\alpha_\tau) \geq \frac{k_0-2+k_0^v}{2}$, so we have that $k_0 - 2 + k_0^v \leq \frac{k_0-2+k_0^v}{2} + 1$, or equivalently,

$$k_0 - 2 + k_0^v \leq 2.$$

As we assumed $k_0 > 2$, this proves the proposition. □

We are now ready to prove Theorem 1.1.

Theorem 1.1. *Let f be as in Proposition* 4.4 *and suppose that it does not have CM and that it is not a twist of a base change of a Hilbert newform on $E \subsetneq F$. Then $H^2(G_{F,S}, \mathrm{ad}^0\, \bar\rho_{f,\lambda}) = 0$ for almost all primes λ of K_f. That is, $\mathbf{D}^{\det=\delta}_{\bar\rho_{f,\lambda}}$ is unobstructed for almost all λ.*

Proof. We verify that the hypotheses (1) – (3) of Proposition 2.4 hold for almost all λ. Combining the results of Corollary 3.2, Proposition 3.6 and Proposition 4.4 shows that $H^0(G_v, \bar\varepsilon \otimes \mathrm{ad}^0\, \bar\rho_{f,\lambda}) = 0$ for almost all λ.

The Selmer group $H^1_f(G_F, A_{\rho_{f,\lambda}})$ vanishes for almost all λ because of [5, Theorem 6.6], [6, Theorem 2.1] and the fact that (in Dimitrov's notation)

$$H^1_f(G_F, A_{\rho_{f,\lambda}}) \subset H^1_\Sigma(G_F, A_{\rho_{f,\lambda}})$$

for any finite set of primes Σ.

Theorem B(i) of [5] and [6, Theorem 2.1] tell us that $H^1_f(G_F, V_{\rho_{f,\lambda}}) = 0$ for almost all λ. To show the vanishing of $H^1_f(G_F, V_{\rho_{f,\lambda}}(1))$, we define for a place $v|\ell$ of F the tangent space

$$t_V = ((B_{\mathrm{crys}}/B^+_{\mathrm{crys}}) \otimes_{\mathbf{Q}_\ell} V)^{G_{F_v}}$$

of a crystalline E-linear representation V where E is a finite extension of $\mathbf{Q}_\ell$ containing $K_{f,\lambda}$ and F_v. Then [10, Proposition I.2.2.2(ii)] tells us that

$$t_V \cong D_{\mathrm{crys}}(V)/D_{\mathrm{crys}}(V)^0.$$

In particular, for $V_{\rho_{f,\lambda}}$, we extend scalars to a finite extension E of $\mathbf{Q}_\ell$ such that E contains F_v for all places $v|\ell$ in F and set

$$t_{V_{\rho_{f,\lambda}}} = \bigoplus_{v|\ell} t_{V_{\rho_{f,\lambda}|G_v}}.$$

By Schur's lemma, $H^0(G_F, \varepsilon \otimes \mathrm{ad}^0\, \rho_{f,\lambda}) = 0$, so [10, Remark II.2.2.2] implies that

$$\dim_E H^1_f(G_F, V_{\rho_{f,\lambda}}(1)) - \dim_E H^1_f(G_F, V_{\rho_{f,\lambda}}) = -\dim_E t_{V_{\rho_{f,\lambda}}} + \sum_{v|\infty} \dim_E H^0(G_v, V_{\rho_{f,\lambda}}).$$

A straightforward computation using the Hodge-filtration on $V_{\rho_{f,\lambda}}$ shows that the right-hand-side vanishes, so

$$\dim_E H^1_f(G_F, V_{\rho_{f,\lambda}}(1)) = \dim_E H^1_f(G_F, V_{\rho_{f,\lambda}}).$$

Thus Proposition 2.4 implies the unobstructedness of $\mathbf{D}^{\det=\delta}_{\bar\rho_{f,\lambda}}$ for almost all λ. □

5. Explicit computations

The methods we used to prove Theorem 1.1 are essentially effective in the sense that given enough information about the Hecke eigenvalues of a given Hilbert newform as well as the eigenvalues of the other newforms of the same level, one can find an explicit lower bound B such that for all $\ell \geq B$, the deformation problem $\mathbf{D}^{\det=\delta}_{\bar\rho_{f,\lambda}}$ is unobstructed for all λ over these ℓ. We illustrate this with an example.

Let $F = \mathbf{Q}(\sqrt{5})$, $k = (4, 8)$, and $\mathfrak{n} = (1)$. Then using MAGMA we computed that the space of cuspforms $S_k(\mathfrak{n})$ is one dimensional. Thus $S_k(\mathfrak{n})$ is generated by a newform f whose first few Hecke eigenvalues $c(f, \mathfrak{p})$ we computed in MAGMA and list in Table 1.

Remark 5.1. It can be shown that $K_f = \mathbf{Q}(\sqrt{5})$ in this case.

TABLE 5.1. Hecke eigenvalues of f

$\mathrm{N}_{F/\mathbf{Q}}(\pi)$	$c(f,\mathfrak{p})$
4	-160
5	150
9	-270
11	$-1800\sqrt{5}+2172$
11	$1800\sqrt{5}+2172$
19	$9000\sqrt{5}-21340$
19	$-9000\sqrt{5}-21340$

Remark 5.2. In what follows, note that $S=\{\lambda\}\cup\{v|\infty\}$ for $\bar{\rho}_{f,\lambda}$.

Proposition 5.3. *The deformation problem* $\mathbf{D}^{\det=\delta}_{\bar{\rho}_{f,\lambda}}$ *is unobstructed for all primes* λ *of* K_f *over* $\ell\geq 17$.

Proof. Our approach is to give a lower bound on ℓ for which the residual representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible and for which the three hypotheses of Proposition 2.4 hold. We begin with absolute irreducibility. Let $\omega=\frac{3+\sqrt{5}}{2}$. By [4, Proposition 3.1(ii)], since ω is a totally positive unit, we conclude that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all λ not dividing $\mathrm{N}_{F/\mathbf{Q}}(\omega-1)=-1$, $\mathrm{N}_{F/\mathbf{Q}}(\omega^2-1)=-5$, $\mathrm{N}_{F/\mathbf{Q}}(\omega^8-1)=-2205$, and $\mathrm{N}_{F/\mathbf{Q}}(\omega^9-1)=-5776$. More concretely, computing the prime factors of the principal ideals generated by these elements, [4, Proposition 3.1(ii)] tells us that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all λ over $\ell\geq 17$ except possibly the primes over 19.

Note that since $\bar{\rho}_{f,\lambda}$ is an odd representation, it is absolutely irreducible if and only if it is irreducible. Thus to prove absolute irreducibility of $\bar{\rho}_{f,\lambda}$ for the remaining $\lambda|19$, it suffices to provide a prime $\mathfrak{p}$ over $p\neq 19$ such that the characteristic polynomial of $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})$ is irreducible over $k_{f,\lambda}$. Recall that the characteristic polynomial for $\bar{\rho}_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})$ is $X^2-c(f,\mathfrak{p})X+p^3$ if p splits in F. In particular, for each λ over 19, we found a prime $\mathfrak{p}$ over 11 such that the polynomial $X^2-c(f,\mathfrak{p})X+11^3$ is irreducible over $k_{f,\lambda}$. Thus $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all λ over $\ell\geq 17$.

We now check hypotheses (1) – (3) of Proposition 2.4. For (1), we know that $H^0(G_\lambda,\bar{\varepsilon}\otimes\mathrm{ad}^0\,\bar{\rho}_{f,\lambda})=0$ for all λ over $\ell\geq 15$ by Proposition 4.4. Regarding (2), as we discussed in the proof of Theorem 1.1, the vanishing of $H^1_f(G_F,V_{\bar{\rho}_{f,\lambda}}(1))$ is equivalent to the condition that $H^1_f(G_F,V_{\bar{\rho}_{f,\lambda}})=0$. Furthermore, Dimitrov [5, Theorem B] showed that $H^1_f(G_F,V_{\bar{\rho}_{f,\lambda}})$ vanishes as long as $\ell\geq 13$, and the image of $\mathrm{Ind}_F^{\mathbf{Q}}\,\bar{\rho}_{f,\lambda}$ is "large." (We will give more details about this large image condition in the next paragraph.) This means that for the desired set of primes λ, whenever this "large image" condition

holds, hypothesis (2) of Proposition 2.4 also holds. Moreover, let η_f denote the congruence ideal obtained from the $\mathfrak{o}$-algebra homomorphism $\mathcal{T}_{\mathfrak{m}} \to \mathfrak{o}$ by $T_{\mathfrak{a}} \mapsto \iota_\lambda(c(f,\mathfrak{a}))$ where $\mathfrak{o} = \mathcal{O}_{K_f,\lambda}$, $\mathcal{T}$ is the Hecke algebra $\mathfrak{o}[T_{\mathfrak{a}} | \mathfrak{a} \subset \mathcal{O}_F]$ and ι_λ is the fixed isomorphism $\mathbf{C} \to \bar{K}_{f,\lambda}$ extending the embedding $\mathcal{O}_{K_f} \hookrightarrow \mathfrak{o}$. (See [5, Definition 3.1] as well as the discussions before [5, Theorems 1.4 and 3.6] for more details about η_f.) Then [5, Theorem 3.6] implies that for $\ell \geq 13$, and $\mathrm{Ind}_F^{\mathbf{Q}} \bar{\rho}_{f,\lambda}$ satisfying the same large image hypothesis, the Selmer group $H^1_f(G_F, A_{\bar{\rho}_{f,\lambda}}) = 0$ if and only if $\eta_f = \mathfrak{o}$. That is, if and only if λ does not divide η_f. By definition, however, λ divides η_f if and only if there is another newform g of the same weight, level and character such that $c(f,\mathfrak{a}) \equiv c(g,\mathfrak{a}) \bmod \lambda$ for all $\mathfrak{a} \subset \mathcal{O}_F$. As f is the only newform in $S_k(\mathfrak{n})$, this means that $H^1_f(G_F, A_{\bar{\rho}_{f,\lambda}}) = 0$ for all such λ. That is, hypothesis (3) also holds for all λ over $\ell \geq 13$, satisfying the large image condition. Thus we are reduced to checking that this large image condition on $\mathrm{Ind}_F^{\mathbf{Q}} \bar{\rho}_{f,\lambda}$ holds for all λ over $\ell \geq 17$.

The large image condition on $\mathrm{Ind}_F^{\mathbf{Q}} \bar{\rho}_{f,\lambda}$ that we referred to throughout the previous paragraph is a somewhat technical hypothesis that Dimitrov uses for Theorem 1.4 of [5]. We refer the interested reader to [5, Theorem A] for a detailed statement of this large image hypothesis on $\mathrm{Ind}_F^{\mathbf{Q}} \bar{\rho}_{f,\lambda}$. In our case, however, since the weight (4,8) is non-induced in the sense of [4, Definition 3.11] and we assume that $\ell \geq 17$, we may instead use the large image condition on $\bar{\rho}_{f,\lambda}$ that $\mathrm{Im}(\bar{\rho}_{f,\lambda})$ contains a conjugate of $\mathrm{SL}_2(k_{f,\lambda})$ (see [4, Proposition 3.13]). Moreover, since we have already shown that $\bar{\rho}_{f,\lambda}$ is irreducible for all λ over $\ell \geq 17$, we can use Dickson's classification of subgroups of $\mathrm{GL}_2(k_{f,\lambda})$ in such cases. In particular, this classification states that an irreducible subgroup of $\mathrm{GL}_2(k_{f,\lambda})$ that does *not* contain a conjugate of $\mathrm{SL}_2(k_{f,\lambda})$ is isomorphic to either a dihedral group or one of A_4, S_4, or A_5. Thus we need to show that the projective image of $\mathrm{Im}(\bar{\rho}_{f,\lambda}) \subset \mathrm{GL}_2(k_{f,\lambda})$ is not isomorphic to a dihedral group nor any of the groups A_4, S_4, and A_5.

To check that the projective image of $\mathrm{Im}(\bar{\rho}_{f,\lambda})$ is not dihedral, we use [4, Lemma 3.4]. More specifically, assume that the image of $\bar{\rho}_{f,\lambda}$ in $\mathrm{PGL}_2(k_{f,\lambda})$ is dihedral, meaning $\bar{\rho}_{f,\lambda} \cong \bar{\rho}_{f,\lambda} \otimes \chi_{K/F}$ where $\chi_{K/F}$ is the character of a quadratic extension K/F. Then supposing that $\ell \neq 2k_{\tau_i} - 1$ for all i where $k = (k_{\tau_1}, \ldots, k_{\tau_d})$ is the weight of f and $d = [F : \mathbf{Q}]$, this lemma says that K/F is unramified outside of $\mathfrak{n}$. In our case, $\mathfrak{n} = \mathcal{O}_F$. Since F has class number one, we conclude that no such K exists. Hence the image of $\bar{\rho}_{f,\lambda}$ in $\mathrm{PGL}_2(k_{f,\lambda})$ is not dihedral.

Finally, to show that the projective image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to A_4, S_4, or A_5, we use Section 3.2 of [4]. The main result of this section is

that if

$$\ell - 1 > \frac{5}{d}\sum_{i=1}^{d} k_{\tau_i} - 1,$$

then the projective image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to any of the groups A_4, S_4 or A_5. In our case, $d = 2$ and $k = (4, 8)$ so it is easy to conclude that the image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to A_4, S_4, or A_5 for λ over $\ell \geq 25$. For λ over 19, a closer analysis of [4, Section 3.2] shows that if the projective image of $\bar{\rho}_{f,\lambda}$ is A_4, S_4, or A_5 then the arguments there imply that either ± 3 or ± 7 has order ≤ 5 in $\mathbf{Z}/18\mathbf{Z}$, which is a contradiction. Similarly for λ over 17, these same arguments imply that 58, 116, 122, or 244 would have order at most five in $\mathbf{Z}/288\mathbf{Z}$ if the projective image of $\bar{\rho}_{f,\lambda}$ is A_4, S_4, or A_5, which is false. For λ over 23, if the projective image of $\bar{\rho}_{f,\lambda}$ is A_4, S_4, or A_5 then 76, 164, 370, or 466 would have order at most five in $\mathbf{Z}/528\mathbf{Z}$, which is also a contradiction. □

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