JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Malte WITTE

Unit L-Functions for étale sheaves of modules over noncommutative rings

Tome 28, nº 1 (2016), p. 89-113.

<http://jtnb.cedram.org/item?id=JTNB_2016__28_1_89_0>

Journal de Théorie des Nombres
de Bordeaux **28** (2016), 89–113

Unit *L*-Functions for étale sheaves of modules over noncommutative rings

par Malte WITTE

Résumé. Soit $s\colon X \to \operatorname{Spec}\mathbb{F}$ un schéma séparé de type fini sur un corps fini $\mathbb{F}$ de caractéristique p, soit Λ une $\mathbb{Z}_p$-algèbre avec un nombre fini d'éléments, non nécessairement commutative, et soit $\mathcal{F}^\bullet$ un complexe parfait de Λ-faisceaux sur le site étale de X. Nous prouvons que le quotient $L(\mathcal{F}^\bullet, T)/L(\mathrm{R}\, s_!\mathcal{F}^\bullet, T)$, qui est a priori un élément de $\mathrm{K}_1(\Lambda[[T]])$, a un antécédent canonique dans $\mathrm{K}_1(\Lambda[T])$. Nous utilisons cela pour prouver une version de la conjecture principale d'Iwasawa non-commutative pour des revêtements de Lie p-adiques de X.

Abstract. Let $s\colon X \to \operatorname{Spec}\mathbb{F}$ be a separated scheme of finite type over a finite field $\mathbb{F}$ of characteristic p, let Λ be a $\mathbb{Z}_p$-algebra with finitely many elements, not necessarily commutative, and let $\mathcal{F}^\bullet$ be a perfect complex of Λ-sheaves on the étale site of X. We show that the ratio $L(\mathcal{F}^\bullet, T)/L(\mathrm{R}\, s_!\mathcal{F}^\bullet, T)$, which is a priori an element of $\mathrm{K}_1(\Lambda[[T]])$, has a canonical preimage in $\mathrm{K}_1(\Lambda[T])$. We use this to prove a version of the noncommmutative Iwasawa main conjecture for p-adic Lie coverings of X.

1. Introduction

Let p be a prime number and $\mathbb{F}$ the finite field with $q = p^v$ elements, and $s\colon X \to \operatorname{Spec}\mathbb{F}$ a separated finite type $\mathbb{F}$-scheme. Let further Λ be an adic $\mathbb{Z}_p$-algebra, i. e. Λ is compact for the topology defined by the powers of its Jacobson radical $\operatorname{Jac}(\Lambda)$. For any perfect complex of Λ-sheafs $\mathcal{F}^\bullet$ on the étale site of X we have defined in [16] an L-function $L(\mathcal{F}^\bullet, T)$ attached to $\mathcal{F}^\bullet$. This is an element in the first K-group $\mathrm{K}_1(\Lambda[[T]])$ of the power series ring $\Lambda[[T]]$ in the formal variable T commuting with the elements of Λ. The total higher direct image $\mathrm{R}\, f_!\mathcal{F}^\bullet$ is again a perfect complex of Λ-sheaves and so we may form the L-function $L(\mathrm{R}\, f_!\mathcal{F}^\bullet, T)$, which is also an element of $\mathrm{K}_1(\Lambda[[T]])$. Different from the situation where Λ is an adic $\mathbb{Z}_\ell$-algebra

Manuscrit reçu le 22 octobre 2013, accepté le 29 mai 2015.
Mathematics Subject Classification. 14G10, 11G25, 14G15, 11R23.
Mots-clefs. Unit L-Functions, Grothendieck trace formula, Iwasawa main conjecture, varieties over finite fields.

with $\ell \neq p$ dicussed in [16], the ratio $L(\mathcal{F}^\bullet, T)/L(\mathrm{R}\, f_! \mathcal{F}^\bullet, T)$ does not need to be 1 in $\mathrm{K}_1(\Lambda[[T]])$.

Let

$$\Lambda\langle T\rangle = \varprojlim_k \Lambda/\operatorname{Jac}(\Lambda)^k[T]$$

denote the $\operatorname{Jac}(\Lambda)$-adic completion of the polynomial ring $\Lambda[T]$ and let

$$\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) = \varprojlim_k \mathrm{K}_1(\Lambda/\operatorname{Jac}(\Lambda)^k[T])$$

denote its first completed K-group. If Λ is commutative, then

$$\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) = \mathrm{K}_1(\Lambda\langle T\rangle) = \Lambda\langle T\rangle^\times$$

is a subgroup of $\mathrm{K}_1(\Lambda[[T]]) = \Lambda[[T]]^\times$ and Emerton and Kisin [6] show that $L(\mathcal{F}^\bullet, T)/L(\mathrm{R}\, f_! \mathcal{F}^\bullet, T) \in \Lambda\langle T\rangle^\times$. If Λ is not commutative, the canonical homomorphism

$$\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \to \mathrm{K}_1(\Lambda[[T]])$$

is no longer injective in general. Nevertheless, we shall prove:

Theorem 1.1 (see Thm. 5.1)**.** *There exists a unique way to associate to each separated $\mathbb{F}$-scheme $s\colon X \to \operatorname{Spec} \mathbb{F}$ of finite type, each adic $\mathbb{Z}_p$-algebra Λ, and each perfect complex of Λ-sheaves $\mathcal{F}^\bullet$ on X an element $Q(\mathcal{F}^\bullet, T) \in \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$ such that*

(1) *the image of $Q(\mathcal{F}^\bullet, T)$ in $\mathrm{K}_1(\Lambda[[T]])$ is the ratio*

$$L(\mathcal{F}^\bullet, T)/L(\mathrm{R}\, s_! \mathcal{F}^\bullet, T),$$

(2) *$Q(\mathcal{F}^\bullet, T)$ is multiplicative on exact sequences of perfect complexes and depends only on the quasi-isomorphism class of $\mathcal{F}^\bullet$,*

(3) *$Q(\mathcal{F}^\bullet, T)$ is compatible with changes of the ring Λ.*

Aside from the result of Emerton and Kisin, a central ingredient for the proof is the recent work of Chinburg, Pappas, and Taylor [2, 3] on the first K-group of p-adic group rings. In fact, the main strategy of the proof is to reduce the assertion first to the case $\Lambda = \mathbb{Z}_p[G]$ for a finite group G and then use the results of Chinburg, Pappas, and Taylor to reduce it further to the case already treated by Emerton and Kisin. In particular, we use almost exclusively methods from representation theory, whereas the result of Emerton and Kisin itself may be considered as the geometric input.

As an application, we deduce the following version of a noncommutative Iwasawa main conjecture for varieties over finite fields. Assume for the moment that X is geometrically connected and let G be a factor group of the fundamental group of X such that $G \cong H \rtimes \Gamma$ where H is a compact p-adic Lie group and

$$\Gamma = \operatorname{Gal}(\mathbb{F}_{q^{p^\infty}}/\mathbb{F}) \cong \mathbb{Z}_p.$$

We write

$$\mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G/U]$$

for the Iwasawa algebra of G. Let

$$S = \{f \in \mathbb{Z}_p[[G]] \colon \mathbb{Z}_p[[G]]/\mathbb{Z}_p[[G]]f \text{ is finitely generated over } \mathbb{Z}_p[[H]]\}$$

denote Venjakob's canonical Ore set and write $\mathbb{Z}_p[[G]]_S$ for the localisation of $\mathbb{Z}_p[[G]]$ at S. We turn $\mathbb{Z}_p[[G]]$ into a smooth $\mathbb{Z}_p[[G]]$-sheaf $\mathcal{M}(G)$ on X by letting the fundamental group $\pi^1_{\text{ét}}(X)$ of X act contragrediently on $\mathbb{Z}_p[[G]]$, i. e. $\sigma \in \pi^1_{\text{ét}}(X)$ acts on $f \in \mathbb{Z}_p[[G]]$ by $f[\sigma]^{-1}$, where $[\sigma]$ denotes the image of σ in G. Let $\mathrm{R}\,\Gamma_{\mathrm{c}}(X, \mathcal{F})$ denote the étale cohomology with proper support of a flat constructible $\mathbb{Z}_p$-sheaf $\mathcal{F}$ on X.

For every continuous $\mathbb{Z}_p$-representation ρ of G, there exists a homomorphism

$$\rho\colon\ \mathrm{K}_1(\mathbb{Z}_p[[G]]_S) \to Q(\mathbb{Z}_p[[\Gamma]])^\times$$

into the units $Q(\mathbb{Z}_p[[\Gamma]])^\times$ of the field of fractions of $\mathbb{Z}_p[[\Gamma]]$. It is induced by sending $g \in G$ to $\det([g]\rho(g)^{-1})$, with $[g]$ denoting the image of g in Γ (see [4] for the explicit construction, but note the difference in the sign convention). On the other hand, ρ gives rise to a flat and smooth $\mathbb{Z}_p$-sheaf $\mathcal{M}(\rho)$ on X.

Theorem 1.2.

(1) $\mathrm{R}\,\Gamma_{\mathrm{c}}(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_p} \mathcal{F})$ *is a perfect complex of* $\mathbb{Z}_p[[G]]$*-modules whose cohomology groups are* S*-torsion. In particular, it gives rise to a class*

$$[\mathrm{R}\,\Gamma_{\mathrm{c}}(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_p} \mathcal{F})]^{-1}$$

in the relative K*-group* $\mathrm{K}_0(\mathbb{Z}_p[[G]], \mathbb{Z}_p[[G]]_S)$.

(2) *There exists an element* $\widetilde{\mathcal{L}}_G(X/\mathbb{F}, \mathcal{F}) \in \mathrm{K}_1(\mathbb{Z}_p[[G]]_S)$ *with the following properties:*

(a) (Characteristic element) *The image of* $\widetilde{\mathcal{L}}_G(X/\mathbb{F}, \mathcal{M}(\rho) \otimes_{\mathbb{Z}_p} \mathcal{F})$ *under the boundary homomorphism*

$$d\colon\ \mathrm{K}_1(\mathbb{Z}_p[[G]]_S) \to \mathrm{K}_0(\mathbb{Z}_p[[G]], \mathbb{Z}_p[[G]]_S)$$

is

$$[\mathrm{R}\,\Gamma_{\mathrm{c}}(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_p} \mathcal{F})]^{-1}.$$

(b) (Interpolation with respect to all continuous representations) *Assume that* ρ *is a continuous* $\mathbb{Z}_p$*-representation of* G*. We let* γ *denote the image of the geometric Frobenius in* Γ*. Then*

$$\rho(\widetilde{\mathcal{L}}_G(X/\mathbb{F}, \mathcal{F})) = L(\mathcal{M}(\rho) \otimes_{\mathbb{Z}_p} \mathcal{F}, \gamma^{-1})$$

in $Q(\mathbb{Z}_p[[\Gamma]])^\times$.

This enhances the main result of [1], which also asserts the existence of $\widetilde{\mathcal{L}}_G(X/\mathbb{F},\mathcal{F})$, but requires only that it satisfies the interpolation property with respect to finite order representations. In fact, we will prove in Section 6 an even more general version of this theorem in the style of [17], replacing $\mathbb{Z}_p$ by arbitrary adic $\mathbb{Z}_p$-algebras and allowing schemes and coverings which are not necessarily connected.

We refer to [1] and [13] for applications of Thm. 1.2.

2. Preliminaries on completed K-Theory

For any topological ring R (associative and with unity) we let $\mathfrak{I}_R$ denote the lattice of all two-sided open ideals of R. For any $n \geq 0$ we call

$$\widehat{\mathrm{K}}_n(R) = \varprojlim_{I\in\mathfrak{I}_R} \mathrm{K}_n(R/I)$$

the n-th completed K-group of R. The group $\widehat{\mathrm{K}}_n(R)$ becomes a topological group by equipping each $\mathrm{K}_n(R/I)$ with the discrete topology.

Recall that we call R an adic ring if R is compact and the Jacobson radical $\mathrm{Jac}(R)$ is open and finitely generated, or equivalently,

$$R = \varprojlim_k R/\mathrm{Jac}(R)^k$$

with $R/\mathrm{Jac}(R)^k$ a finite ring. Fukaya and Kato showed that for adic rings the canonical homomorphism $\mathrm{K}_1(R) \to \widehat{\mathrm{K}}_1(R)$ is an isomorphism [7, Prop. 1.5.1]. The same is true if $R = A[G]$ with G a finite group and A a commutative, p-adically complete, Noetherian integral domain with fraction field of characteristic 0 [3, Thm. 1.2]. We can add the following rings to the list.

Proposition 2.1. *Let R be a commutative topological ring such that*

(1) *the topology of R is an ideal topology, i. e. $\mathfrak{I}_R$ is a basis of open neighbourhoods of* 0,
(2) $R = \varprojlim_{I\in\mathfrak{I}_R} R/I$,
(3) *the Jacobson radical* $\mathrm{Jac}(R)$ *is open.*

Then $\mathrm{K}_1(R) \to \widehat{\mathrm{K}}_1(R)$ *is an isomorphism.*

Proof. For any commutative ring A we have an exact sequence

$$1 \to 1 + \mathrm{Jac}(A) \to \mathrm{K}_1(A) \to \mathrm{K}_1(A/\mathrm{Jac}(A)) \to 1.$$

Since $R/\mathrm{Jac}(R)$ carries the discrete topology we have

$$\mathrm{K}_1(R/\mathrm{Jac}(R)) = \widehat{\mathrm{K}}_1(R/\mathrm{Jac}(R)).$$

Because of the completeness of R and because $\operatorname{Jac}(R)$ is open we have

$$1 + \operatorname{Jac}(R) = \varprojlim_{I \in \mathfrak{I}_R} 1 + \operatorname{Jac}(R)/\operatorname{Jac}(R) \cap I.$$

The claim now follows from the snake lemma. □

Let Λ be an adic ring. We are mainly interested in the topological rings

$$\Lambda\langle T\rangle = \varprojlim_{I \in \mathfrak{I}_\Lambda} \Lambda/I[T]$$

where we give the polynomial ring $\Lambda/I[T]$ the discrete topology. As every open two-sided ideal I of Λ is finitely generated as left or right ideal, we note that $I\Lambda\langle T\rangle = \ker(\Lambda\langle T\rangle \to \Lambda/I[T])$ and $\operatorname{Jac}(\Lambda\langle T\rangle) = \operatorname{Jac}(\Lambda)\Lambda\langle T\rangle$. In particular, the open ideals of $\Lambda\langle T\rangle$ are again finitely generated.

We suspect that for all these rings $\mathrm{K}_1(\Lambda\langle T\rangle)$ and $\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$ agree, but we were not able to prove this in general. By the above results they do agree if either Λ is commutative or $\Lambda = \mathbb{Z}_p[G]$ for a finite group G and this is all we need for our purposes. The following result is therefore only for the reader's edification.

Proposition 2.2. *Let Λ be an adic ring. Then the homomorphisms*

$$\Lambda\langle T\rangle^\times \to \mathrm{K}_1(\Lambda\langle T\rangle) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$$

are surjective.

Proof. We note that $\Lambda/\operatorname{Jac}(\Lambda)$ is a finite product of finite-dimensional matrix rings over finite fields. In particular,

$$\mathrm{K}_n(\Lambda/\operatorname{Jac}(\Lambda)[T]) = \mathrm{K}_n(\Lambda/\operatorname{Jac}(\Lambda))$$

for all n by a celebrated result of Quillen. Since $\Lambda/\operatorname{Jac}(\Lambda)$ is semi-simple, the homomorphism $\Lambda/\operatorname{Jac}(\Lambda)^\times \to \mathrm{K}_1(\Lambda/\operatorname{Jac}(\Lambda))$ is surjective. Moreover, $\mathrm{K}_2(\Lambda/\operatorname{Jac}(\Lambda)) = 0$ since this is true for all finite fields.

By a result of Vaserstein (see [9, Thm. 1.5]) we have for any ring R and any two-sided ideal $I \subset \operatorname{Jac}(R)$ an exact sequence

$$1 \to V(R, I) \to 1 + I \to \mathrm{K}_1(R, I) \to 1$$

with $V(R, I)$ the subgroup of $1 + I$ generated by the elements $(1 + ri)(1 + ir)^{-1}$ with $r \in R$ and $i \in I$. Choosing $R = \Lambda\langle T\rangle$ and $I = \operatorname{Jac}(\Lambda)\Lambda\langle T\rangle$ we conclude that $\Lambda\langle T\rangle^\times \to \mathrm{K}_1(\Lambda\langle T\rangle)$ is surjective.

Since the homomorphisms

$$V(\Lambda\langle T\rangle, \operatorname{Jac}(\Lambda\langle T\rangle)) \to V(\Lambda/I[T], \operatorname{Jac}(\Lambda/I[T]))$$

are surjective for any open two-sided ideal $I \subset \operatorname{Jac}(\Lambda)$ of Λ, we conclude that

$$\mathrm{R}^1 \varprojlim_k V(\Lambda/\operatorname{Jac}(\Lambda)^k[T], \operatorname{Jac}(\Lambda)\Lambda/\operatorname{Jac}(\Lambda)^k[T]) = 0.$$

Hence,

$$\mathrm{K}_1(\Lambda\langle T\rangle, \mathrm{Jac}(\Lambda\langle T\rangle)) \to \varprojlim_k \mathrm{K}_1(\Lambda/\mathrm{Jac}(\Lambda)^k[T], \mathrm{Jac}(\Lambda)\Lambda/\mathrm{Jac}(\Lambda)^k[T])$$

is surjective. Passing to the limit over the exact sequences

$$1 \to \mathrm{K}_1(\Lambda/\mathrm{Jac}(\Lambda)^k[T], \mathrm{Jac}(\Lambda)\Lambda/\mathrm{Jac}(\Lambda)^k[T]) \\ \to \mathrm{K}_1(\Lambda/\mathrm{Jac}(\Lambda)^k[T]) \to \mathrm{K}_1(\Lambda/\mathrm{Jac}(\Lambda)[T]) \to 1$$

and comparing it to the corresponding sequence for $\Lambda\langle T\rangle$ we conclude that

$$\mathrm{K}_1(\Lambda\langle T\rangle) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$$

is surjective. □

If Λ is an adic ring and P a finitely generated, projective left Λ-module, we set

$$P[T] = \Lambda[T] \otimes_\Lambda P, \quad P\langle T\rangle = \Lambda\langle T\rangle \otimes_\Lambda P, \quad P[[T]] = \Lambda[[T]] \otimes_\Lambda P$$

Note that

$$P\langle T\rangle = \varprojlim_k P/\mathrm{Jac}(\Lambda)^k P[T], \qquad P[[T]] = \varprojlim_k P/\mathrm{Jac}(\Lambda)^k P[[T]],$$

and that $\Lambda[[T]]$ is again an adic ring.

If Λ' is another adic ring acting on P from the right such that P becomes a Λ-Λ'-bimodule, then $P/\mathrm{Jac}(\Lambda)^k P[T]$ is annihilated by some power $\mathrm{Jac}(\Lambda')^{m(k)}$ of the Jacobson radical of Λ'. This shows that $P\langle T\rangle$ is a $\Lambda\langle T\rangle$-$\Lambda'\langle T\rangle$-bimodule and therefore induces homomorphisms

$$\Psi_{P\langle T\rangle}\colon \mathrm{K}_n(\Lambda'\langle T\rangle) \to \mathrm{K}_n(\Lambda\langle T\rangle).$$

At the same time, this shows that the system $(P/\mathrm{Jac}(\Lambda)^k P[T])_{k\geq 1}$ of $\Lambda/\mathrm{Jac}(\Lambda)^k$-$\Lambda'/\mathrm{Jac}(\Lambda')^{m(k)}$-bimodules induces homomorphisms

$$\Psi_{P\langle T\rangle}\colon \widehat{\mathrm{K}}_n(\Lambda'\langle T\rangle) \to \widehat{\mathrm{K}}_n(\Lambda\langle T\rangle),$$

which are compatible with the above homomorphisms. The construction of $\Psi_{P\langle T\rangle}$ extends in the obvious manner to complexes $P^\bullet$ of Λ-Λ'-bimodules which are strictly perfect as complexes of Λ-modules. By a similar reasoning we also obtain change-of-ring homomorphisms

$$\Psi_{P[[T]]^\bullet}\colon \mathrm{K}_n(\Lambda'[[T]]) \to \mathrm{K}_n(\Lambda[[T]]),$$

as well as the corresponding versions for the completed K-theory.

3. On $\mathbf{K}_1(\mathbb{Z}_p[G]\langle T\rangle)$

In this section, we use the results of Chinburg, Pappas, and Taylor [2, 3] to analyse $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle)$ for a finite group G.

For a noetherian integral domain of finite Krull dimension R with field of fractions $Q(R)$ of characteristic 0 we set

$$\mathrm{SK}_1(R[G]) = \ker(\mathrm{K}_1(R[G]) \to \mathrm{K}_1(\overline{Q(R)}[G])),$$
$$\mathrm{Det}(R[G]^\times) = \mathrm{im}(R[G]^\times \to \mathrm{K}_1(R[G])/\,\mathrm{SK}_1(R[G]).$$

Here, $\overline{Q(R)}$ denotes a fixed algebraic closure of $Q(R)$. For any subgroup U of $R[G]^\times$ we write $\mathrm{Det}(U)$ for its image in $\mathrm{Det}(R[G]^\times)$. We are mainly interested in the cases $R = \mathcal{O}_K\langle T\rangle$ or $R = \mathcal{O}_K[[T]]$ with $\mathcal{O}_K$ the valuation ring of a finite extension $K/\mathbb{Q}_p$.

Assume that L/K is a finite extension such that L is a splitting field for G, i. e. $L[G]$ is a finite product of matrix algebras over L. Let $\mathcal{M}$ be a maximal $\mathbb{Z}_p$-order inside $L[G]$ containing $\mathcal{O}_L[G]$. Then $\mathcal{M}$ is a finite product of matrix algebras over $\mathcal{O}_L$ such that

$$\mathrm{K}_1(\mathcal{M}\langle T\rangle) \to \mathrm{K}_1(Q(\mathcal{O}_K\langle T\rangle)[G])$$

is injective. In particular, we have

$$\mathrm{SK}_1(\mathcal{O}_K[G]\langle T\rangle) = \ker(\mathrm{K}_1(\mathcal{O}_K[G]\langle T\rangle) \to \mathrm{K}_1(\mathcal{M}\langle T\rangle)).$$

The same reasoning also applies to $\mathrm{SK}_1(\mathcal{O}_K[G][[T]])$. Moreover, note that the group $\mathrm{K}_1(\mathcal{M}\langle T\rangle)$ injects into $\mathrm{K}_1(\mathcal{M}[[T]])$, such that we also have an injection $\mathrm{Det}(\mathcal{O}_K[G]\langle T\rangle^\times) \subset \mathrm{Det}(\mathcal{O}_K[G][[T]]^\times)$.

Lemma 3.1. *For any finite group G and any finite field extension $K/\mathbb{Q}_p$, the inclusion $\mathcal{O}_K[G] \to \mathcal{O}_K[G][[T]]$ induces an isomorphism*

$$\mathrm{SK}_1(\mathcal{O}_K[G]) \cong \mathrm{SK}_1(\mathcal{O}_K[G][[T]]).$$

In particular,

$$\mathrm{K}_1(\mathcal{O}_K[G][[T]], T\mathcal{O}_K[G][[T]]) = \mathrm{Det}(1 + T\mathcal{O}_K[G][[T]]).$$

Proof. The first equality is proved in [16, Prop. 5.4]. Since $T \in \mathrm{Jac}(\mathcal{O}_K[G][[T]])$, the map

$$1 + T\mathcal{O}_K[G][[T]] \to \mathrm{K}_1(\mathcal{O}_K[G][[T]], T\mathcal{O}_K[G][[T]])$$

is surjective by the result of Vaserstein [9, Thm. 1.5] that we already used in the proof of Prop. 2.2. The decomposition

$$\mathrm{K}_1(\mathcal{O}_K[G][[T]]) = \mathrm{K}_1(\mathcal{O}_K[G]) \times \mathrm{K}_1(\mathcal{O}_K[G][[T]], T\mathcal{O}_K[G][[T]])$$

induced by the inclusion $\mathcal{O}_K[G] \to \mathcal{O}_K[G][[T]]$ and the evaluation $T \mapsto 0$ shows that

$$\mathrm{SK}_1(\mathcal{O}_K[G][[T]]) \cap \mathrm{K}_1(\mathcal{O}_K[G][[T]], T\mathcal{O}_K[G][[T]]) = 1. \qquad \square$$

Lemma 3.2. *Let $L/K/\mathbb{Q}_p$ be finite extensions such that L is a splitting field for G and let $\mathcal{M}$ be a maximal $\mathbb{Z}_p$-order inside $L[G]$ containing $\mathcal{O}_L[G]$. Assume that f is in the intersection of* $\mathrm{K}_1(\mathcal{M}\langle T\rangle, \mathrm{Jac}(\mathcal{M}\langle T\rangle))$ *and* $\mathrm{Det}(1+\mathrm{Jac}(\mathcal{O}_K[G])\mathcal{O}_K[G][[T]])$ *inside* $\mathrm{K}_1(\mathcal{M}[[T]])$*. Then there exists $n \geq 0$ such that $f^{p^n} \in \mathrm{Det}(\mathcal{O}_K[G]\langle T\rangle)$.*

Proof. Let p^k be the order of a p-Sylow subgroup of G. Since $\mathcal{M}/p^{k+2}\mathcal{M}$ is a finite ring, some power of $\mathrm{Jac}(\mathcal{M})$ is contained in $p^{k+2}\mathcal{M}$. Now $p^k\mathcal{M} \subset \mathcal{O}_L[G]$ [9, Thm. 1.4]. For large n we thence have

$$f^{p^n} \in \mathrm{Det}(1 + p^2\mathcal{O}_L[G]\langle T\rangle) \cap \mathrm{Det}(1 + p^2\mathcal{O}_K[G][[T]]).$$

By [2, Prop. 2.4] the p-adic logarithm induces R-linear isomorphisms

$$v\colon\ \mathrm{Det}(1 + p^2 R[G]) \to p^2 R[C_G]$$

where C_G is the set of conjugacy classes of G and R is equal to either $\mathcal{O}_K\langle T\rangle$ or $\mathcal{O}_K[[T]]$ for any K. In particular,

$$v(f^{p^n}) \in p^2\mathcal{O}_L[C_G]\langle T\rangle \cap p^2\mathcal{O}_K[[T]][C_G] = p^2\mathcal{O}_K\langle T\rangle[C_G].$$

Hence, $f^{p^n} \in \mathrm{Det}(1 + p^2\mathcal{O}_K[G]\langle T\rangle)$. □

For any finite group G we let $[G,G]$ denote the commutator subgroup of G and set $G^{\mathrm{ab}} = G/[G,G]$.

Lemma 3.3. *Let G be a finite p-group and $K/\mathbb{Q}_p$ a finite unramified extension. Let further A denote the kernel of $\mathcal{O}_K[G] \to \mathcal{O}_K[G^{\mathrm{ab}}]$. The abelian group*

$$\mathrm{Det}(1 + A\mathcal{O}_K[G][[T]])/\,\mathrm{Det}(1 + A\mathcal{O}_K[G]\langle T\rangle)$$

is torsionfree.

Proof. Let R be equal to either $\mathcal{O}_K\langle T\rangle$ or $\mathcal{O}_K[[T]]$. Write C_G for set of the conjugacy classes of G and let $\phi(AR[G])$ denote the kernel of the natural R-linear map $R[C_G] \to R[G^{\mathrm{ab}}]$. Note that $\phi(AR[G])$ is finitely generated and free as an R-module. For any choice of Frobenius lifts compatible with the inclusion $\mathcal{O}_K[G]\langle T\rangle \subset \mathcal{O}_K[[T]]$ we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Det}(1 + A\mathcal{O}_K[G]\langle T\rangle) & \longrightarrow & \mathrm{Det}(1 + A\mathcal{O}_K[G][[T]]) \\ \nu \downarrow \cong & & \nu \downarrow \cong \\ p\phi(A\mathcal{O}_K[G]\langle T\rangle) & \longrightarrow & p\phi(A\mathcal{O}_K[G][[T]]) \end{array}$$

with the horizontal arrows induced by the natural inclusion and the vertical isomorphisms induced by the integral group logarithm [2, Thm. 3.16]. Since $\mathcal{O}_K[[T]]/\mathcal{O}_K\langle T\rangle$ is a torsionfree abelian group, the same is true for the group $p\phi(A\mathcal{O}_K[G][[T]])/p\phi(A\mathcal{O}_K[G]\langle T\rangle)$. □

Recall that a semi-direct product $G = \mathbb{Z}/s\mathbb{Z} \rtimes P$ with s prime to p is called p-$\mathbb{Q}_p$-elementary if P is a p-group and the image of $t\colon P \to (\mathbb{Z}/s\mathbb{Z})^\times$ given by the action of P on $\mathbb{Z}/s\mathbb{Z}$ lies in $\mathrm{Gal}(\mathbb{Q}_p(\zeta_s)/\mathbb{Q}_p) \subset (\mathbb{Z}/s\mathbb{Z})^\times$. For any divisor m of s, and $R = \mathbb{Z}_p$, $R = \mathbb{Z}_p\langle T\rangle$, or $R = \mathbb{Z}_p[[T]]$ we set

$$R[m] = \mathbb{Z}[\zeta_m] \otimes_{\mathbb{Z}} R$$

and let $R[m][P;t]$ denote the twisted group ring for t, i. e. $\sigma r = t(\sigma)(r)\sigma$ for elements $r \in R[m]$, $\sigma \in P$. Set

$$H_m = \ker(t\colon P \to \mathrm{Gal}(\mathbb{Q}_p(\zeta_m)/\mathbb{Q}_p)), \qquad B_m = P/H_m.$$

We may write

$$R[G] = \prod_{m|s} R[m][P;t]$$

[9, Prop. 11.6]. We then see that $R[G]$ is a finitely generated, projective module over the subring

$$\prod_{m|s} R[m][H_m] \subset \prod_{m|s} R[m][P;t].$$

We let

$$r\colon \mathrm{Det}(R[G]^\times) \to \prod_{m|s} \mathrm{Det}(R[m][H_m]^\times)$$

denote the corresponding restriction map. We further set

$$A = \ker(\mathbb{Z}_p[G] \to \prod_{m|s} \mathbb{Z}_p[m][P/[H_m,H_m];t]),$$

$$A_m = \ker(\mathbb{Z}_p[m][H_m] \to \mathbb{Z}_p[m][H_m^{\mathrm{ab}}]),$$

and let b_m denote the order of B_m.

Lemma 3.4. *With the notation as above,*

(1) $R[m][P/[H_m,H_m];t]$ *is isomorphic to the ring of* $b_m \times b_m$ *matrices over its centre* $R[m][H_m^{\mathrm{ab}}]^{B_m}$*,*

(2) r *induces an isomorphism*

$$r\colon \mathrm{Det}(1 + AR[G]) \to \prod_{m|s} \mathrm{Det}(1 + A_m R[m][H_m])^{B_m}.$$

(3) $\mathrm{Det}(1 + A\mathbb{Z}_p[G][[T]])/\,\mathrm{Det}(1 + A\mathbb{Z}_p[G]\langle T\rangle)$ *is a torsionfree abelian group.*

Proof. Assertion (1) is a theorem of Wall [10, Thm. 8.3]. Assertion (2) follows from [2, Thm 6.2 and Diagram (6.7)]. Assertion (3) is then a consequence of Lemma 3.3 and (2). □

We now return to the case that G is an arbitrary finite group. For $R = \mathbb{Z}_p$, $R = \mathbb{Z}_p\langle T\rangle$, and $R = \mathbb{Z}_p[[T]]$ and any subgroup $H \subset G$ we write Res^G_H for the change-of-ring homomorphism $\mathrm{K}_1(R[G]) \to \mathrm{K}_1(R[H])$ induced by the $\mathbb{Z}_p[H]$-$\mathbb{Z}_p[G]$-bimodule $\mathbb{Z}_p[G]$.

Lemma 3.5. *Let $f \in \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$ such that*

1. *for any p-$\mathbb{Q}_p$-elementary group H, $\mathrm{Res}^G_H f$ is in the image of the homomorphism $\mathrm{K}_1(\mathbb{Z}_p[H]\langle T\rangle) \to \mathrm{K}_1(\mathbb{Z}_p[H][[T]])$,*
2. *f^{p^n} is in the image of $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle) \to \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$ for some $n \geq 0$.*

Then f is in the image of $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle) \to \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$.

Proof. The homomorphisms $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle) \to \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$ for each finite group G constitute a homomorphism of Green modules over the Green ring $G \mapsto \mathrm{G}_0(\mathbb{Z}_p[G])$ [3, §4.3]. Hence, [3, Thm. 4.3] implies that f^ℓ is in the image of $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle) \to \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$ for some integer ℓ prime to p. Because of (2) we may choose $\ell = 1$. □

Finally, we need the following vanishing result for SK_1, which is a variant of [7, Prop. 2.3.7].

Proposition 3.6. *Let $K/\mathbb{Q}_p$ be unramified and $R = \mathcal{O}_K\langle T\rangle$ or $R = \mathcal{O}_K[[T']]\langle T\rangle$ for some indeterminate T'. (More generally, R can be any ring satisfying the standing hypothesis of* [3]*.) Let further $\mathcal{G}$ be a profinite group with cohomological p-dimension $\mathrm{cd}_p\,\mathcal{G} \leq 1$. For any open normal subgroup $U \subset \mathcal{G}$ there exists an open subgroup $V \subset U$ normal in $\mathcal{G}$ such that the natural homomorphism*

$$\mathrm{SK}_1(R[\mathcal{G}/V]) \to \mathrm{SK}_1(R[\mathcal{G}/U])$$

is the zero map.

Proof. For any finite group G, let G_r denote the set of p-regular elements, i. e. those elements in G of order prime to p. The group G acts on G_r via conjugation. Write $\mathbb{Z}_p[G_r]$ for the free $\mathbb{Z}_p$-module generated by G_r. By [3, Thm. 1.7] there exists a natural surjection

$$R \otimes_{\mathbb{Z}_p} \mathrm{H}_2(G, \mathbb{Z}_p[G_r]) \to \mathrm{SK}_1(R[G]).$$

Since $\mathrm{H}_2(G, \mathbb{Z}_p[G_r])$ is finite for all finite groups G, it suffices to show that

$$\varprojlim_{U \subset \mathcal{G}} \mathrm{H}_2(\mathcal{G}/U, \mathbb{Z}_p[(\mathcal{G}/U)_r]) = 0,$$

where the limit extends over all open normal subgroups of $\mathcal{G}$. After taking the Pontryagin dual we deduce the latter from

$$\varinjlim_{U \subset \mathcal{G}} \mathrm{H}^2(\mathcal{G}, \mathrm{Map}((\mathcal{G}/U)_r, \mathbb{Q}_p/\mathbb{Z}_p)) = 0.$$

□

4. *L*-functions of perfect complexes of adic sheaves

We will briefly recall some notation from [16] (see also [14] and [17]). Let $\mathbb{F}$ be the finite field with $q = p^v$ elements and fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. For any scheme X in the category $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ of separated schemes of finite type over $\mathbb{F}$ and any adic ring Λ we introduced a Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ of perfect complexes of adic sheaves on X [16, Def. 4.3]. The objects of this category are certain inverse systems over the index set $\mathfrak{I}_\Lambda$ such that for $I \in \mathfrak{I}_\Lambda$ the I-th level is a perfect complex of étale sheaves of Λ/I-modules.

Let us write $\mathrm{K}_0(X, \Lambda)$ for the zeroth Waldhausen K-group of the Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$, i. e. the abelian group generated by quasi-isomorphism classes of objects in the category $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ modulo the relations

$$[\mathcal{G}^\bullet][\mathcal{F}^\bullet]^{-1}[\mathcal{H}^\bullet]^{-1}$$

for any sequence

$$0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$$

in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ which is exact (in each level $I \in \mathfrak{I}_\Lambda$).

For any morphism $f\colon X \to Y$ in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ we have Waldhausen exact functors

$$f^*\colon \mathbf{PDG}^{\mathrm{cont}}(Y, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda),$$
$$\mathrm{R}\, f_!\colon \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \to \mathbf{PDG}^{\mathrm{cont}}(Y, \Lambda)$$

that correspond to the usual inverse image and the direct image with proper support. As Waldhausen exact functors they induce homomorphisms

$$f^*\colon \mathrm{K}_0(Y, \Lambda) \to \mathrm{K}_0(X, \Lambda), \qquad \mathrm{R}\, f_!\colon \mathrm{K}_0(X, \Lambda) \to \mathrm{K}_0(Y, \Lambda).$$

If Λ' is a second adic ring we let Λ^{op}-$\mathbf{SP}(\Lambda')$ denote the Waldhausen category of complexes of Λ'-Λ-bimodules which are strictly perfect as complexes of Λ'-modules. For each such complex $P^\bullet$ we have a change-of-ring homomorphism

$$\Psi_{P^\bullet}\colon \mathrm{K}_0(X, \Lambda) \to \mathrm{K}_0(X, \Lambda').$$

The compositions of these homomorphisms behave as expected. In particular, $\Psi_{P^\bullet}$ commutes with f^* and $\mathrm{R}\, f_!$, for $f\colon X \to Y$, $g\colon Y \to Z$ we have $(g \circ f)^* = f^* \circ g^*$, $\mathrm{R}(f \circ g)_! = \mathrm{R}\, f_! \circ \mathrm{R}\, g_!$, and $\mathrm{R}\, f'_! g'^* = g^* \,\mathrm{R}\, f_!$ if

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

is a cartesian square [16, §4].

For any $A \in \mathrm{K}_0(X,\Lambda)$ we define the L-function $L(A,T) \in \mathrm{K}_1(\Lambda[[T]])$ as follows. First, assume that $X = \operatorname{Spec}\mathbb{F}'$ for a finite field extension $\mathbb{F}'/\mathbb{F}$ of degree d, that Λ is finite and that A is the class of a locally constant flat étale sheaf of Λ-modules on X. This sheaf corresponds to a finitely generated, projective Λ-module P with a continuous action of the absolute Galois group of $\mathbb{F}'$, which is topologically generated by the geometric Frobenius automorphism $\mathfrak{F}_{\mathbb{F}'}$ of $\mathbb{F}'$. We then let $L(A,T)$ be the inverse of the class of the automorphism $\mathrm{id} - \mathfrak{F}_{\mathbb{F}'}T^d$ of $\Lambda[[T]] \otimes_\Lambda P$ in $\mathrm{K}_1(\Lambda[[T]])$. If A is the class of any perfect complex of étale sheaves of Λ-modules on $\operatorname{Spec}\mathbb{F}'$, we replace it by a quasi-isomorphic, strictly perfect complex $\mathcal{P}^\bullet$ and define $L(A,T)$ as the alternating product

$$L(A,T) = \prod_k L(\mathcal{P}^k, T)^{(-1)^k}.$$

This then extends to a group homomorphism $\mathrm{K}_0(\operatorname{Spec}\mathbb{F}',\Lambda) \to \mathrm{K}_1(\Lambda[[T]])$. If Λ is an arbitrary adic ring and $A \in \mathrm{K}_0(\operatorname{Spec}\mathbb{F}',\Lambda)$, then $L(A,T)$ is given by the system $(L(\Psi_{A/I}(A),T))_{I\in\mathfrak{I}_\Lambda}$ in

$$\mathrm{K}_1(\Lambda[[T]]) = \varprojlim_{I\in\mathfrak{I}_\Lambda} \mathrm{K}_1(\Lambda/I[[T]]).$$

Finally, if X is any separated scheme over $\mathbb{F}$, we let X^0 denote the set of closed points of X and

$$x\colon \operatorname{Spec} k(x) \to X$$

the closed immersion corresponding to any $x \in X^0$. We set

$$L(A,T) = \prod_{x\in X^0} L(x^*(A),T) \in \mathrm{K}_1(\Lambda[[T]]).$$

The product converges in the topology of $\mathrm{K}_1(\Lambda[[T]])$ induced by the adic topology of $\Lambda[[T]]$ because $T \in \operatorname{Jac}(\Lambda[[T]])$ and for any d there are only finitely many closed points of degree d in X. We have thus constructed a group homomorphism

$$L\colon \mathrm{K}_0(X,\Lambda) \to \mathrm{K}_1(\Lambda[[T]]), \qquad A \mapsto L(A,T),$$

for any X in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ and any adic ring Λ.

This construction agrees with [16, Def. 6.4]. If Λ is commutative, such that $\mathrm{K}_1(\Lambda[[T]]) \cong \Lambda[[T]]^\times$ via the determinant map, it is also seen to agree with the classical definition used in [6]. Moreover, we note that for any pair of adic rings Λ and Λ' and any complex $P^\bullet$ of bimodules in Λ^{op}-$\mathbf{SP}(\Lambda')$, we have

$$L(\Psi_{P^\bullet}(A),T) = \Psi_{P[[T]]^\bullet}(L(A,T))$$

for $A \in \mathrm{K}_1(X,\Lambda)$.

Remark 4.1. Note that $L(A,T)$ depends on the base field $\mathbb{F}$. If $\mathbb{F}' \subset \mathbb{F}$ is a subfield, then the L-function of A with respect to $\mathbb{F}'$ is $L(A, T^{[\mathbb{F}:\mathbb{F}']})$ [16, Rem. 6.5].

5. The construction of unit *L*-functions

In this section, we prove the following theorem.

Theorem 5.1. *Let $s\colon X \to \operatorname{Spec} \mathbb{F}$ be a separated $\mathbb{F}$-scheme of finite type. There exists a unique way to associate to each adic $\mathbb{Z}_p$-algebra Λ a homomorphism*

$$Q\colon \mathrm{K}_0(X,\Lambda) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle), \qquad A \mapsto Q(A,T),$$

such that for $A \in \mathrm{K}_0(X,\Lambda)$

(1) *the image of $Q(A,T)$ in $\mathrm{K}_1(\Lambda[[T]])$ is the ratio $L(A,T)/L(\mathrm{R}\, s_! A, T)$,*
(2) *if Λ' is a second adic ring and $P^\bullet$ is in Λ'^{op}-$\mathbf{SP}(\Lambda)$, then*

$$\Psi_{P\langle T\rangle^\bullet}(Q(A,T)) = Q(\Psi_{P^\bullet}(A), T).$$

Proof. If we restrict to the class of adic rings Λ which are full matrix algebras over commutative adic $\mathbb{Z}_p$-algebras, then the natural homomorphism $\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \to \mathrm{K}_1(\Lambda[[T]])$ is injective and the existence of $Q\colon \mathrm{K}_0(X,\Lambda) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$ follows from [6, Cor. 1.8] and Morita invariance. In fact, we even know that the image of Q lies in the subgroup $\mathrm{K}_1(\Lambda\langle T\rangle, \mathrm{Jac}(\Lambda)T\Lambda\langle T\rangle)$. (The result of Emerton and Kisin is stated only for $\mathbb{F} = \mathbb{F}_p$, but the result for general $\mathbb{F}$ follows from Remark 4.1 and the simple observation that $\lambda(T) \in \Lambda[[T]]$ is in $\Lambda\langle T\rangle$ if and only if for some $n > 0$ $\lambda(T^n) \in \Lambda\langle T\rangle$.) We note further that it suffices to prove the assertion of the theorem for the class of finite $\mathbb{Z}_p$-algebras. The general case then follows by taking projective limits.

We proceed by induction on $\dim X$, starting with the empty scheme $\emptyset$ of dimension -1. Since $K_0(\emptyset, \Lambda) = 1$, there is nothing to prove in this case. So, we assume that the theorem has already been proved for all schemes in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ of dimension less than $\dim X$. Note that any open subscheme $j\colon U \to X$ with closed complement $i\colon Z \to X$ induces a decomposition

$$\mathrm{K}_0(U,\Lambda) \times \mathrm{K}_0(Z,\Lambda) \xrightarrow{\cong} \mathrm{K}_0(X,\Lambda), \qquad (A,B) \mapsto (\mathrm{R}\, j_! A)(\mathrm{R}\, i_! B)$$

and if the theorem is true for U and Z, then it is also true for X. In particular, we may reduce to the case that X is an integral scheme (if $\dim X = 0$, this means X is a point).

If Λ is finite then each object $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ is quasi-isomorphic to a strictly perfect complex $\mathcal{P}^\bullet$ of étale sheaves of Λ-modules on X. This means, $\mathcal{P}^n$ is flat and constructible and for $|n|$ sufficiently large, $\mathcal{P}^n = 0$. In particular, we may choose $j\colon U \to X$ open and dense such that $j^*\mathcal{P}^n$ is locally constant for each n (if $\dim X = 0$, this means $U = X$). We may then find a finite connected Galois covering $g\colon V \to U$ with Galois group G and a complex $P^\bullet$ in $\mathbb{Z}_p[G]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda)$ such that

$$j^*\mathcal{P}^\bullet \cong \Psi_{P^\bullet}(g_!g^*\mathbb{Z}_p)$$

[16, Lemma 4.12]. Here, we consider $g_!g^*\mathbb{Z}_p$ as an object in the category $\mathbf{PDG}^{\mathrm{cont}}(U, \mathbb{Z}_p[G])$. Since the function field of X is of characteristic p, the cohomological p-dimension of its absolute Galois group is 1. So, we may apply Prop. 3.6 and choose G (after possibly shrinking U) large enough such that

$$\Psi_{P\langle T\rangle^\bullet}\colon \mathrm{K}_0(\mathbb{Z}_p[G]\langle T\rangle) \to \mathrm{K}_0(\Lambda\langle T\rangle)$$

factors through $\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times)$.

Let $s\colon U \to \operatorname{Spec}\mathbb{F}$ be the structure map. We will now show that

$$\alpha = L(g_!g^*\mathbb{Z}_p, T)/L(\mathrm{R}\, s_!(g_!g^*\mathbb{Z}_p), T) \in \mathrm{K}_1(\mathbb{Z}_p[G][[T]])$$

has a preimage in $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle)$. First, we note that by construction, α is in the subgroup $\mathrm{K}_1(\mathbb{Z}_p[G][[T]], T\mathbb{Z}_p[G][[T]])$. Further, by [5, Fonction L mod ℓ^n, Theorem 2.2.(b)], we know that the image of α in the group $\mathrm{K}_1(\mathbb{Z}_p[G]/\operatorname{Jac}(\mathbb{Z}_p[G])[[T]])$ vanishes. Hence, using Lemma 3.1, we may view α as an element of the group $\mathrm{Det}(1 + \operatorname{Jac}(\mathbb{Z}_p[G])T\mathbb{Z}_p[G][[T]])$. From [6, Cor. 1.8] and Lemma 3.2 we conclude that α^{p^n} is in $\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times)$ for some $n \geq 0$. By Lemma 3.5 it thence suffices to show that for every $\mathbb{Q}_p$-p-elementary subgroup $H \subset G$, the element

$$\operatorname{Res}^H_G \alpha = L(\operatorname{Res}^H_G g_!g^*\mathbb{Z}_p, T)/L(\mathrm{R}\, s_! \operatorname{Res}^H_G g_!g^*\mathbb{Z}_p, T) \in \mathrm{K}_1(\mathbb{Z}_p[H][[T]])$$

has a preimage in $\mathrm{K}_1(\mathbb{Z}_p[H]\langle T\rangle)$. Choosing the ideal $A \subset \mathbb{Z}_p[H]$ as in Lemma 3.4 and noting that by part (1) of this lemma, $\mathbb{Z}_p[H]/A$ is a finite product of full matrix rings over commutative adic rings, we can again apply [6, Cor. 1.8] to see that the image of $\operatorname{Res}^H_G \alpha$ in

$$\begin{aligned}\mathrm{K}_1(\mathbb{Z}_p[H]/A[[T]])/\,\mathrm{K}_1(\mathbb{Z}_p[H]/A\langle T\rangle)\\ = \mathrm{Det}(\mathbb{Z}_p[H]/A[[T]]^\times)/\,\mathrm{Det}(\mathbb{Z}_p[H]/A\langle T\rangle^\times)\end{aligned}$$

vanishes. From the exact sequence

$$\begin{aligned}&\mathrm{Det}(1 + A\mathbb{Z}_p[H][[T]])/\,\mathrm{Det}(1 + A\mathbb{Z}_p[H]\langle T\rangle)\\ &\quad\to \mathrm{Det}(\mathbb{Z}_p[H][[T]]^\times)/\,\mathrm{Det}(\mathbb{Z}_p[H]\langle T\rangle^\times)\\ &\quad\quad\to \mathrm{Det}(\mathbb{Z}_p[H]/A[[T]]^\times)/\,\mathrm{Det}(\mathbb{Z}_p[H]/A\langle T\rangle^\times) \to 0\end{aligned}$$

and part (3) of Lemma 3.4 we conclude that $\operatorname{Res}^H_G \alpha$ does indeed have a preimage in $\mathrm{K}_1(\mathbb{Z}_p[H]\langle T\rangle)$. Thus, the element α has a preimage $\widetilde{\alpha}$ in $\mathrm{K}_1(\mathbb{Z}_p[G]\langle T\rangle)$.

We return to the complex $\mathcal{F}^\bullet$ and define

$$Q(\mathcal{F}^\bullet, T) = Q(i^*\mathcal{F}^\bullet, T) \cdot \Psi_{P\langle T\rangle^\bullet}(\widetilde{\alpha}).$$

We need to check that this definition does not depend on the choices of U, V, $P^\bullet$, and $\widetilde{\alpha}$. For this, let $j'\colon U' \to U$ be an open dense subscheme with closed complement $i'\colon Z' \to U$. Then $g\colon U' \times_U V \to U'$ is still a finite connected Galois covering with Galois group G. Let $h\colon V' \to U' \times_U V$ be a finite connected Galois covering with Galois group H and assume that $g' = h \circ g$ is Galois with Galois group G' such that $G'/H = G$. Assume further that $P'^\bullet$ is a complex in $\mathbb{Z}_p[G']^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda)$ such that $\Psi_{P'^\bullet}(g'_! g'^* \mathbb{Z}_p)$ is quasi-isomorphic to $(j' \circ j)^* \mathcal{F}^\bullet$. By taking the stalks of $\Psi_{P'^\bullet}(g'_! g'^* \mathbb{Z}_p)$ and $\Psi_{P^\bullet}(g_! g^* \mathbb{Z}_p)$ at any geometric point of U', we see that $P'^\bullet \otimes_{\mathbb{Z}_p[G']} \mathbb{Z}_p[G]$ is quasi-isomorphic to $P^\bullet$ in $\mathbb{Z}_p[G]^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda)$. In particular,

$$\Psi_{P'\langle T\rangle^\bullet} = \Psi_{P\langle T\rangle^\bullet} \circ \Psi_{\mathbb{Z}_p[G]\langle T\rangle}\colon \mathrm{K}_1(\mathbb{Z}_p[G']\langle T\rangle) \to \mathrm{K}_1(\Lambda\langle T\rangle).$$

As above, we construct a preimage $\widetilde{\alpha}'$ of

$$L(g'_! g'^* \mathbb{Z}_p, T)/L(\mathrm{R}(s \circ j')_!(g'_! g'^* \mathbb{Z}_p), T)$$

in $\mathrm{K}_1(\mathbb{Z}_p[G']\langle T\rangle)$. Since

$$\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times) \to \mathrm{Det}(\mathbb{Z}_p[G][[T]]^\times)$$

is injective, we see that the image of $\widetilde{\alpha}/\Psi_{\mathbb{Z}_p[G]\langle T\rangle}(\widetilde{\alpha}')$ in $\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times)$ must agree with the unique preimage of

$$L(i'^* g_! g^* \mathbb{Z}_p, T)/L(\mathrm{R}(s \circ i')_! i'^* g_! g^* \mathbb{Z}_p, T),$$

which in turn agrees with the image of $Q(i'^* g_! g^* \mathbb{Z}_p, T)$ by the induction hypothesis. Hence,

$$\Psi_{P\langle T\rangle^\bullet}(\widetilde{\alpha}) = \Psi_{P\langle T\rangle^\bullet}(Q(i'^* g_! g^* \mathbb{Z}_p, T)\Psi_{\mathbb{Z}_p[G]\langle T\rangle}(\widetilde{\alpha}')) = Q(i'^* \mathcal{F}^\bullet, T)\Psi_{P'^\bullet}(\widetilde{\alpha}').$$

Let $i''\colon Z'' \to X$ be the closed complement of U' in X. By the uniqueness of Q for Z'' we conclude

$$Q(i''^* \mathcal{F}^\bullet, T) = Q(i'^* \mathcal{F}^\bullet, T) Q(i^* \mathcal{F}^\bullet, T)$$

and so, the above definition of $Q(\mathcal{F}^\bullet, T)$ is independent of all choices.

Assume now that Λ' is a second finite $\mathbb{Z}_p$-algebra and that $W^\bullet$ is in $\Lambda^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda')$. Then

$$\begin{aligned}
\Psi_{W\langle T\rangle^\bullet}(Q(\mathcal{F}^\bullet, T)) &= \Psi_{W\langle T\rangle^\bullet}(Q(i^* \mathcal{F}^\bullet, T)\Psi_{P\langle T\rangle^\bullet}(\widetilde{\alpha})) \\
&= Q(i^* \Psi_{W^\bullet}(\mathcal{F}^\bullet), T)\Psi_{W \otimes_\Lambda P\langle T\rangle^\bullet}(\widetilde{\alpha}) \\
&= Q(\Psi_{W^\bullet}(\mathcal{F}^\bullet), T).
\end{aligned}$$

It is immediately clear from the definition that $Q(\mathcal{F}^\bullet, T)$ does only depend on the quasi-isomorphism class of $\mathcal{F}^\bullet$. Moreover, if Λ is finite, any exact sequence in $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ can be completed to a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_1^\bullet & \longrightarrow & \mathcal{P}_2^\bullet & \longrightarrow & \mathcal{P}_3^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1^\bullet & \longrightarrow & \mathcal{F}_2^\bullet & \longrightarrow & \mathcal{F}_3^\bullet & \longrightarrow & 0 \end{array}$$

where the top row is an exact sequence of strictly perfect complexes, the bottom row is the original exact sequence, the downward pointing arrows are quasi-isomorphisms, and the squares commute up to homotopy. Then one finds $j\colon U \to X$, $i\colon Z \to X$, $g\colon V \to U$, G as above and an exact sequence

$$0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0$$

of complexes in $\mathbb{Z}_p[G]^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda)$ such that for $k = 1, 2, 3$,

$$\Psi_{P_k^\bullet}(g_! g^* \mathbb{Z}_p) \cong j^* \mathcal{P}_k^\bullet.$$

Choosing $\widetilde{\alpha}$ as above, we conclude

$$\begin{aligned} Q(\mathcal{F}_2^\bullet, T) &= Q(i^*\mathcal{F}_1^\bullet, T) Q(i^*\mathcal{F}_3^\bullet, T) \Psi_{P\langle T\rangle_1{}^\bullet}(\widetilde{\alpha}) \Psi_{P\langle T\rangle_3{}^\bullet}(\widetilde{\alpha}) \\ &= Q(\mathcal{F}_1^\bullet, T) Q(\mathcal{F}_3^\bullet, T). \end{aligned}$$

Thus, Q is well-defined as homomorphism $\mathrm{K}_0(X,\Lambda) \to \mathrm{K}_1(\Lambda\langle T\rangle)$ and satisfies properties (1) and (2) of the theorem.

It remains to prove uniqueness. So let Q' be a second family of homomorphisms $\mathrm{K}_0(X,\Lambda) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$ ranging over all adic rings Λ such that (1) and (2) are satisfied. If $i\colon Z \to X$ is any closed subscheme, then

$$\mathrm{K}_0(Z,\Lambda) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle), \qquad A \mapsto Q'(i_* A, T),$$

still satisfies (1) and (2). Thus, by the induction hypothesis, we must have

$$Q'(i_* A, T) = Q(A, T).$$

If Λ is finite and G is a finite group and $P^\bullet$ a complex in $\mathbb{Z}_p[G]^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda)$ such that $\Psi_{P\langle T\rangle^\bullet}$ factors through $\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times)$, then by the injectivity of

$$\mathrm{Det}(\mathbb{Z}_p[G]\langle T\rangle^\times) \to \mathrm{Det}(\mathbb{Z}_p[G][[T]]^\times)$$

and properties (1) and (2) we must have

$$Q'(\Psi_{P\langle T\rangle^\bullet}(B), T) = Q(\Psi_{P\langle T\rangle^\bullet}(B), T)$$

for all $B \in \mathrm{K}_0(X, \mathbb{Z}_p[G])$. By the construction of Q we thus see that $Q = Q'$. This completes the proof of the theorem. $\square$

We conclude with some ancillary observations. First, we note that Q depends in the same way as the L-function on the choosen base field $\mathbb{F}$:

Proposition 5.2. *Let X be a separated scheme of finite type over $\mathbb{F}$ and $\mathbb{F}' \subset \mathbb{F}$ be a subfield. Write $Q'\colon \mathrm{K}_0(X,\Lambda) \to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$ for the homomorphism resulting from applying Thm. 5.1 to X considered as $\mathbb{F}'$-scheme. Then*

$$Q'(A,T) = Q(A, T^{[\mathbb{F}:\mathbb{F}']})$$

for every $A \in \mathrm{K}_0(X,\Lambda)$.

Proof. By Remark 4.1, the given equality holds for the images in $\mathrm{K}_1(\Lambda[[T]])$. Now one proceeds as in the proof of the uniqueness part of Thm. 5.1 to show that it also holds in $\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$. □

As in [6], we can also define a version of Q relative to a morphism $f\colon X \to Y$ in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ by setting

$$\begin{aligned} Q(f)\colon \mathrm{K}_0(X,\Lambda) &\to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \\ A &\mapsto Q(f,A,T) = Q(A,T)/Q(\mathrm{R}\, f_! A, T), \end{aligned}$$

and extend [6, Lemma 2.4, 2.5] as follows.

Proposition 5.3. *Let A be an object in $\mathrm{K}_0(X,\Lambda)$.*

(1) *If $f\colon X \to Y$, $g\colon Y \to Z$ are two morphisms in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$, then*

$$Q(g \circ f, A, T) = Q(g, \mathrm{R}\, f_! A, T) Q(f, A, T).$$

(2) *If $f\colon X \to Y$ is quasi-finite, then*

$$Q(f, A, T) = 1.$$

Proof. Assertion (1) follows immediately from the definition. For the second assertion, we note that for $f\colon X \to Y$ quasi-finite,

$$\begin{aligned} L(\mathrm{R}\, f_! A, T) &= \prod_{y \in Y^0} L(y^* f_! A, T) \\ &= \prod_{y \in Y^0} \prod_{x \in f^{-1}(y)^0} L(x^* A, T) \\ &= \prod_{x \in X^0} L(x^* A, T) = L(A,T). \end{aligned}$$

In particular, the images of $Q(A,T)$ and $Q(\mathrm{R}\, f_! A, T)$ in $\mathrm{K}_1(\Lambda[[T]])$ agree. By the uniqueness of Q, we must have

$$Q(A,T) = Q(\mathrm{R}\, f_! A, T)$$

in $\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle)$. □

Finally, setting

$$\widehat{\mathrm{K}}_0(X,\Lambda) = \varprojlim_{I \in \mathfrak{I}_\Lambda} \mathrm{K}_0(X, \Lambda/I),$$

we observe that the constructions of the L-function and of Q extends to

$$L\colon \widehat{\mathrm{K}}_0(X,\Lambda)\to \mathrm{K}_1(\Lambda[[T]]),\qquad Q\colon \widehat{\mathrm{K}}_0(X,\Lambda)\to \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle).$$

We cannot say much about the canonical homomorphism

$$\mathrm{K}_0(X,\Lambda)\to \widehat{\mathrm{K}}_0(X,\Lambda),$$

but we suspect that it might be neither injective nor surjective in general.

6. A noncommutative main conjecture for separated schemes over $\mathbb{F}$

In this section, we will complete the $\ell = p$ case of the version of the noncommutative Iwasawa main conjecture for separated schemes X over $\mathbb{F}=\mathbb{F}_q$ considered in [17]. We will briefly recall the terminology of the cited article.

Recall from [17, Def. 2.1] that a principal covering $(f\colon Y\to X, G)$ of X with G a profinite group is an inverse system $(f_U\colon Y_U\to X)_{U\in\mathbf{NS}(G)}$ of finite principal G/U-coverings (not necessarily connected), where U runs through the lattice $\mathbf{NS}(G)$ of open normal subgroups of G. As a particular case, for k prime to p and $\Gamma_{kp^\infty}=\mathrm{Gal}(\mathbb{F}_q^{kp^\infty}/\mathbb{F}_q)$, we have the cyclotomic Γ_{kp^∞}-covering

$$(X_{kp^\infty}=X\times_{\mathrm{Spec}\,\mathbb{F}_q}\mathrm{Spec}\,\mathbb{F}_{q^{kp^\infty}}\to X,\Gamma_{kp^\infty})$$

[17, Def. 2.5]. We will only consider principal coverings $(f\colon Y\to X, G)$ such that there exists a closed normal subgroub $H\subset G$ which is a topologically finitely generated virtual pro-p-group and such that the quotient covering $(f_H\colon Y_H\to X, G/H)$ is the cyclotomic Γ_{p^∞}-covering. These coverings will be called admissible coverings for short [17, Def. 2.6]. For such a group $G=H\rtimes\Gamma_{p^\infty}$, if Λ is any adic $\mathbb{Z}_p$-algebra, then the profinite group ring $\Lambda[[G]]$ is again an adic $\mathbb{Z}_p$-algebra [17, Prop. 3.2].

For any admissible covering $(f\colon Y\to X, G)$ we constructed in [17, Prop. 6.2] a Waldhausen exact functor

$$f_!f^*\colon \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)\to \mathbf{PDG}^{\mathrm{cont}}(X,\Lambda[[G]])$$

by forming the inverse system over the intermediate finite coverings. As before, we will denote the induced homomorphism

$$f_!f^*\colon\ \mathrm{K}_0(X,\Lambda)\to \mathrm{K}_0(X,\Lambda[[G]])$$

by the same symbol.

We also constructed a localisation sequence

$$1\to \mathrm{K}_1(\Lambda[[G]])\to \mathrm{K}_1^{\mathrm{loc}}(H,\Lambda[[G]])\xrightarrow{d}\mathrm{K}_0^{\mathrm{rel}}(H,\Lambda[[G]])\to 1$$

with

$$\mathrm{K}_1^{\mathrm{loc}}(H, \Lambda[[G]]) = \mathrm{K}_1(w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])),$$
$$\mathrm{K}_0^{\mathrm{rel}}(H, \Lambda[[G]]) = \mathrm{K}_0(\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]))$$

and certain Waldhausen categories

$$w_H\mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]) \text{ and } \mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]]),$$

respectively [17, §4], [15, Cor. 3.3]. Whenever $\Lambda[[H]]$ is noetherian, there exists a left denominator set S such that

$$\mathrm{K}_1^{\mathrm{loc}}(H, \Lambda[[G]]) = \mathrm{K}_1(\Lambda[[G]]_S),$$
$$\mathrm{K}_0^{\mathrm{rel}}(H, \Lambda[[G]]) = \mathrm{K}_0(\Lambda[[G]], \Lambda[[G]]_S),$$

where $\Lambda[[G]]_S$ is the localisation of $\Lambda[[G]]$ at S and the K-groups on the right-hand side are the usual groups appearing in the corresponding localisation sequence of higher K-theory [15, Thm. 2.18, Prop. 2.20, Rem 2.22]. In particular, if Λ is commutative (hence, noetherian) and $G = \mathit{\Gamma}_{kp^\infty}$, then the Ore set S is given by

$$S = \{\lambda \in \Lambda[[\mathit{\Gamma}_{kp^\infty}]] \colon \lambda \text{ is a nonzerodivisor in } \Lambda/\operatorname{Jac}(\Lambda)[[\mathit{\Gamma}_{kp^\infty}]]\}$$

and

$$\mathrm{K}_1^{\mathrm{loc}}(H, \Lambda[[G]]) = \mathrm{K}_1(\Lambda[[G]]_S) = \Lambda[[\mathit{\Gamma}_{kp^\infty}]]_S^\times.$$

Let Λ' be a second adic ring and $M^\bullet$ a complex in $\Lambda[[G]]^{\mathrm{op}}-\mathbf{SP}(\Lambda')$. Then

$$\widetilde{M}^\bullet = \Psi_{M^\bullet}(f_! f^* \Lambda)$$

is a perfect complex of smooth Λ'-adic sheaves on X with an additional right Λ-module structure. Using this complex we obtain a homomorphism

$$\Psi_{\widetilde{M}^\bullet} \colon \mathrm{K}_0(X, \Lambda) \to \mathrm{K}_0(X, \Lambda').$$

Furthermore, we can form the complex

$$M[[G]]^{\delta^\bullet} = \Lambda'[[G]] \otimes_\Lambda M^\bullet$$

in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda[[G]])$ with the canonical left G-action and the diagonal right G-action.

If G acts trivially on $M^\bullet$, then $\Psi_{\widetilde{M}^\bullet}$ is just the homomorphism

$$\Psi_{M^\bullet} \colon \mathrm{K}_0(X, \Lambda) \to \mathrm{K}_0(X, \Lambda')$$

that we have already used above. Moreover, we have

$$\Psi_{M[[G]]^{\delta\bullet}} f_! f^* A = f_! f^* \Psi_{\widetilde{M}^\bullet}(A)$$

[17, Prop. 6.7]. Note that $M[[G]]^{\delta^\bullet}$ also induces compatible homomorphisms on the groups $\mathrm{K}_1(\Lambda[[G]])$, $\mathrm{K}_1^{\mathrm{loc}}(H, \Lambda[[G]])$ and $\mathrm{K}_0^{\mathrm{rel}}(H, \Lambda[[G]])$ [17, Prop. 4.6].

As a special case we may take $\Lambda = \Lambda' = \mathbb{Z}_p$ and let $\rho \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ be a continuous left G-representation as in the introduction. We may then choose M to be the $\mathbb{Z}_p$-$\mathbb{Z}_p[[G]]$-module obtained from ρ by letting G act contragrediently on $\mathbb{Z}_p^n$ from the right. The sheaf $\widetilde{M}$ is then just the smooth Λ-adic sheaf $\mathcal{M}(\rho)$ associated to ρ and $\Psi_{\widetilde{M}}$ corresponds to taking the (completed) tensor product with this sheaf over $\mathbb{Z}_p$.

In [17, Thm. 8.1] we have already shown that for each complex $\mathcal{F}^\bullet$ in the Waldhausen category $\mathbf{PDG}^{\mathrm{cont}}(X,\Lambda)$ and each admissible covering $(f\colon Y \to X, G)$ the complex of Λ-adic cohomology with proper support

$$\mathrm{R}\,\Gamma_{\mathrm{c}}(X, f_! f^* \mathcal{F}^\bullet)$$

is an object of $\mathbf{PDG}^{\mathrm{cont},w_H}(\Lambda[[G]])$. Hence, we obtain a homomorphism

$$\mathrm{K}_0(X,\Lambda) \to \mathrm{K}_0^{\mathrm{rel}}(H,\Lambda[[G]]), \qquad A \mapsto \mathrm{R}\,\Gamma_{\mathrm{c}}(X, f_! f^* A).$$

We have also contructed an explicit homomorphism

$$\mathrm{K}_0(X,\Lambda) \to \mathrm{K}_1^{\mathrm{loc}}(H,\Lambda[[G]]), \qquad A \mapsto \mathcal{L}_G(X/\mathbb{F}, A),$$

such that

$$d\mathcal{L}_G(X/\mathbb{F}, A) = \mathrm{R}\,\Gamma_c(X, f_! f^* A)^{-1}$$

[17, Def. 8.3].

We let γ denote the image of the geometric Frobenius automorphism $\mathfrak{F}_{\mathbb{F}}$ in Γ_{kp^∞}. If Ω is a commutative adic $\mathbb{Z}_p$-algebra, we write $\widetilde{S} \subset \Omega\langle T\rangle$ for the denominator set consisting of those elements which become a unit in $\Omega[[T]]$. The proof of [17, Lemma 8.5] shows that the evaluation $T \mapsto \gamma^{-1}$ extends to a ring homomorphism

$$\Omega\langle T\rangle_{\widetilde{S}} \to \Omega[[\Gamma_{kp^\infty}]]_S, \qquad \omega(T) \mapsto \omega(\gamma^{-1}).$$

Note that $\Omega\langle T\rangle_{\widetilde{S}}$ is a semilocal ring, hence

$$\Omega\langle T\rangle_{\widetilde{S}}^\times = \mathrm{K}_1(\Omega\langle T\rangle_{\widetilde{S}}).$$

Let $s\colon X \to \operatorname{Spec}\mathbb{F}$ be the structure map. Then the proof of [17, Thm. 8.6] shows that for every $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Omega)$ and every $A \in \mathrm{K}_0(X,\Lambda)$, we have

$$L\left(\mathrm{R}\, s_! \Psi_{\widetilde{M}^\bullet}(f_! f^* A), T\right) \in \mathrm{K}_1(\Omega\langle T\rangle_{\widetilde{S}})$$

and that

$$\Psi_{\Omega[[\Gamma_{kp^\infty}]]}\Psi_{M[[G]]^{\delta\bullet}}(\mathcal{L}_G(X/\mathbb{F}, A)) = L\left(\mathrm{R}\, s_! \Psi_{\widetilde{M}^\bullet}(f_! f^* A), \gamma^{-1}\right)$$

in

$$\mathrm{K}_1^{\mathrm{loc}}(H, \Omega[[\Gamma_{kp^\infty}]]) = \Omega[[\Gamma_{kp^\infty}]]_S^\times.$$

So, $\mathcal{L}_G(X/\mathbb{F}, A)$ satisfies the desired interpolation property with respect to the functions $L\left(\mathrm{R}\, s_! \Psi_{\widetilde{M}^\bullet}(f_! f^* A), \gamma^{-1}\right)$, but not with respect to the functions $L(\Psi_{\widetilde{M}^\bullet}(f_! f^* A), \gamma^{-1})$. We will construct a modification of $\mathcal{L}_G(X/\mathbb{F}, A)$ below.

For any adic ring Λ, the evaluation map $T \mapsto 1$ induces a homomorphism

$$\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \to \mathrm{K}_1(\Lambda), \qquad \lambda(T) \mapsto \lambda(1).$$

We also obtain an evaluation map

$$\widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \to \mathrm{K}_1(\Lambda[[\mathit{\Gamma}_{kp^\infty}]]), \qquad \lambda(T) \mapsto \lambda(\gamma^{-1}),$$

as composition of $T \mapsto 1$ with the automorphism of $\widehat{\mathrm{K}}_1(\Lambda[[\mathit{\Gamma}_{kp^\infty}]]\langle T\rangle)$ induced by $T \mapsto \gamma^{-1}T$ and the injection

$$\Psi_{\Lambda[[\mathit{\Gamma}_{kp^\infty}]]\langle T\rangle} \colon \widehat{\mathrm{K}}_1(\Lambda\langle T\rangle) \to \widehat{\mathrm{K}}_1(\Lambda[[\mathit{\Gamma}_{kp^\infty}]]\langle T\rangle).$$

Definition 6.1. For any admissible covering $(f \colon Y \to X, G)$ and any $A \in \mathrm{K}_0(X, \Lambda)$ we set

$$\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A) = \mathcal{L}_G(X/\mathbb{F}, A)Q(f_! f^* A, 1).$$

Since $Q(f_! f^* A, 1) \in \mathrm{K}_1(\Lambda[[G]])$, we still have

Theorem 6.2.

$$d\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A) = \mathrm{R}\,\Gamma_c(X, f_! f^* A)^{-1}$$

in $\mathrm{K}_0^{\mathrm{loc}}(H, \Lambda[[G]])$.

We will now investigate the transformation properties of $\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A)$.

Theorem 6.3. *Consider a separated scheme X of finite type over a finite field $\mathbb{F}$. Let Λ be any adic $\mathbb{Z}_p$ algebra and let A be in* $\mathrm{K}_0(X, \Lambda)$.

(1) *Let Λ' be another adic $\mathbb{Z}_p$-algebra. For any complex $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda')$, we have*

$$\Psi_{M[[G]]^{\delta\bullet}}(\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A)) = \widetilde{\mathcal{L}}_G(X/\mathbb{F}, \Psi_{\widetilde{M}^\bullet}(A))$$

in $\mathrm{K}_1^{\mathrm{loc}}(H, \Lambda'[[G]])$.

(2) *Let H' be a closed virtual pro-p-subgroup of H which is normal in G. Then*

$$\Psi_{\Lambda[[G/H']]}(\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A)) = \widetilde{\mathcal{L}}_{G/H'}(X/\mathbb{F}, A)$$

in $\mathrm{K}_1^{\mathrm{loc}}(H', \Lambda[[G/H']])$.

(3) *Let U be an open subgroup of G and let $\mathbb{F}'$ be the finite extension corresponding to the image of U in $\mathit{\Gamma}_{p^\infty}$. Then*

$$\Psi_{\Lambda[[G]]}(\widetilde{\mathcal{L}}_G(X/\mathbb{F}, A)) = \widetilde{\mathcal{L}}_U(Y_U/\mathbb{F}', f_U^* A)$$

in $\mathrm{K}_1(H \cap U, (\Lambda[[U]]))$.

Proof. In [17, Thm. 8.4] we have already proved that $\mathcal{L}_G(X/\mathbb{F}, A)$ satisfies the given transformation properties. To prove the same properties for

$Q(f_* f^* A, 1)$, one uses the general transformation rule for $Q(A,T)$ and Ψ and the equalities

$$\begin{aligned}\Psi_{M[[G]]^{\delta\bullet}} f_! f^* A &= f_! f^* \Psi_{\widetilde{M}^\bullet}(A)\\ \Psi_{\Lambda[[G/H']]} f_! f^* A &= f_{H'!} f^*_{H'} A\\ \Psi_{\Lambda[[G]]} f_! f^* A &= \mathrm{R}\, f_{U!} f^U_! f^{U*} f^*_U A\end{aligned}$$

with $(f^U \colon Y \to Y_U, U)$ the restriction of the covering to the subscheme U [17, Prop. 6.5, 6.7]. For (3) it remains to notice that the evaluation $Q(A,1)$ does not depend on the base field $\mathbb{F}$ by Prop. 5.2. □

Proposition 6.4. *Consider the admissible covering* $(f\colon X_{kp^\infty} \to X, \Gamma_{kp^\infty})$. *For any adic ring* Λ *and any* $A \in \mathrm{K}_1(X,\Lambda)$,

$$Q(f_! f^* A, T) = Q(\Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(A), \gamma^{-1} T)$$

in $\widehat{\mathrm{K}}_1(\Lambda[[\Gamma_{kp^\infty}]]\langle T\rangle)$.

Proof. We may assume that Λ is finite. Let $s\colon X \to \mathbb{F}$ denote the structure map. From [17, Prop. 7.2] it follows that

$$L(\mathrm{R}\, s_! f_! f^* A, T) = L(\mathrm{R}\, s_! \Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(A), \gamma^{-1} T).$$

By applying this to $x^* A$ for each closed point $x\colon \operatorname{Spec} k(x) \to X$ of X and using

$$f_! f^* x^* A = x^* f_! f^* A$$

[17, Prop. 6.4.(1)] we see that

$$L(f_! f^* A, T) = L(\Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(A), \gamma^{-1} T).$$

Hence, the images of $Q(f_! f^* A, T)$ and $Q(\Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(A), \gamma^{-1} T)$ agree in the group $\mathrm{K}_1(\Lambda[[\Gamma_{kp^\infty}]][[T]])$.

If $a\colon X' \to X$ is a morphism in $\mathbf{Sch}^{\mathrm{sep}}_{\mathbb{F}}$ and $(f\colon X' \to X, \Gamma_{kp^\infty})$ is the cyclotomic Γ_{kp^∞}-covering of X', then

$$\mathrm{R}\, a_! f_! f^* A = f_! f^* \,\mathrm{R}\, a_! A$$

by [17, Prop. 6.4.(2)]. In particular, this applies to open and closed immersions. By induction on the dimension of X we can thus reduce to the case X integral and $A = \Psi_{P^\bullet}(g_! g^* \mathbb{Z}_p)$ for some finite connected Galois covering $(g\colon Y \to X, G)$ and some complex $P^\bullet$ in $\mathbb{Z}_p[G]^{\mathrm{op}}$-$\mathbf{SP}(\Lambda)$. With the complex

$$P[[\Gamma_{kp^\infty}]]^\bullet = \Lambda[[\Gamma_{kp^\infty}]] \otimes_\Lambda P^\bullet$$

in $\mathbb{Z}_p[[G \times \Gamma_{kp^\infty}]]$-$\mathbf{SP}(\Lambda[[\Gamma_{kp^\infty}]])$ we have

$$\begin{aligned}\Psi_{P[[\Gamma_{kp^\infty}]]^\bullet}(f_! f^* g_! g^* \mathbb{Z}_p) &= f_! f^* A,\\ \Psi_{P[[\Gamma_{kp^\infty}]]^\bullet}(\Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(g_! g^* \mathbb{Z}_p)) &= \Psi_{\Lambda[[\Gamma_{kp^\infty}]]}(A).\end{aligned}$$

Applying Prop. 3.6 to

$$R = \mathbb{Z}_p[[\Gamma_{kp^\infty}]]\langle T\rangle \cong \mathbb{Z}_p[\mathbb{Z}/k\mathbb{Z}][[T']]\langle T\rangle$$

we may assume that $\Psi_{P[[\Gamma_{kp^\infty}]]\langle T\rangle^\bullet}$ factors through $\operatorname{Det}(\mathbb{Z}_p[[G\times\Gamma_{kp^\infty}]]\langle T\rangle^\times)$. We conclude

$$\begin{aligned} Q(f_!f^*A,T) &= \Psi_{P[[\Gamma_{kp^\infty}]]\langle T\rangle^\bullet}(Q(f_!f^*g_!g^*\mathbb{Z}_p,T)) \\ &= \Psi_{P[[\Gamma_{kp^\infty}]]\langle T\rangle^\bullet}(Q(g_!g^*\mathbb{Z}_p,\gamma^{-1}T)) \\ &= Q(A,\gamma^{-1}T). \end{aligned}$$

□

The following theorem shows that $\widetilde{\mathcal{L}}_G(X/\mathbb{F},A)$ satisfies the right interpolation property.

Theorem 6.5. *Let X be a scheme in $\mathbf{Sch}_{\mathbb{F}}^{\mathrm{sep}}$ and let $(f\colon Y\to X,G)$ be an admissible principal covering containing the cyclotomic Γ_{kp^∞}-covering. Furthermore, let Λ and Ω be adic $\mathbb{Z}_p$-algebras with Ω commutative. For every $A\in \mathrm{K}_0(X,\Lambda)$ and every $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}$-$\mathbf{SP}(\Omega)$, we have*

$$L(\Psi_{\widetilde{M}^\bullet}(A),T)\in \mathrm{K}_1(\Omega\langle T\rangle_{\widetilde{S}})$$

and

$$\Psi_{\Omega[[\Gamma_{kp^\infty}]]}\Psi_{M[[G]]^\delta{}^\bullet}(\widetilde{\mathcal{L}}_G(X/\mathbb{F},A)) = L(\Psi_{\widetilde{M}^\bullet}(A),\gamma^{-1})$$

in $\mathrm{K}_1(\Omega[[\Gamma_{kp^\infty}]]_S)$.

Proof. As remarked above, the corresponding statement for the elements $L(\mathrm{R}\,s_!\Psi_{\widetilde{M}^\bullet}(A),T)$ and $\mathcal{L}_G(X/\mathbb{F},A)$ follows from [17, Thm. 8.6]. Since

$$L(\Psi_{\widetilde{M}^\bullet}(A),T) = Q(\Psi_{\widetilde{M}^\bullet}(A),T)L(\mathrm{R}\,s_!\Psi_{\widetilde{M}^\bullet}(A),T),$$

an application of Prop. 6.4 concludes the proof of the theorem. □

As in the case where p is not equal to the characteristic of $\mathbb{F}$, the element $L(A,\gamma^{-1})$ interpolates the values $L(A(\epsilon^n),1)$, where $A(\epsilon^n)$ denotes the twist of A by the n-th power of the cyclotomic character ϵ for $n\in\mathbb{Z}$. However, different from the situation that $p\neq \operatorname{char}\mathbb{F}$, we do not expect a direct relation between $L(A(\epsilon^n),1)$ and the values $L(A,q^{-n})$ for $n\neq 0$ in the $p=\operatorname{char}\mathbb{F}$ case. In the case $\Lambda=\mathbb{Z}_p$ and smooth and proper schemes $X/\mathbb{F}_q$, the p-adic valuation of $L(\mathbb{Z}_p,q^{-n})$ may instead be connected to the Euler characteristics of the étale cohomology of the logarithmic deRham-Witt sheaves $\mathbb{Z}_p(n)$ and of truncated deRham complexes by a classical result of Milne [8]. So, one might hope to find an analogous formula for any adic $\mathbb{Z}_p$-algebra Λ and any Λ-sheaf $\mathcal{F}$, describing $L(\mathcal{F},q^{-n})$ in terms of the étale cohomology of $\mathcal{F}\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(n)$ and an error term coming from the deRham complex of X. In the case of a smooth curve X, $n=1$, $g\colon Y\to X$ a at most tamely ramified finite Galois covering with Galois group G, $\Lambda=\mathbb{Z}_p[G]$ and

$\mathcal{F} = g_! g^* \mathbb{Z}_p$, such a formula has already been considered in the thesis [12]. However, Wan shows in [11] that there exist $\mathbb{Z}_p$-sheafs $\mathcal{F}$ such that $L(\mathcal{F}, T)$ may not be meromorphically continued to the entire p-adic plane, so that we are not even able to give a definition of $L(\mathcal{F}, q^{-n})$ for positive n in general. Note that for negative n, we have $L(\mathcal{F}, q^{-n}) \in \mathbb{Z}_p^\times$, such that the p-adic valuation is trivial.

References

[1] D. Burns, "On main conjectures of geometric Iwasawa theory and related conjectures", preprint (version 6) `www.mth.kcl.ac.uk/staff/dj_burns/gmcrc-vers6.pdf`, 2011.

[2] T. Chinburg, G. Pappas & M. J. Taylor, "K_1 of a p-adic group ring I. The determinantal image", *J. Algebra* **326** (2011), p. 74-112.

[3] ———, "K_1 of a p-adic group ring II. The determinantal kernel SK_1", *J. Pure Appl. Algebra* **219** (2015), no. 7, p. 2581-2623.

[4] J. Coates, T. Fukaya, K. Kato, R. Sujatha & O. Venjakob, "The GL_2 main conjecture for elliptic curves without complex multiplication", *Publ. Math. Inst. Hautes Études Sci.* (2005), no. 101, p. 163-208.

[5] P. Deligne, *Cohomologie étale*, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, iv+312pp pages.

[6] M. Emerton & M. Kisin, "Unit L-functions and a conjecture of Katz", *Ann. of Math. (2)* **153** (2001), no. 2, p. 329-354.

[7] T. Fukaya & K. Kato, "A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory", in *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., Providence, RI, 2006, p. 1-85.

[8] J. S. Milne, "Values of zeta functions of varieties over finite fields", *Amer. J. Math.* **108** (1986), no. 2, p. 297-360.

[9] R. Oliver, *Whitehead groups of finite groups*, London Mathematical Society Lecture Note Series, vol. 132, Cambridge University Press, Cambridge, 1988, viii+349 pages.

[10] C. T. C. Wall, "Norms of units in group rings", *Proc. London Math. Soc. (3)* **29** (1974), p. 593-632.

[11] D. Wan, "Meromorphic continuation of L-functions of p-adic representations", *Ann. of Math. (2)* **143** (1996), no. 3, p. 469-498.

[12] C. Ward, "On geometric zeta functions, epsilon constants and canonical classes", PhD Thesis, King's College London, 2011.

[13] M. Witte, "On a noncommutative Iwasawa main conjecture for function fields", preprint, 2013.

[14] ———, "Noncommutative Iwasawa main conjectures for varieties over finite fields", PhD Thesis, Universität Leipzig, 2008, `http://d-nb.info/995008124/34`.

[15] ———, "On a localisation sequence for the K-theory of skew power series rings", *J. K-Theory* **11** (2013), no. 1, p. 125-154.

[16] ———, "Noncommutative L-functions for varieties over finite fields.", in *Iwasawa theory 2012. State of the art and recent advances. Selected papers based on the presentations at the conference, Heidelberg, Germany, July 30 – August 3, 2012*, Berlin: Springer, 2014, p. 443-469 (English).

[17] ———, "On a noncommutative Iwasawa main conjecture for varieties over finite fields", *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 2, p. 289-325.

Malte WITTE
Ruprecht-Karls-Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
D-69120 Heidelberg
GERMANY
E-mail: witte@mathi.uni-heidelberg.de