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Criteria for Irreducibility of mod p Representations of Frey Curves

par NUNO FREITAS et SAMIR SIKSEK

RÉSUMÉ. Soit K un corps de nombres galoisien totalement réel, et soit \mathcal{E} un ensemble de courbes elliptiques sur K . Nous donnons des conditions suffisantes pour l'existence d'un ensemble calculable de nombres premiers \mathcal{P} tels que, pour $p \notin \mathcal{P}$ et $E \in \mathcal{E}$, la représentation $\text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(E[p])$ soit irréductible. Nos conditions sont en général satisfaites par les courbes de Frey associées à des solutions d'équations diophantiennes. Dans ce contexte, l'irréductibilité de la représentation mod p est une hypothèse requise pour l'application des théorèmes d'abaissement du niveau. Comme illustration de notre approche, nous avons amélioré le résultat de [6] pour les équations de Fermat de signature $(13, 13, p)$.

ABSTRACT. Let K be a totally real Galois number field and let \mathcal{E} be a set of elliptic curves over K . We give sufficient conditions for the existence of a finite computable set of rational primes \mathcal{P} such that for $p \notin \mathcal{P}$ and $E \in \mathcal{E}$, the representation $\text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(E[p])$ is irreducible. Our sufficient conditions are often satisfied for Frey elliptic curves associated to solutions of Diophantine equations; in that context, the irreducibility of the mod p representation is a hypothesis needed for applying level-lowering theorems. We illustrate our approach by improving on a result of [6] for Fermat-type equations of signature $(13, 13, p)$.

1. Introduction

The ‘modular approach’ is a popular method for attacking Diophantine equations using Galois representations of elliptic curves; see [1], [21] for recent surveys. The method relies on three important and difficult theorems.

- (i) Wiles et al.: elliptic curves over \mathbb{Q} are modular [3], [23], [22].
- (ii) Mazur: if E/\mathbb{Q} is an elliptic curve and $p > 167$ is a prime, then the Galois representation on the p -torsion of E is irreducible [15] (and variants of this result).
- (iii) Ribet’s level-lowering theorem [19].

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The strategy of the method is to associate to a putative solution of certain Diophantine equations a Frey elliptic curve, and apply Ribet's level-lowering theorem to deduce a relationship between the putative solution and a modular form of relatively small level. Modularity (i) and irreducibility (ii) are necessary hypotheses that need to be verified in order to apply level-lowering (iii).

Attention is now shifting towards Diophantine equations where the Frey elliptic curves are defined over totally real fields (for example [2], [6], [7], [8]). One now finds in the literature some of the necessary modularity (e.g. [9]) and level-lowering theorems (e.g. [10], [12] and [18]) for the totally real setting. Unfortunately, there is as of yet no analogue of Mazur's Theorem over any number field $K \neq \mathbb{Q}$, which does present an obstacle for applying the modular approach over totally real fields.

Let K be a number field, and write $G_K = \text{Gal}(\overline{K}/K)$. Let E an elliptic curve over K . Let p be a rational prime, and write $\overline{\rho}_{E,p}$ for the associated representation of G_K on the p -torsion of E :

$$(1.1) \quad \overline{\rho}_{E,p} : G_K \rightarrow \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p).$$

Mazur's Theorem asserts that if $K = \mathbb{Q}$ and $p > 167$ then $\overline{\rho}_{E,p}$ is irreducible. For a general number field K , it is expected that there is some B_K , such that for all elliptic curves E/K without complex multiplication, and all $p > B_K$, the mod p representation $\overline{\rho}_{E,p}$ is irreducible. Several papers, including those by Momose [17], Kraus [13], [14] and David [5], establish a bound B_K depending on the field K , under some restrictive assumptions on E , such as semistability. The Frey elliptic curves one deals with in the modular approach are close to being semistable [21, Section 15.2.4]. The purpose of this note is to prove the following theorem, which should usually be enough to supply the desired irreducibility statement in that setting.

Theorem 1. *Let K be a totally real Galois number field of degree d , with ring of integers \mathcal{O}_K and Galois group $G = \text{Gal}(K/\mathbb{Q})$. Let $S = \{0, 12\}^G$, which we think of as the set of sequences of values 0, 12 indexed by $\tau \in G$. For $\mathbf{s} = (s_\tau) \in S$ and $\alpha \in K$, define the **twisted norm associated to \mathbf{s}** by*

$$\mathcal{N}_{\mathbf{s}}(\alpha) = \prod_{\tau \in G} \tau(\alpha)^{s_\tau}.$$

Let $\epsilon_1, \dots, \epsilon_{d-1}$ be a basis for the unit group of K , and define

$$(1.2) \quad A_{\mathbf{s}} := \text{Norm}(\text{gcd}((\mathcal{N}_{\mathbf{s}}(\epsilon_1) - 1)\mathcal{O}_K, \dots, (\mathcal{N}_{\mathbf{s}}(\epsilon_{d-1}) - 1)\mathcal{O}_K)).$$

Let B be the least common multiple of the $A_{\mathbf{s}}$ taken over all $\mathbf{s} \neq (0)_{\tau \in G}$, $(12)_{\tau \in G}$. Let $p \nmid B$ be a rational prime, unramified in K , such that $p \geq 17$

or $p = 11$. Let E/K be an elliptic curve, and $\mathfrak{q} \nmid p$ be a prime of good reduction for E . Let

$$P_{\mathfrak{q}}(X) = X^2 - a_{\mathfrak{q}}(E)X + \text{Norm}(\mathfrak{q})$$

be the characteristic polynomial of Frobenius for E at \mathfrak{q} . Let $r \geq 1$ be an integer such that \mathfrak{q}^r is principal. If E is semistable at all $\mathfrak{p} \mid p$ and $\bar{\rho}_{E,p}$ is reducible then

$$(1.3) \quad p \mid \text{Res}(P_{\mathfrak{q}}(X), X^{12r} - 1)$$

where Res denotes resultant.

We will see in due course that B above is non-zero. It is easy to show that the resultant in (1.3) is also non-zero. The theorem therefore does give a bound on p so that $\bar{\rho}_{E,p}$ is reducible.

The main application we have in mind is to Frey elliptic curves associated to solutions of Fermat-style equations. In such a setting, one usually knows that the elliptic curve in question has semistable reduction outside a given set of primes, and one often knows some primes of potentially good reduction. We illustrate this, by giving an improvement to a recent theorem of Dieulefait and Freitas [6] on the equation $x^{13} + y^{13} = Cz^p$. In a forthcoming paper [2], the authors apply our Theorem 1 together with modularity and level-lowering theorems to completely solve the equation $x^{2n} \pm 6x^n + 1 = y^2$ in integers x, y, n with $n \geq 2$, after associating this to a Frey elliptic curve over $\mathbb{Q}(\sqrt{2})$. In another paper [7], the first-named author uses our Theorem 1 as part of an investigation that associates solutions of equations $x^r + y^r = Cz^p$ with (r, p) prime) with Frey elliptic curves over real subfields of $\mathbb{Q}(\zeta_r)$.

The following is closely related to a result of David [5, Theorem 2], but formulated in a way that is more suitable for attacking specific examples.

Theorem 2. *Let K be a totally real Galois field of degree d . Let B be as in the statement of Theorem 1. Let $p \nmid B$ be a rational prime, unramified in K , such that $p \geq 17$ or $p = 11$. If E is an elliptic curve over K which is semistable at all $\mathfrak{p} \mid p$ and $\bar{\rho}_{E,p}$ is reducible then $p < (1 + 3^{6dh})^2$, where h is the class number of K .*

2. Preliminaries

We shall henceforth fix the following notation and assumptions.

K	a Galois number field,
d_K	the degree of K/\mathbb{Q} ,
G_K	$\text{Gal}(\bar{K}/K)$,
\mathfrak{q}	a finite prime of K ,
$I_{\mathfrak{q}}$	the inertia subgroup of G_K corresponding to \mathfrak{q} ,
G	$\text{Gal}(K/\mathbb{Q})$,
p	a rational prime unramified in K satisfying $p \geq 17$, or $p = 11$,
χ_p	the mod p cyclotomic character $G_K \rightarrow \mathbb{F}_p^*$,
E	an elliptic curve semistable at all places \mathfrak{p} of K above p ,
$\bar{\rho}_{E,p}$	the mod p representation associated to E as in (1.1).

Suppose $\bar{\rho}_{E,p}$ is reducible. With an appropriate choice of basis for $E[p]$ we can write

$$(2.1) \quad \bar{\rho}_{E,p} \sim \begin{pmatrix} \lambda & * \\ 0 & \lambda' \end{pmatrix},$$

where $\lambda, \lambda' : G_K \rightarrow \mathbb{F}_p^*$ are characters. Thus $\lambda\lambda' = \det(\bar{\rho}_{E,p}) = \chi_p$. The character λ is known as the **isogeny character** of $E[p]$.

As in the aforementioned works of Momose, Kraus and David, our approach relies on controlling the ramification of the characters λ, λ' at places above p .

Proposition 2.1. (David [5, Propositions 1.2, 1.3]) *Suppose $\bar{\rho}_{E,p}$ is reducible and let λ, λ' be as above. Let $\mathfrak{p} \mid p$ be a prime of K . Then*

$$\lambda^{12}|_{I_{\mathfrak{p}}} = (\chi_{\mathfrak{p}}|_{I_{\mathfrak{p}}})^{s_{\mathfrak{p}}}$$

where $s_{\mathfrak{p}} \in \{0, 12\}$.

Proof. Indeed, by [5, Propositions 1.2, 1.3],

- (i) if \mathfrak{p} is a prime of potentially multiplicative reduction or potentially good ordinary reduction for E then $s_{\mathfrak{p}} = 0$ or $s_{\mathfrak{p}} = 12$;
- (ii) if \mathfrak{p} is a prime of potentially good supersingular reduction for E then $s_{\mathfrak{p}} = 4, 6, 8$.

However, we have assumed that E is semistable at $\mathfrak{p} \mid p$ and that p is unramified in K . By Serre [20, Proposition 12], if E has good supersingular reduction at \mathfrak{p} , then the image of $\bar{\rho}_{E,p}$ contains a non-split Cartan subgroup of $\text{GL}_2(\mathbb{F}_p)$ and is therefore irreducible, contradicting the assumption that $\bar{\rho}_{E,p}$ is reducible. Hence E has multiplicative or good ordinary reduction at \mathfrak{p} . \square

Remark. The order of $\chi_{\mathfrak{p}}|_{I_{\mathfrak{p}}}$ is $p - 1$. Hence the value of $s_{\mathfrak{p}}$ in the above proposition is well-defined modulo $p - 1$. Of course, since $0 \leq s_{\mathfrak{p}} \leq 12$, it follows for $p \geq 17$ that $s_{\mathfrak{p}}$ is unique.

As K is Galois, G acts transitively on $\mathfrak{p} \mid p$. Fix $\mathfrak{p}_0 \mid p$. For each $\tau \in G$ write s_τ for the number $s_{\mathfrak{p}}$ associated to the ideal $\mathfrak{p} := \tau^{-1}(\mathfrak{p}_0)$ by the previous proposition. We shall refer to $\mathbf{s} := (s_\tau)_{\tau \in G}$ as the **isogeny signature** of E at p . The set $S := \{0, 12\}^G$ shall denote the set of all possible sequences of values 0, 12 indexed by elements of G . For an element $\alpha \in K$, we define the **twisted norm associated to $\mathbf{s} \in S$** by

$$\mathcal{N}_{\mathbf{s}}(\alpha) = \prod_{\tau \in G} \tau(\alpha)^{s_\tau}.$$

Proposition 2.2. (David [5, Proposition 2.6]) *Suppose $\bar{\rho}_{E,p}$ is reducible with isogeny character λ , having isogeny signature $\mathbf{s} \in S$. Let $\alpha \in K$ be non-zero. Suppose $v_{\mathfrak{p}}(\alpha) = 0$ for all $\mathfrak{p} \mid p$. Then*

$$\mathcal{N}_{\mathbf{s}}(\alpha) \equiv \prod \left(\lambda^{12}(\sigma_{\mathfrak{q}}) \right)^{v_{\mathfrak{q}}(\alpha)} \pmod{\mathfrak{p}_0},$$

where the product is taken over all prime \mathfrak{q} in the support of α .

3. A bound in terms of a prime of potentially good reduction

Let \mathfrak{q} be a prime of potentially good reduction for E . Denote by $P_{\mathfrak{q}}(X)$ the characteristic polynomial of Frobenius for E at \mathfrak{q} .

Lemma 3.1. *Let \mathfrak{q} be a prime of potentially good reduction for E , and suppose $\mathfrak{q} \nmid p$. Let $r \geq 1$ be such that \mathfrak{q}^r is principal, and write $\alpha \mathcal{O}_K = \mathfrak{q}^r$. Let $\mathbf{s} = (s_\tau)_{\tau \in G}$ be the isogeny signature of E at p . Then*

$$\mathfrak{p}_0 \mid \text{Res}(P_{\mathfrak{q}}(X), X^{12r} - \mathcal{N}_{\mathbf{s}}(\alpha)),$$

where Res denotes the resultant.

Proof. From (2.1), it is clear that

$$P_{\mathfrak{q}}(X) \equiv (X - \lambda(\sigma_{\mathfrak{q}}))(X - \lambda'(\sigma_{\mathfrak{q}})) \pmod{p}.$$

Moreover, from Proposition 2.2, $\lambda(\sigma_{\mathfrak{q}})$ is a root modulo \mathfrak{p}_0 of the polynomial $X^{12r} - \mathcal{N}_{\mathbf{s}}(\alpha)$. As $\mathfrak{p}_0 \mid p$, the lemma follows. \square

We note the following surprising consequence.

Corollary 3.2. *Let ϵ be a unit of \mathcal{O}_K . If the isogeny signature of E at p is \mathbf{s} then $\mathcal{N}_{\mathbf{s}}(\epsilon) \equiv 1 \pmod{\mathfrak{p}_0}$.*

Proof. Let $\mathfrak{q} \nmid p$ be any prime of good reduction of E . Let h be the class number of K . Choose any $\alpha \in \mathcal{O}_K$ so that $\alpha \mathcal{O}_K = \mathfrak{q}^h$. By Proposition 2.2,

$$\mathcal{N}_{\mathbf{s}}(\alpha) \equiv (\lambda(\sigma_{\mathfrak{q}}))^{12h} \pmod{\mathfrak{p}_0}.$$

However, if ϵ is unit, then $\epsilon \alpha \mathcal{O}_K = \mathfrak{q}^h$ too. So

$$\mathcal{N}_{\mathbf{s}}(\epsilon \alpha) \equiv (\lambda(\sigma_{\mathfrak{q}}))^{12h} \pmod{\mathfrak{p}_0}.$$

Taking ratios we have $\mathcal{N}_{\mathbf{s}}(\epsilon) \equiv 1 \pmod{\mathfrak{p}_0}$. \square

Corollary 3.2 is only useful in bounding p for a given signature \mathbf{s} , if there is some unit ϵ of K such that $\mathcal{N}_{\mathbf{s}}(\epsilon) \neq 1$. Of course, if \mathbf{s} is either of the constant signatures $(0)_{\tau \in G}$ or $(12)_{\tau \in G}$ then

$$\mathcal{N}_{\mathbf{s}}(\epsilon) = (\text{Norm } \epsilon)^0 \text{ or } 12 = 1.$$

Given a non-constant signature $\mathbf{s} \in S$, does there exist a unit ϵ such that $\mathcal{N}_{\mathbf{s}}(\epsilon) \neq 1$? It is easy to construct examples where the answer is no. The following lemma gives a positive answer when K is totally real.

Lemma 3.3. *Let K be totally real of degree $d \geq 2$. Suppose $\mathbf{s} \neq (0)_{\tau \in G}$, $\neq (12)_{\tau \in G}$. Then there exists a unit ϵ of K such that $\mathcal{N}_{\mathbf{s}}(\epsilon) \neq 1$.*

Proof. Let τ_1, \dots, τ_d be the elements of G . Fix an embedding $\sigma : K \hookrightarrow \mathbb{R}$, and denote $\sigma_i = \sigma \circ \tau_i$. Rearranging the elements of G , we may suppose that

$$s_{\tau_1} = \dots = s_{\tau_r} = 12, \quad s_{\tau_{r+1}} = \dots = s_{\tau_d} = 0$$

where $1 \leq r \leq d-1$. Suppose that $\mathcal{N}_{\mathbf{s}}(\epsilon) = 1$ for all $\epsilon \in U(K)$, where $U(K)$ is the unit group of K . Then the image of $U(K)$ under the Dirichlet embedding

$$U(K)/\{\pm 1\} \hookrightarrow \mathbb{R}^{d-1}, \quad \epsilon \mapsto (\log |\sigma_1(\epsilon)|, \dots, \log |\sigma_{d-1}(\epsilon)|)$$

is contained in the hyperplane $x_1 + x_2 + \dots + x_r = 0$. This contradicts the fact the image must be a lattice in \mathbb{R}^{d-1} of rank $d-1$. \square

4. Proof of Theorem 1

We now prove Theorem 1. Suppose $\bar{\rho}_{E,p}$ is reducible and let \mathbf{s} be the isogeny signature. Let $A_{\mathbf{s}}$ be as in (1.2). By Corollary 3.2, $p \mid A_{\mathbf{s}}$. If $\mathbf{s} \neq (0)_{\tau \in G}$, $(12)_{\tau \in G}$ then $A_{\mathbf{s}} \neq 0$ by Lemma 3.3. Now, suppose $p \nmid B$, where B is as in the statement of Theorem 1. Then $\mathbf{s} = (0)_{\tau \in G}$, $(12)_{\tau \in G}$.

Suppose first that $\mathbf{s} = (0)_{\tau \in G}$. Then $\mathcal{N}_{\mathbf{s}}(\alpha) = 1$ for all α . Let $\mathfrak{q} \nmid p$ be a prime of good reduction for E . It follows from Lemma 3.1 that \mathfrak{p}_0 divides the resultant of $P_{\mathfrak{q}}(X)$ and $X^{12r} - 1$. As both polynomials have coefficients in \mathbb{Z} , the resultant belongs to \mathbb{Z} , and so is divisible by p . This completes the proof for $\mathbf{s} = (0)_{\tau \in G}$.

Finally, we deal with the case $\mathbf{s} = (12)_{\tau \in G}$. Let $C \subset E[p]$ be the subgroup of order p corresponding to λ . Replacing E by the isogenous curve $E' = E/C$ has the effect of swapping λ and λ' in (2.1). As $\lambda\lambda' = \chi_p$, the isogeny signature for E' at p is $(0)_{\tau \in G}$. The theorem follows.

5. Proof of Theorem 2

Suppose $\bar{\rho}_{E,p}$ is reducible with signature \mathbf{s} . As in the proof of Theorem 1, we may suppose $\mathbf{s} = (0)_{\tau \in G}$ or $\mathbf{s} = (12)_{\tau \in G}$. Moreover, replacing E by an isogenous elliptic curve we may suppose that $\mathbf{s} = (0)_{\tau \in G}$. By definition of \mathbf{s} , we have λ^{12} is unramified at all $\mathfrak{p} \mid p$. As is well-known (see for example [5,

Proposition 1.4 and Proposition 1.5]), λ^{12} is unramified at the finite places outside p ; λ^{12} is clearly unramified at the infinite places because of the even exponent 12. Thus λ^{12} is everywhere unramified. Thus λ has order dividing $12 \cdot h$, where h is the class number of K . Let L/K be the extension cut out by λ ; this has degree dividing $12 \cdot h$. Then E/L has a point of order p . Applying Merel's bounds [16], we conclude that

$$p < (1 + 3^{[L:\mathbb{Q}]/2})^2 \leq (1 + 3^{6dh})^2.$$

6. An Example: Frey Curves Attached to Fermat Equations of Signature $(13, 13, p)$

In [6], Dieulefait and Freitas, used the modular method to attack certain Fermat-type equations of the form $x^{13} + y^{13} = Cz^p$, for infinitely many values of C . They attach Frey curves (independent of C) over $\mathbb{Q}(\sqrt{13})$ to primitive solutions of these equations, and prove irreducibility of the mod p representations attached to these Frey curves, for $p > 7$ and $p \neq 13, 37$ under the assumption that the isogeny signatures are $(0, 0)$ or $(12, 12)$. Here we improve on the argument by dealing with the isogeny signature $(0, 12)$, $(12, 0)$ and also by dealing with $p = 37$. More precisely, we prove the following.

Theorem 3. *Let $d = 3, 5, 7$ or 11 and let γ be an integer divisible only by primes $\ell \not\equiv 1 \pmod{13}$. Let p be a prime satisfying $p \geq 17$ or $p = 11$. Let $(a, b, c) \in \mathbb{Z}^3$ satisfy*

$$a^{13} + b^{13} = d\gamma c^p, \quad \gcd(a, b) = 1, \quad abc \neq 0, \pm 1.$$

Write $K = \mathbb{Q}(\sqrt{13})$; this has class number 1. Let $E = E_{(a,b)}/K$ be the Frey curve defined in [6]. Then, the Galois representation $\bar{\rho}_{E,p}$ is irreducible.

Proof. Suppose $\bar{\rho}_{E,p}$ is reducible. For a quadratic field such as K , the set S of possible isogeny signatures $(s_\tau)_{\tau \in G}$ is

$$S = \{(12, 12), (12, 0), (0, 12), (0, 0)\}.$$

Note that $(13) = (\sqrt{13})^2$ is the only prime ramifying in K . In [6] it is shown that the curves E have additive reduction only at 2 and $\sqrt{13}$. Moreover, E has good reduction at all primes $\mathfrak{q} \nmid 26$ above rational primes $q \not\equiv 1 \pmod{13}$. Furthermore, the trace $a_{\mathfrak{q}}(E_{(a,b)})$ depends only on the values of a, b modulo q . By the assumption $\gcd(a, b) = 1$, we have $(a, b) \not\equiv 0 \pmod{q}$.

The fundamental unit of K is $\epsilon = (3 + \sqrt{13})/2$. Then

$$\text{Norm}(\epsilon^{12} - 1) = -2^6 \cdot 3^4 \cdot 5^2 \cdot 13.$$

As $p \geq 17$ or $p = 11$, it follows from Corollary 3.2 that the isogeny signature of E at p is either $(0, 0)$ or $(12, 12)$. As in the proof of Theorem 1, we may suppose that the isogeny signature is $(0, 0)$.

Let q be a rational prime $\not\equiv 1 \pmod{13}$ that splits as $(q) = \mathfrak{q}_1 \cdot \mathfrak{q}_2$ in K . By the above, $\mathfrak{q}_1, \mathfrak{q}_2$ must be primes of good reduction. The trace $a_{\mathfrak{q}_i}(E_{(a,b)})$ depends only on the values of a, b modulo q . For each non-zero pair $(a, b) \pmod{q}$, let

$$(6.1) \quad P_{\mathfrak{q}_1}^{(a,b)}(X) = X^2 - a_{\mathfrak{q}_1}(E_{(a,b)})X + q \quad P_{\mathfrak{q}_2}^{(a,b)}(X) = x^2 - a_{\mathfrak{q}_2}(E_{(a,b)})x + q,$$

be the characteristic polynomials of Frobenius at $\mathfrak{q}_1, \mathfrak{q}_2$. Let

$$R_q^{a,b} = \gcd(\text{Res}(P_{\mathfrak{q}_1}^{a,b}(X), X^{12} - 1), \text{Res}(P_{\mathfrak{q}_2}^{a,b}(X), X^{12} - 1)).$$

Let

$$R_q = \text{lcm}\{R_q^{a,b} : 0 \leq a, b \leq q-1, (a, b) \neq (0, 0)\}.$$

By the proof of Theorem 1, we have that p divides R_q . Using a short SAGE script we computed the values of R_q for $q = 3, 17$. We have

$$R_3 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 37, \quad R_{17} = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 19 \cdot 23 \cdot 53 \cdot 97 \cdot 281 \cdot 21481 \cdot 22777.$$

As $p \geq 17$ or $p = 11$ we see that $\bar{\rho}_{E,p}$ is irreducible. \square

We will now use the improved irreducibility result (Theorem 3) to correctly restate Theorem 1.3 in [6]. Furthermore, we will also add an argument using the primes above 17 that actually allows us to improve it. More precisely, we will prove.

Theorem 4. *Let $d = 3, 5, 7$ or 11 and let γ be an integer divisible only by primes $\ell \not\equiv 1 \pmod{13}$. Let also $\mathcal{L} := \{2, 3, 5, 7, 11, 13, 19, 23, 29, 71\}$.*

If p is a prime not belonging to \mathcal{L} , then:

(I) *The equation $x^{13} + y^{13} = d\gamma z^p$ has no solutions (a, b, c) such that*

$$\gcd(a, b) = 1, \quad abc \neq 0, \pm 1 \quad \text{and} \quad 13 \nmid c.$$

(II) *The equation $x^{26} + y^{26} = 10\gamma z^p$ has no solutions (a, b, c) such that*

$$\gcd(a, b) = 1, \quad \text{and} \quad abc \neq 0, \pm 1.$$

Proof. Suppose there is a solution (a, b, c) , satisfying $\gcd(a, b) = 1$, to the equation in part (I) of the theorem for $p \geq 17$ or $p = 11$. Let $E = E_{(a,b)}$ be the Frey curves attached to it in [6]. As explained in [6], but now using Theorem 3 above instead of Theorem 4.1 in *loc. cit.*, we obtain an isomorphism

$$(6.2) \quad \bar{\rho}_{E,p} \sim \bar{\rho}_{f,\mathfrak{p}},$$

where $\mathfrak{p} \mid p$ and $f \in S_2(2^i w^2)$ for $i = 3, 4$. In *loc. cit* the newforms are divided into the sets

S1: The newforms in $S_2(2^i w^2)$ for $i = 3, 4$ such that $\mathbb{Q}_f = \mathbb{Q}$,

S2: The newforms in the same levels with \mathbb{Q}_f strictly containing \mathbb{Q} .

We eliminate the newforms in S1 with the same argument as in [6]. Suppose now that isomorphism (6.2) holds with a form in S2. Also in [6], using the primes dividing 3, a contradiction is obtained if we assume that

$$p \notin \mathcal{P} = \{2, 3, 5, 7, 11, 13, 19, 23, 29, 71, 191, 251, 439, 1511, 13649\}.$$

Going through analogous computations, using the two primes dividing 17, gives a contradiction if p does not belong to

$$\begin{aligned} \mathcal{P}' = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 37, 41, 43, 47, 59, 71, 73, 79, 83, 89, 109, \\ 113, 157, 167, 197, 227, 229, 239, 281, 359, 431, 461, 541, 1429, 5237, 253273, \\ 271499, 609979, 6125701, 93797731, 530547937, 733958569, 6075773983, \\ 11740264873\}. \end{aligned}$$

Thus, we have a contradiction as long as p is not in the intersection

$$\mathcal{P} \cap \mathcal{P}' = \{2, 3, 5, 7, 11, 13, 19, 23, 29, 71\}.$$

Thus part (I) of the theorem follows. Part (II) follows exactly as in [6] \square

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