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## Constructions of some perfect integral lattices with minimum 4

par ROLAND BACHER

RÉSUMÉ. Cet article présente des constructions de plusieurs familles de sous-réseaux parfaits de  $\mathbb{Z}^d$  avec minimum 4. En particulier, le nombre de tels réseaux parfaits de dimension  $d$  croît plus vite que tout polynôme en  $d$ .

ABSTRACT. We construct several families of perfect sublattices with minimum 4 of  $\mathbb{Z}^d$ . In particular, the number of  $d$ -dimensional perfect integral lattices with minimum 4 grows faster than  $d^k$  for every exponent  $k$ .

### 1. Perfection and perfect lattices

A subset  $\mathcal{S}$  of a real  $d$ -dimensional vector space  $V$  is a *perfect subset of  $V$*  (or perfect in  $V$ ) if the span of the set  $\{v \otimes v\}_{v \in \mathcal{S}}$  is the full  $\binom{d+1}{2}$ -dimensional vector space  $\sum_{v,w \in V} v \otimes w + w \otimes v$  of all symmetric tensor products in  $V \otimes V$ . In the sequel we speak simply of perfect sets if the ambient vector space is obvious.

A choice of a basis  $x_1, \dots, x_d$  of  $V$  identifies  $V$  with the vector space  $\{a_1x_1 + \dots + a_dx_d \mid a_1, \dots, a_d \in \mathbb{R}\}$  of all homogeneous 1-forms contained in  $\mathbb{R}[x_1, \dots, x_d]$ . Perfection of  $\mathcal{S}$  is equivalent to the fact that the set

$$\left\{ \left( \sum_{i=1}^d a_i x_i \right)^2 \right\}_{\sum_{i=1}^d a_i x_i \in \mathcal{S}}$$

of all quadratic forms associated to elements in  $\mathcal{S}$  spans the full  $\binom{d+1}{2}$ -dimensional vector space of all quadratic forms (homogeneous polynomials of degree 2). Equivalently,  $\mathcal{S}$  is perfect (in  $V$ ) if and only if the set of symmetric matrices  $\{(a_i a_j)_{1 \leq i, j \leq d}\}_{\sum a_i x_i \in \mathcal{S}}$  spans the vector space of all symmetric square-matrices of size  $d$ . The matrix  $(a_i a_j)_{1 \leq i, j \leq d}$  is, up to a

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scalar multiple, the orthogonal projection of  $V$  onto  $\sum a_i x_i$  with respect to the scalar product with orthonormal basis  $x_1, \dots, x_n$ .

Perfect sets of vector spaces over real fields determine scalar products uniquely in the following way: A scalar product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  on  $V \times V$  is uniquely defined by the set  $\{\langle v, v \rangle\}_{v \in \mathcal{S}}$  of norms of elements in  $\mathcal{S}$  if and only if  $\mathcal{S}$  is perfect.

**Remark.** Perfection can be generalized as follows: A subset  $\mathcal{S}$  of a vector space  $V$  over a field of characteristic 0 or larger than  $k$  is *k-perfect* in  $V$  if the elements of the set  $\{v \otimes v \otimes \dots \otimes v \in V^{\otimes k} \mid v \in \mathcal{S}\}$  span the  $\binom{n+k-1}{k}$ -dimensional subspace spanned by all symmetric  $k$ -fold tensor powers  $\{\sum_{\sigma \in S_k} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(k)} \mid v_1, \dots, v_k \in V\}$  of  $V$ . A  $k$ -perfect subset of  $V$  is  $k'$ -perfect for  $k' \leq k$ . Given a subset  $\mathcal{S}$  of a  $d$ -dimensional vector space  $V$  over a field  $\mathbb{K}$  of characteristic 0, we denote by  $A_k$  the vector space spanned by the set  $\{v \otimes v \otimes \dots \otimes v \in V^{\otimes k} \mid v \in \mathcal{S}\}$  and by  $\alpha_k = \dim(A_k)$  the dimension of  $A_k$ . We use the convention  $A_0 = \mathbb{K}$  and  $\alpha_0 = 1$ . The generating series  $\sum_{k=0}^{\infty} \alpha_k t^k \in \mathbb{N}[t]$  is always a rational function of the form  $\frac{P_{\mathcal{S}}(t)}{1-t}$  for  $P_{\mathcal{S}} \in \mathbb{N}[t]$  a polynomial with non-negative integral coefficients. It would be interesting to understand all possible polynomials arising in this way. In dimension  $d = 2$  we have  $P_{\mathcal{S}} = 1 + t + t^2 + \dots + t^{a-1} = \frac{1-t^a}{1-t}$  where  $a$  is the number of distinct lines  $\{\mathbb{K}v\}_{v \in \mathcal{S}}$  defined by all elements of  $\mathcal{S}$ .

We will make repeated use of the following trivial observation which is assertion 1 of Proposition 3.5.3 in [3]:

**Proposition 1.1.** *Let  $\mathcal{S}$  be a set of non-zero elements in a  $d$ -dimensional vector space  $V$ . Suppose that  $V$  contains a hyperplane  $\mathcal{H}$  such that  $\mathcal{S} \cap \mathcal{H}$  is perfect in  $\mathcal{H}$  and suppose that the elements  $\mathcal{S} \setminus (\mathcal{S} \cap \mathcal{H})$  of  $\mathcal{S}$  in the complement  $V \setminus \mathcal{H}$  of  $\mathcal{H}$  generate  $V$ . Then  $\mathcal{S}$  is perfect.*

We apply Proposition 1.1 always in the case where  $V$  is a Euclidean vector space. The hyperplane  $\mathcal{H}$  can then be described as the orthogonal subspace  $\mathcal{H} = v^\perp$  of a non-zero element  $v$  in  $V$ .

*Proof of Proposition 1.1.* We extend a basis  $b_1, \dots, b_{d-1}$  of  $\mathcal{H}$  to a basis  $b_1, \dots, b_d$  of  $V$ . The vector space spanned by  $\{v \otimes v\}_{v \in \mathcal{S} \cap \mathcal{H}}$  contains the vector space of all symmetric tensor products in  $\mathcal{H} \otimes \mathcal{H}$  by perfection of  $\mathcal{S} \cap \mathcal{H}$ . The fact that  $\mathcal{S} \setminus (\mathcal{S} \cap \mathcal{H})$  generates  $V$  implies that the vector space spanned by  $\{v \otimes v\}_{v \in \mathcal{S} \setminus (\mathcal{S} \cap \mathcal{H})}$  contains all symmetric tensors  $b_i \otimes b_d + b_d \otimes b_i$  for  $i = 1, \dots, d$ . □

**1.1. Perfect lattices.** A *Euclidean lattice* (or *lattice* in the sequel) is a discrete subgroup of a finite-dimensional Euclidean vector space  $\mathbb{E}$ . A lattice  $\Lambda$  of rank  $d = \dim(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$  is isomorphic to  $\mathbb{Z}^d$  as a group.  $\Lambda$  is *integral* if the scalar product  $\mathbb{E} \times \mathbb{E} \ni (u, v) \mapsto \langle u, v \rangle \in \mathbb{R}$  has an integral restriction

$\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ . The *norm* of a lattice-element  $\lambda$  is in the sequel always the squared Euclidean length  $\langle \lambda, \lambda \rangle$  of  $\lambda$ . An integral lattice is *even* if all its elements have even norm. We denote by  $\Lambda_{\min}$  the set of shortest non-zero elements, called *minimal elements*, in  $\Lambda$  and by  $\min(\Lambda)$  the *minimal norm*  $\langle v, v \rangle$  of a minimal element  $v$  in  $\Lambda_{\min}$ . The *determinant*  $\det(\Lambda)$  of a lattice is the squared volume of a fundamental domain for the action (by translations) of  $\Lambda$  on  $(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ . The determinant  $\det(\Lambda)$  is given by  $\det(G)$  with  $(G)_{i,j} = \langle b_i, b_j \rangle$  a Gram matrix defined by scalar products between basis elements  $b_1, \dots, b_d$  of the  $d$ -dimensional lattice  $\Lambda = \bigoplus_{i=1}^d \mathbb{Z}b_i$ . The *density*

$$\frac{\sqrt{\min(\Lambda)}^d \pi^{d/2}}{2^d \sqrt{\det(\Lambda)} (d/2)!}$$

of a  $d$ -dimensional lattice  $\Lambda$  is the density of the associated sphere-packing obtained by packing the space  $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$  with spheres having equal radius  $\sqrt{\min(\Lambda)}/2$  (and delimiting balls of volume  $\left(\frac{\sqrt{\min(\Lambda)}}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}$ ) centered at all lattice points. *Extreme lattices* are lattices whose density is locally maximal (with respect to the obvious natural topology on the space of lattices of given dimension). Extreme lattices are perfect and eutactic (a positivity condition), cf. Theorem 3.4.6 in [3]. Perfection and eutaxy are however independent in the sense that one property does not necessarily imply the other. Thus there exist perfect lattices which are not extreme. All perfect lattices can be realized, up to similarity, as integral lattices (cf. Proposition 3.2.11 of [3]) and there are only finitely many of them (up to similarity and isometry) in any given dimension, cf. Theorem 3.5.4 in [3]. The following definition provides a measure for perfection: Given a lattice  $\Lambda$  of rank  $d$ , we denote by  $\text{pd}(\Lambda)$  its *perfection-default* (called co-rank in the monograph [3] devoted to perfect lattices) defined as  $\binom{d+1}{2} - \dim(\mathcal{A})$  with  $\mathcal{A} = \sum_{v \in \Lambda_{\min}} \mathbb{R} v \otimes v$  denoting the vector space spanned by  $\{v \otimes v\}_{v \in \Lambda_{\min}}$ . A lattice is perfect if and only if its perfection-default is zero.

The aim of this paper is the construction of a few integral lattices with minimum 4 (we describe also a family with minimum 3). All considered lattices are sublattices of  $\mathbb{Z}^n$  and are thus kernels of morphisms  $\varphi : \mathbb{Z}^n \rightarrow A$  onto a suitable abelian group  $A$ . The specific form of  $\varphi$  is of crucial importance since it allows the deduction of perfection from combinatorial properties. Our construction is very flexible and gives rise to many inequivalent perfect lattices. In particular, we show in Theorem 2.5 that the number of inequivalent perfect integral lattices of minimum 4 and dimension  $d$  has no polynomial upper bound as a function of  $d$ .

The sequel of this paper is organized as follows: Section 2 describes the main construction and its generalization, obtained by considering suitable  $d$ -dimensional sublattices of the  $(d + 1)$ -dimensional root lattice of type  $A$ .

The rest of the paper is essentially a variation on this theme. Section 3 avoids the use of the root lattice of type  $A$  by considering the orthogonal of an integral vector having only odd coefficients. Section 4 replaces the root lattice of type  $A$  by the root lattice of type  $D$ . Section 5 considers sublattices of finite index in root lattices of type  $A$ . Section 6 considers sublattices of finite index in root lattices of type  $D$ . Section 7 discusses briefly a family of perfect lattices having minimum 3 related to projective spaces over the field  $\mathbb{F}_2$  of 2 elements. The rest of the paper deals with other variations based on finite abelian groups and generalizations.

### 2. A sequence of perfect lattices

We denote by  $L_d$  the even integral lattice of rank  $d$  defined by all vectors of  $\mathbb{Z}^{d+2}$  orthogonal to both elements  $(1, 1, \dots, 1)$  and  $(1, 2, \dots, d + 2)$  of  $\mathbb{Z}^{d+2}$ .

**Theorem 2.1.** *The lattice  $L_d$  has determinant  $\frac{1}{12}(d + 1)(d + 2)^2(d + 3)$  and contains no roots (vectors of norm 2). It has  $\frac{1}{24}d(d + 2)(2d - 1)$  pairs of opposite vectors of (squared Euclidean) norm 4 if  $d$  is even and  $\frac{1}{24}(d - 1)(d + 1)(2d + 3)$  pairs of opposite vectors of norm 4 if  $d$  is odd. The lattice  $L_d$  is perfect for  $d \geq 7$ .*

**Remark.** The lattice  $L_6$  has 22 pairs of minimal elements. The associated set  $\{v \otimes v\}_{v \in \min(L_6)}$  spans a vector space of dimension 20. The lattice  $L_6$  has thus perfection-default  $\binom{7}{2} - 20 = 1$  and is not perfect.

The seven rows of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

span the perfect lattice  $L_7$ . The associated Gram matrix  $AA^t$  with determinant  $\frac{2^3 \cdot 3^4 \cdot 2 \cdot 5}{2^2 \cdot 3} = 2^2 \cdot 3^3 \cdot 5$  is the reduced Gram matrix

$$P_7^7 = \begin{pmatrix} 4 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 & 1 & 1 & 0 \\ 2 & 2 & 4 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 4 & -1 & 0 & -1 \\ 2 & 1 & 0 & -1 & 4 & 0 & 2 \\ 2 & 1 & 2 & 0 & 0 & 4 & 2 \\ 2 & 0 & 1 & -1 & 2 & 2 & 4 \end{pmatrix}$$

at page 382 of [2]. Gram-matrices for all perfect lattices up to dimension 7 are only given in the original French version [2]. They are unfortunately missing in the English translation [3].

*Proof of Theorem 2.1.* The determinant of  $L_d$  is equal to the determinant of the 2-dimensional lattice  $\mathbb{Z}^{d+2} \cap (L_d \otimes_{\mathbb{Z}} \mathbb{R})^\perp$  of  $\mathbb{Z}^{d+2}$  which is orthogonal to  $L_d$ . Thus the determinant of  $L_d$  is given by

$$\det \begin{pmatrix} \langle u, u \rangle & \langle u, v \rangle \\ \langle v, u \rangle & \langle v, v \rangle \end{pmatrix}$$

with  $u = (1, 1, \dots, 1), v = (1, 2, \dots, d+2) \in \mathbb{Z}^{d+2}$ , and checking the formula is straightforward.

The lattice  $L_d$  is obviously integral and even. Vectors of norm 2 in  $\mathbb{Z}^{d+2}$  are of the form  $\pm e_i \pm e_j$  (with  $e_1, \dots, e_{d+2}$  denoting the natural orthonormal basis of  $\mathbb{Z}^{d+2}$ ) and are never orthogonal to both elements  $(1, \dots, 1)$  and  $(1, \dots, d+2)$  of  $\mathbb{Z}^{d+2}$ .

Vectors of norm 4 in  $L_d$  are of the form

$$e_i - e_{i+\alpha} - e_{i+\alpha+\beta} + e_{i+2\alpha+\beta}$$

with  $i \in \{1, \dots, d-1\}$  and  $\alpha, \beta$  two natural numbers greater than 0 such that  $i + 2\alpha + \beta \leq d + 2$ . The lattice  $L_d$  contains thus

$$\sum_{i=1}^{d-1} \sum_{\alpha=1}^{\lfloor (d+1-i)/2 \rfloor} d + 2 - i - 2\alpha$$

pairs of minimal vectors. This formula, restricted to even, respectively odd, natural integers, defines a polynomial function of degree 3. Explicit expressions can be found by interpolation of four values.

We prove perfection of  $L_d$  by induction on  $d$ . Perfection of the lattice  $L_7$  considered above establishes the result for  $d = 7$ .

The identity  $(1, 2, 3, \dots, d+3) - (1, 1, 1, \dots, 1) = (0, 1, 2, \dots, d+2)$  shows that the sublattice of  $L_{d+1}$  orthogonal to  $(1, 0, 0, \dots, 0)$  is the lattice  $L_d$  which is perfect by assumption. By Proposition 1.1 it is enough to show that the vector space spanned by the set of minimal vectors with first coordinate non-zero (i.e. with first coordinate  $\pm 1$ ) has dimension  $d + 1$ . We set  $u = e_1 - e_3 - e_4 + e_6$  and  $v_i = e_1 - e_2 - e_{i-1} + e_i$  for  $i = 4, \dots, d + 3$ . Consideration of the last index  $i$  with non-zero coefficient of the vector  $v_i$  shows linear independency of the  $d$  vectors  $v_4, \dots, v_{d+3}$ . Computation of

$$v_5 + v_6 - u = e_1 - 2e_2 + e_3$$

shows linear independency of  $u$  from  $v_4, \dots, v_{d+3}$  and ends the proof.  $\square$

**2.1. A generalization.** To a strictly increasing sequence  $1 \leq a_1 < a_2 < \dots < a_k$  of  $k$  natural integers  $a_1, \dots, a_k$  and an integer  $n$  we associate the set  $\mathcal{I}_n(a_1, \dots, a_k)$  defined by the smallest  $n$  elements of  $\{1, 2, \dots\} \setminus \{a_1, \dots, a_k\}$ . We consider now the sequence of lattices  $L_d(a_1, \dots, a_k)$  consisting of all elements of  $\mathbb{Z}^{d+2}$  which are orthogonal to  $(1, \dots, 1) \in \mathbb{Z}^{d+2}$  and to the vector of  $\mathbb{Z}^{d+2}$  with increasing coefficients given by the elements of  $\mathcal{I}_{d+2}(a_1, \dots, a_k)$ . Equivalently,  $L_d(a_1, \dots, a_k)$  can be defined for  $d$  large enough as the sublattice of  $L_{d+k}$  defined by all vectors with zero coefficients for indices in  $\{a_1, a_2, \dots, a_k\}$ .

The lattices  $L_d(a_1, \dots, a_k)$  and  $L_d(d+2+k-a_k, d+2+k-a_{k-1}, \dots, d+2+k-a_1)$  are obviously isomorphic for  $d > a_k - k - 2$ .

Theorem 2.1 has the following generalization:

**Theorem 2.2.** *The lattice  $L_d(a_1, \dots, a_k)$  is perfect for  $d \geq \max(7, 2(k+1)^3 - 1)$ .*

Theorem 2.2 is an easy consequence of the following two results:

**Proposition 2.3.** *The lattice  $L_d(a_1, \dots, a_k)$  with  $d \geq 7$  is perfect if  $a_1 \geq 2(k+1)^2 + 2$ .*

**Proposition 2.4.** *The  $d$ -dimensional lattice  $L_d(a_1, \dots, a_k)$  is perfect if the subset  $\mathcal{I}_d(a_1, \dots, a_k)$  defined by the  $d+2$  smallest elements of  $\{1, 2, \dots\} \setminus \{a_1, \dots, a_k\}$  contains  $\max(2(k+1)^2 + 1, 9)$  consecutive elements.*

*Proof of Theorem 2.2.* The result holds for  $k = 0$  by Theorem 2.1. Removing a non-empty set of  $k$  integers from  $\{1, 2, \dots\}$  leaves (at most)  $k+1$  subsets of consecutive integers. A partition of  $2(k+1)^3 - 1 + 2 = 2(k+1)^3 + 1$  elements into (at most)  $k+1$  subsets of consecutive integers contains thus a subset having at least  $2(k+1)^2 + 1$  consecutive elements and the result follows from Proposition 2.4. □

*Proof of Proposition 2.3.* Theorem 2.1 shows that the result holds for  $d \in \{7, \dots, 2(k+1)^2 - 1\}$ . As in the proof of Theorem 2.1 we use induction on  $d$  establishing the induction-step through Proposition 1.1. The sublattice of  $L_{d+1}(a_1, \dots, a_k)$  consisting of all elements with last coefficient zero is the lattice  $L_d(a_1, \dots, a_k)$  which is perfect by assumption. We have thus only to prove that the set of all minimal vectors in  $L_{d+1}(a_1, \dots, a_k)$  with last coefficient non-zero spans a vector space of dimension  $d+1$ . For simplicity, we work with  $\mathcal{I}_{d+3}(a_1, \dots, a_k)$  as the set of indices for the  $d+3$  coefficients of elements in  $L_{d+1}(a_1, \dots, a_k)$ . We denote by  $\omega \leq d+3+k$ , respectively  $\psi$  the largest, respectively second-largest, element of  $\mathcal{I}_{d+3}(a_1, \dots, a_k)$ . For every index  $i \in \mathcal{I}_{d+3}(a_1, \dots, a_k)$  with  $i < \psi$  we construct a linear combination  $u(i) = \sum \alpha_v v$  of minimal elements  $v = (v_1, \dots, v_\omega)$  ending with last non-zero coefficient  $v_\omega = 1$  such that  $i$  is the index of the first non-zero coefficient in  $u(i)$ . If  $i < \omega - 2k - 3$ , there exists an integer  $j = j(i)$  in  $\{1, \dots, k+1\}$

such that both integers  $i + j$  and  $\omega - j$  belong to  $\mathcal{I}_{d+3}(a_1, \dots, a_k)$  and we can take  $u(i) = e_i - e_{i+j} - e_{\omega-j} + e_\omega$ . For  $i \in \mathcal{I}_{d+3}(a_1, \dots, a_k)$  such that  $i \in \{\omega - 2k - 2, \dots, \psi - 1\}$  we set  $\alpha = \omega - i$  and  $\beta = \omega - \psi$ . We have  $1 \leq \beta \leq k + 1$  and  $\beta \leq \alpha \leq 2(k + 1)$ . Since  $a_1 \geq 2(k + 1)^2 + 2$ , all integers  $1, 2, \dots, 1 + \alpha\beta \leq 2(k + 1)^2 + 1$  are in  $\mathcal{I}$  and we can consider

$$\begin{aligned} u(i) &= (e_1 - e_{1+\alpha} - e_i + e_\omega) + (e_{1+\alpha} - e_{1+2\alpha} - e_i + e_\omega) \\ &\quad + \dots + (e_{1+(\beta-1)\alpha} - e_{1+\beta\alpha} - e_i + e_\omega) \\ &\quad - (e_1 - e_{1+\beta} - e_\psi + e_\omega) - (e_{1+\beta} - e_{1+2\beta} - e_\psi + e_\omega) \\ &\quad - \dots - (e_{1+(\alpha-1)\beta} - e_{1+\alpha\beta} - e_\psi + e_\omega) \\ &= -\beta e_i + \alpha e_\psi + (\beta - \alpha)e_\omega \end{aligned}$$

which ends the proof. □

*Proof of Proposition 2.4.* Theorem 2.1 shows the result for  $k = 0$ . We assume henceforth  $k \geq 1$  and  $\max(2(k + 1)^2 + 1, 9) = 2(k + 1)^2 + 1$ .

We denote by  $\alpha$  the smallest integer such that  $\{\alpha, \alpha + 1, \dots, \alpha + 2(k + 1)^2\}$  is contained in  $\{1, 2, \dots\} \setminus \{a_1, \dots, a_k\}$ . For  $\alpha = 1$  the proof follows from Proposition 2.3. Otherwise we establish the result by induction on  $k$  and  $d$  using Proposition 1.1. We assume  $\alpha > 1$  and we consider  $L_{d+1}(a_1, \dots, a_k)$ . Since  $L_{d+1}(a_1, \dots, a_k)$  with  $a_1 = 1$  is isomorphic to  $L_{d+1}(a_2 - 1, a_3 - 1, \dots, a_k - 1)$  (which is perfect by induction on  $k$ , the case  $k = 0$  being covered by Theorem 2.1) we can assume  $a_1 > 1$ . Since  $\alpha > 1$ , the sublattice  $L_d(a_1 - 1, a_2 - 1, \dots, a_k - 1)$  consisting of all vectors of  $L_{d+1}(a_1, \dots, a_k)$  with first coefficient zero is perfect by induction on  $d$ . By Proposition 1.1 it is thus enough to show that the set of minimal elements of  $L_{d+1}(a_1, \dots, a_k)$  with first coefficient non-zero spans a  $(d + 1)$ -dimensional vector space. The proof is analogous to the proof of Proposition 2.3 except that we work with small coordinates instead of large ones and that we use the  $2(k + 1)^2 + 1$  indices  $\alpha, \dots, \alpha + 2(k + 1)^2$  instead of the set  $\{1, \dots, 2(k + 1)^2 + 1\}$ . □

**2.2. Examples for Theorem 2.2.** Below we list a few lattices of dimension 7 or 8 illustrating Theorem 2.2 for  $k = 1, 2$ . For  $k = 1$  we indicate the relevant integer  $a_1$  missing in  $1, 2, \dots, d + 3$  together with the determinant, the perfection-default  $\text{pd}(L_d(a_1)) = \binom{d+1}{2} - \dim\left(\sum_{v \in L_d(a_1)} \mathbb{R} v \otimes v\right)$  and the number  $\text{mp}$  of pairs of minimal vectors. The lattice  $L_d(1)$  is obviously isomorphic to the lattice  $L_d = L_d(\emptyset)$  studied previously. Since  $L_d(i)$  and  $L_d(d + 4 - i)$  are isomorphic, it is enough to consider the four cases



$i = 2, 3, 4, 5$  for  $d = 7$ .

lattice	det	pd	mp
$L_7(2)$	$2^2 \cdot 5 \cdot 31$	1	31
$L_7(3)$	$2^3 \cdot 5 \cdot 17$	1	29
$L_7(4)$	$2^4 \cdot 3^2 \cdot 5$	0	28
$L_7(5)$	$2^2 \cdot 5 \cdot 37$	4	28

The perfect lattice  $L_7(4)$  of determinant  $2^4 \cdot 3^2 \cdot 5$  (and defined as the set of all integral vectors in  $\mathbb{Z}^9$  which are orthogonal to  $(1, 1, 1, 1, 1, 1, 1, 1, 1)$  and  $(1, 2, 3, 5, 6, 7, 8, 9, 10)$ ) has a basis given by the rows of

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

with Gram matrix  $AA^t$  given by the matrix

$$P_7^{31} = \begin{pmatrix} 4 & 2 & 2 & 2 & -1 & 2 & 1 \\ 2 & 4 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 & 1 & 2 & 2 \\ -1 & 1 & 1 & 1 & 4 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 4 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 4 \end{pmatrix}$$

at page 383 of [2].

The list of lattices of the form  $L_8(a_i)$  (with  $a_i \in \{2, \dots, 6\}$  in order to avoid duplicates) with a few properties is given by

lattice	det	pd	mp
$L_8(2)$	$2^2 \cdot 3 \cdot 7 \cdot 11$	0	46
$L_8(3)$	$7 \cdot 11 \cdot 13$	0	44
$L_8(4)$	$2^5 \cdot 3 \cdot 11$	0	42
$L_8(5)$	$3^2 \cdot 11^2$	0	42
$L_8(6)$	$2^2 \cdot 5^2 \cdot 11$	3	42

The following table lists all six perfect non-isomorphic lattices of the form  $L_8(a_1, a_2)$  obtained by removing two elements  $a_1, a_2$  from  $\{2, \dots, 11\}$

(we exclude  $a_1 = 1$  in order to avoid perfect lattices of the form  $L_8(a_1)$ ):

lattice	det	pd	mp
$L_8(2, 3)$	$3 \cdot 347$	0	43
$L_8(2, 5)$	$7 \cdot 167$	0	40
$L_8(2, 6)$	$2^4 \cdot 3 \cdot 5^2$	0	39
$L_8(2, 9)$	$3^3 \cdot 43$	0	40
$L_8(2, 10)$	$2^4 \cdot 3 \cdot 23$	0	41
$L_8(3, 5)$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	0	37

**2.3. Bounds for perfection.** Every finite sequence  $a_1 < a_2 < \dots < a_k$  determines a maximal subset  $\mathcal{P}(a_1, \dots, a_k)$  of  $\mathbb{N}$  such that  $L_d(a_1, \dots, a_k)$  has minimum 4 and is perfect for  $d \in \mathcal{P}(a_1, \dots, a_k)$ . We denote now by  $D(a_1, \dots, a_k)$  the successor of the largest missing integer in  $\mathcal{P}(a_1, \dots, a_k)$ . We have  $D(a_1, \dots, a_k) \leq \max(7, 2(k + 1)^3 - 1)$  by Theorem 2.2. Since  $L_d(a_1, \dots, a_k)$  is perfect for every  $d \geq \max(7, 2(k + 1)^3 - 1) \geq D(a_1, \dots, a_k)$  there exists a smallest integer  $d_k = \max_{a_1, \dots, a_k} D(a_1, \dots, a_k)$  (bounded above by  $\max(7, 2(k + 1)^3 - 1)$ ) such that  $L_d(a_1, \dots, a_k)$  is perfect for every  $d \geq d_k$  and for every  $\{a_1, \dots, a_k\}$  in  $\{1, 2, \dots\}$ . We have  $d_0 = 7$  by Theorem 2.1.

For  $k = 1$  we get the numbers

$a_1$	1	2	3	4	5	6	7	8	9	$\geq 10$
$D(a_1)$	7	8	8	7	8	9	7	8	8	7

showing  $d_1 = 9$ .

**Remark.** Analogues of the above numbers and bounds exist of course also for most subsequent constructions.

**2.4. Automorphisms and growth.** The aim of this Section is to sketch a proof of the following result:

**Theorem 2.5.** *The number of non-isomorphic perfect integral lattices of dimension  $d$  and minimum 4 grows faster than any polynomial in  $d$ .*

The proof of Theorem 2.5 is based on Theorem 2.2 and gives an explicit lower bound on the number of perfect integral lattices of dimension  $d$  and minimum 4. This lower bound is unlikely to be sharp: The construction underlying Theorem 2.2 yields probably only a small fraction of all non-isomorphic integral perfect lattices with minimum 4. Moreover, the bounds in Theorem 2.2 are certainly far from optimal.

Two minimal vectors  $v, w \in (L_d)_{\min}$  are *neighbours* if  $\langle v, w \rangle = 2$ .

We call the real number  $\gamma(v) = \frac{2(i+\alpha)+\beta}{2(d+1)} \in (0, 1)$  the (*normalized*) *center* and  $\delta(v) = \frac{2\alpha+\beta}{d+1} \in (0, 1]$  the (*normalized*) *diameter* of a minimal vector

$$v = \pm(e_i - e_{i+\alpha} - e_{i+\alpha+\beta} + e_{i+2\alpha+\beta})$$

in  $L_d$ . We have  $\delta(v) \leq 2 \min(\gamma(v), 1 - \gamma(v))$ .

**Lemma 2.6.** *The number of neighbours in  $L_d$  of a minimal vector  $v$  in  $L_d$  is given by*

$$(2.1) \quad 2d(\min(\gamma(v), 1 - \gamma(v)) + 2 - \delta(v)) + O(1) .$$

Formula (2.1) is bounded above by  $5d$  with asymptotic equality for  $\gamma(v) = \frac{1}{2}$  and  $\delta(v) = 0$  and bounded below by  $3d$  with asymptotic equality for  $\gamma(v) = \frac{1}{2}$  and  $\delta(v) = 1$ .

*Proof of Lemma 2.6.* Neighbours of a minimal vector  $v \in \min(L_d)$  are partitioned into  $6 = \binom{4}{2}$  families according to their two common non-zero coefficients. We denote these families by

$$\mathcal{F}_{**00}, \mathcal{F}_{*0*0}, \mathcal{F}_{*00*}, \mathcal{F}_{0**0}, \mathcal{F}_{0*0*}, \mathcal{F}_{00**}$$

where  $*$  stands for a common non-zero coefficient with respect to the obvious linear order  $i < i + \alpha < i + \alpha + \beta < i + 2\alpha + \beta$  on the indices of the four coefficients of  $v = e_i - e_{i+\alpha} - e_{i+\alpha+\beta} + e_{i+2\alpha+\beta}$ . Neighbours in  $\mathcal{F}_{*00*}$  or in  $\mathcal{F}_{0**0}$  share the center with  $v$  and there are roughly  $d \min(\gamma(v), 1 - \gamma(v))$  possibilities for the remaining smallest nonzero coordinate in each of these families.

The number of neighbours in  $\mathcal{F}_{**00}$  (or in  $\mathcal{F}_{00**}$ ) is roughly given by  $d - \alpha$  and the number of neighbours in  $\mathcal{F}_{*0*0}$  (or in  $\mathcal{F}_{0*0*}$ ) is roughly given by  $d - \alpha - \beta$ .

All errors are bounded by an absolute constant (which is small). Summing over all families and using the definitions of the normalized center and diameter ends the proof. □

*Sketch of proof for Theorem 2.5.* We partition neighbours of a minimal vector  $v$  into six families as in the proof of Lemma 2.6. An element  $u$  of such a family  $\mathcal{F} \in \{\mathcal{F}_{**00}, \mathcal{F}_{*0*0}, \mathcal{F}_{*00*}, \mathcal{F}_{0**0}, \mathcal{F}_{0*0*}, \mathcal{F}_{00**}\}$  is adjacent to all other elements of  $\mathcal{F}$  except perhaps for two elements  $u', u''$  for which we have  $\langle u, u' \rangle = \langle u, u'' \rangle = -1$ . In  $\mathcal{F}_{*00*}$  and  $\mathcal{F}_{0**0}$  there are no exceptions: two distinct elements of  $\mathcal{F}_{*00*}$  or of  $\mathcal{F}_{0**0}$  are always adjacent. Families associated to complementary pairs of indices, like  $\mathcal{F}_{*00*}, \mathcal{F}_{0**0}$  or  $\mathcal{F}_{**00}, \mathcal{F}_{00**}$  or the remaining two sets  $\mathcal{F}_{0*0*}, \mathcal{F}_{*0*0}$  are called *complementary*. Complementary sets are related by a natural involution  $\iota$  defined by  $\iota(u) = u'$  for  $u \in \mathcal{F}$  and  $u' \in \bar{\mathcal{F}}$  such that  $\langle u, u' \rangle = -2$ . Complementary sets have the same number of elements, given by  $d \min(\gamma(v), 1 - \gamma(v)) + O(1)$  for  $\mathcal{F}_{*00*}$  or for  $\mathcal{F}_{0**0}$ . The sets  $\mathcal{F}_{**00}, \mathcal{F}_{00**}$  associated to  $v = e_i - e_{i+\alpha} - e_{i+\alpha+\beta} + e_{i+2\alpha+\beta}$

have  $d - \alpha + O(1)$  elements and the sets  $\mathcal{F}_{0*0*}, \mathcal{F}_{*0*0}$  have  $d - \alpha - \beta + O(1)$  elements. An element  $u \in \mathcal{F}$  is orthogonal to every element  $v$  of the complementary pair  $\overline{\mathcal{F}}$ , except for  $\iota(u)$  and perhaps at most two other elements in  $\overline{\mathcal{F}}$ . The behaviour is simpler in the complementary pair  $\mathcal{F}_{*00*}, \mathcal{F}_{0**0}$ : every element  $u$  of  $\mathcal{F}_{*00*}$  is orthogonal to every element of  $\mathcal{F}_{0**0} \setminus \{\iota(u)\}$ .

These properties allow to reconstruct (at least approximately, however exact coordinates can be found) the coordinates, up to the obvious symmetry  $e_i \mapsto e_{d+3-i}$ , of a minimal element  $v$  from the knowledge of all scalar products between elements of  $(L_d)_{\min}$ . This shows that the lattice  $L_d$  has at most four automorphisms:  $\pm 1$ , perhaps followed by reversal of all coordinates. The same holds for the lattices  $L_d(a_1, \dots, a_k)$ : Knowledge of all scalar products between the set  $(L_d(a_1, \dots, a_k))_{\min}$  of minimal elements determines, up to signs and global reversal of all coordinates, the coefficients (and thus also the missing integers  $a_1, \dots, a_k$ ). In particular, all lattices  $L_d(a_1, \dots, a_k)$  have at most four automorphisms. In the generic case, only the two trivial automorphisms  $\pm 1$  occur. Thus the number of different lattices  $L_d(a_1, \dots, a_k)$  of large dimension  $d$  is at least equal to  $\frac{1}{2} \binom{d+2+k}{k} \geq \frac{d^k}{2 \cdot k!}$ . Since  $k$  is arbitrary, the number of perfect integral lattices of dimension  $d$  with minimum 4 grows faster than any polynomial in  $d$ .  $\square$

### 3. The odd construction

We denote by  $O_d$  the  $d$ -dimensional lattice of all integral vectors in  $\mathbb{Z}^{d+1}$  which are orthogonal to  $(1, 3, 5, \dots, 2d + 1)$ . The lattice  $O_d$  is even and contains no roots. Since  $(1, 3, 5, 7, \dots) = 2(1, 2, 3, 4, \dots) - (1, 1, 1, 1, \dots)$  the lattice  $O_d$  contains  $L_{d-1}$  as a sublattice. Pairs of minimal vectors of  $O_d$  not contained in the sublattice  $L_{d-1}$  are of the form  $e_{2a-1} + e_{2b-1} + e_{2c-1} - e_{2k-1}$  (with indices given by the coefficients of  $(1, 3, 5, \dots)$ ) corresponding to sums  $2k - 1 = (2a - 1) + (2b - 1) + (2c - 1)$  of three distinct odd natural integers  $2a - 1, 2b - 1, 2c - 1 \in \{1, 3, 5, \dots, 2d - 3\}$  adding up to  $2k - 1 \in \{9, 11, \dots, 2d + 1\}$ .

**Theorem 3.1.** *The lattice  $O_d$  has determinant  $\frac{1}{3}(d + 1)(2d + 1)(2d + 3)$  and minimum 4 (for  $d \geq 3$ ). It has*

$$\frac{1}{18} (2d^3 - 3d^2 - 3d + c_d)$$

*pairs of minimal vectors of norm 4 where*

$$c_d = \begin{cases} 0 & \text{if } d \equiv 0 \pmod{3}, \\ 4 & \text{if } d \equiv 1 \pmod{3}, \\ 2 & \text{if } d \equiv 2 \pmod{3}. \end{cases}$$

*The lattice  $O_d$  is perfect for  $d \geq 8$ .*

The lattice  $O_7$  (with 29 pairs of minimal vectors) has perfection default 1 and is thus not perfect.

*Proof of Theorem 3.1.* The squared Euclidean norm of  $(1, 3, 5, \dots, 2d + 1)$  is polynomial of degree 3 in  $d$  and the formula for the determinant of  $O_d$  (given by  $1^2 + 3^2 + 5^2 + \dots + (2d + 1)^2$ ) can thus be checked using 4 values.

The lattice  $O_d$  cannot contain roots and its minimum is obviously 4 (realized e.g. by the vector  $(1, -1, -1, 1, 0, 0, \dots)$ ) if  $d \geq 3$ . The number of pairs of minimal vectors are polynomial functions of degree 3 for  $d$  in arithmetic progressions of length 6. Computation of small examples gives enumerative formulae.

For proving perfection we use Proposition 1.1 with  $\mathcal{H} = (1, 1, 1, \dots, 1)^\perp \subset \mathbb{R}^{d+1}$ . The inclusion of  $L_{d-1} = \mathcal{H} \cap O_d$  in  $O_d$  and perfection of  $L_{\geq 7}$  (see Theorem 2.1) implies that it is enough to show that minimal vectors with non-zero coordinate-sum span a  $d$ -dimensional vector space. As already mentioned, such minimal elements are of the form  $\pm(e_a + e_b + e_c - e_k)$  with  $(2a - 1) + (2b - 1) + (2c - 1) = (2k - 1)$  for four distinct elements  $a, b, c, k$  in  $\{1, \dots, d + 1\}$ . The seven minimal vectors given by the rows of

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

of  $O_7$  are linearly independent and span thus the full 7-dimensional vector space orthogonal to  $(1, 3, 5, 7, 9, 11, 13, 15)$ . The union of these vectors (extended to elements of  $\mathbb{Z}^{d+1}$  by appending zeros) with vectors

$$\begin{aligned} &(1, 1, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, \dots) \\ &(1, 1, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0, \dots) \\ &(1, 1, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, \dots) \\ &\vdots \end{aligned}$$

is a basis of the  $d$ -dimensional vector space  $O_d \otimes_{\mathbb{Z}} \mathbb{R} = (1, 3, \dots, 2d+1)^\perp$ .  $\square$

For a finite increasing sequence  $1 \leq a_1 < a_2 < \dots < a_k$  of  $k$  odd natural integers, we denote by  $O_d(a_1, \dots, a_k) \subset \mathbb{Z}^{d+1}$  the  $d$ -dimensional lattice of all integral vectors orthogonal to  $(1, 3, \dots, a_1 - 2, \widehat{a_1}, a_1 + 2, \dots, a_k - 2, \widehat{a_k}, a_k + 2, \dots, 2(d + k) + 1)$  (elements  $\widehat{a_k}$  carrying a magical hat are removed) with increasing coefficients given by the  $d + 1$  smallest elements of  $\{1, 3, 5, 7, \dots\} \setminus \{a_1, \dots, a_k\}$ .

The following analogue of Theorem 2.2 holds:

**Theorem 3.2.** *Given a strictly increasing sequence  $1 \leq a_1 < a_2 < \dots < a_k$  of  $k$  odd natural integers, the  $d$ -dimensional lattice  $O_d(a_1, \dots, a_k)$  is perfect for  $d \geq \max(10(k + 1)^3 + 5, 22(k + 1) + 2)$ .*

*Proof.* We apply again Proposition 1.1. Since  $O_d(a_1, \dots, a_k)$  contains the lattice  $L_{d-1}((a_1 + 1)/2, (a_2 + 1)/2, \dots, (a_k + 1)/2)$  defined by all elements of  $O_d(a_1, \dots, a_k)$  which are orthogonal to both vectors  $(1, 1, \dots, 1)$  and  $\frac{1}{2}((1, 1, \dots, 1) + (1, 3, 5, \dots, \widehat{a_1}, \dots))$ , we suppose  $d$  large enough in order to ensure perfection of the sublattice  $L_{d-1}((a_1 + 1)/2, (a_2 + 1)/2, \dots, (a_k + 1)/2)$  (which can be done using Theorem 2.2). We show now that minimal vectors  $e_{2a+1} + e_{2b+1} + e_{2c+1} - e_{2l+1}$  (with indices in  $\{1, 3, 5, \dots\} \setminus \{a_1, \dots, a_k\}$ ) of  $O_d(a_1, \dots, a_k)$  corresponding to sums  $2l + 1 = 2a + 1 + 2b + 1 + 2c + 1$  with  $2l + 1, 2a + 1, 2b + 1, 2c + 1 \notin \{a_1, \dots, a_k\}$  span the  $d$ -dimensional vector space orthogonal to  $(1, 3, \dots, a_1 - 2, \widehat{a_1}, a_1 + 2, \dots, a_k - 2, \widehat{a_k}, a_k + 2, \dots, 2(d + k) + 1)$ . This is done in two steps. First we show that for every possible index  $2l + 1 \geq 12(k + 1) + 1$  occurring in elements of  $O_d(a_1, \dots, a_k)$  there exists a minimal vector  $e_{2a+1} + e_{2b+1} + e_{2c+1} - e_{2l+1}$  in  $O_d(a_1, \dots, a_k)$ . The second step deals with the remaining vectors involving only small indices.

Every even integer  $2m \geq 4(k + 1)$  can be written in  $\lfloor m/2 \rfloor \geq k + 1$  different ways as a sum of two odd natural numbers. For any odd integer  $2l + 1 \geq 12(k + 1) + 1$  there exist thus three distinct odd natural numbers  $2a + 1, 2b + 1, 2c + 1 \notin \{a_1, \dots, a_k\}$  with  $2a + 1, 2b + 1 < 2c + 1$  and  $2c + 1 \in \{2l + 1 - 6(k + 1), 2l + 1 - 4(k + 1)\}$  such that  $2l + 1 = 2a + 1 + 2b + 1 + 2c + 1$ : Indeed, we start by choosing an odd integer  $2c + 1 \in \{2l + 1 - 6(k + 1), 2l + 1 - 4(k + 1)\} \setminus \{a_1, \dots, a_k\}$ . The even integer  $2l + 1 - (2c + 1) \in \{4(k + 1), \dots, 6(k + 1)\}$  can now be written as a sum of two distinct odd integers  $2a + 1, 2b + 1$  not in  $\{a_1, \dots, a_k\}$ . Since  $2l + 1 \geq 12(k + 1) + 1$ , we have  $2c + 1 > \frac{2l + 1}{2}$  and thus all three integers  $2a + 1, 2b + 1, 2c + 1 \in \{1, 3, 5, \dots\} \setminus \{a_1, \dots, a_k\}$  are distinct. This completes the proof of the first step.

We end now the proof by showing that the span of minimal vectors as above (i.e. of the form  $e_{2a+1} + e_{2b+1} + e_{2c+1} - e_{2l+1}$ ) with  $2l + 1 \geq 12(k + 1) + 1$  contains the vector space  $V$  of all vectors which are orthogonal to  $(1, 3, \dots, \widehat{a_1}, \dots)$  and which involve only non-zero coefficients with indices  $\leq A = \max(6(k + 1)^3 + 3, 14(k + 1) + 1)$ . We have  $2a + 1 + 2b + 1 \leq 2A - 2$  for two distinct odd integers  $a, b \leq A$ . There exists thus an odd integer  $2c + 1 \in \{A + 3, \dots, A + 2(k + 1) + 3\}$  such that  $2c + 1$  and  $2l + 1 = 2a + 1 + 2b + 1 + 2c + 1 \leq 3A + 2(k + 1) + 1 \leq \max(20(k + 1)^3 + 10, 44(k + 1) + 4) \leq 2d + 1$  are different from  $a_1, \dots, a_k$ . Given four distinct odd integers  $2a + 1, 2b + 1, 2\alpha + 1, 2\beta + 1$  in  $\{1, 3, \dots, A\} \setminus \{a_1, a_2, \dots, a_k\}$  such that  $2a + 1 + 2b + 1 = 2\alpha + 1 + 2\beta + 1$ , we can consider the vector

$$\begin{aligned} & (e_{2a+1} + e_{2b+1} + e_{2c+1} - e_{2(a+b+c)+3}) - (e_{2\alpha+1} + e_{2\beta+1} + e_{2c+1} - e_{2(a+b+c)+3}) \\ & = e_{2a+1} - e_{2\alpha+1} - e_{2\beta+1} + e_{2b+1}. \end{aligned}$$

Since the vector space orthogonal to  $(1, 3, \dots, \hat{a}_1, \dots)$  involving no indices exceeding  $A$  is at least of dimension

$$\frac{A-1}{2} - k - 1 \geq \frac{6(k+1)^3 + 2}{2} - k - 1 \geq 2(k+1)^3,$$

minimal elements of the form  $e_{2\alpha+1} - e_{2\alpha+1} - e_{2\beta+1} + e_{2b+1}$  with all indices  $\leq A$  span a perfect lattice by Theorem 2.2. Adding a vector of the form  $e_{2a+1} + e_{2b+1} + e_{2c+1} - e_{2l+1}$  with  $2l+1 \in \{12(k+1)+1, \dots, 14(k+1)+1\}$  which exists by the discussion of step 1 we get a generating set of  $V$ . This completes the proof of the second step and establishes Theorem 3.2.  $\square$

**Remark.** The bound in Theorem 3.2 (and similar bounds occurring elsewhere) is not optimal and can be improved by more careful arguments.

A few data for the lattices  $O_8(i)$  with pd indicating the perfection default and with mp indicating the number of pairs of minimal vectors are

lattice	det	pd	mp
$O_8(1)$	$3 \cdot 443$	4	38
$O_8(3)$	1321	3	37
$O_8(5)$	$3^2 \cdot 5 \cdot 29$	1	37
$O_8(7)$	$3 \cdot 7 \cdot 61$	0	38
$O_8(9)$	1249	2	38
$O_8(11)$	$3 \cdot 13 \cdot 31$	1	39
$O_8(13)$	$3^3 \cdot 43$	0	40
$O_8(15)$	$5 \cdot 13 \cdot 17$	1	41
$O_8(17)$	$3 \cdot 347$	0	43

For the lattices  $O_9(i)$  the data are

lattice	det	pd	mp
$O_9(1)$	$2 \cdot 3 \cdot 5 \cdot 59$	2	59
$O_9(3)$	$2 \cdot 881$	0	56
$O_9(5)$	$2 \cdot 3^2 \cdot 97$	0	56
$O_9(7)$	$2 \cdot 3 \cdot 7 \cdot 41$	0	56
$O_9(9)$	$2 \cdot 5 \cdot 13^2$	0	57
$O_9(11)$	$2 \cdot 3 \cdot 5^2 \cdot 11$	0	58
$O_9(13)$	$2 \cdot 3^2 \cdot 89$	0	59
$O_9(15)$	$2 \cdot 773$	0	60
$O_9(17)$	$2 \cdot 3 \cdot 13 \cdot 19$	0	62
$O_9(19)$	$2 \cdot 3 \cdot 5 \cdot 47$	0	64

The lattice  $O_{10}(1)$  (with determinant  $11^2 \cdot 19$  and 81 pairs of minimal vectors) is perfect.

### 4. The even-sublattice construction

The even-sublattice construction is defined as the  $d$ -dimensional lattice  $M_d$  consisting of all integral vectors  $(x_0, \dots, x_d) \in \mathbb{Z}^{d+1}$  which are orthogonal to  $(0, 1, 2, \dots, d)$  and have even coordinate-sum  $\sum_{i=0}^d x_i \equiv 0 \pmod{2}$ . Its minimal vectors are given by the pair  $\pm(2, 0, 0, \dots)$ , vectors  $\pm(e_i - e_{i+\alpha} - e_{i+\alpha+\beta} + e_{i+2\alpha+\beta})$  with  $i \in \{0, \dots, d\}$  and  $\alpha, \beta \geq 1$  such that  $i + 2\alpha + \beta \leq d$  together with vectors  $\pm(e_h + e_i + e_j - e_k)$  where  $h, i, j, k$  are four distinct integers in  $\{0, \dots, d\}$  such that  $h + i + j = k$ .

**Theorem 4.1.** *The lattice  $M_d$  has determinant  $\frac{2}{3}d(d + 1)(2d + 1)$  and minimum 4. It has*

$$\frac{1}{36} (4d^3 - 3d^2 - 6d + c_d)$$

*pairs of minimal vectors of norm 4 where  $c_d$  depends only on  $d \pmod{6}$  and is given by*

$d \pmod{6}$	0	1	2	3	4	5
$c_d$	36	41	28	45	32	37

*The lattice  $M_d$  is perfect for  $d \geq 8$ .*

The vector space  $\sum_{v \in (M_7)_{\min}} \mathbb{R} v \otimes v$  associated to the 34 pairs of minimal vectors in  $M_7$  is of dimension 27. The lattice  $M_7$  with perfection-default  $28 - 27 = 1$  is thus not perfect.

*Proof of Theorem 4.1.*  $M_d$  contains  $L_{d-1}$  as a sublattice. Since  $d \geq 8$ , the lattice  $L_{d-1}$  is perfect by Theorem 2.1. Hence Proposition 1.1 shows that it is enough to prove that minimal vectors with non-zero coordinate-sum span the full  $d$ -dimensional space  $M_d \otimes_{\mathbb{Z}} \mathbb{R} = (0, 1, 2, \dots, d)^\perp \subset \mathbb{R}^{d+1}$ . This holds for  $d \geq 5$  by linear independency of the five rows

$$\begin{matrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & -1 & 0 & 1 \end{matrix}$$

(with a suitable number of additional zero-coordinates) together with minimal elements of the form  $-e_0 - e_1 - e_{i-1} + e_i$  for  $i = 6, 7, \dots, d$ . □

For a strictly increasing sequence  $0 \leq a_1 < a_2 < \dots < a_k$  of  $k$  natural integers, we denote by  $M_d(a_1, \dots, a_k)$  the  $d$ -dimensional lattice of all integral vectors with even coordinate-sum which are orthogonal to  $(0, 1, \dots, a_1 - 1, \widehat{a_1}, a_1 + 1, \dots, a_k - 1, \widehat{a_k}, a_k + 1, \dots, d + k) \in \mathbb{Z}^{d+1}$ .



The following analogue of Theorem 4.1 holds:

**Theorem 4.2.** *Given a finite strictly increasing sequence  $0 \leq a_1 < a_2 < \dots < a_k$  of  $k$  natural integers, the  $d$ -dimensional lattice  $M_d(a_1, \dots, a_k)$  is perfect for  $d$  large enough.*

The proof, similar to the proof of Theorem 3.2, is left to the reader.

A few data with pd indicating the perfection default and with mp indicating the number of minimal pairs in lattices  $M_8(i)$  are

lattice	det	pd	mp
$M_8(0)$	$2^2 \cdot 3 \cdot 5 \cdot 19$	1	41
$M_8(1)$	$2^4 \cdot 71$	1	42
$M_8(2)$	$2^2 \cdot 281$	1	42
$M_8(3)$	$2^4 \cdot 3 \cdot 23$	0	42
$M_8(4)$	$2^2 \cdot 269$	3	44
$M_8(5)$	$2^4 \cdot 5 \cdot 13$	2	44
$M_8(6)$	$2^2 \cdot 3 \cdot 83$	0	45
$M_8(7)$	$2^4 \cdot 59$	0	47
$M_8(8)$	$2^2 \cdot 13 \cdot 17$	1	49

For the lattices  $M_9(i)$  the data are

lattice	det	pd	mp
$M_9(0)$	$2^2 \cdot 5 \cdot 7 \cdot 11$	0	61
$M_9(1)$	$2^9 \cdot 3$	0	61
$M_9(2)$	$2^2 \cdot 3 \cdot 127$	0	62
$M_9(3)$	$2^5 \cdot 47$	0	61
$M_9(4)$	$2^2 \cdot 3^2 \cdot 41$	0	64
$M_9(5)$	$2^5 \cdot 3^2 \cdot 5$	1	64
$M_9(6)$	$2^2 \cdot 349$	0	65
$M_9(7)$	$2^6 \cdot 3 \cdot 7$	0	66
$M_9(8)$	$2^2 \cdot 3 \cdot 107$	0	69
$M_9(9)$	$2^6 \cdot 19$	0	70

## 5. A construction using finite abelian groups

To a finite abelian group  $A$  indexing the coordinates of  $\mathbb{Z}^A$  we associate the integral lattice  $L(A)$  consisting of all elements  $v = (v_a)_{a \in A} \in \mathbb{Z}^A$  such that  $\sum_{a \in A} v_a = 0 \in \mathbb{Z}$  and  $\sum_{a \in A} v_a a = 0 \in A$  (i.e. vectors  $v \in \mathbb{Z}^A$  of coefficient-sum zero such that the element  $\sum_a v_a a$  of  $A$  is the identity 0 of

the finite additive group  $A$ ). Equivalently,  $L(A)$  is the set of all elements in the kernel of the augmentation ideal in the group-algebra  $\mathbb{Z}[A]$  of  $A$  over  $\mathbb{Z}$ . The lattice  $L(A)$  is even and without roots. It has rank  $|A| - 1$  and determinant  $|A|^3$ . The semidirect product  $\text{Aut}(A) \ltimes A$  acts isometrically on  $L(A)$  in the obvious way. Vectors of norm 4 in  $L(A)$  determine the group  $A$  uniquely as follows: An arbitrary index of a basis element can be chosen as the identity 0 of  $A$ . A vector  $e_0 - e_a - e_b + e_c$  yields the identity  $a + b = c$  in  $A$ .

The number of pairs of minimal vectors of norm 4 in  $L(A)$  is given by the following result:

**Proposition 5.1.** *The number of pairs or vectors of norm 4 in  $L(A)$  is given by*

$$|A| \left(1 - \frac{1}{2^c}\right) \binom{|A|/2}{2} + \frac{|A|}{2^c} \binom{(|A| - 2^c)/2}{2}$$

where  $c$  is the minimal number of generators of the 2-torsion subgroup in  $A$ . Equivalently,  $c$  is the largest integer such that  $A$  contains a subgroup isomorphic to the  $c$ -dimensional vector space  $\mathbb{F}_2^c$  over the field  $\mathbb{F}_2$  of two elements.

*Proof.* We count for each element  $a$  of  $A$  the number  $N_a$  of solutions of the equation  $x + y = a$  with  $x, y$  two different elements in  $A$ . The total number of pairs of vectors of norm 4 in  $L(A)$  is then given by  $\sum_{a \in A} \binom{N_a/2}{2}$  since such pairs are given by  $\pm(e_{x_1} + e_{y_1} - (e_{x_2} + e_{y_2}))$  with  $\{x_1, y_1\} \neq \{x_2, y_2\}$  such that we have the equality  $x_1 + y_1 = x_2 + y_2$  in  $A$ .

The kernel of the endomorphism of  $A$  defined by  $x \mapsto 2x$  is an  $\mathbb{F}_2$ -vector space of dimension  $c$ . We denote by  $2A$  its image (of size  $\frac{|A|}{2^c}$ ) in  $A$ . For an element  $a \in A \setminus (2A)$  there are  $N_a = |A|$  solutions to the equation  $x + y = a$  with  $x \neq y$  and there are  $|A| \left(1 - \frac{1}{2^c}\right)$  elements in  $A \setminus (2A)$ .

If  $a$  is one of the  $\frac{|A|}{2^c}$  elements in  $2A$ , there are  $2^c$  solutions of  $2x = a$  and the equation  $x + y = a$  has thus only  $N_a = |A| - 2^c$  solutions with  $x$  different from  $y$ . □

**Remark.** The algebraic identity

$$|A| \left(1 - \frac{1}{2^c}\right) |A|/2 + \frac{|A|}{2^c} (|A| - 2^c)/2 = \binom{|A|}{2}$$

encodes the fact that  $A$  contains  $\binom{|A|}{2}$  pairs of distinct elements.

**Theorem 5.2.** *The lattice  $L(A)$  associated to an abelian group having at least 9 elements is perfect.*

Some lattices  $L(A)$  associated to abelian groups  $A$  with less than 9 elements are perfect. The lattice  $L((\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$  is however not perfect

(the other two abelian groups with 8 elements and the cyclic group with 7 elements give rise to perfect lattices, see Sections 5.2.2 and 5.2.1).

Given a subset  $\mathcal{A}$  of a finite abelian group  $A$ , we define the lattice  $L(\mathcal{A})$  as the sublattice of  $L(A)$  generated by all vectors of  $L(A)$  involving no elements of  $A \setminus \mathcal{A}$ .

We have the following generalization of Theorem 5.2:

**Theorem 5.3.** *For a fixed integer  $k$  there are only finitely many isomorphism classes of pairs  $(\mathcal{A} \subset A)$  where  $A$  is a finite abelian group and where  $\mathcal{A}$  is a subset of  $A$  with  $A \setminus \mathcal{A}$  containing at most  $k$  elements such that  $L(\mathcal{A})$  is not perfect.*

**5.1. Proofs.**

*Proof of Theorem 5.2.* We establish Theorem 5.2 first for cyclic groups. It holds for  $A = \mathbb{Z}/7\mathbb{Z}$  and  $A = \mathbb{Z}/8\mathbb{Z}$  by a direct computation left to the reader (see also Sections 5.2.1 and 5.2.2). For  $N \geq 9$  the  $(N-1)$ -dimensional lattice  $L(\mathbb{Z}/N\mathbb{Z})$  contains the perfect lattice  $L_{N-2}$  as a sublattice, see Theorem 2.1. By Proposition 1.1 we need to show that minimal vectors of  $L(\mathbb{Z}/N\mathbb{Z})$  not orthogonal to  $(0, 1, 2, \dots, N-1)$  span the  $(N-1)$ -dimensional vector space  $(1, 1, \dots, 1)^\perp$ .

We consider first the  $N-3$  minimal vectors

$$\begin{aligned} v_2 &= e_0 + e_1 - e_2 - e_{N-1}, \\ v_3 &= e_0 + e_2 - e_3 - e_{N-1}, \\ v_4 &= e_0 + e_3 - e_4 - e_{N-1}, \\ &\vdots \\ v_{N-3} &= e_0 + e_{N-4} - e_{N-3} - e_{N-1}, \\ v_{N-2} &= e_0 + e_{N-3} - e_{N-2} - e_{N-1} \end{aligned}$$

defining the rows of the  $(N-3) \times N$  matrix

$$M = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ \vdots & & & & \ddots & \ddots & & & & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

which has obviously rank  $N-3$  (column indices are the representatives  $0, 1, \dots, N-1$  of  $\mathbb{Z}/N\mathbb{Z}$ ). It is easy to check that  $M$  (acting on row-vectors) has a kernel spanned by the all one vector  $(1, 1, \dots, 1, 1) \in \mathbb{Z}^N$  and by the

two elements

$$\begin{aligned} w_1 &= (1, 0, 0, 0, 0, \dots, 0, 0, 0, 1), \\ w_2 &= (1, 2, 3, \dots, N - 2, N - 1, 0) \end{aligned}$$

of  $\mathbb{Z}^N$ . We consider now two additional minimal vectors with signed index-sum  $N$  given by

$$\begin{aligned} v_0 &= e_0 + e_1 - e_3 - e_{N-2}, \\ v_1 &= e_1 + e_2 - e_4 - e_{N-1}. \end{aligned}$$

Since

$$\begin{pmatrix} \langle w_1, v_0 \rangle & \langle w_1, v_1 \rangle \\ \langle w_2, v_0 \rangle & \langle w_2, v_1 \rangle \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 - N & 0 \end{pmatrix}$$

is invertible, the vectors  $v_0, \dots, v_{N-1}$  are linearly independent.

In the general case we have to show that linear combinations of rank 1 matrices with coefficients  $v_a v_b, a, b \in A$  for  $v = (v_a)_{a \in A} \in L(A)_{\min}$  have arbitrary off-diagonal coefficients. Let  $(a, b)$  be the index of such an off-diagonal coefficient. By translation-invariance we can suppose  $a = 0$ . If  $b$  is contained in a cyclic group of order  $\geq 7$  we are in the previous case. We can thus assume that the cardinality of  $A$  is divisible only by primes  $\leq 5$ .

If 5 and either 2 or 3 divide the cardinality of  $A$ , then every non-zero element of  $A$  is contained in a cyclic subgroup of order at least 10 and we are done. Otherwise, a non-trivial element of  $A$  is either contained in a cyclic group of order 25 (and we are done) or in  $(\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z})$  and  $L((\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}))$  is perfect by a direct computation (using a Computer-Algebra system).

We are left with the remaining cases where every cyclic subgroup containing  $b$  is of order 2, 3, 4 or 6. If  $b$  is only contained in a cyclic group of order 2, the result follows from perfection of the two groups  $L((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$  and  $L((\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}))$ . If  $b$  is only contained in a cyclic group of order 3, the result follows from perfection of  $L((\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}))$ . If  $b$  is contained in a cyclic group of order 6, the result follows from perfection of  $L((\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$  and  $L((\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}))$ . □

**Proposition 5.4.** *If  $N$  is large enough then  $L(\mathbb{Z}/N\mathbb{Z} \setminus \{a_1, \dots, a_k\})$  is perfect for every subset  $\{a_1, \dots, a_k\}$  of  $k$  elements in  $\mathbb{Z}/N\mathbb{Z}$ .*

*Proof.*  $L(\mathbb{Z}/N\mathbb{Z} \setminus \{a_1, \dots, a_k\})$  contains the lattice  $L_{N-k-2}(a_1 + 1, \dots, a_k + 1)$  as a sublattice (we represent elements of  $\mathbb{Z}/N\mathbb{Z}$  by natural integers in  $\{0, \dots, N - 1\}$ ) and this sublattice is perfect for  $N \geq \max(9 + k, 2(k + 1)^3 + k + 1)$  by Theorem 2.2. It is thus enough to show that minimal vectors with signed indices summing up to  $N$  generate the whole vector-space  $L(\mathbb{Z}/N\mathbb{Z} \setminus \{a_1, \dots, a_k\}) \otimes_{\mathbb{Z}} \mathbb{R}$ . This can be done (with an effective lower bound on  $N$ ) as in the proof of Theorem 3.2. □

**Proposition 5.5.** *There exists an integer  $N = N_k$  such that  $L(\mathcal{A})$  is perfect if the finite abelian group  $A$  containing  $\mathcal{A}$  has an element of prime-order at least  $N$  and if  $A \setminus \mathcal{A}$  has at most  $k$  elements.*

*Proof.* We identify tensor products  $v \otimes v$  defined by elements  $v$  in  $L(\mathcal{A})$  with symmetric matrices whose rows and columns are indexed by  $\mathcal{A}$ . It is enough to show that all such matrices with exactly two non-zero diagonal entries and two off-diagonal non-zero entries defining a symmetric submatrix of the form  $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$  are sums of symmetric matrices associated to minimal elements in  $L(\mathcal{A})$ . Up to a translation (of  $\mathcal{A}$  and all indices) we can assume that the first diagonal entry is associated to the trivial element 0 in  $A$ . The second diagonal element is then associated to a certain non-zero element  $b \in A$  contained in a cyclic group of order at least  $N$  and we are done by Proposition 5.4. □

*Proof of Theorem 5.3.* As in the proof of Proposition 5.5 we want to realize a symmetric matrix corresponding to  $-e_0 \otimes e_0 + e_0 \otimes e_b + e_b \otimes e_0 - e_b \otimes e_b$  (up to a suitable translation), perhaps modulo diagonal matrices. In particular, we can suppose that  $\mathcal{A}$  contains the trivial element 0. Proposition 5.4 shows that we can assume that every cyclic group containing  $b$  is small. The group  $A$  (if it is huge) has then a huge number of distinct subgroups. In particular, we can suppose that it contains a non-trivial translate  $b + B \neq B$  of a group  $B$  containing  $a$  with  $L(B)$  perfect (this is the case if  $B$  has at least 9 elements by Theorem 5.2). We may now consider the symmetric matrix  $P$  associated to the tensor-product

$$v_1 \otimes v_1 + v_2 \otimes v_2 + \dots + v_\alpha \otimes v_\alpha$$

where  $\alpha$  is the order of  $a$  and where

$$v_i = e_0 - e_a + e_{b+(i+1)a} - e_{b+ia}$$

for  $i = 1, \dots, \alpha$ . We have  $P_{0,a} = P_{a,0} = \alpha$  and all other non-zero coefficients of  $P$  are either diagonal or have both indices in  $b + B$ . Coefficients of the last form can be killed using perfection of  $L(B)$ . □

**Remark.** Our proof of Theorem 5.3 can be unravelled in order to yield effective bounds on the size of  $A$ .

**5.2. Examples.** There are no interesting examples in dimension  $< 6$ .

**5.2.1. Dimension 6.** The 6-dimensional lattice  $L(\mathbb{Z}/7\mathbb{Z})$  associated to the unique group with seven elements has 21 pairs of minimal elements and

is perfect. A basis is given by the six rows of

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

(with columns indexed by the representatives 0, 1, 2, 3, 4, 5, 6 of  $\mathbb{Z}/7\mathbb{Z}$ ). Its Gram matrix is the matrix

$$P_6^5 = \begin{pmatrix} 4 & 2 & 2 & 1 & 2 & 2 \\ 2 & 4 & 2 & 2 & 1 & 2 \\ 2 & 2 & 4 & 0 & 2 & 1 \\ 1 & 2 & 0 & 4 & 1 & 2 \\ 2 & 1 & 2 & 1 & 4 & 0 \\ 2 & 2 & 1 & 2 & 0 & 4 \end{pmatrix}$$

at page 381 in Chapter XIV of [2].

**5.2.2. Dimension 7.** There are 3 groups with 8 elements.

For the cyclic group  $\mathbb{Z}/8\mathbb{Z}$  we get

$$\left( 8 \binom{4}{2} + 4 \binom{3}{2} \right) = 36$$

pairs of minimal vectors in the associated 7-dimensional lattice  $L(\mathbb{Z}/8\mathbb{Z})$  which is perfect and has a basis given by the seven rows of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$$

with associated Gram matrix  $AA^t$  the matrix

$$P_7^5 = \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & -1 & 1 \\ 2 & 4 & 2 & 2 & -1 & 1 & -1 \\ 2 & 2 & 4 & 1 & -1 & -1 & 1 \\ 2 & 2 & 1 & 4 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 4 & -2 & 0 \\ -1 & 1 & -1 & 1 & -2 & 4 & -2 \\ 1 & -1 & 1 & -1 & 0 & -2 & 4 \end{pmatrix}$$

of page 382 in [2].

The lattice  $L((\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}))$  (with 38 pairs of minimal vectors) has perfection-default 2 and is thus not perfect.

The lattice  $L((\mathbb{Z}/2) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})) = L(\mathbb{F}_2^3)$  with 42 pairs of minimal vectors has a basis given by the seven last rows of the table

000	001	010	011	100	101	110	111
0	0	1	-1	1	-1	0	0
-1	1	0	0	1	-1	0	0
0	1	1	0	0	-1	-1	0
0	0	1	-1	0	0	-1	1
-1	0	1	0	1	0	-1	0
0	1	0	-1	1	0	-1	0
0	0	0	0	1	-1	-1	1

(with the first row showing all elements of  $\mathbb{F}_2^3$  corresponding to column-indices). The associated Gram matrix has only even entries. Dividing it by 2 we get the matrix

$$P_7^A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

(see page 382 in [2]) defining the root lattice  $\mathbb{D}_7$ .

**Remark.** Even parity of all scalar products between minimal vectors fails for the lattices  $L(\mathbb{F}_2^k)$  with  $k \geq 4$ .

**5.2.3. Dimension 8.** Both 8-dimensional lattices  $L(\mathbb{Z}/9\mathbb{Z})$  and  $L((\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}))$  have 54 pairs of minimal vectors and are perfect. They are non-isomorphic: Every pair of minimal vectors is orthogonal to exactly 15 pairs of minimal vectors in  $L(\mathbb{Z}/9\mathbb{Z})$  and every such pair is orthogonal to exactly 9 pairs of minimal vectors in  $L((\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}))$ .

**5.3. Examples with one missing element.** The obvious action of  $A$  on itself shows that all the lattices  $L(\mathcal{A})$  are isomorphic if  $\mathcal{A}$  is obtained by removing a unique element from  $A$ . The lattice  $L(A \setminus \{0\})$  has

$$|A| \binom{1 - \frac{1}{2^c}}{2} \binom{|A|/2 - 1}{2} + \frac{|A| - 2^c}{2^c} \binom{(|A| - 2^c)/2 - 1}{2} + \binom{(|A| - 2^c)/2}{2}$$

pairs of minimal vectors with norm 4 (where  $c$  denotes the dimension of the maximal  $\mathbb{F}_2$ -vector space occurring as a subgroup in  $A$ ).

**5.4. The root lattice  $A_6$ .** Working with the set  $\mathcal{A}$  consisting of all seven non-zero elements 001, 010, 011, 100, 101, 110, 111 in  $\mathbb{F}_2^3$  we get the perfect rescaled root lattice  $A_6$  generated by the last seven rows (with the first row indicating the index-set  $\mathcal{A}$ ) of

001	010	011	100	101	110	111
0	0	0	1	1	-1	-1
0	1	1	0	0	-1	-1
-1	0	1	1	0	-1	0
-1	1	0	0	1	-1	0
-1	1	0	1	0	0	-1
-1	0	1	0	1	0	-1

Identifying the seven elements of  $\mathcal{A}$  in the obvious way with the seven points of the Fano plane (projective plane over  $\mathbb{F}_2$ ) we can consider pairs of minimal vectors of  $L(\mathcal{A})$  (i.e. pairs of opposite roots of  $A_6$ ) as projective lines endowed with marked points (or, dually, as points together with incident lines) as follows: The two coordinates corresponding to coefficients 1 and the two coordinates corresponding to coefficients  $-1$  of a minimal vector define projective lines which meet at a point on the projective line associated to the three coordinates corresponding to coefficients 0. Up to multiplication by  $-1$ , this construction is one-to-one and yields the  $21 = 7 \times 3$  pairs of roots of  $A_6$ .

The Gram matrix associated to the basis of  $L(\mathcal{A})$  given above is twice the matrix

$$P_6^7 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

(which is the Gram matrix with respect to the basis  $e_0 - e_1, \dots, e_0 - e_6$  of  $A_6$ ) in Chapter XIV of [2].

**5.5. Two perfect examples of dimension 7.** The subset  $\mathcal{A} = \{1, \dots, 8\}$  of  $\mathbb{Z}/9\mathbb{Z}$  defines a perfect 7-dimensional lattice  $L(\mathcal{A})$  with 30 pairs of minimal vectors. A basis is given by the seven rows of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$



(with column-indices representing  $1, \dots, 8 \in \mathbb{Z}/9\mathbb{Z}$ ) with associated Gram matrix  $AA^t$  given by the matrix

$$P_7^{28} = \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 1 & 1 & 0 & 2 \\ 2 & 2 & 4 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 4 & 0 & 2 \\ 2 & 0 & 1 & 2 & 0 & 4 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 4 \end{pmatrix}$$

(with determinant  $2^3 \cdot 3^4$ ) of Chapter XIV in [2].

The last seven rows of the table

01	02	10	11	12	20	21	22
1	-1	1	0	-1	0	0	0
1	0	1	0	0	-1	-1	0
0	0	1	-1	0	0	-1	1
1	0	0	-1	-1	0	0	1
1	-1	0	0	0	0	-1	1
0	0	1	0	-1	-1	0	1
0	0	0	-1	1	-1	0	1

(the first row displays the column indices  $\alpha\beta$  with  $(\alpha, \beta) \in (\mathbb{Z}/3\mathbb{Z})^2 \setminus \{0, 0\}$ ) define the perfect 7-dimensional lattice  $L((\mathbb{Z}/3\mathbb{Z}) + (\mathbb{Z}/3\mathbb{Z}) \setminus \{0, 0\})$  with 30 pairs of minimal vectors. Its Gram matrix is

$$P_7^{27} = \begin{pmatrix} 4 & 2 & 1 & 2 & 2 & 2 & -1 \\ 2 & 4 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 4 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 4 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 4 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 4 & 1 \\ -1 & 1 & 2 & 1 & 1 & 1 & 4 \end{pmatrix}$$

in Chapter XIV of [2].

## 6. The even sublattice construction for abelian groups

Given a finite abelian group  $A$  indexing the coordinates of  $\mathbb{Z}^A$ , we denote by  $M(A/(\pm 1))$  the even sublattice of  $\mathbb{Z}^{A/(\pm 1)}$  consisting of all elements  $v = (v_a)_{a \in A/(\pm a)}$  such that  $\sum_{a \in A/(\pm 1)} v_a \equiv 0 \pmod{2}$  (this ensures evenness of  $M(A/(\pm 1))$ ) and such that  $\sum_{a \in A/(\pm 1)} v_a a = 0 \in A$  where  $A/(\pm 1)$  denotes (somewhat abusively) a set of representatives of  $A$  under the involutive automorphism  $a \mapsto -a$ . The lattice  $M(A/(\pm 1))$  is without roots. It has rank  $|A/(\pm 1)|$  and determinant  $4|A|^2$ . Vectors of norm 4 in  $M(A/(\pm 1))$  are of the form  $\pm 2e_a$  if  $2a = 0$  in  $A$  for  $a \in A/(\pm 1)$  or of the form  $\pm e_{a_1} \pm e_{a_2} \pm$

$e_{a_3} \pm e_{a_4}$  if  $\pm a_1 \pm a_2 \pm a_3 \pm a_4 = 0$  in  $A$  for four distinct elements  $a_1, \dots, a_4$  of  $A/(\pm 1)$  with  $\pm$  denoting suitable choices of signs. The subgroup of all elements of order at most 2 acts by isometries on  $M(A/(\pm 1))$  and the group  $A$  can be recovered (up to isometries) from the set of minimal vectors of norm 4 in  $M(A/(\pm 1))$ .

**Theorem 6.1.** *The lattice  $M((\mathbb{Z}/N\mathbb{Z})/(\pm 1))$  associated to a cyclic group of order  $\geq 15$  is perfect.*

Theorem 6.1 can probably be generalized to arbitrary finite abelian groups which are sufficiently large. It should have a further generalization obtained by removing  $k$  elements from  $A/(\pm 1)$ .

*Proof of Theorem 6.1.* We consider first a cyclic group  $A = \mathbb{Z}/N\mathbb{Z}$  of even order  $N = 2m$ . Representatives of  $A/(\pm 1)$  are  $\{0, 1, \dots, m\}$ . For  $N \geq 16$ , the lattice  $M(A/(\pm 1))$  contains the perfect sublattice

$$M_{m-1} = M(A/(\pm 1)) \cap (0, 1, 2, \dots, m)^\perp,$$

see Theorem 4.1.

We set  $v_i = -e_i + e_{i+1} + e_{m-1} + e_m$  for  $i = 0, \dots, m - 3$ . The minimal elements  $v_0, \dots, v_{m-3}$  together with  $2e_0 = (e_0 + e_1 + e_{m-1} + e_m) - (-e_0 + e_1 + e_{m-1} + e_m)$ ,  $2e_m$  and  $e_1 + e_2 + e_{m-2} + e_{m-1}$  are linearly independent. Since the corresponding signed index-sum  $-i + i + (m - 1) + m = 2m$  (respectively  $0 + 1 + (m - 1) + m = 2m$  and  $2m$ ) is non-zero they are not orthogonal to  $(0, 1, 2, \dots, m)$ . Perfection of  $M((\mathbb{Z}/N\mathbb{Z})/(\pm 1))$  for even  $N \geq 16$  follows now from Proposition 1.1.

For a cyclic group  $N = 2m + 1$  of odd order  $2m + 1$  we proceed as follows: The  $m - 3$  linearly independent minimal elements  $-e_i + e_{i+2} + e_{m-1} + e_m, i = 0, \dots, m - 4$  can be completed to a base by adjoining the following four elements

$$\begin{aligned} u_1 &= (1, 0, 1, 0, 1, 0, 1, 0, \dots, 0, 0) \\ u_2 &= (1, 1, 1, 1, \dots, 1, 1, 0, 0) \\ u_3 &= (0, 1, 2, 3, 4, 5, \dots, m - 3, m - 2, -1, -1) \\ u_4 &= (0, 0, 0, 0, 0, \dots, 0, 0, 0, 1, -1) \end{aligned}$$

( $u_1$  has alternating coefficients 0, 1 except for the last two coefficients which are both zero) which are orthogonal to  $-e_i + e_{i+2} + e_{m-1} + e_m$  for  $i \in \{0, \dots, m - 4\}$ . We consider now four minimal vectors given by

$$\begin{aligned} w_1 &= e_0 + e_2 + e_{m-1} + e_m \\ w_2 &= e_0 + e_3 + e_{m-2} + e_m \\ w_3 &= e_0 + e_4 + e_{m-2} + e_{m-1} \\ w_4 &= e_1 + e_3 + e_{m-2} + e_{m-1} \end{aligned}$$

The matrix  $S$  of scalar products  $S_{i,j} = \langle w_i, u_j \rangle$  equals

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ 1 + \epsilon & 3 & m & -1 \\ 2 + \epsilon & 3 & m + 1 & 1 \\ 3 + \epsilon & 3 & m + 1 & 1 \end{pmatrix}$$

where  $\epsilon = 0$  if  $m$  is odd and  $\epsilon = 1$  if  $m$  is even. The matrix  $S$  has non-zero determinant  $8m + 4$  which ends the proof by Proposition 1.1. □

**6.1. A non-cyclic example giving  $E_8$ .** All elements of the additive group  $\mathbb{F}_2^3$  are their own inverses and  $M((\mathbb{F}_2)^3)/(\pm 1)$  is obtained from the lattice  $L(\mathbb{F}_2^3)$  by considering  $L(\mathbb{F}_2^3) + (2\mathbb{Z})^{\mathbb{F}_2^3}$ . The resulting lattice is the (rescaled) exceptional root-lattice  $E_8$  with basis the last eight rows of

000	001	010	011	100	101	110	111
2	0	0	0	0	0	0	0
-1	-1	-1	-1	0	0	0	0
0	2	0	0	0	0	0	0
0	-1	1	0	-1	0	0	-1
0	0	0	0	1	-1	1	1
0	0	0	0	0	2	0	0
0	0	0	0	-1	-1	-1	1
0	0	-1	1	0	0	-1	-1

having twice the Dynkin matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

of  $E_8$  as its Gram matrix .

**6.2. Removing an element.** (One can in fact remove an arbitrary element from  $\mathbb{F}_2^3$ .) The even lattice associated to all 7 non-zero elements

$\mathbb{F}_2^3 \setminus \{0\}$  of  $\mathbb{F}_2^3$  is the lattice generated by the seven vectors

001	010	011	100	101	110	111
0	0	0	1	1	1	-1
0	0	0	2	0	0	0
1	0	1	1	0	1	0
0	0	0	1	1	-1	-1
0	0	0	1	1	1	1
0	1	1	1	1	0	0
0	-1	1	1	1	0	0.

The associated Gram matrix is twice the matrix

$$P_7^1 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

see [2], page 382, defining the exceptional root lattice  $E_7$ .

### 7. A construction with minimum 3 using $\mathbb{F}_2^c$

Given a finite-dimensional vector space  $\mathbb{F}_2^c$  of dimension  $c$  over the field  $\mathbb{F}_2$  of two elements, the lattice  $T(\mathbb{F}_2^c)$  is the integral sublattice of  $\mathbb{Z}^{\mathbb{F}_2^c \setminus \{0\}}$  consisting of all vectors  $v = (v_a)_{a \in \mathbb{F}_2^c \setminus \{0\}}$  such that  $\sum_{a \in \mathbb{F}_2^c \setminus \{0\}} v_a a = 0$  in  $\mathbb{F}_2^c$ . Minimal vectors have norm 3 (except in the trivial case  $c = 1$ ) and are given by  $\epsilon_1 e_a + \epsilon_2 e_b + \epsilon_3 e_c$  with  $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$  and with  $a, b, c = a + b \in \mathbb{F}_2^c$  defining a projective line of the  $(c - 1)$ -dimensional projective space over  $\mathbb{F}_2$ .

**Theorem 7.1.** *The lattice  $T(\mathbb{F}_2^c)$  has no roots, determinant  $4^c$  and  $\frac{4}{3} \binom{2^c - 1}{2}$  pairs of vectors of norm 3. It is perfect for  $c \geq 3$ .*

*Proof.* The lattice  $T(\mathbb{F}_2^c)$  is the kernel of the augmentation-map. It is thus of index  $2^c$  in  $\mathbb{Z}^{2^c - 1}$  and has determinant  $(2^c)^2 = 4^c$ . There are  $\frac{(2^c - 1)(2^c - 2)}{3 \cdot 2} = \frac{1}{3} \binom{2^c - 1}{2}$  projective lines in  $\mathbb{F}_2^c \setminus \{0\}$  and every projective line determines 4 pairs of minimal vectors.

In order to prove perfection, we consider a symmetric matrix  $S$  with  $2^c - 1$  rows and columns indexed by all non-zero elements of  $\mathbb{F}_2^c$ . A non-zero diagonal coefficient  $c_{a,b}$  of  $S$  can be eliminated by subtracting

$$\frac{c_{a,b}}{4} \left( -v_{+++} v_{+++}^t - v_{+-+} v_{+-+}^t + v_{++-} v_{++-}^t + v_{+--} v_{+--}^t \right)$$

from  $S$  where  $v_{\epsilon_1, \epsilon_2 \epsilon_3} = \epsilon_1 e_a + \epsilon_2 e_b + \epsilon_3 e_c$  with  $c = a + b \in \mathbb{F}_2^c$ .

The orthogonal projector

$$\frac{1}{4} \left( v_{+++} v_{+++}^t + v_{+-+} v_{+-+}^t + v_{++-} v_{++-}^t + v_{+--} v_{+--}^t \right)$$

has only three non-zero coefficients on the diagonal corresponding to rows (and columns) indexed by  $a, b$  and  $c = a + b$ . It is thus associated to the diagonal coefficient of a projective line over  $\mathbb{F}_2$ . The matrix  $A$  defined by the last seven rows (with the first row indicating the seven points of the projective plane over  $\mathbb{F}_2$ ) of

001	010	011	100	101	110	111
1	1	1	0	0	0	0
1	0	0	1	1	0	0
1	0	0	0	0	1	1
0	1	0	1	0	1	0
0	1	0	0	1	0	1
0	0	1	1	0	0	1
0	0	1	0	1	1	0

has determinant  $-24$  and is thus invertible. This shows that we can get rid of diagonal coefficients using the “diagonal” projectors onto projective lines by embedding them into projective planes. More precisely, given a point  $a \in \mathbb{F}_c$  of a projective plane  $\Pi$ , the projector

$$\frac{1}{6} \left( 2 \sum_{a \in L \subset \Pi} P_L - \sum_{a \notin L \subset \Pi} P_L \right)$$

is the diagonal projector onto the diagonal element indexed by  $a$  where  $P_L$  is the projector

$$\frac{1}{4} \left( v_{+++} v_{+++}^t + v_{+-+} v_{+-+}^t + v_{++-} v_{++-}^t + v_{+--} v_{+--}^t \right)$$

(with  $v_{e_1 e_2 e_3}$  as above) associated to projective line  $\{a, b, c = a + b\} \subset \Pi$ .  $\square$

**Remark.** No elements (except subsets leaving the non-zero elements of a subgroup containing at least 8 elements) can be removed from the set  $\mathbb{F}_2^c \setminus \{0\}$  in the construction of  $T(\mathbb{F}_2^c)$  without destroying perfection of the associated lattice.

The construction cannot be adapted to other finite abelian groups (with  $\mathbb{F}_2^c \setminus \{0\}$  replaced by representatives of all non-zero orbits of a finite abelian group  $A$  under the automorphism  $x \mapsto -x$ ) without losing perfection.

**7.1. Digression: The equiangular system of the perfect lattice  $T(\mathbb{F}_2^3)$  and the Schläfli graph.** The 7-dimensional perfect lattice  $T(\mathbb{F}_2^3)$  with 28 pairs of minimal vectors has a basis given by the last seven rows of

001	010	011	100	101	110	111
0	0	1	0	1	-1	0
0	1	0	0	1	0	-1
1	0	0	0	0	-1	1
1	0	0	-1	1	0	0
0	0	1	-1	0	0	1
0	0	1	1	0	0	-1
-1	0	0	1	1	0	0

with Gram matrix

$$P_7^2 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 3 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 3 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 3 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 3 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 3 \end{pmatrix},$$

see page 382 of [2]. Up to rescaling, this is the dual lattice  $E_7^*$  of the root lattice  $E_7$ .

Its 28 pairs of minimal vectors define a system of 28 equiangular lines (meeting two-by-two in a common angle given by  $\arccos(1/3)$ ) in  $\mathbb{R}^7$ . Supports of minimal vectors define projective lines in the Fano plane (projective plane over  $\mathbb{F}_2$ ). The automorphism group of  $T(\mathbb{F}_2^3)$  acts transitively on the set of minimal vectors. Fixing a first minimal vector, say  $w = (1, 1, 1, 0, 0, 0, 0)$ , we chose representatives  $v_1, \dots, v_{27}$  of the 27 minimal pairs different from  $\pm w$  such that  $\langle w, v_i \rangle = 1$ . We encode the angles between  $v_1, \dots, v_{27}$  by a graph  $\Gamma$  with vertices  $v_1, \dots, v_{27}$  and edges  $v_i, v_j$  if  $\langle v_i, v_j \rangle = -1$ . The characteristic polynomial of the adjacency matrix  $A$  of  $\Gamma$  (with diagonal zero and off-diagonal coefficients  $\frac{1 - \langle v_i, v_j \rangle}{2}$ ) is given by

$$(t - 10)(t - 1)^{20}(t + 5)^6$$

and the graph  $\Gamma$  is thus a strongly regular graph on 27 vertices with parameters  $(v, k, \lambda, \mu) = (27, 10, 1, 5)$ .

Otherwise stated, the graph  $\Gamma$  has  $v = 27$  vertices. It is of degree  $k = 10$  and diameter 2 such that two adjacent vertices in  $\Gamma$  have always  $\lambda = 1$  common neighbours and two non-adjacent vertices of  $\Gamma$  have always  $\mu = 5$  common neighbours.  $\lambda = 1$  is equivalent to the fact that every edge of  $\Gamma$  is contained in a unique triangle (complete graph on 3 vertices) of  $\Gamma$ .

Such a graph is unique and it (or sometimes its complement) is called the Schläfli graph.

**Remark.**

(i) The even sublattice of the lattice  $T(\mathbb{F}_3^2)$  is (up to rescaling) the root lattice  $E_7$  consisting of all vectors of the lattice  $M(\mathbb{F}_3^3)$  (see Section 6.2) not involving the basis vector  $e_0$  associated to the identity 0 of the additive group  $\mathbb{F}_3^3$ . Its 63 pairs of minimal vectors can be described as follows: Every line  $\{i, j, k\}$  gives rise to  $2^3 = 8$  pairs of minimal vectors by considering a vector with zero coordinates corresponding to  $i, j, k$  and with coordinates  $\pm 1$  associated to points not in  $\{i, j, k\}$ . This gives  $7 \cdot 8 = 56$  pairs of minimal vectors (of norm 4). Seven additional pairs are given by  $\pm 2e_i$  and are associated to the seven points of the projective plane.

(ii) Restricting to vectors with zero coordinate-sum of the even sublattice of  $T(\mathbb{F}_3^2)$ , we get the rescaled root lattice  $A_6$  of Section 5.4.

**8. Generalizations**

All lattices constructed in this paper are of the form  $\Lambda = \ker(\varphi(\mathbb{Z}^{d+a}))$  for a surjective morphism  $\varphi$  from  $\mathbb{Z}^{d+a}$  onto an abelian group  $\mathbb{Z}^a \oplus A$  with  $A$  finite. A suitable choice of  $\varphi$  ensures nice combinatorial properties of small elements in  $\Lambda$ . Up to this point (except in Section 7), we have worked with even lattices containing no roots and we have used properties of  $\varphi$  for proving perfection of the set  $\Lambda_4$  of minimal vectors in  $\Lambda$ . It is of course tempting to consider  $\varphi$  such that the norm  $\lambda_1^2 + \dots + \lambda_{d+a}^2$  of every non-zero element  $(\lambda_1, \dots, \lambda_{d+a}) \in \Lambda$  is at least equal to some larger integer  $m > 4$ . Sidon sets provide examples leading to minimum 6 (but do not ensure perfection) as follows: A *Sidon set* in an additive group  $A$  is a subset  $\mathcal{S}$  such that  $x_1 + y_1 = x_2 + y_2$  implies  $\{x_1, y_1\} = \{x_2, y_2\}$  as multisets for  $x_1, y_1, x_2, y_2 \in \mathcal{S}$ . The sublattice of all elements in  $\mathbb{Z}^{\mathcal{S}}$  with zero coefficient-sum  $\sum_{x \in \mathcal{S}} \lambda_x = 0$  such that  $\sum_{x \in \mathcal{S}} \lambda_x x = 0 \in A$  is then even and without roots or vectors of norm 4. More generally, one might consider subsets  $\mathcal{S}$  which have the  $m$ -lattice Sidon property: every non-zero vector in the lattice of all elements in  $\mathbb{Z}^{\mathcal{S}}$  with zero coefficient-sum  $\sum_{x \in \mathcal{S}} \lambda_x = 0$  such that  $\sum_{x \in \mathcal{S}} \lambda_x x = 0 \in A$  has (squared Euclidean) norm at least  $2(m + 1)$ . As a variation, one can drop the requirement  $\sum_{x \in \mathcal{S}} \lambda_x = 0$  by replacing it with the evenness condition  $\sum_{x \in \mathcal{S}} \lambda_x \equiv 0 \pmod{2}$  or by dropping it without any other requirement altogether (this puts of course an even stronger constraint on  $\mathcal{S}$ ).

**8.1. Craig lattices.** Given a finite field  $\mathbb{F}_q$  with  $q = p^e$  a prime power and an integer  $k$ , we can consider the lattice  $C_{q-1,k}$  defined by all vectors of  $\mathbb{Z}^{\mathbb{F}_q}$  with zero coefficient sum  $\sum_{x \in \mathbb{F}_q} \lambda_x = 0$  and such that  $\sum_{x \in \mathbb{F}_q} \lambda_x x^i = 0 \in \mathbb{F}_q$  for  $i = 1, \dots, k$  (equality holds of course also for  $i = 0$ ). For  $q = p$  a prime

number, the lattice  $C_{q-1,k}$  is a Craig lattice. The lattice  $C_{q-1,k}$  is even and has determinant  $q^{2k+1}$ .

**Proposition 8.1.** *The lattice  $C_{q-1,k}$  has minimum  $\geq 2k + 2$  if  $k$  is smaller than the characteristic  $p$  of  $\mathbb{F}_q$ .*

*Proof.* Symmetric power-sums of degree up to  $p - 1$  define elementary symmetric polynomials of degree up to  $p - 1$ . A minimal vector with strictly positive coefficients of indices  $a_1, \dots, a_l$  (with indices repeated according to the value of the associated integral coefficient) and strictly negative coefficients of indices  $b_1, \dots, b_l$  gives rise to two polynomials  $\prod_{i=1}^l (x - a_i)$  and  $\prod_{i=1}^l (x - b_i)$ . Since symmetric power-sums of degree up to  $p - 1$  define elementary symmetric polynomials of degree up to  $p - 1$  this implies either of  $l > k$  or  $k \geq p$ . □

**Proposition 8.2.** *For  $k$  smaller than the characteristic  $p$  of  $\mathbb{F}_q$ , the number of pairs of elements of norm  $2(k + 1)$  in  $C_{q-1,k}$  is given by*

$$(8.1) \quad \sum_{(a_1, \dots, a_k) \in \mathbb{F}_q^k} \binom{N(a_1, \dots, a_k)}{2}$$

where  $N(a_1, \dots, a_k) \leq q$  is the number of constants  $a_0 \in \mathbb{F}_q$  such that the polynomial  $x^{k+1} + \sum_{i=0}^k a_i x^i$  has exactly  $k + 1$  distinct roots in  $\mathbb{F}_q$ .

*Proof.*  $N(a_1, \dots, a_k)$  is also the number of subsets  $\{x_1, \dots, x_{k+1}\}$  of  $k + 1$  distinct elements in  $\mathbb{F}_q$  such that  $\sum_{i=1}^{k+1} x_i^j = b_j$  with  $b_1, \dots, b_k$  the power-sums corresponding to the elementary symmetric functions  $a_k, a_{k-1}, \dots, a_1$ . Such subsets are disjoint and pairs of two such subsets define indices of coefficients 1 and  $-1$  in minimal vectors. □

**Corollary 8.3.** *The lattice  $C_{q-1,k}$  (for  $k$  smaller than the characteristic  $p$  of  $\mathbb{F}_q$ ) has at least*

$$q^k \binom{\frac{1}{q^k} \binom{q}{k+1}}{2}$$

*pairs of vectors of norm  $2(k + 1)$ .*

*In particular, for a fixed value of  $k$ , the lattice  $C_{q-1,k}$  has asymptotically at least  $\frac{q^{k+2}}{2((k+1)!)^2}$  pairs of minimal vectors of norm  $2(k + 1)$ .*

*Proof.* Since every subset of  $k + 1$  elements in  $\mathbb{F}_q$  contributes 1 to exactly one of the numbers  $N(a_1, \dots, a_k)$  we have

$$\binom{q}{k + 1} = \sum_{(a_1, \dots, a_k) \in \mathbb{F}_q^k} N(a_1, \dots, a_k).$$

Convexity properties of the polynomial  $\binom{x}{2} = \frac{x(x-1)}{2}$  imply that (8.1) is minimal if all  $q^k$  numbers  $N(a_1, \dots, a_k)$  are equal. □



**Theorem 8.4.** *For  $k = 2$ , the number of minimal pairs in  $C_{q-1,2}$  is given by*

$$\frac{1}{72}q(q-1)(q^2 - 10q + 33)$$

for  $q$  a prime-power congruent to 1 modulo 6 and by

$$\frac{1}{72}q(q-1)(q-5)^2$$

for  $q$  a prime power congruent to 5 modulo 6.

*Sketch of Proof.* We have to evaluate Formula (8.1) for  $k = 2$ . Substituting  $x$  with  $x - \frac{a_2}{3}$  we get  $N(a_1, a_2) = N(a_1 - \frac{1}{3}a_2^2, 0)$ . Formula (8.1) for  $k = 2$  is thus given by

$$q \sum_{a \in \mathbb{F}_q} \binom{N(a, 0)}{2}$$

if  $q$  is not a power of 3. Since  $N(a, 0)$  depends only on the value  $\left(\frac{a}{q}\right)$  of the quadratic character extending the Jacobi symbol, we have to compute  $N(a, 0)$  for  $a = 0, 1$  and for a non-square of  $\mathbb{F}_q$ . These computations boil down to classical properties of binary quadratic forms over finite fields. (One can alternatively use a result of Stickelberger, as observed by the reviewer.) □

**Remark.** A close relative of the lattice  $C_{q-1,2}$  is the lattice associated to the Sidon set  $\{(x, x^{-1})\}_{x \in \mathbb{F}_q^*} \subset \mathbb{F}_q^2$  for  $\mathbb{F}_q$  a finite field of odd characteristic. It is of dimension  $q - 2$ , has minimum 6 (except for a few small values of  $q$ ) and consist of all elements  $(\lambda_x)_{x \in \mathbb{F}_q^*} \in \mathbb{Z}^{\mathbb{F}_q^*}$  (integral vectors with indices in  $\mathbb{F}_q^*$ ) such that  $\sum_{x \in \mathbb{F}_q^*} \lambda_x = 0$  and  $\sum_{x \in \mathbb{F}_q^*} \lambda_x x = \sum_{x \in \mathbb{F}_q^*} \lambda_x x^{-1} = 0 \in \mathbb{F}_q$ .

For  $k = 3$ , let  $c_q$  be such that the number of pairs of minimal vectors (of norm 8) in  $C_{q-1,3}$  is given by

$$\frac{1}{1152}q(q-1)(q^3 - 21q^2 + 171q - c_q).$$

Writing  $c_k$  as

$$c_k = 483 + 36 \left(\frac{-1}{q}\right) + 64 \left(\frac{-3}{q}\right) + \delta_q,$$

we have the following result due to Noam D. Elkies, see [1] (a preliminary draft of the present paper proposed the values corresponding to  $\delta_q = 0$  conjecturally):

**Theorem 8.5.** *If  $q$  is a prime  $\leq 5$ , then  $\delta_q = 0$  if  $\left(\frac{-2}{q}\right) = -1$  (yielding the values*

$$c_q = \begin{cases} 455 & \text{if } q \equiv 5 \pmod{24}, \\ 511 & \text{if } q \equiv 7 \pmod{24}, \\ 583 & \text{if } q \equiv 13 \pmod{24}, \\ 383 & \text{if } q \equiv 23 \pmod{24} \end{cases}$$

for  $c_q$  in these cases) and

$$\delta_q = 24(m^2 - 2n^2) + 192 + 72\left(\frac{-1}{q}\right)$$

where  $m$  and  $n$  are the unique natural integers such that  $q = m^2 + 2n^2$  otherwise (i.e. for  $q \geq 11$  a prime such that  $\left(\frac{-2}{q}\right) = 1$ ).

See [1] for the fairly sophisticated proof.

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