Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 27, n° 1 (2015), p. 1-31.

<http://jtnb.cedram.org/item?id=JTNB_2015__27_1_1_0>

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de Bordeaux **27** (2015), 1–31

Non-planarity and metric Diophantine approximation for systems of linear forms

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Résumé. Dans cet article, nous développons la théorie métrique générale des approximations diophantiennes pour les systèmes de formes linéaires. Nous introduisons et puis étudions une nouvelle notion de "non-planéité faible" des variétés, et plus généralement des mesures sur l'espace $M_{m,n}$ des matrices $m \times n$ avec coefficients dans $\mathbb{R}$. Cette notion généralise celle de non-planéité dans $\mathbb{R}^n$. Nous utilisons cette notion pour établir une extrémalité forte (au sens diophantien) des variétés et des mesures de $M_{m,n}$. Ainsi, nos résultats contribuent à la résolution d'un problème mentionné dans [20, §9.1] concernant l'extrémalité forte des variétés dans $M_{m,n}$. Outre ce thème principal, nous développons aussi la théorie inhomogène et la théorie des approximations diophantiennes pondérées. En particulier, nous étendons les résultats récents sur le principe de transfert inhomogène du premier auteur et de Velani [11] et utilisons ce nouveau résultat pour mettre la théorie inhomogène en équilibre avec son homologue homogène.

Abstract. In this paper we develop a general theory of metric Diophantine approximation for systems of linear forms. A new notion of 'weak non-planarity' of manifolds and more generally measures on the space $M_{m,n}$ of $m \times n$ matrices over $\mathbb{R}$ is introduced and studied. This notion generalizes the one of non-planarity in $\mathbb{R}^n$ and is used to establish strong (Diophantine) extremality of manifolds and measures in $M_{m,n}$. Thus our results contribute to resolving a problem stated in [20, §9.1] regarding the strong extremality of manifolds in $M_{m,n}$. Beyond the above main theme of the paper, we also develop a corresponding theory of inhomogeneous and weighted Diophantine approximation. In particular, we extend the recent inhomogeneous transference results of the first named author and Velani [11] and use them to bring the inhomogeneous theory in balance with its homogeneous counterpart.

Manuscrit reçu le 21 octobre 2013, révisé le 12 mars 2014, accepté le 15 mars 2014.
Mathematics Subject Classification. 11J83, 11J13, 11K60.

1. Introduction

Throughout $M_{m,n}$ denotes the set of $m \times n$ matrices over $\mathbb{R}$ and $\|\cdot\|$ stands for a norm on $\mathbb{R}^k$ which, without loss of generality, will be taken to be Euclidean. Thus $\|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_k^2}$ for a k-tuple $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$. We also define the following two functions of $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ that are particularly convenient for introducing the multiplicative form of Diophantine approximation:

$$\Pi(\mathbf{x}) = \prod_{i=1}^{k} |x_i| \qquad \text{and} \qquad \Pi_+(\mathbf{x}) = \prod_{i=1}^{k} \max\{1, |x_i|\}.$$

We begin by recalling some fundamental concepts from the theory of Diophantine approximation. Let $Y \in M_{m,n}$. If there exists $\varepsilon > 0$ such that the inequality

$$\|Y\mathbf{q} - \mathbf{p}\|^m < \|\mathbf{q}\|^{-(1+\varepsilon)n} \tag{1.1}$$

holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$, where $\mathbf{q}$ is regarded as a column, then Y is called *very well approximable (VWA)*. Further, if there exists $\varepsilon > 0$ such that the inequality

$$\Pi(Y\mathbf{q} - \mathbf{p}) < \Pi_+(\mathbf{q})^{-1-\varepsilon} \tag{1.2}$$

holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ then Y is called *very well multiplicatively approximable (VWMA)*. See Lemma 6.3 for an equivalent (and new) characterization of this property within a more general inhomogeneous setting.

One says that a measure μ on $M_{m,n}$ is *extremal* (resp., *strongly extremal*) if μ-almost all $Y \in M_{m,n}$ are not VWA (resp., not VWMA). It will be convenient to say that Y itself is (strongly) extremal if so is the atomic measure supported at Y; in other words, if Y is not very well (multiplicatively) approximable.

It is easily seen that

$$\Pi_+(\mathbf{q}) \le \|\mathbf{q}\|^n \qquad \text{and} \qquad \Pi(Y\mathbf{q} - \mathbf{p}) \le \|Y\mathbf{q} - \mathbf{p}\|^m \tag{1.3}$$

for any $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$. Therefore, (1.1) implies (1.2) and thus strong extremality implies extremality. It is worth mentioning that if $\varepsilon = 0$ then (1.1) as well as (1.2) holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$. The latter fact showing the optimality of exponents in (1.1) and (1.2) is due to Minkowski's theorem on linear forms – see, *e.g.*, [35].

The property of being strongly extremal is generic in $M_{m,n}$. Indeed, it is a relatively easy consequence of the Borel-Cantelli lemma that Lebesgue measure on $M_{m,n}$ is strongly extremal. However, when the entries of Y are restricted by some functional relations (in other words Y lies on a submanifold of $M_{m,n}$) investigating the corresponding measure for extremality or strong extremality becomes much harder. The study of manifolds for extremality goes back to the problem of Mahler [33] that almost all points on the Veronese curves $\{(x, \dots, x^n)\}$ (viewed as either row or column matrices) are extremal. The problem was studied in depth for over 30 years and eventually settled by Sprindžuk in 1965 – see [37] for a full account. The far more delicate conjecture that the Veronese curves in $\mathbb{R}^n$ are *strongly extremal* (that is almost all points on the curves are not VWMA) has been stated by Baker [1] and generalized by Sprindžuk [38].

It will be convenient to introduce the following definition (cf. [29, §4]): say that a subset $\mathcal{M}$ of $\mathbb{R}^n$ is *non-planar* if whenever U is an open subset of $\mathbb{R}^n$ containing at least one point of $\mathcal{M}$, the intersection $\mathcal{M} \cap U$ is not entirely contained in any affine hyperplane of $\mathbb{R}^n$. Clearly the curve parametrized by $(x, \dots, x^n)$ is non-planar; more generally, if $\mathcal{M}$ is immersed into $\mathbb{R}^n$ by an analytic map $\mathbf{f} = (f_1, \dots, f_n)$, then the non-planarity of $\mathcal{M}$ exactly means that the functions $1, f_1, \dots, f_n$ are linearly independent over $\mathbb{R}$. Sprindžuk conjectured in 1980 that non-planar analytic submanifolds of $\mathbb{R}^n$ are strongly extremal. There has been a sequence of partial results regarding the Baker-Sprindžuk problem but the complete solution was given in [27]. In fact, a more general result was established there: strong extremality of sufficiently smooth non-degenerate submanifolds. Namely, a submanifold $\mathcal{M}$ is said to be *non-degenerate* if for almost every (with respect to the volume measure) point $\mathbf{x}$ of $\mathcal{M}$ there exists $k \in \mathbb{N}$ such that $\mathcal{M}$ is C^k on a neighborhood of $\mathbf{x}$ and

$$\mathbb{R}^n = T_{\mathbf{x}}^{(k)}\mathcal{M}\,, \tag{1.4}$$

where $T_{\mathbf{x}}^{(k)}\mathcal{M}$ is the k-th order tangent space to $\mathcal{M}$ at $\mathbf{x}$ (the span of partial derivatives of a parameterizing map of orders up to k). It is not hard to see that any non-degenerate submanifold is non-planar while any non-planar analytic submanifold is non-degenerate. (In a way, non-degeneracy is an infinitesimal analog of the notion of non-planarity.)

The paper [27] also opened up the new avenues for investigating submanifolds of $M_{m,n}$ for extremality and strong extremality. The following explicit problem was subsequently stated by Gorodnik as Question 35 in [20]:

Problem 1: *Find reasonable and checkable conditions for a smooth submanifold $\mathcal{M}$ of $M_{m,n}$ which generalize non-degeneracy of vector-valued maps and impliy that almost every point of $\mathcal{M}$ is extremal (strongly extremal).*

One can also pose a problem of generalizing the notion of non-planarity of subsets of $\mathbb{R}^n$ to those of $M_{m,n}$, so that, when $\mathcal{M}$ is an analytic submanifold, its non-planarity implies that almost every point of $\mathcal{M}$ is extremal (strongly extremal). It is easy to see, e.g. from examples considered in [28], that being locally not contained in proper affine subspaces of $M_{m,n}$ is not the right condition to consider.

Until recently the only examples of extremal manifolds of $M_{m,n}$ with $\min\{m,n\} \ge 2$ have been those found by Kovalevskaya [30, 31]. She has considered submanifolds $\mathcal{M}$ of $M_{m,n}$ of dimension m immersed by the map

$$(x_1, \dots, x_m) \mapsto \begin{pmatrix} f_{1,1}(x_1) & \dots & f_{1,n}(x_1) \\ \vdots & \ddots & \vdots \\ f_{m,1}(x_m) & \dots & f_{m,n}(x_m) \end{pmatrix}, \tag{1.5}$$

where $f_{i,j} : I_i \to \mathbb{R}$ are C^{n+1} functions defined on some intervals $I_i \subset \mathbb{R}$ such that every row in (1.5) represents a non-degenerate map. Assuming that $m \ge n(n-1)$ Kovalevskaya has shown that $\mathcal{M}$ is extremal. In the case $n = 2$ and $m \ge 2$ Kovalevskaya [32] has also established a stronger statement, which treats the inequality $\|Y\mathbf{q} - \mathbf{p}\|^m < \Pi_+(\mathbf{q})^{-(1+\varepsilon)n}$ – a mixture of (1.1) and (1.2).

In principle, manifolds (1.5) are natural to consider but within the above results the dimensions m and n are bizarrely confined. The overdue general result regarding Kovalevskaya-type manifolds has been recently established in [28]. More precisely, it has been shown that any manifold of the form (1.5) is strongly extremal provided that every row $(f_{i,1}, \dots, f_{i,n})$ in (1.5) is a non-degenerate map into $\mathbb{R}^n$ defined on an open subset of $\mathbb{R}^{d_i}$.

Working towards the solution of Problem 1 the following more general result has been established in [28]. Let $\mathbf{d}$ be the map defined on $M_{m,n}$ that, to a given $Y \in M_{m,n}$, assigns the collection of all minors of Y in a certain fixed order. Thus $\mathbf{d}$ is a map from $M_{m,n}$ to $\mathbb{R}^N$, where $N = \binom{m+n}{n} - 1$ is the number of all possible minors of an $m \times n$ matrix. According to [28, Theorem 2.1] any smooth submanifold $\mathcal{M}$ of $M_{m,n}$ such that $\mathbf{d}(\mathcal{M})$ is non-degenerate is strongly extremal. The result also treats pushforwards of Federer measures – see Theorem 2.2 for further details.

In the present paper we introduce a weaker (than in [28]) version of non-planarity of a subset of $M_{m,n}$ which naturally extends the one for subsets of vector spaces and, in the smooth manifold case, is implied by the non-degeneracy of $\mathbf{d}(\mathcal{M})$. Then we use results of [28] to conclude (Corollary 2.4) that weakly non-planar analytic submanifolds of $M_{m,n}$ are strongly extremal. See Theorem 2.3 for a more general statement.

Towards the end of the paper (§6) we also investigate generalizations of the main results to weighted and inhomogeneous forms of approximation. The weighted form of approximation is a modification of (1.1) which is obtained by introducing weights (exponents) of approximation for each linear form (see §6.2 for details). In the inhomogeneous case, the system of linear forms $\mathbf{q} \mapsto Y\mathbf{q}$ given by $Y \in M_{m,n}$ is replaced by the system of affine forms $\mathbf{q} \mapsto Y\mathbf{q} + \mathbf{z}$ given by the pairs $(Y; \mathbf{z})$, where $Y \in M_{m,n}$ and $\mathbf{z} \in \mathbb{R}^m$. Naturally, one can consider inhomogeneous versions of extremality in the standard, weighted and multiplicative settings (see §6 for precise definitions). In this paper we establish that within all the three settings inhomogeneous and homogeneous forms of extremality are equivalent under certain natural conditions. This extends recent results of [11]. Our approach is bases on the general framework developed in [11] and the equivalence of strong and weighted forms of extremality established in the present paper.

The structure of the paper is as follows: we formally introduce the weak non-planarity condition and state our main results in §2. In the next section we compare our new condition with the one introduced in [28]. The main theorem is proved in §4, while §5 is devoted to some further features of the concept of weak non-planarity; §6 discusses inhomogeneous and weighted extensions of our main results, and the last section contains several concluding remarks and open questions.

Acknowledgements. The authors are grateful to the University of Bielefeld for providing a stimulative research environment during their visits supported by SFB701. We gratefully acknowledge the support of the National Science Foundation through grants DMS-0801064, DMS-0801195, DMS-1101320 and DMS-1265695, and of EPSRC through grants EP/C54076X/1 and EP/J018260/1. Thanks are also due to the reviewer for useful comments.

2. Main results

Let us begin by introducing some terminology and stating some earlier results. Let X be a Euclidean space. Given $x \in X$ and $r > 0$, let $B(x, r)$ denote the open ball of radius r centered at x. If $V = B(x, r)$ and $c > 0$, let cV stand for $B(x, cr)$. Let μ be a measure on X. All the measures within this paper will be assumed to be Radon. Given $V \subset X$ such that $\mu(V) > 0$ and a function $f : V \to \mathbb{R}$, let

$$\|f\|_{\mu,V} = \sup_{x \in V \cap \operatorname{supp} \mu} |f(x)|.$$

A Radon measure μ will be called *D-Federer on U*, where $D > 0$ and U is an open subset of X, if $\mu(3V) < D\mu(V)$ for any ball $V \subset U$ centered in the support of μ. The measure μ is called *Federer* if for μ-almost every point $x \in X$ there is a neighborhood U of x and $D > 0$ such that μ is D-Federer on U.

Given $C, \alpha > 0$ and an open subset $U \subset X$, we say that $f : U \to \mathbb{R}$ is *(C, α)-good on U with respect to the measure μ* if for any ball $V \subset U$

centered in $\operatorname{supp}\mu$ and any $\varepsilon > 0$ one has

$$\mu\big(\{x \in V : |f(x)| < \varepsilon\}\big) \le C\left(\frac{\varepsilon}{\|f\|_{\mu,V}}\right)^{\alpha}\mu(V)\,.$$

Given $\mathbf{f} = (f_1, \dots, f_N) : U \to \mathbb{R}^N$, we say that the pair $(\mathbf{f}, \mu)$ is *good* if for μ-almost every $x \in U$ there is a neighborhood $V \subset U$ of x and $C, \alpha > 0$ such that any linear combination of $1, f_1, \dots, f_N$ over $\mathbb{R}$ is (C, α)-good on V with respect to μ. The pair $(\mathbf{f}, \mu)$ is called *non-planar* if

$$\begin{gathered}\text{for any ball } V \subset U \text{ centered in } \operatorname{supp}\mu,\\ \text{the set } \mathbf{f}(V \cap \operatorname{supp}\mu) \text{ is not contained in any affine hyperplane of } \mathbb{R}^N.\end{gathered} \tag{2.1}$$

Clearly it generalizes the definition of non-planarity given in the introduction: $\operatorname{supp}\mu$ is non-planar iff so is the pair (Id, μ).

Basic examples of good and nonplanar pairs $(\mathbf{f}, \mu)$ are given by $\mu = \lambda$ (Lebesgue measure on $\mathbb{R}^d$) and $\mathbf{f}$ smooth and nondegenerate, see [27, Proposition 3.4]. The paper [26] introduces a class of *friendly* measures: a measure μ on $\mathbb{R}^n$ is friendly if and only if it is Federer and the pair (Id, μ) is good and nonplanar. In the latter paper the approach to metric Diophantine approximation developed in [27] has been extended to maps and measures satisfying the conditions described above. One of its main results is the following statement, implicitly contained in [26]:

Theorem 2.1. [23, Theorem 4.2] *Let μ be a Federer measure on $\mathbb{R}^d$, $U \subset \mathbb{R}^d$ open, and $\mathbf{f} : U \to \mathbb{R}^n$ a continuous map such that $(\mathbf{f}, \mu)$ is good and nonplanar; then $\mathbf{f}_*\mu$ is strongly extremal.*

Here and hereafter $\mathbf{f}_*\mu$ is the *pushforward* of μ by $\mathbf{f}$, defined by $\mathbf{f}_*\mu(\cdot) \overset{\text{def}}{=} \mu\big(\mathbf{f}^{-1}(\cdot)\big)$. When μ is Lebesgue measure and $\mathbf{f}$ is smooth and nonsingular, $\mathbf{f}_*\mu$ is simply (up to equivalence) the volume measure on the manifold $\mathbf{f}(U)$.

The next development came in the paper by Kleinbock, Margulis and Wang in 2011. Given $F : U \to M_{m,n}$, let us say that (F, μ) is *good* if $(\mathbf{d} \circ F, \mu)$ is good, where $\mathbf{d}$ is the imbedding of $M_{m,n}$ to $\mathbb{R}^N$ defined in §1, where $N = \binom{m+n}{n} - 1$. Also we will say that (F, μ) is *strongly non-planar* if

$$(\mathbf{d} \circ F, \mu) \text{ is non-planar.} \tag{2.2}$$

Clearly $\mathbf{d}$ is the identity map when $\min\{n, m\} = 1$, thus in both row-matrix and column-matrix cases (2.2) is equivalent to (2.1). Therefore the following general result, established in [28], generalizes the above theorem:

Theorem 2.2. [28, Theorem 2.1] *Let U be an open subset of $\mathbb{R}^d$, μ be a Federer measure on U and $F : U \to M_{m,n}$ be a continuous map such* *that* (F,μ) *is* (i) *good, and* (ii) *strongly non-planar. Then* $F_*\mu$ *is strongly extremal.*

In this paper we introduce a broader class of strongly extremal measures on $M_{m,n}$ by relaxing condition (ii) of Theorem 2.2. To introduce a weaker notion of non-planarity, we need the following notation: given

$$A \in M_{n,m}(\mathbb{R}) \text{ and } B \in M_{n,n}(\mathbb{R}) \text{ with } \operatorname{rank}(A|B) = n\,, \tag{2.3}$$

define

$$\mathcal{H}_{A,B} \stackrel{\text{def}}{=} \{Y \in M_{m,n} : \det(AY + B) = 0\}. \tag{2.4}$$

These sets will play the role of proper affine subspaces of vector spaces. It will be convenient to introduce notation $\mathbf{H}_{m,n}$ for the collection of all sets $\mathcal{H}_{A,B}$ as above. Here and elsewhere $(A|B)$ stands for the matrix obtained by joining A and B, A being its left block and B being its right block. Then for F and μ as above, let us say that (F,μ) is *weakly non-planar* if

$$\begin{aligned} &F(V \cap \operatorname{supp}\mu) \not\subset \mathcal{H} \text{ for any ball } V \subset U \\ &\text{centered in } \operatorname{supp}\mu \text{ and any } \mathcal{H} \in \mathbf{H}_{m,n}\,. \end{aligned} \tag{2.5}$$

Obviously any $\mathcal{H}_{A,B} \in \mathbf{H}_{m,n}$ with $A = 0$ is empty. Otherwise, $\det(AY + B)$ is a non-constant polynomial and $\mathcal{H}_{A,B}$ is a hypersurface in $M_{m,n}$. Thus, the weak non-planarity of (F,μ) simply requires that $F(\operatorname{supp}\mu)$ does not locally lie entirely inside such a hypersurface. We shall see in the next section that in both row-matrix and column-matrix cases the weak non-planarity defined above is again equivalent to (2.1) (hence to strong non-planarity), and that in general strong non-planarity implies weak non-planarity but not vice versa. Thus the following theorem is a nontrivial generalization of Theorem 2.2:

Theorem 2.3 (Main Theorem)**.** *Let* U *be an open subset of* $\mathbb{R}^d$, μ *a Federer measure on* U *and* $F : U \to M_{m,n}$ *a continuous map such that* (F,μ) *is* (i) *good, and* (ii) *weakly non-planar. Then* $F_*\mu$ *is strongly extremal.*

Specializing to the case of submanifolds of $M_{m,n}$, we can call a smooth submanifold $\mathcal{M}$ of $M_{m,n}$ *weakly non-planar* if

$$V \cap \mathcal{M} \not\subset \mathcal{H} \text{ for any ball } V \text{ centered in } \mathcal{M} \text{ and any } \mathcal{H} \in \mathbf{H}_{m,n}\,. \tag{2.6}$$

Then Theorem 2.3 readily implies

Corollary 2.4. *Any analytic weakly non-planar submanifold of* $M_{m,n}$ *is strongly extremal.*

Proof. Without loss of generality we can assume that $\mathcal{M}$ is immersed in $M_{m,n}$ by an analytic map F defined on $\mathbb{R}^d$. Let μ be the d-dimensional Lebesgue measure; then, saying that $\mathcal{M}$ is strongly extremal is the same as saying that $F_*\mu$ is strongly extremal. To see that (F,μ) is weakly non-planar in the sense of (2.1) provided that $\mathcal{M}$ is weakly non-planar in the sense of (2.6), take a ball $V \subset \mathbb{R}^d$ and assume that $F(V) = F(V \cap \operatorname{supp}\mu) \subset \mathcal{H}_{A,B}$ for some choice of $A \in M_{n,m}$ and $B \in M_{n,n}$ with $\operatorname{rank}(A|B) = n$. Clearly there exists a ball U in $M_{m,n}$ centered in $\mathcal{M}$ and a ball $V' \subset V$ such that $U \cap \mathcal{M} \subset F(V')$, contradicting to (2.6). Finally, the fact that (F,μ) is good is due to the analyticity of F – see [28] or [21]. □

We remark that if $\mathcal{M}$ is a connected analytic submanifold of $M_{m,n}$, then (2.6) is simply equivalent to $\mathcal{M}$ not being contained in $\mathcal{H}$ for any $\mathcal{H} \in \mathbf{H}_{m,n}$.

We postpone the proof of Theorem 2.3 until §4, after we compare the two (strong and weak) nonplanarity conditions introduced above.

3. Weak vs. strong non-planarity

Throughout this section $F : U \to M_{m,n}$ denotes a map from an open subset U of a Euclidean space X, and μ is a measure on X.

The first result of the section shows that Theorem 2.2 is a consequence of Theorem 2.3:

Lemma 3.1. *If (F,μ) is strongly non-planar, then it is weakly non-planar.*

Proof. Let $(\mathbf{d} \circ F, \mu)$ be non-planar. Let $A \in M_{n,m}$ and $B \in M_{n,n}$ with $\operatorname{rank}(A|B) = n$ and let $V \subset U$ be a ball centered in $\operatorname{supp}\mu$. Observe that for any $Y \in M_{m,n}$

$$\begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix}\begin{pmatrix} I_m & Y \\ -A & B \end{pmatrix} = \begin{pmatrix} I_m & Y \\ 0 & AY+B \end{pmatrix}.$$

Therefore,

$$\det(AY+B) = \det\begin{pmatrix} I_m & Y \\ -A & B \end{pmatrix}. \tag{3.1}$$

We will expand the right hand side of (3.1) using the Laplace expansion formula from linear algebra that implies that the determinant of an $(m+n)\times(m+n)$ matrix can be written as the sum of all minors from the first m rows multiplied by their co-factors. It is easily seen that each minor in question (that arises from the right hand side of (3.1)) is either a minor of Y or simply the number 1. Their co-factors are the minors of $(-A|B)$ taken with appropriate signs. Since $\operatorname{rank}(-A|B) = \operatorname{rank}(A|B) = n$, these cofactors are not all zeros. Hence, since $\mathcal{H}_{A,B}$ is defined by the equation $\det(AY+B) = 0$, we have that $\mathbf{d}(\mathcal{H}_{A,B})$ lies in either a hyperplane of $\mathbb{R}^N$, where $N = \binom{m+n}{n} - 1$ is the same as in §1, or an empty set. Since $(\mathbf{d}\circ F,\mu)$ is non-planar, it follows that $F(V \cap \operatorname{supp}\mu) \not\subset \mathcal{H}_{A,B}$. This verifies that (F,μ) is weakly non-planar and completes the proof. □

The converse to the above lemma is in general not true; here is a counterexample:

Proposition 3.2. *Let*

$$Y = F(x,y,z) = \begin{pmatrix} x & y \\ z & x \end{pmatrix}, \tag{3.2}$$

and let μ *be Lebesgue measure on* $\mathbb{R}^3$*. Then* (F,μ) *is weakly but not strongly non-planar.*

Proof. The fact that $\mathcal{M} = F(\mathbb{R}^3)$ is not strongly non-planar is trivial because there are two identical minors (elements) in every $Y \in \mathcal{M}$. Now let $A, B \in M_{2,2}$ with $\operatorname{rank}(A|B) = 2$. By (2.6) and in view of the analyticity of F, it suffices to verify that

$$\det(AY + B) \neq 0 \qquad \text{for some } Y \text{ of the form (3.2).} \tag{3.3}$$

If $\det B \neq 0$ then taking $Y = 0$ proves (3.3). Also if $\det A \neq 0$ then ensuring (3.3) is very easy. Indeed, take Y of the form (3.2) with $y = z = 0$ and x sufficiently large. Then

$$\det(AY + B) = \det(xA + B) = x \det A \det(I_n + \tfrac{1}{x} BA^{-1}) \neq 0$$

if and onely if

$$\det(I_n + \tfrac{1}{x} BA^{-1}) \neq 0\,.$$

The latter condition is easily met for sufficiently large x because $\frac{1}{x}BA^{-1} \to 0$ as $x \to \infty$. Thus for the rest of the proof we can assume that $\det A = \det B = 0$. Then without loss of generality we can also assume that

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ \beta_1 & \beta_2 \end{pmatrix},$$

otherwise we can use Gaussian elimination method to replace A and B with the matrices of the above form. For $\operatorname{rank}(A|B) = 2$ we have that at least one of α_1 and α_2 is non-zero and at least one of β_1 and β_2 is non-zero. For Y is of the form (3.2), we have

$$AY + B = \begin{pmatrix} \alpha_1 x + \alpha_2 z & \alpha_1 y + \alpha_2 x \\ \beta_1 & \beta_2 \end{pmatrix}.$$

If $\alpha_1 \neq 0$ and $\beta_1 \neq 0$ then taking $x = 0$, $y = 1$, $z = 0$ ensures (3.3).
If $\alpha_2 \neq 0$ and $\beta_1 \neq 0$ while $\alpha_1 = 0$ then taking $x = 1$ and $z = 0$ ensures (3.3).
If $\alpha_2 \neq 0$ and $\beta_2 \neq 0$ then taking $x = 0$, $y = 0$ and $z = 1$ ensures (3.3).
If $\alpha_1 \neq 0$ and $\beta_2 \neq 0$ while $\alpha_2 = 0$ then taking $x = 1$ and $y = 0$ ensures (3.3). $\square$

Note however that in the case when matrices are rows/columns, conditions (2.1) and (2.2) are equivalent. This readily follows from

Lemma 3.3. *Let* $\min\{n,m\}=1$*. Then for any* $A\in M_{n,m}$ *and* $B\in M_{n,n}$ *such that* $\operatorname{rank}(A|B)=n$*, the equation* $\det(AY+B)=0$ *defines either a hyperplane or an empty set.*

Proof. First consider the case $n=1$. Then $A=(a_1,\dots,a_m)\in M_{1,m}$, $B=(b)\in M_{1,1}$ and $Y=(y_1,\dots,y_m)^t\in M_{1,1}$. Obviously, $AY+B=0$ becomes $\sum_{i=1}^m a_iy_i+b=0$. Since $\operatorname{rank}(A|B)=1$, one of the coefficients is non-zero, and the claim follows.

Consider now the case $m=1$. Then $A=(a_1,\dots,a_n)^t\in M_{1,n}$, $B\in M_{n,n}$ and $Y=(y_1,\dots,y_n)\in M_{1,n}$. By (3.1),

(3.4)
$$\det(AY+B)=0\qquad\iff\qquad\det\begin{pmatrix}1 & Y\\ -A & B\end{pmatrix}=\det B+\sum_{i=1}^n c_iy_i=0,$$

where c_i is the cofactor of y_i. Since $\operatorname{rank}(-A|B)=\operatorname{rank}(A|B)=n$, as least one of the numbers $\det B,c_1,\dots,c_n$ is non-zero. If $c_1=\dots=c_n=0$ then $\det B\neq 0$ and (3.4) defines an empty set. Otherwise, (3.4) obviously defines a hyperplane. $\square$

4. Proof of Theorem 2.3

Let us first express subsets $\mathcal{H}_{A,B}$ of $M_{m,n}$ in several equivalent ways. It will be convenient to introduce the following notation: we let $W=\mathbb{R}^{m+n}$, denote by $\mathbf{e}_1,\dots,\mathbf{e}_{m+n}$ the standard basis of W, and, for $i=1,\dots,m+n$, by E_i^+ (resp., E_i^-) the span of the first (resp., the last) i vectors of this basis, and by π_i^+ (resp., π_i^-) the orthogonal projection of W onto E_i^+ (resp., E_i^-). Also, if $I=\{i_1,\dots,i_\ell\}\subset\{1,\dots,m+n\}$ (written in the increasing order), we denote $\mathbf{e}_I\stackrel{\text{def}}{=}\mathbf{e}_{i_1}\wedge\dots\wedge\mathbf{e}_{i_\ell}\in\bigwedge^\ell(W)$. For A,B as in (2.3), let $W_{A,B}$ be the subspace of W spanned by the columns of the matrix $\begin{pmatrix}A^t\\ B^t\end{pmatrix}=(A|B)^t$. (Here and hereafter the superscript t stands for transposition.) Note that $\dim(W_{A,B})=n$ due to the assumption on the rank of $(A|B)$.

Given $Y\in M_{m,n}$, let us denote

$$u_Y\stackrel{\text{def}}{=}\begin{pmatrix}I_m & Y\\ 0 & I_n\end{pmatrix}.\tag{4.1}$$

Then we have the following elementary

Lemma 4.1. *Let* A *and* B *satisfy* (2.3)*. Then the following are equivalent:*

(i) $Y\in\mathcal{H}_{A,B}$*;*
(ii) $\dim\left(\pi_n^-(u_Y^tW_{A,B})\right)<n$*;*
(iii) $u_Y^tW_{A,B}\cap E_m^+\neq\{0\}$*;*

(iv) $(u_Y^t W_{A,B})^\perp \cap E_n^- \neq \{0\}$;

(v) $\dim\left(\pi_m^+\left((u_Y^t W_{A,B})^\perp\right)\right) < m$;

(vi) $Y^t \in \mathcal{H}_{D,-C}$, *where* $C \in M_{m,m}$ *and* $D \in M_{m,n}$ *are such that the columns of* $(C|D)^t$ *form a basis for* $W_{A,B}^\perp$.

Proof. Note that $u_Y^t W_{A,B}$ is spanned by the columns of the matrix

$$u_Y^t \begin{pmatrix} A^t \\ B^t \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ Y^t & I_n \end{pmatrix} \begin{pmatrix} A^t \\ B^t \end{pmatrix} = \begin{pmatrix} A^t \\ (AY+B)^t \end{pmatrix},$$

and $\pi_n^-(u_Y^t W_{A,B})$ is therefore spanned by the columns of $(AY+B)^t$. Since the latter matrix has rank less than n if and only if (i) holds, the equivalence between (i) and (ii) follows. The equivalence (ii) $\iff$ (iii), together with (iv) $\iff$ (v), is a simple exercise in linear algebra. To derive (iii) $\iff$ (iv), observe that dimensions of $(u_Y^t W_{A,B})^\perp$ and E_n^- add up to $\dim(W)$, therefore these two subspaces have trivial intersection if and only if the same is true for their orthogonal complements.

Finally, to establish (v) $\iff$ (vi), it suffices to note that

$$(u_Y^t W_{A,B})^\perp = \left((u_Y^t)^t\right)^{-1} W_{A,B}^\perp = u_{-Y} W_{A,B}^\perp$$

is spanned by the (linearly independent) columns of the matrix

$$u_{-Y} \begin{pmatrix} C^t \\ D^t \end{pmatrix} = \begin{pmatrix} I_m & -Y \\ 0 & I_n \end{pmatrix} \begin{pmatrix} C^t \\ D^t \end{pmatrix} = \begin{pmatrix} C^t - YD^t \\ D^t \end{pmatrix}$$

and its orthogonal projection onto E_m^+ is therefore spanned by the columns of

$$C^t - YD^t = -(DY^t - C)^t .$$

Hence (v) holds if and only if $\det(DY^t - C) = 0$. $\square$

Now let $F : U \to M_{m,n}$ be a map from an open subset U of a Euclidean space X, μ a measure on X, and denote by $F^t : U \to M_{n,m}$ the map given by $F^t(x) = \big(F(x)\big)^t$.

Corollary 4.2. (F,μ) *is weakly non-planar if and only if* (F^t,μ) *is weakly non-planar.*

Proof. Suppose that (F,μ) is not weakly non-planar, that is (2.5) does not hold. Then there exists a ball V centered in $\operatorname{supp}\mu$ such that $F(V \cap \operatorname{supp}\mu) \subset \mathcal{H}_{A,B}$ for some $A \in M_{n,m}$ and $B \in M_{n,n}$ with $\operatorname{rank}(A|B) = n$. Using the equivalence (i) $\iff$ (vi) of the previous lemma, we conclude that there exist $C \in M_{m,m}$ and $D \in M_{m,n}$ such that

$$\operatorname{rank}(-C|D) = \operatorname{rank}(C|D) = m \quad \text{and} \quad F^t(V \cap \operatorname{supp}\mu) \subset \mathcal{H}_{D,-C}\,;$$

hence F^t is not weakly non-planar. Converse is proved similarly. $\square$

The main theorem will be derived using the approach based on dynamics on the space of lattices, which was first developed by Kleinbock and Margulis in [27] and then extended in [28]. The key observation here is the fact that Diophantine properties of $Y \in M_{m,n}$ can be expressed in terms of of the action of diagonal matrices in $\mathrm{SL}_{m+n}(\mathbb{R})$ on

$$u_Y \mathbb{Z}^{m+n} = \left\{ \begin{pmatrix} Y\mathbf{q} - \mathbf{p} \\ \mathbf{q} \end{pmatrix} : \mathbf{p} \in \mathbb{Z}^m,\ \mathbf{q} \in \mathbb{Z}^n \right\}.$$

The latter object is a lattice in W which is viewed as a point of the homogeneous space $\mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z})$ of unimodular lattices in W. However we are able to use the final outcome of the techniques developed in [28] and preceding papers, thus in this paper there is no need to state the quantitative nondivergence estimates ([27, Theorem 5.2], [26, Theorem 4.3]) and the correspondence between Diophantine approximation and dynamics on the space of lattices [28, Proposition 3.1]. The reader is referred to the aforementioned paper, as well as to survey papers [23, 24] for more details.

Now let us introduce some more notation. For an $(m+n)$-tuple $\mathbf{t} = (t_1, \ldots, t_{m+n})$ of real numbers, define

$$g_{\mathbf{t}} \stackrel{\mathrm{def}}{=} \mathrm{diag}(e^{t_1}, \ldots, e^{t_m}, e^{-t_{m+1}}, \ldots, e^{-t_{m+n}}).$$

We will denote by $\mathcal{A}$ the set of $(m+n)$-tuples $\mathbf{t}$ such that

$$t_1, \ldots, t_{m+n} > 0 \quad \text{and} \quad \sum_{i=1}^{m} t_i = \sum_{j=1}^{n} t_{m+j}. \tag{4.2}$$

For a fixed $\mathbf{t} \in \mathcal{A}$ let us denote by $\mathcal{E}_{\mathbf{t}}^{+}$ the span of all the eigenvectors of $g_{\mathbf{t}}$ in $\bigwedge(W)$ with eigenvalues greater or equal to one (in other words, those which are not contracted by the $g_{\mathbf{t}}$-action). It is easy to see that $\mathcal{E}_{\mathbf{t}}^{+}$ is spanned by elements $\mathbf{e}_I \wedge \mathbf{e}_J$ where $I \subset \{1, \ldots, m\}$ and $J \subset \{m+1, \ldots, m+n\}$ are such that

$$\sum_{i \in I} t_i \ge \sum_{j \in J} t_j. \tag{4.3}$$

We will let $\pi_{\mathbf{t}}^{+}$ be the orthogonal projection onto $\mathcal{E}_{\mathbf{t}}^{+}$.

For $1 \le \ell \le m+n-1$, let us denote by $\mathcal{W}^{\ell}$ the set of *decomposable elements* of $\bigwedge^{\ell}(W)$ (that is, elements which can be written as $\mathbf{w} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{\ell}$, where $\mathbf{v}_i \in W$), and denote $\mathcal{W} \stackrel{\mathrm{def}}{=} \bigcup_{\ell=1}^{m+n-1} \mathcal{W}^{\ell}$. Up to a nonzero factor the nonzero elements of $\mathcal{W}^{\ell}$ can be identified with subgroups of W of rank ℓ. A special attention will be paid to decomposable elements with integer coordinates: we will let

$$\mathcal{W}_{\mathbb{Z}}^{\ell} \stackrel{\mathrm{def}}{=} \mathcal{W}^{\ell} \cap \bigwedge(\mathbb{Z}^{m+n}) \quad \text{and} \quad \mathcal{W}_{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \bigcup_{\ell=1}^{m+n-1} \mathcal{W}_{\mathbb{Z}}^{\ell} \subset \bigwedge(\mathbb{Z}^{m+n}).$$

The next statement is a simplified version of Corollary 5.1 from [28]:

Theorem 4.3. *Let an open subset U of $\mathbb{R}^d$, a continuous map $F : U \to M_{m,n}$ and a Federer measure μ on U be given. Suppose that (F,μ) is good, and also that for any ball $V \subset U$ with $\mu(V) > 0$ there exists positive c such that*

$$\|\pi_{\mathbf{t}}^{+} u_{F(\cdot)}\mathbf{w}\|_{\mu,V} \geq c \quad \text{for all } \mathbf{w} \in \mathcal{W}_{\mathbb{Z}} \smallsetminus \{\mathbf{0}\} \text{ and } \mathbf{t} \in \mathcal{A}\,. \tag{4.4}$$

Then $F_*\mu$ is strongly extremal.

See also [28, Theorem 4.3] for a necessary and sufficient condition for strong extremality in the class of good pairs (F,μ).

Now we can proceed with the proof of our main theorem.

Proof of Theorem 2.3. For F and μ as in Theorem 2.3, we need to take a ball $V \subset U$ with $\mu(V) > 0$ (which we can without loss of generality center at a point of $\operatorname{supp}\mu$) and find $c > 0$ such that (4.4) holds. Since (F,μ) is weakly non-planar, from the equivalence (i) $\iff$ (vi) of Lemma 4.1 we conclude that for any $C \in M_{m,m}$ and $D \in M_{m,n}$ with $\operatorname{rank}(D| -C) = m$ one has $\det\big(DF(x)^t - C\big) \neq 0$ for some $x \in \operatorname{supp}\mu \cap V$. Equivalently, for any $\mathbf{w} \in \mathcal{W}^m \smallsetminus \{\mathbf{0}\}$, which we take to be the exterior product of columns of $\begin{pmatrix} -C^t \\ D^t \end{pmatrix}$, the orthogonal projection of $u_{F(x)}\mathbf{w}$ onto $\bigwedge^m(E_m^+)$, which is equal to the exterior product of columns of

$$\begin{pmatrix} I & F(x) \end{pmatrix}\begin{pmatrix} -C^t \\ D^t \end{pmatrix} = -C^t + F(x)D^t = \big(DF(x)^t - C\big)^t\,,$$

is nonzero for some $x \in \operatorname{supp}\mu \cap V$.

Our next goal is to treat $\mathbf{w} \in \mathcal{W}^\ell$ with $\ell \neq m$ in a similar way. For this, let us consider the subspace $\mathcal{E}^+$ of $\bigwedge(W)$ defined by

$$\mathcal{E}^+ \stackrel{\text{def}}{=} \operatorname{span}\left\{\mathbf{e}_I, \mathbf{e}_{\{1,\dots,m\}} \wedge \mathbf{e}_J : I \subset \{1,\dots,m\}, J \subset \{m+1,\dots,m+n\}\right\}, \tag{4.5}$$

or, equivalently, by

$$\mathcal{E}^+ \cap \textstyle\bigwedge^\ell(W) = \begin{cases} \bigwedge^\ell(E_m^+) & \text{if } \ell \leq m\,, \\ \mathbb{R}\mathbf{e}_{\{1,\dots,m\}} \wedge \bigwedge^{\ell-m}(E_n^-) & \text{if } \ell \geq m\,. \end{cases}$$

In particular, $\mathcal{E}^+ \cap \bigwedge^m(W) = \bigwedge^m(E_m^+)$ is one-dimensional and is spanned by $\mathbf{e}_{\{1,\dots,m\}}$.

The relevance of the space $\mathcal{E}^+$ to our set-up is highlighted by

Lemma 4.4. $\mathcal{E}^+ = \bigcap_{\mathbf{t}\in\mathcal{A}} \mathcal{E}_{\mathbf{t}}^+$.

Proof. The direction $\subset$ is clear from (4.5) and the validity of (4.3) when either $J = \varnothing$ or $I = \{1,\dots,m\}$. Conversely, take $\mathbf{w} \in \bigwedge(W)$ and suppose that there exist a proper subset I of $\{1,\dots,m\}$ and a nonempty subset J

of $\{m+1,\dots,m+n\}$ such that the orthogonal projection of $\mathbf{w}$ onto $\mathbf{e}_I\wedge\mathbf{e}_J$ is not zero. Then choose $\mathbf{t}\in\mathcal{A}\smallsetminus\{\mathbf{0}\}$ such that $t_i=0$ when $i\in I$, and $t_j\neq 0$ when $j\in J$; this way $\mathbf{e}_I\wedge\mathbf{e}_J$ is contracted by $g_{\mathbf{t}}$, which implies that $\mathbf{w}$ is not contained in $\mathcal{E}^+_{\mathbf{t}}$. □

Denote by π^+ the orthogonal projection $\bigwedge(W)\to\mathcal{E}^+$; thus we have shown that

$$\|\pi^+u_{F(\cdot)}\mathbf{w}\|_{\mu,V}>0\quad\forall\,\mathbf{w}\in\mathcal{W}^m\smallsetminus\{\mathbf{0}\}\,. \tag{4.6}$$

We now claim that the same is true for all $\mathbf{w}\in\mathcal{W}\smallsetminus\{\mathbf{0}\}$. Indeed, take

$$\mathbf{w}=\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_\ell\neq 0\,,$$

where $\ell<m$, and choose arbitrary $\mathbf{v}_{\ell+1},\dots,\mathbf{v}_m$ such that $\mathbf{v}_1,\dots,\mathbf{v}_m$ are linearly independent. Then $\pi^+\big(u_{F(x)}(\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_m)\big)$ being nonzero is equivalent to $\pi^+(u_{F(x)}\mathbf{v}_1),\dots,\pi^+(u_{F(x)}\mathbf{v}_m)$ being linearly independent, which implies $\pi^+(u_{F(x)}\mathbf{v}_1),\dots,\pi^+(u_{F(x)}\mathbf{v}_\ell)$ being linearly independent, i.e. $\pi^+(u_{F(x)}\mathbf{w})\neq 0$.

The case $\ell>m$ can be treated in a dual fashion: if $\mathbf{w}=\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_\ell\neq 0$ is such that $\pi^+(u_{F(x)}\mathbf{w})=0$, then there exists $\mathbf{v}\in E^+_m$ which is orthogonal to all of $\pi^+(u_{F(x)}\mathbf{v}_1),\dots,\pi^+(u_{F(x)}\mathbf{v}_\ell)$, hence to all of $\pi^+(u_{F(x)}\mathbf{v}_1),\dots,\pi^+(u_{F(x)}\mathbf{v}_m)$, and the latter amounts to saying that $\pi^+\big(u_{F(x)}(\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_m)\big)=0$, contradicting (4.6).

Notice that we have proved that for any ball $V\subset U$ centered in $\operatorname{supp}\mu$, the (continuous) function

$$\mathbf{w}\mapsto\|\pi^+(u_{F(\cdot)}\mathbf{w})\|_{\mu,V}$$

is nonzero on the intersection of $\mathcal{W}$ with the unit sphere in $\bigwedge(W)$, hence, by compactness, it has a uniform lower bound. Since $\|\mathbf{w}\|\geq 1$ for any $\mathbf{w}\in\mathcal{W}_{\mathbb{Z}}\smallsetminus\{\mathbf{0}\}$, it follows that for any V as above there exists $c>0$ such that

$$\|\pi^+u_{F(\cdot)}\mathbf{w}\|_{\mu,V}\geq c\quad\text{ for all }\mathbf{w}\in\mathcal{W}_{\mathbb{Z}}\smallsetminus\{\mathbf{0}\}\,.$$

This, in view of Lemma 4.4, finishes the proof of (4.4).

5. More about weak non-planarity

The set of strongly extremal matrices in $M_{m,n}$ is invariant under various natural transformations. For example, it is invariant under non-singular rational transformations, in particular, under the permutations of rows and columns, and, in view of Khintchine's Transference Principle [36], under transpositions. Also, if a matrix $Y\in M_{m,n}$ is strongly extremal then any submatrix of Y is strongly extremal. We have already shown in §4 that weak non-planarity is invariant under transposition; in this section we demonstrate some additional invariance properties.

As before, throughout this section $F : U \to M_{m,n}$ denotes a map from an open subset U of a Euclidean space X and μ is a measure on X. The following statement shows the invariance of weak non-planarity under non-singular transformations.

Lemma 5.1. *Assume that (F, μ) is weakly non-planar. Let $L \in \mathrm{GL}_m(\mathbb{R})$ and $R \in \mathrm{GL}_n(\mathbb{R})$ be given and let $\widetilde{F} : U \to M_{m,n}$ be a map given by $\widetilde{F}(x) = LF(x)R$ for $x \in U$. Then $(\widetilde{F}, \mu)$ is weakly non-planar.*

Proof. Take any $\widetilde{A} \in M_{n,m}$ and $\widetilde{B} \in M_{n,n}$ such that $\mathrm{rank}(\widetilde{A}|\widetilde{B}) = n$ and let $V \subset U$ be a ball centered in $\mathrm{supp}\,\mu$. Define $A = \widetilde{A}L$ and $B = \widetilde{B}R^{-1}$. It is easily seen that

$$(A|B) = (\widetilde{A}|\widetilde{B}) \begin{pmatrix} L & 0 \\ 0 & R^{-1} \end{pmatrix},$$

that is, the product of $(\widetilde{A}|\widetilde{B})$ by a non-singular matrix; thus $\mathrm{rank}(A|B) = \mathrm{rank}(\widetilde{A}|\widetilde{B}) = n$. Since (F, μ) is weakly non-planar, $F(V \cap \mathrm{supp}\,\mu) \not\subset \mathcal{H}_{A,B}$. Therefore, there exists $x \in V \cap \mathrm{supp}\,\mu$ such that $\det(AF(x) + B) \neq 0$. Then

$$\begin{aligned} AF(x) + B &= A(L^{-1}\widetilde{F}(x)R^{-1}) + B \\ &= ((AL^{-1})\widetilde{F}(x) + BR)R^{-1} \\ &= (\widetilde{A}\widetilde{F}(x) + \widetilde{B})R^{-1}. \end{aligned} \tag{5.1}$$

Since $\det R \neq 0$ and $\det(AF(x) + B) \neq 0$, (5.1) implies that $\det(\widetilde{A}\widetilde{F}(x) + \widetilde{B}) \neq 0$. This means that $\widetilde{F}(V \cap \mathrm{supp}\,\mu) \not\subset \mathcal{H}_{\widetilde{A},\widetilde{B}}$. The proof is complete. □

Taking L and R to be I_m and I_n with permuted columns/rows readily implies (as a corollary of Lemma 5.1) that *weak non-planarity in invariant under permutations of rows and/or columns in F*. The next statement is a natural generalization of Lemma 5.1.

Lemma 5.2. *Assume that (F, μ) is weakly non-planar. Let $\widetilde{n} \le n$, $\widetilde{m} \le m$ and $L \in M_{\widetilde{m},m}$ and $R \in M_{n,\widetilde{n}}$ and let $\widetilde{F} : U \to M_{\widetilde{m},\widetilde{n}}$ be a map given by $\widetilde{F}(x) = LF(x)R$ for $x \in U$. If* $\mathrm{rank}\,L = \widetilde{m}$ *and* $\mathrm{rank}\,R = \widetilde{n}$ *then $(\widetilde{F}, \mu)$ is also weakly non-planar.*

Proof. Since $\mathrm{rank}\,L = \widetilde{m}$ and $\mathrm{rank}\,R = \widetilde{n}$, there are $C \in \mathrm{GL}_m(\mathbb{R})$, $\widetilde{C} \in \mathrm{GL}_{\widetilde{m}}(\mathbb{R})$, $D \in \mathrm{GL}_n(\mathbb{R})$ and $\widetilde{D} \in \mathrm{GL}_{\widetilde{n}}(\mathbb{R})$ such that $L = \widetilde{C}L_0C$ and $R = DR_0\widetilde{D}$, where $L_0 = (I_{\widetilde{m}}|0)$ and $R_0 = (I_{\widetilde{n}}|0)^t$. By Lemma 5.1, (F_1, μ) is weakly non-planar, where $F_1(x) = CF(x)D$. Obviously, $\widetilde{F} = \widetilde{C}F_2(x)\widetilde{D}$, where $F_2(x) = L_0F_1(x)R_0$. Therefore, by Lemma 5.1 again, the fact that $(\widetilde{F}, \mu)$ is weakly non-planar would follow from the fact that (F_2, μ) is weakly non-planar. Thus, without loss of generality, within this proof we can simply assume that $L = L_0$ and $R = R_0$.

Take any $\widetilde{A} \in M_{\widetilde{m},\widetilde{n}}$ and $\widetilde{B} \in M_{\widetilde{n},\widetilde{n}}$ such that $\operatorname{rank}(\widetilde{A}|\widetilde{B}) = \widetilde{n}$. Let

$$A = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix} \in M_{n,m} \qquad \text{and} \qquad B = \begin{pmatrix} \widetilde{B} & 0 \\ 0 & I_{n-\widetilde{n}} \end{pmatrix} \in M_{n,n}. \tag{5.2}$$

It is easily seen $\operatorname{rank}(A|B) = \operatorname{rank}(\widetilde{A}|\widetilde{B}) + n - \widetilde{n} = n$. Take any ball V centered in $\operatorname{supp}\mu$. Since (F, μ) is weakly non-planar, there is $x \in V \cap \operatorname{supp}\mu$ such that $\det(AF(x) + B) \neq 0$. It is easily seen that $F(x)$ has the form

$$F(x) = \begin{pmatrix} \widetilde{F}(x) & * \\ * & * \end{pmatrix},$$

where $\widetilde{F}(x) = L_0 F(x) R_0 \in M_{\widetilde{m},\widetilde{n}}$. Then using (5.2) we get

$$AF(x) + B = \begin{pmatrix} \widetilde{A}\widetilde{F}(x) + \widetilde{B} & * \\ 0 & I_{n-\widetilde{n}} \end{pmatrix}.$$

It follows that $\det(AF(x) + B) = \det(\widetilde{A}\widetilde{F}(x) + \widetilde{B}) \neq 0$, whence the claim of the lemma readily follows. □

Taking L to be L_0 with permuted columns and R to be R_0 with permuted rows readily implies (as a corollary of Lemma 5.2) that *any submatrix in a weakly non-planar F is weakly non-planar*. Note that, combined with Proposition 3.2, this shows that for any m, n with $\min\{m, n\} > 1$ there exists a submanifold of $M_{m,n}$ which is weakly but not strongly non-planar.

In the final part of this section we will talk about products of weakly non-planar measures. In essence, strongly non-planar (and thus weakly non-planar) manifolds given by (1.5) are products of non-planar rows. One can generalize this construction by considering products of matrices with arbitrary dimensions. For the rest of the section we will assume that X_1 and X_2 are two Euclidean spaces and μ_1 and μ_2 are Radon measures on X_1 and X_2 respectively.

Lemma 5.3. *For $i = 1, 2$ let U_i be an open set is X_i and let $F_i : U_i \to M_{m_i,n}(\mathbb{R})$ be given. Let $\mu = \mu_1 \times \mu_2$ be the product measure over $X = X_1 \times X_2$ and let $F : U \to M_{m,n}$, where $U = U_1 \times U_2$ and $m = m_1 + m_2$, be given by*

$$F(x_1, x_2) \stackrel{\text{def}}{=} \begin{pmatrix} F_1(x_1) \\ F_2(x_2) \end{pmatrix}. \tag{5.3}$$

Assume that (F_1, μ_1) and (F_2, μ_2) are weakly non-planar. Then (F, μ) is weakly non-planar.

In view of Corollary 4.2 the following statement is equivalent to Lemma 5.3.

Lemma 5.4. *For $i = 1, 2$ let U_i be an open set is X_i and let $F_i : U_i \to M_{m,n_i}(\mathbb{R})$ be given. Let $\mu = \mu_1 \times \mu_2$ be the product measure over $X = X_1 \times X_2$ and let $F : U \to M_{m,n}$, where $U = U_1 \times U_2$ and $n = n_1 + n_2$, be given by*

$$F(x_1, x_2) \stackrel{\text{def}}{=} \Big(\; F_1(x_1) \; \mid \; F_2(x_2) \; \Big).$$

Assume that (F_1, μ_1) and (F_2, μ_2) are weakly non-planar. Then (F, μ) is weakly non-planar.

In order to prove Lemma 5.3 we will use the following auxiliary statement.

Lemma 5.5. *Let (F, μ) be weakly non-planar, $r \le n$, $A \in M_{r,m}$, $B \in M_{r,n}$ and let* $\operatorname{rank}(A|B) = r$. *Then for any ball $V \subset U$ centered in* $\operatorname{supp} \mu$ *there is $x \in V \cap \operatorname{supp} \mu$ such that* $\operatorname{rank}(AF(x) + B) = r$.

Proof. Let $V \subset U$ be a ball centered in $\operatorname{supp} \mu$. Since $\operatorname{rank}(A|B) = r$, there are matrices $\widetilde{A} \in M_{n-r,m}$ and $\widetilde{B} \in M_{n-r,n}$ such that

$$\operatorname{rank} \begin{pmatrix} A & B \\ \widetilde{A} & \widetilde{B} \end{pmatrix} = n. \tag{5.4}$$

Let

$$A^* = \begin{pmatrix} A \\ \widetilde{A} \end{pmatrix} \in M_{n,m} \qquad \text{and} \qquad B^* = \begin{pmatrix} B \\ \widetilde{B} \end{pmatrix} \in M_{n,n}.$$

Then, by (5.4) and the weak non-planarity of (F, μ), there is a $x \in V \cap \operatorname{supp} \mu$ such that $\det(A^*F(x) + B^*) \ne 0$. Therefore, $\operatorname{rank}(A^*F(x) + B^*) = n$. Clearly

$$A^*F(x) + B^* = \begin{pmatrix} AF(x) + B \\ \widetilde{A}F(x) + \widetilde{B} \end{pmatrix}.$$

Then, the fact that the rank of this matrix is n implies that $\operatorname{rank}(AF(x) + B) = r$. $\square$

Proof of Lemma 5.3. For $i = 1, 2$ let $V_i \subset U_i$ be a ball centered in $\operatorname{supp} \mu_i$. The ball $V = V_1 \times V_2 \subset U$ is then centered in $\operatorname{supp} \mu$. Let $A \in M_{n,m}$, $B \in M_{n,n}$ and $\operatorname{rank}(A|B) = n$. Our goal is to show that there is a point $(x_1, x_2) \in V \cap \operatorname{supp} \mu$ such that $\det(AF(x_1, x_2) + B) \ne 0$.

Split A into $A_1 \in M_{n,m_1}$ and $A_2 \in M_{n,m_2}$ so that $A = (A_1|A_2)$. By (5.3), we have that $AF(x_1, x_2) + B = A_1F_1(x_1) + A_2F_2(x_2) + B$. Assume for the moment that we have shown that

$$\exists \; x_2 \in V_2 \cap \operatorname{supp} \mu_2 \quad \text{such that} \quad \operatorname{rank}(A_1 | A_2F_2(x_2) + B) = n. \tag{5.5}$$

Then, since (F_1,μ_1) is weakly non-planar, there would be an $x_1\in V_1\cap \operatorname{supp}\mu_1$ such that $\det\big(A_1F_1(x_1)+(A_2F_2(x_2)+B)\big)\neq 0$ and the proof would be complete. Thus, it remains to show (5.5).

Let $r=\operatorname{rank}(A_2|B)$. Using the Gauss method eliminate the last $n-r$ rows from $(A_2|B)$. This means that without loss of generality we can assume that $(A_1|A_2|B)$ is of the following form

$$(A_1|A_2|B)=\left(\begin{array}{c|c|c} * & A_2' & C\\ \hline A_1' & 0 & 0\end{array}\right),$$

where $A_1'\in M_{n-r,m_1}$, $A_2'\in M_{r,m_2}$ and $C\in M_{r,n}$. Observe that $\operatorname{rank}(A_2'|C)=r$. Since $\operatorname{rank}(A|B)=n$, we necessarily have that $\operatorname{rank}A_1'=n-r$. Now verify that

$$(A_1|A_2F_2(x_2)+B)=\left(\begin{array}{c|c} * & A_2'F_2(x_2)+C\\ \hline A_1' & 0\end{array}\right) \tag{5.6}$$

Since $\operatorname{rank}(A_2'|C)=r$ and (F_2,μ_2) is weakly non-planar, by Lemma 5.5, there is an $x_2\in V_2\cap\operatorname{supp}\mu_2$ such that $\operatorname{rank}(A_2'F_2(x_2)+C)=r$. This together with the fact that $\operatorname{rank}A_1'=n-r$ immediately implies that matrix (5.6) is of rank n. Thus (5.5) is established and the proof is complete. ⊠

Using Lemmas 5.3 alongside [29, Lemma 2.2] and [26, Theorem 2.4] one relatively straightforwardly obtains the following generalizations of Theorem 6.3 from [28].

Theorem 5.6. *For $i=1,\dots,l$ let an open subset U_i of $\mathbb{R}^{d_i}$, a continuous map $F_i:U_i\to M_{m_i,n}$ and a Federer measure μ_i on U_i be given. Assume that for every i the pair (F_i,μ_i) is good and weakly non-planar. Let $\mu=\mu_1\times\cdots\times\mu_l$ be the product measure on $U=U_1\times\cdots\times U_l$, $m=m_1+\cdots+m_l$ and let $F:U\to M_{m,n}$ be given by*

$$F(x_1,\dots,x_l)\stackrel{\text{def}}{=}\begin{pmatrix}F_1(x_1)\\ \vdots\\ F_l(x_l)\end{pmatrix}. \tag{5.7}$$

Then (a) *μ is Federer,* (b) *(F,μ) is good and* (c) *(F,μ) is weakly non-planar.*

A similar analogue can be deduced from Lemma 5.4 for the transpose of (5.7).

6. Inhomogeneous and weighted extremality

6.1. Inhomogeneous approximation. In the inhomogeneous case, instead of the systems of linear forms $\mathbf{q}\mapsto Y\mathbf{q}$ given by $Y\in M_{m,n}$, one considers systems of affine forms $\mathbf{q}\mapsto Y\mathbf{q}+\mathbf{z}$ given by the pairs $(Y;\mathbf{z})$, where $Y\in M_{m,n}$ and $\mathbf{z}\in\mathbb{R}^m$. The homogeneous case corresponds to

$(Y;\mathbf{z}) = (Y;\mathbf{0})$. Let us say that $(Y;\mathbf{z})$ is *VWA* (*very well approximable*) if there exists $\varepsilon > 0$ such that for arbitrarily large $Q > 1$ there are $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying

$$\|Y\mathbf{q} + \mathbf{z} - \mathbf{p}\|^m < Q^{-1-\varepsilon} \qquad \text{and} \qquad \|\mathbf{q}\|^n \le Q\,. \tag{6.1}$$

Let us say that $(Y;\mathbf{z})$ is *VWMA* (*very well multiplicatively approximable*) if there exists $\varepsilon > 0$ such that for arbitrarily large $Q > 1$ there are $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying

$$\Pi(Y\mathbf{q} + \mathbf{z} - \mathbf{p}) < Q^{-1-\varepsilon} \qquad \text{and} \qquad \textstyle\prod_+(\mathbf{q}) \le Q\,. \tag{6.2}$$

The above definitions are consistent with those used in other papers (see, e.g., [11, 16]). It is easy to see that in the homogeneous case ($\mathbf{z} = \mathbf{0}$) these definitions are equivalent to those given in §1. Note that, in general, $(Y;\mathbf{z})$ is VWA if either $Y\mathbf{q} + \mathbf{z} \in \mathbb{Z}^m$ for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$, or there is $\varepsilon > 0$ such that the inequality

$$\|Y\mathbf{q} + \mathbf{z} - \mathbf{p}\|^m < \|\mathbf{q}\|^{-(1+\varepsilon)n} \tag{6.3}$$

holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$. Similarly, $(Y;\mathbf{z})$ is VWMA if either $Y\mathbf{q} + \mathbf{z}$ has an integer coordinate for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$, or there is $\varepsilon > 0$ such that the inequality

$$\Pi(Y\mathbf{q} + \mathbf{z} - \mathbf{p}) < \Pi_+(\mathbf{q})^{-1-\varepsilon} \tag{6.4}$$

holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$.

One says that a measure μ on $M_{m,n}$ is *inhomogeneously extremal* (resp., *inhomogeneously strongly extremal*) if for every $\mathbf{z} \in \mathbb{R}^m$ the pair $(Y;\mathbf{z})$ is VWA (resp., VWMA) for μ-almost all $Y \in M_{m,n}$. This property holds e.g. for Lebesgue measure on $M_{m,n}$ as an easy consequence of the Borel-Canteli Lemma – see also [34] for a far more general result. Clearly, any inhomogeneously (strongly) extremal measure μ is (strongly) extremal. However, the converse is not generally true. For example, Remark 2 in [11, p. 826] contains examples of lines in $M_{2,1}$ that are strongly extremal but NOT inhomogeneously strongly extremal. More to the point, *no atomic measure can be inhomogeneously extremal.* This readily follows from the fact that for any extremal Y and $v > 1$ the set

$$\mathcal{V}_Y(v) \stackrel{\text{def}}{=} \left\{ \mathbf{z} \in [0,1)^m : \begin{array}{l} \|Y\mathbf{q} + \mathbf{z} - \mathbf{p}\|^m < \|\mathbf{q}\|^{-vn} \text{ holds for} \\ \text{infinitely many } \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\} \text{ and } \mathbf{p} \in \mathbb{Z}^m \end{array} \right\}$$

is non-empty, and in fact has Hausdorff dimension

$$\dim \mathcal{V}_Y(v) = \frac{m}{v}.$$

The proof of this fact is analogous to that of Theorem 6 from [17] and will not be considered here. The extremality of Y is not necessary to ensure that $\mathcal{V}_Y(v) \ne \emptyset$. For example, using the effective version of Kronecker's theorem

[18, Theorem VI, p. 82] and the Mass Transference Principle of [8] one can easily show the following: *if for some* $\varepsilon > 0$ *inequality* (1.1) *has only finitely many solutions* $\mathbf{q} \in \mathbb{Z}^n$ *and* $\mathbf{p} \in \mathbb{Z}^m$, *then* $\dim \mathcal{V}_Y(v) > 0$ for any $v > 1$.

The main goal of this section is to prove an inhomogeneous generalization of Theorem 2.3 (see Corollary 6.2 below). This is based on establishing an inhomogeneous transference akin to Theorem 1 in [11]. In short, the transference enables us to deduce the inhomogeneous (strong) extremality of a measure once we know it is (strongly) extremal. As we have discussed above, such a transference is impossible for arbitrary measures and would require some conditions on the measures under consideration. In [11], the notion of contracting measures on $M_{m,n}$ has been introduced and used to establish such a transference. Our following result makes use of the notion of good and non-planar rows which is much easier to verify, thus simplifying and in a sense generalizing the result of [11].

Theorem 6.1. *Let* U *be an open subset of* $\mathbb{R}^d$, μ *a Federer measure on* U *and* $F : U \to M_{m,n}$ *a continuous map. Let* $F_j : U \to \mathbb{R}^n$ *denote the* j*-th row of* F. *Assume that the pair* (F_j, μ) *is good and non-planar for each* j. *Then we have the following two equivalences*

$$F_*\mu \textit{ is extremal} \iff F_*\mu \textit{ is inhomogeneously extremal},$$

$$F_*\mu \textit{ is strongly extremal} \iff F_*\mu \textit{ is inhomogeneously strongly extremal.}$$

Observe that (F_j, μ) is good and non-planar for each j whenever (F, μ) is good and weakly non-planar. Hence, Theorems 2.3 and 6.1 imply the following

Corollary 6.2. *Let* U *be an open subset of* $\mathbb{R}^d$, μ *a Federer measure on* U *and* $F : U \to M_{m,n}$ *a continuous map such that* (F, μ) *is* (i) *good, and* (ii) *weakly non-planar. Then* $F_*\mu$ *is inhomogeneously strongly extremal.*

Remark. The assumptions of being good and non-planar imposed on each row in Theorem 6.1 are required to meet the conditions of the inhomogeneous transference of [11, §5], which is used in the proof of the result. Although it is possible to relax these assumptions, it does not seem to be feasible to drop any of them altogether. On the other hand, let us note that the assumptions of Theorem 6.1 are generally weaker that those of Theorem 2.3. For example, if $F \in C^1$, μ is Lebesgue measure and $n = 1$ (simultaneous Diophantine approximations), applying Theorem 6.1 only requires that the gradient of F has non-zero coordinates almost everywhere (see also [12] for even weaker conditions in the latter case).

6.2. Weighted approximation. Weighted extremality is a modification of the standard (non-multiplicative) case obtained by introducing weights of approximation for each linear form. Formally, let $\mathbf{r} = (r_1, \ldots, r_{m+n})$ be an $(m+n)$-tuple of real numbers such that

$$r_i \geq 0 \ (1 \leq i \leq m+n) \text{ and } r_1 + \ldots + r_m = r_{m+1} + \ldots + r_{m+n} = 1. \tag{6.5}$$

One says that $(Y; \mathbf{z})$ is $\mathbf{r}$-*VWA* ($\mathbf{r}$-*very well approximable*) if there exists $\varepsilon > 0$ such that for arbitrarily large $Q > 1$ there are $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying

$$|Y_j\mathbf{q}+z_j-p_j| < Q^{-(1+\varepsilon)r_j} \ (1 \leq j \leq m) \text{ and } |q_i| < Q^{r_{m+i}} \ (1 \leq i \leq n)\,, \tag{6.6}$$

where Y_j is the j-th row of Y. A measure μ on $M_{m,n}$ will be called $\mathbf{r}$-*extremal* if $(Y; \mathbf{0})$ is $\mathbf{r}$-VWA for μ-almost all $Y \in M_{m,n}$; a measure μ on $M_{m,n}$ will be called *inhomogeneously* $\mathbf{r}$-*extremal* if for every $\mathbf{z} \in \mathbb{R}^m$ the pair $(Y; \mathbf{z})$ is $\mathbf{r}$-VWA for μ-almost all $Y \in M_{m,n}$.

It is readily seen that $(Y; \mathbf{z})$ is VWA if and only if it is $(\frac{1}{m}, \ldots, \frac{1}{m}, \frac{1}{n}, \ldots, \frac{1}{n})$-VWA. Thus, (inhomogeneous) extremality is a special case of (inhomogeneous) $\mathbf{r}$-extremality. In fact, the strong extremality is also encompassed by $\mathbf{r}$-extremality as follows from the following

Lemma 6.3. $(Y; \mathbf{z})$ *is VWMA* $\iff$ $(Y; \mathbf{z})$ *is* $\mathbf{r}$-*VWA for some* $\mathbf{r}$ *satisfying* (6.5)*. Furthermore, each VWMA pair* $(Y; \mathbf{z})$ *is* $\mathbf{r}$-*VWA for some* $\mathbf{r} \in \mathbb{Q}^{m+n}$.

We remark that the equivalence given by Lemma 6.3 is specific to the notion of extremality and cannot be obtained in relation to the more 'fine tuned' forms of Diophantine approximation appearing in Khintchine type theorems or the theory of badly approximable points. Note that, although the argument given below has been used previously in one form or another, the above equivalence is formally new even in the 'classical' case $\mathbf{z} = \mathbf{0}$ and $\min\{m, n\} = 1$.

Proof. The sufficiency is an immediate consequence of the obvious fact that (6.6) implies (6.2). For the necessity consider the following two cases.

Case (a): There exists $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and j_0 such that $Y_{j_0}\mathbf{q} + z_{j_0} = p_{j_0} \in \mathbb{Z}$. Then it readily follows from the definitions that $(Y; \mathbf{z})$ is both VWMA and $\mathbf{r}$-VWA with $r_{j_0} = 1$, $r_j = 0$ for $1 \leq j \leq m$, $j \neq j_0$, and $r_{m+i} = \frac{1}{n}$ for $1 \leq i \leq n$.

Case (b): $Y_j\mathbf{q} + z_j \notin \mathbb{Z}$ for all $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and $1 \leq j \leq m$. We are given that for some $\varepsilon \in (0, 1)$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying (6.4). Without loss of generality we may also assume that

$$\max_{1\leq j\leq m} |Y_j\mathbf{q} + z_j - p_j| < 1. \tag{6.7}$$

Let $0<\varepsilon'<\varepsilon$. Fix any positive parameters δ and δ' such that

$$\frac{1+\varepsilon}{(1+\delta')(1+\varepsilon')}-m\delta\geq 1,\qquad \frac{1}{1+\delta'}+n\delta\leq 1\qquad\text{and}\qquad \frac{1}{\delta}\in\mathbb{Z}. \tag{6.8}$$

The existence of δ and δ' is easily seen. For each $(\mathbf{q},\mathbf{p})$ satisfying (6.4) and (6.7) define $Q=\Pi_+(\mathbf{q})^{1+\delta'}$ and the unique $(m+n)$-tuple $\mathbf{u}=(u_1,\dots,u_{m+n})$ of integer multiples of δ such that

$$\begin{aligned}
&Q^{-(1+\varepsilon')(u_j+\delta)}\leq|Y_j\mathbf{q}+z_j-p_j|<Q^{-(1+\varepsilon')u_j}\ (1\leq j\leq m),\\
&Q^{u_{m+i}-\delta}\leq|q_i|<Q^{u_{m+i}}\ (1\leq i\leq n,\ q_i\neq 0),\\
&u_{m+i}=0\ (1\leq i\leq n,\ q_i=0).
\end{aligned}\tag{6.9}$$

Let $u=\sum_{i=1}^n u_{m+i}$. Then, by (6.9), we have that $Q^{u-n\delta}\leq Q^{1/(1+\delta')}\leq Q^u$. Therefore, $1/(1+\delta')\leq u\leq 1/(1+\delta')+n\delta$. By (6.8), we have that

$$1/(1+\delta')\leq u\leq 1. \tag{6.10}$$

Next, by (6.4) and (6.9),

$$\prod_{j=1}^m Q^{-(1+\varepsilon')(u_j+\delta)}\times Q^{(1+\varepsilon)/(1+\delta')}\leq \Pi(Y\mathbf{q}+\mathbf{z}-\mathbf{p})\times\textstyle\prod_+(\mathbf{q})^{1+\varepsilon}<1\,. \tag{6.11}$$

Let $\widehat{u}=\sum_{j=1}^m u_j$. Then, by (6.11), we get

$$Q^{-m(1+\varepsilon')\delta}Q^{-(1+\varepsilon')\widehat{u}}Q^{(1+\varepsilon)/(1+\delta')}<1,$$

whence

$$-m(1+\varepsilon')\delta-(1+\varepsilon')\widehat{u}+(1+\varepsilon)/(1+\delta')<0.$$

Hence, by (6.8), we get

$$\widehat{u}>(1+\varepsilon)/(1+\delta')(1+\varepsilon')-m\delta\geq 1. \tag{6.12}$$

By (6.10), (6.12) and the fact that $\delta^{-1}\in\mathbb{Z}$, we can find an $(m+n)$-tuple $\mathbf{r}$ of integer multiples of δ satisfying (6.5) such that $r_j\leq u_j$ for $1\leq j\leq m$ and $r_{m+i}\geq u_{m+i}$ for $1\leq i\leq n$. Then, by (6.9), we get that

$$\begin{aligned}
&|Y_j\mathbf{q}+z_j-p_j|<Q^{-(1+\varepsilon')r_j}\quad(1\leq j\leq m),\\
&|q_i|<Q^{r_{m+i}}\ (1\leq i\leq n).
\end{aligned}\tag{6.13}$$

This holds for infinitely many $\mathbf{q}$, $\mathbf{p}$ and arbitrarily large Q. Since the components of $\mathbf{r}$ are integer multiples of δ, there is only a finite number of choices for $\mathbf{r}$. Therefore, there is a $\mathbf{r}$ satisfying (6.5) such that (6.13) holds for some $\mathbf{q}\in\mathbb{Z}^n\smallsetminus\{\mathbf{0}\}$ and $\mathbf{p}\in\mathbb{Z}^m$ for arbitrarily large Q. The furthermore part of the lemma is also established as, by construction, $\mathbf{r}\in\mathbb{Q}^{m+n}$. $\square$

In view of Lemma 6.3, Theorem 6.1 is a consequence of the following transference result regarding $\mathbf{r}$-extremality.

Theorem 6.4. *Let U be an open subset of $\mathbb{R}^d$, μ a Federer measure on U and $F : U \to M_{m,n}$ a continuous map. Let $F_j : U \to \mathbb{R}^n$ denote the j-th row of F. Let $\mathbf{r}$ be an $(m+n)$-tuple of real numbers satisfying* (6.5)*. Assume that the pair (F_j, μ) is good and non-planar for each j. Then*

$$F_*\mu \text{ is } \mathbf{r}\text{-extremal} \iff F_*\mu \text{ is } \mathbf{r}\text{-inhomogeneously extremal}.$$

Another consequence of Lemma 6.3 and Theorems 6.4 and 2.3 is the following

Theorem 6.5. *Let U be an open subset of $\mathbb{R}^d$, μ a Federer measure on U and $F : U \to M_{m,n}$ a continuous map such that (F, μ) is* (i) *good, and* (ii) *weakly non-planar. Then $F_*\mu$ is inhomogeneously $\mathbf{r}$-extremal for any $(m+n)$-tuple $\mathbf{r}$ of real numbers satisfying* (6.5)*.*

For the rest of §6 we will be concerned with proving Theorem 6.4. This will be done by using the Inhomogeneous Transference of [11, §5] that is now recalled.

6.3. Inhomogeneous Transference framework. In this section we recall the general framework of Inhomogeneous Transference of [11, §5]. Let $\mathcal{A}$ and $\mathbf{T}$ be two countable indexing sets. For each $\alpha \in \mathcal{A}$, $\mathbf{t} \in \mathbf{T}$ and $\varepsilon > 0$ let $\mathrm{H}_{\mathbf{t}}(\alpha, \varepsilon)$ and $\mathrm{I}_{\mathbf{t}}(\alpha, \varepsilon)$ be open subsets of $\mathbb{R}^d$ (more generally the framework allows one to consider any metric space instead of $\mathbb{R}^d$). Let Ψ be a set of functions $\psi : \mathbf{T} \to \mathbb{R}^+$. Let μ be a non-atomic finite Federer measure supported on a bounded subset of $\mathbb{R}^d$. The validity of the following two properties is also required.

The Intersection Property. For any $\psi \in \Psi$ there exists $\psi^* \in \Psi$ such that for all but finitely many $\mathbf{t} \in \mathbf{T}$ and all distinct α and α' in $\mathcal{A}$ we have that

$$\mathrm{I}_{\mathbf{t}}(\alpha, \psi(\mathbf{t})) \cap \mathrm{I}_{\mathbf{t}}(\alpha', \psi(\mathbf{t})) \subset \bigcup_{\alpha'' \in \mathcal{A}} \mathrm{H}_{\mathbf{t}}(\alpha'', \psi^*(\mathbf{t})) \,. \tag{6.14}$$

The Contraction Property. For any $\psi \in \Psi$ there exists $\psi^+ \in \Psi$ and a sequence of positive numbers $\{k_{\mathbf{t}}\}_{\mathbf{t} \in \mathbf{T}}$ satisfying

$$\sum_{\mathbf{t} \in \mathbf{T}} k_{\mathbf{t}} < \infty, \tag{6.15}$$

such that for all but finitely $\mathbf{t} \in \mathbf{T}$ and all $\alpha \in \mathcal{A}$ there exists a collection $\mathcal{C}_{\mathbf{t},\alpha}$ of balls B centered at $\operatorname{supp}\mu$ satisfying the following conditions:

$$\operatorname{supp}\mu \cap \mathrm{I}_{\mathbf{t}}(\alpha, \psi(\mathbf{t})) \ \subset \bigcup_{B \in \mathcal{C}_{\mathbf{t},\alpha}} B \,, \tag{6.16}$$

$$\operatorname{supp}\mu \cap \bigcup_{B \in \mathcal{C}_{\mathbf{t},\alpha}} B \ \subset \ \mathrm{I}_{\mathbf{t}}(\alpha, \psi^+(\mathbf{t})) \tag{6.17}$$

and

$$\mu(5B\cap \mathrm{I}_{\mathbf{t}}(\alpha,\psi(\mathbf{t})))\ \le\ k_{\mathbf{t}}\,\mu(5B)\ . \tag{6.18}$$

For $\psi\in\Psi$, consider the lim sup sets

$$\Lambda_{\mathrm{H}}(\psi)=\limsup_{\mathbf{t}\in\mathbf{T}}\bigcup_{\alpha\in\mathcal{A}}\mathrm{H}_{\mathbf{t}}(\alpha,\psi(\mathbf{t}))\quad\text{and}\quad \Lambda_{\mathrm{I}}(\psi)=\limsup_{\mathbf{t}\in\mathbf{T}}\bigcup_{\alpha\in\mathcal{A}}\mathrm{I}_{\mathbf{t}}(\alpha,\psi(\mathbf{t}))\,. \tag{6.19}$$

The following statement from [11] will be all that we need to give a proof of Theorem 6.4.

Theorem 6.6 (Theorem 5 in [11])**.** *Suppose $\mathcal{A}$, $\mathbf{T}$, $\mathrm{H}_{\mathbf{t}}(\alpha,\varepsilon)$, $\mathrm{I}_{\mathbf{t}}(\alpha,\varepsilon)$, Ψ and μ as above are given and the intersection and contraction properties are satisfied. Then*

$$\forall\ \psi\in\Psi\quad \mu(\Lambda_{\mathrm{H}}(\psi))=0\qquad\Longrightarrow\qquad \forall\ \psi\in\Psi\quad \mu(\Lambda_{\mathrm{I}}(\psi))=0. \tag{6.20}$$

6.4. Proof of Theorem 6.4. While proving Theorem 6.4 there is no loss of generality in assuming that $r_1,\dots,r_m>0$ as otherwise we would consider the smaller system of forms that correspond to $r_j>0$.

From now on fix any $\mathbf{z}\in\mathbb{R}^m$. With the aim of using Theorem 6.6 define $\mathbf{T}=\mathbb{Z}_{\ge 0}$, $\mathcal{A}=(\mathbb{Z}^n\smallsetminus\{\mathbf{0}\})\times\mathbb{Z}^m$ and $\Psi=(0,+\infty)$, that is the functions $\psi\in\Psi$ are constants. Further for $t\in\mathbf{T}$, $\alpha=(\mathbf{q},\mathbf{p})\in\mathcal{A}$ and $\varepsilon>0$, let

$$\mathrm{I}_t(\alpha,\varepsilon)=\left\{x\in U:\ \begin{array}{ll}|F_j(x)\mathbf{q}+z_j-p_j|<\frac12\cdot 2^{-(1+\varepsilon)r_jt} & (1\le j\le m)\\ |q_i|<\frac12\cdot 2^{r_{m+i}t}\ (1\le i\le n) & \end{array}\right\} \tag{6.21}$$

and

$$\mathrm{H}_t(\alpha,\varepsilon)=\left\{x\in U\ :\ \begin{array}{ll}|F_j(x)\mathbf{q}-p_j|<2^{-(1+\varepsilon)r_jt} & (1\le j\le m)\\ |q_i|<2^{r_{m+i}t}\ (1\le i\le n) & \end{array}\right\}. \tag{6.22}$$

Proposition 6.7. *Let $x\in U$. Then*

(i) $(F(x);\mathbf{z})$ *is* $\mathbf{r}$-*VWA* $\iff$ $x\in\Lambda_{\mathrm{I}}(\psi)$ *for some* $\psi>0$;
(ii) $(F(x);\mathbf{0})$ *is* $\mathbf{r}$-*VWA* $\iff$ $x\in\Lambda_{\mathrm{H}}(\psi)$ *for some* $\psi>0$.

Proposition 6.7 and Theorem 6.6 would imply Theorem 6.4 upon establishing the intersection and contraction properties. While postponing the verification of these properties until the end of the section, we now give a proof of Proposition 6.7.

Proof. We consider the proof of part (i) as that of part (ii) is similar (and in a sense simpler). Assume that $(F(x),\mathbf{z})$ is $\mathbf{r}$-VWA. Then there exists $\varepsilon>0$ such that for arbitrarily large $Q>1$ there are $\mathbf{q}\in\mathbb{Z}^n\smallsetminus\{\mathbf{0}\}$ and $\mathbf{p}\in\mathbb{Z}^m$ satisfying (6.6) with $Y=F(x)$. For each such Q define $t\in\mathbb{N}$ such that

$2^{t-1} < 2^{1/r'}Q \le 2^t$, where $r' = \min\{r_{m+i} > 0 : 1 \le i \le n\}$. Let $0 < \psi < \varepsilon$. Then, by (6.6) with $Y = F(x)$, we have that

$$|F_j(x)\mathbf{q} + z_j - p_j| < 2^{(1+\varepsilon)r_j} 2^{-(1+\varepsilon)r_j t} < \tfrac{1}{2} \cdot 2^{-(1+\psi)r_j t} \qquad \text{for } 1 \le j \le m$$

when t is sufficiently large. Here we use the fact that $r_j > 0$. Also when $r_{m+i} > 0$ we have that $Q^{r_{m+i}} \le \frac{1}{2} \cdot 2^{r_{m+i}t}$. This is a consequence of the definition of t. Hence by (6.6) with $Y = F(x)$, we have that

$$|q_i| < \tfrac{1}{2} \cdot 2^{r_{m+i}t} \qquad \text{for } 1 \le i \le n \tag{6.23}$$

when $r_{m+i} > 0$. If $r_{m+i} = 0$, then we have that $|q_i| < Q^{r_{m+i}} = 1$. Since $q_i \in \mathbb{Z}$ we necessarily have that $q_i = 0$. Consequently (6.23) also holds when $r_{m+i} = 0$. Thus, $x \in \mathrm{I}_t(\alpha, \psi)$ and furthermore this holds for infinitely many t. Therefore, $x \in \Lambda_{\mathrm{I}}(\psi)$. The sufficiency is straightforward because the fact that $x \in \Lambda_{\mathrm{I}}(\psi)$ means that with $\varepsilon = \psi$ for arbitrarily large $Q = 2^t$ $(t \in \mathbb{N})$ there are $\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^m$ satisfying (6.6) with $Y = F(x)$. Hence $(F(x); \mathbf{z})$ is $\mathbf{r}$-VWA. □

Verifying the intersection property. Take any $\psi \in \Psi$ and distinct $\alpha = (\mathbf{q}, \mathbf{p})$ and $\alpha' = (\mathbf{q}', \mathbf{p}')$ in $\mathcal{A}$. Take any point $x \in \mathrm{I}_t(\alpha, \psi) \cap \mathrm{I}_t(\alpha', \psi)$. It means that

$$\begin{aligned} &|F_j(x)\mathbf{q} + z_j - p_j| < \tfrac{1}{2} \cdot 2^{-(1+\psi)r_j t}, && |q_i| < \tfrac{1}{2} \cdot 2^{r_{m+i}t}, \\ &|F_j(x)\mathbf{q}' + z_j - p'_j| < \tfrac{1}{2} \cdot 2^{-(1+\psi)r_j t}, && |q'_i| < \tfrac{1}{2} \cdot 2^{r_{m+i}t} \end{aligned} \tag{6.24}$$

for $1 \le j \le m$ and $1 \le i \le n$. Let $\alpha'' = (\mathbf{q}'', \mathbf{p}'')$, where $\mathbf{p}'' = \mathbf{p} - \mathbf{p}' \in \mathbb{Z}^m$ and $\mathbf{q}'' = \mathbf{q} - \mathbf{q}' \in \mathbb{Z}^n$. Using (6.24) and the triangle inequality we obtain that

$$|F_j(x)\mathbf{q}'' - p''_j| < 2^{-(1+\psi)r_j t} \qquad \text{and} \qquad |q''_i| < 2^{r_{m+i}t} \tag{6.25}$$

for $1 \le j \le m$ and $1 \le i \le n$. If $\mathbf{q} = \mathbf{q}'$, then $\mathbf{p}'' \ne \mathbf{0}$ (because $\alpha \ne \alpha'$) and $|p''_j| < 2^{-(1+\psi)r_j t} \le 1$. Since $p''_j \in \mathbb{Z}$ and $|p''_j| < 1$ we must have that $p''_j = 0$ for all j, contrary to $\mathbf{p}'' \ne \mathbf{0}$. Therefore, we must have that $\mathbf{q}'' \ne \mathbf{0}$ and so $\alpha'' \in \mathcal{A}$. By (6.25), we get that $x \in \mathrm{H}_t(\alpha'', \psi)$. This verifies the intersection property with $\psi^* = \psi$.

Verifying the contraction property. Since (F_j, μ) is good for each j, for almost every $x_0 \in \operatorname{supp}\mu \cap U$ there exist positive C_j and α_j and a ball V_j centered at x_0 such that for each $\mathbf{q} \in \mathbb{R}^n$, $p \in \mathbb{R}$ and $1 \le j \le m$ the function $F_j(x)\mathbf{q} + p$ is (C_j, α_j)-good on V_j with respect to μ. Let $C = \max C_j$, $\alpha = \min \alpha_j$ and $V = \cap_j V_j$. Then for each j $(1 \le j \le m)$, $\mathbf{q} \in \mathbb{R}^n$ and $p \in \mathbb{R}$ the function

$$F_j(x)\mathbf{q} + p \quad \text{is } (C, \alpha)\text{-good on } V \text{ with respect to } \mu. \tag{6.26}$$

Since the balls V obtained this way cover μ-almost every point of U without loss of generality we will assume that $U = V$ and that $\operatorname{supp}\mu \subset U$ within our proof of Theorem 6.4. Also since μ is a Radon measure, without loss of generality we can assume that μ is finite and that U is a finite ball such that F is continuous on the closure of U. In particular, this implies that $F(U)$ is a bounded set.

Since (F_j, μ) is non-planar for each j, we have that

$$d_j(\mathbf{q}, p) \stackrel{\text{def}}{=} \|F_j(x)\mathbf{q} + p\|_{\mu,U} > 0$$

for each $\mathbf{q} \in \mathbb{R}^n$ with $\|\mathbf{q}\| = 1$ and $p \in \mathbb{R}$. Since $F(U)$ is bounded, there exists $N > 0$ such that $d_j(\mathbf{q}, p) \geq 1$ for any $p \in \mathbb{R}$ with $|p| \geq N$ and any $\mathbf{q} \in \mathbb{R}^n$ with $\|\mathbf{q}\| = 1$. Further, the quantity $d_j(\mathbf{q}, p)$ is the distance of the furthest point of $F_j(\operatorname{supp}\mu)$ from the hyperplane $\mathbf{y} \cdot \mathbf{q} + p = 0$. Obviously, this is a continuous function of $\mathbf{q}$ and p. Hence it is bounded away from zero on any compact set, in particular, on $\{\mathbf{q} : \|\mathbf{q}\| = 1\} \times [-N, N]$. Hence there is an $r_0 \in (0, 1)$ such that $\|F_j(x)\mathbf{q} + p\|_{\mu,U} \geq r_0$ for all $\mathbf{q} \in \mathbb{R}^n$ with $\|\mathbf{q}\| = 1$, $p \in \mathbb{R}$ and $1 \leq j \leq m$. It follows that

$$\|F_j(x)\mathbf{q} + p\|_{\mu,U} \geq r_0\|\mathbf{q}\| \tag{6.27}$$

for all $\mathbf{q} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\}$, $p \in \mathbb{R}$ and $1 \leq j \leq m$.

Let $\psi > 0$ and $0 < \psi^+ < \psi$. By (6.27) and the assumption that $\min_{1\leq j\leq m} r_j > 0$, for sufficiently large t we have that

$$\operatorname{supp}\mu \not\subset \mathrm{I}_t(\alpha, \psi^+). \tag{6.28}$$

We now construct a collection $\mathcal{C}_{t,\alpha}$ required by the contraction property, where $t \in \mathbb{Z}_{\geq 0}$ is sufficiently large and $\alpha = (\mathbf{q}, \mathbf{p}) \in \mathbb{Z}^n \smallsetminus \{\mathbf{0}\} \times \mathbb{Z}^m$. If $\operatorname{supp}\mu \cap \mathrm{I}_t(\alpha, \psi) = \emptyset$, then taking $C_{t,\alpha} = \emptyset$ does the job. Otherwise, for each $x \in \operatorname{supp}\mu \cap \mathrm{I}_t(\alpha, \psi)$ take any ball $B' \subset \mathrm{I}_t(\alpha, \psi)$ centered at x. Clearly, this is possible because $\mathrm{I}_t(\alpha, \psi)$ is open. Since $\psi^+ < \psi$, we have that $\mathrm{I}_t(\alpha, \psi) \subset \mathrm{I}_t(\alpha, \psi^+)$. Therefore, by (6.28), there exists $\tau \geq 1$ such that

$$5\tau B' \cap \operatorname{supp}\mu \not\subset \mathrm{I}_t(\alpha, \psi^+) \qquad \text{and} \qquad \tau B' \cap \operatorname{supp}\mu \subset \mathrm{I}_t(\alpha, \psi^+)\,. \tag{6.29}$$

Let $B = B(x) = \tau B'$. By the left hand side of (6.29), there exists $j \in \{1, \ldots, m\}$ and $x_0 \in \operatorname{supp}\mu \cap 5B$ such that

$$|f(x_0)| \geq \tfrac{1}{2} \cdot 2^{-(1+\psi^+)r_j t}, \qquad \text{where } f(x) = F_j(x)\mathbf{q} + z_j - p_j.$$

Hence $\|f\|_{\mu,5B} \geq \frac{1}{2} \cdot 2^{-(1+\psi^+)r_j t}$. Observe that

$$5B \cap \mathrm{I}_t(\alpha, \psi) \subset \{x \in 5B : |f(x)| < \tfrac{1}{2} \cdot 2^{-(1+\psi)r_j t}\}.$$

Then, since f is (C,α)-good, we have that

$$\begin{aligned}
\mu(5B\cap \mathrm{I}_t(\alpha,\psi)) &\le \mu\{x\in 5B : |f(x)|<\tfrac{1}{2}\cdot 2^{-(1+\psi)r_jt}\}\\
&\le C\left(\frac{\frac{1}{2}\cdot 2^{-(1+\psi)r_jt}}{\|f\|_{\mu,5B}}\right)^{\alpha}\mu(5B)\\
&\le C\left(\frac{\frac{1}{2}\cdot 2^{-(1+\psi)r_jt}}{\frac{1}{2}\cdot 2^{-(1+\psi^+)r_jt}}\right)^{\alpha}\mu(5B)\\
&\le C\cdot 2^{-(\psi-\psi^+)r_j\alpha t}\,\mu(5B)=k_t\,\mu(5B)
\end{aligned}$$

where

$$k_t = C\cdot 2^{-(\psi-\psi^+)r_j\alpha t}.$$

Clearly, (6.15) holds. Also, by construction, conditions (6.16)–(6.18) are satisfied for the collection $C_{t,\alpha}\stackrel{\text{def}}{=}\{B(x): x\in \operatorname{supp}\mu\cap \mathrm{I}_t(\alpha,\psi)\}$. This completes the proof of Theorem 6.4.

7. Final remarks

7.1. Checking weak non-planarity. The condition of weak non-planarity of pairs (F,μ) has been demonstrated in this paper to have may nice and natural features. But how one can in general show that a given pair is weakly non-planar? This question is tricky even in the analytic category. If $\min\{m,n\}=1$ and $\mathcal{M}$ is immersed into $\mathbb{R}^n$ by an analytic map $\mathbf{f}=(f_1,\dots,f_n)$, its non-planarity can be verified by taking partial derivatives of $\mathbf{f}$, i.e. via (1.4). However, when $\min\{m,n\}>1$ finding an algorithmic way to verify weak non-planarity seems to be an open problem.

Here is a specific example: *a matrix version of Baker's problem.* Let $m,k\in\mathbb{N}$ and $n=mk$. Let

$$\mathcal{M}=\{(X,\dots,X^n)\in M_{m,mn}: X\in M_{m,m}\}.$$

It seems reasonable to conjecture that $\mathcal{M}$ is strongly extremal. In the case $m=1$ this problem reduces to Baker's original problem on strong extremality of the Veronese curves. When $m=n=2$ the manifold $\mathcal{M}$ happens to be non-planar and so weakly non-planar. This is easily verified by writing down all the minors of (X,X^2). It is however unclear how to verify (or disprove) that $\mathcal{M}$ is weakly non-planar (or possibly strongly non-planar) for arbitrary m and n. Note also that the extremality of this manifold has been established in [25], however the argument is not powerful enough to yield strong extremality.

7.2. Beyond weak non-planarity. Let $\mathcal{M}$ be an analytic manifold in $M_{m,n}$, and let

$$\mathcal{H}(\mathcal{M}) = \bigcap_{\substack{\mathcal{H}\in\mathbf{H}_{m,n}\\ \mathcal{M}\subset\mathcal{H}}} \mathcal{H}.$$

If $\mathcal{M} \not\subset \mathcal{H}$ for every $\mathcal{H} \in \mathbf{H}_{m,n}$, then, by definition, we let $\mathcal{H}(\mathcal{M}) = M_{m,n}$. In the case $\min\{m,n\} = 1$ the set $\mathcal{H}(\mathcal{M})$ is simply an affine subspace of $\mathbb{R}^m$ or $\mathbb{R}^n$, depending on which of the dimensions is 1. It is shown in [21] that *if* $\min\{m,n\} = 1$ *then* $\mathcal{M}$ *is (strongly) extremal if and only if so is* $\mathcal{H}(\mathcal{M})$. A natural question is whether a similar characterisation of analytic (strongly) extremal manifolds in $M_{m,n}$ is possible in the case of arbitrary (m,n).

7.3. Hausdorff dimension. Another natural challenge is to investigate the Hausdorff dimension of the exceptional sets of points lying on a non-planar manifold in $M_{m,n}$ such that (1.1) (or (1.2)) has infinitely many solutions (for some fixed $\varepsilon > 1$). The upper bounds for Hausdorff dimension are not fully understood even in the case of manifolds in $\mathbb{R}^n$ – see [4, 9, 13, 15]. However, there has been great success with establishing lower bounds – see [3, 6, 7, 9, 19].

7.4. Khintchine-Groshev type theory. The fact that Lebesgue measure on $M_{m,n}$ is extremal can be thought of as a special case of the convergence part of the Khintchine-Groshev theorem. Specifically, generalizing (1.1), for a function ψ one says that $Y \in M_{m,n}$ is ψ-approximable if the inequality

$$\|Y\mathbf{q} - \mathbf{p}\| < \psi(\|\mathbf{q}\|) \tag{7.1}$$

holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$. A result of Groshev (1938), generalizing Khintchine's earlier work, states that for non-increasing ψ, Lebesgue almost no (resp., almost all) $Y \in M_{m,n}$ are ψ-approximable if the sum

$$\sum_{k=1}^{\infty} k^{n-1}\psi(k)^m \tag{7.2}$$

converges (resp., diverges). The convergence part straightforwardly follows from the Borel-Cantelli Lemma and does not require the monotonicity of ψ; in the divergence part the monotonocity assumption was recently removed in [10] in all cases except $m = n = 1$, where it is known to be necessary.

Proving similar results for manifolds of $M_{m,1}$ and $M_{1,n}$ has been a fruitful activity, see the monograph [15] for some earlier results, and [2, 3, 5, 7, 14] for more recent developments. It seems natural to conjecture that, for a monotonic ψ, almost no (resp., almost all) Y on a weakly non-planar analytic submanifold of $M_{m,n}$ are ψ-approximable if the sum (7.2) converges (resp., diverges). Presently no results are known when $\min\{m,n\} > 1$ except for ψ given by the right hand side of (1.1), or for the manifold being the whole space $M_{m,n}$. One can also study a multiplicative version of the problem, which is much more challenging and where much less is known, see [9].

7.5. Other spaces. The analogue of the Baker-Sprindžuk conjecture has been established in $\mathbb{C}^n$, $\mathbb{Q}_p^n$ and in products of archimedean and non-archimedean spaces – see, e.g., [22, 29]. It would be reasonable to explore similar generalizations of Theorem 2.3.

References

[1] A. Baker, *Transcendental number theory*, Cambridge University Press, London, 1975.

[2] V. Beresnevich, *A Groshev type theorem for convergence on manifolds*, Acta Math. Hungar.,**94**, 1-2 (2002), 99–130.

[3] V. Beresnevich, *Rational points near manifolds and metric Diophantine approximation*, Ann. of Math. (2), **175**, (2012), 187–235.

[4] V. Beresnevich, V. Bernik and M. Dodson, *On the Hausdorff dimension of sets of well-approximable points on nondegenerate curves*, Dokl. Nats. Akad. Nauk Belarusi **46**, 6 (2002), 18–20, (In Russian).

[5] V. Beresnevich, V. Bernik, D. Kleinbock and G. A. Margulis, *Metric Diophantine approximation: the Khintchine-Groshev theorem for nondegenerate manifolds*, Moscow Math. J. **2**, 2 (2002), 203–225.

[6] V. Beresnevich, D. Dickinson and S. Velani, *Measure theoretic laws for lim sup sets*, Mem. Amer. Math. Soc. **179**, 846 (2006), x+91.

[7] V. Beresnevich, D. Dickinson and S. Velani, *Diophantine approximation on planar curves and the distribution of rational points*, Ann. of Math. (2) **166**, 2 (2007), 367–426, With an Appendix II by R. C. Vaughan.

[8] V. Beresnevich and S. Velani, *A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures*, Ann. of Math. (2), **164**, 3 (2006), 971–992.

[9] V. Beresnevich and S. Velani, *A note on simultaneous Diophantine approximation on planar curves*, Math. Ann. **337**, 4 (2007), 769–796.

[10] V. Beresnevich and S. Velani, *Classical metric Diophantine approximation revisited: the Khintchine-Groshev theorem*, Internat. Math. Res. Notices **337**, 1 (2010), 69–86.

[11] V. Beresnevich and S. Velani, *An inhomogeneous transference principle and Diophantine approximation*, Proc. Lond. Math. Soc. (3), **101**, 3 (2010), 821–851.

[12] V. Beresnevich and S. Velani, *Simultaneous inhomogeneous Diophantine approximations on manifolds*. (Russian) Fundam. Prikl. Mat. **16**, 5 (2010), 3–17 ; translation in J. Math. Sci. (N. Y.) **180**, 5 (2012), 531–541.

[13] V. Bernik, *An application of Hausdorff dimension in the theory of Diophantine approximation*, Acta Arith. **42**, 3 (1983), 219–253, (In Russian). English transl. in Amer. Math. Soc. Transl., **140** (1988), 15–44.

[14] V. Bernik, D. Kleinbock and G.A. Margulis, *Khintchine-type theorems on manifolds: the convergence case for standard and multiplicative versions*, Internat. Math. Res. Notices (2001), 9, 453–486.

[15] V. Bernik and M. Dodson, *Metric Diophantine approximation on manifolds*, Cambridge Tracts in Mathematics, **137**, Cambridge University Press, Cambridge, (1999).

[16] Y. Bugeaud, *Multiplicative Diophantine approximation*, in Dynamical systems and Diophantine approximation, Sémin. Congr. **19**, (2009), 105–125.

[17] Y. Bugeaud and N. Chevallier, *On simultaneous inhomogeneous Diophantine approximation*, Acta Arith. **123**, 2 (2006), 97–123.

[18] J.W.S. Cassels, An introduction to Diophantine Approximation, Cambridge University Press, Cambridge, (1957).
[19] H. Dickinson and M. Dodson, *Extremal manifolds and Hausdorff dimension*, Duke Math. J. **101**, 2 (2000), 271–281.
[20] A. Gorodnik, *Open problems in dynamics and related fields*, J. Mod. Dyn. **1**, 1 (2007), 1–35.
[21] D. Kleinbock, *Extremal subspaces and their submanifolds*, Geom. Funct. Anal, **13**, 2 (2003), 437–466.
[22] D. Kleinbock, *Baker-Sprindžuk conjectures for complex analytic manifolds*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, (2004), 539–553.
[23] D. Kleinbock, *Diophantine exponents of measures and homogeneous dynamics*, Pure Appl. Math. Q., **4**, (2008), 81–97.
[24] D. Kleinbock, *Quantitative nondivergence and its Diophantine applications*, Homogeneous flows, moduli spaces and arithmetic. Clay Math. Proc., **10**, 131–153, Amer. Math. Soc., Providence, RI, (2010).
[25] D. Kleinbock, *An 'almost all versus no' dichotomy in homogeneous dynamics and Diophantine approximation*, Geom. Dedicata **149**, (2010), 205–218.
[26] D. Kleinbock, E. Lindenstrauss and B. Weiss, *On fractal measures and Diophantine approximation*, Selecta Math. (N.S.) **10**, 4 (2004), 479–523.
[27] D. Kleinbock and G.A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math. (2), **148**, 1 (1998), 339–360.
[28] D. Kleinbock, G.A. Margulis and J. Wang, *Metric Diophantine approximation for systems of linear forms via dynamics*, Int. J. Number Theory **6**, 5 (2010) 1139–1168.
[29] D. Kleinbock and G. Tomanov, *Flows on S-arithmetic homogeneous spaces and applications to metric Diophantine approximation*, Comment. Math. Helv. **82**, (2007), 519–581.
[30] I. Kovalevskaya, *Simultaneously extremal manifolds*, Mat. Zametki **41**, 1 (1987), 3–8, 119.
[31] I. Kovalevskaya, *Simultaneously extremal manifolds*, Dokl. Akad. Nauk BSSR **31**, 5 (1987), 405–408, 475–476.
[32] I. Kovalevskaya, *Strongly jointly extremal manifolds*, Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk (1987), no. 6, 16–19, 123–124.
[33] K. Mahler, *über das Maßder Menge aller S-Zahlen*, Math. Ann. **106**, (1932), 131–139.
[34] W.M. Schmidt, *Metrical theorems on fractional parts of sequences*, Trans. Amer. Math. Soc. **110**, (1964), 493–518.
[35] W.M. Schmidt, *Diophantine Approximation*, Springer-Verlag, Berlin and New York, (1980).
[36] W.M. Schmidt and Y. Wang, *A note on a transference theorem of linear forms*, Sci. Sinica, **22**, (1979), 276–280.
[37] V.G. Sprindžuk, *Mahler's problem in the metric theory of numbers*, Amer. Math. Soc. **25**, Providence, RI, (1969), Translations of Mathematical Monographs.
[38] V.G. Sprindžuk, *Achievements and problems in Diophantine approximation theory*, Russian Math. Surveys **35**, (1980), 1–80.

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