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A note on the weighted Khintchine-Groshev Theorem

par Mumtaz HUSSAIN et Tatiana YUSUPOVA

RÉSUMÉ. Soit $W(m,n;\underline{\psi})$ l'ensemble des points $\psi_1,\ldots,\psi_n$–approximables dans $\mathbb{R}^{mn}$. Le théorème classique de Khintchine–Groshev suppose une condition de monotonicité sur la fonction approximante $\underline{\psi}$. Différents auteurs ont pu supprimer cette condition pour différents m et n. Mais elle ne peut pas être supprimée quand $m = n = 1$, Duffin et Schaeffer ayant donné un contre-exemple. Nous traitons le seul cas restant $m = 2$, et donc toutes les conditions non-nécessaires dans le théorème de Khintchine–Groshev sont maintenant enlevées.

ABSTRACT. Let $W(m,n;\underline{\psi})$ denote the set of $\psi_1,\ldots,\psi_n$–approximable points in $\mathbb{R}^{mn}$. The classical Khintchine–Groshev theorem assumes a monotonicity condition on the approximating functions $\underline{\psi}$. Removing monotonicity from the Khintchine–Groshev theorem is attributed to different authors for different cases of m and n. It can not be removed for $m = n = 1$ as Duffin–Schaeffer provided the counter example. We deal with the only remaining case $m = 2$ and thereby remove all unnecessary conditions from the Khintchine–Groshev theorem.

1. Introduction

Throughout the paper, m and n are the natural numbers and $\mathbb{I}^{mn}$ is the unit cube $[0,1]^{mn}$ in $\mathbb{R}^{mn}$. Take an mn-dimensional point $\mathbf{X} \in \mathbb{I}^{mn}$, an integer vector $\mathbf{q} \in \mathbb{Z}^m$ and consider their product $\mathbf{qX}$. We may think of $\mathbf{X} = (x_{ij})$ as an $m \times n$ matrix with coefficients in $\mathbb{I}$ and $\mathbf{q} = (q_1,\ldots,q_m)$ as a row vector, allowing this product to be realized as the system

$$q_1x_{1j} + \ldots + q_mx_{mj} \qquad (1 \leqslant j \leqslant n)$$

of n real linear forms in m variables.

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For every $k \in \mathbb{N}$, denote by $|\cdot|$ the standard supremum norm on $\mathbb{R}^k$. Then, given a collection $\underline{\psi}$ of n functions $\psi_1, \ldots, \psi_n : \mathbb{N} \to \mathbb{R}^+$ each tending to 0, let $W(m, n; \underline{\psi})$ denote the set of points $\mathbf{X} \in \mathbb{I}^{mn}$ such that the system of inequalities

$$|q_1 x_{1j} + \ldots + q_m x_{mj} + p_j| < \psi_j(|\mathbf{q}|) \qquad (1 \leqslant j \leqslant n) \tag{1.1}$$

has infinitely many solutions $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^n \times \mathbb{Z}^m \setminus \{\mathbf{0}\}$.The functions $\psi_1, \ldots, \psi_n$ will be referred to as *approximating functions* and the points in $W(m, n; \underline{\psi})$ are said to be $\underline{\psi}$-*approximable*.

The fundamental aim of the paper is to determine the size of the set $W(m, n; \underline{\psi})$ in terms of mn–dimensional Lebesgue measure λ. The measure of $W(m, n; \underline{\psi})$ will necessarily depend on the collection $\underline{\psi}$ and we provide a precise criterion.

In the special case $\psi_1 = \ldots = \psi_n = \psi$ and $m = 1$ the set $W(m, n; \underline{\psi}) = W(1, n; \psi)$ is well studied since the pioneering work of A. Khintchine [15, 16]. Later, Khintchine's work was extended by Groshev [11] to cover the dual cases corresponding to $m > 1$. The following global statement combines both works, often referred to as the Khintchine-Groshev theorem, and provides a criterion relating the Lebesgue measure of the set $W(m, n; \psi)$ to the convergence or divergence of a certain series. This series entirely depends upon the approximating function ψ. We refer the reader to [2, 9, 11, 15, 16, 18] for the proofs as well as the subsequent improvements.

Theorem (Khintchine-Groshev). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$. Then*

$$\lambda\left(W(m, n; \psi)\right) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{m-1} \psi(q)^n < \infty, \\ \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{m-1} \psi(q)^n = \infty \textit{and } \psi \textit{ is monotonic.} \end{cases}$$

The convergence part of the above statement follows immediately from the Borel-Cantelli lemma from probability theory upon using a simple covering argument and is free from any assumption on ψ. The divergence part constitutes the main substance of the Khintchine–Groshev theorem. Due to the latest effort by Beresnevich and Velani [7] it has been shown that the monotonicity condition imposed in the divergence part can be removed from all but the case $m = n = 1$. Here, the Duffin-Schaeffer counterexample [10] shows that monotonicity is vital. We refer the reader to [7] for further details and to [1] for a detailed account of open problems in classical theory of metric Diophantine approximation related to the Khintchine–Groshev theorem.

When $\underline{\psi}$ contains more then one approximating function, not everything is known. The case $m = 1$ (simultaneous approximation) is described by

Harman ([12]), who showed that while the monotonicity assumption allows us to prove stronger results, it can be interchanged with a condition on the relationship between functions and the statement analogous to the Khintchine-Groshev theorem remains true. Schmidt's quantitative theorem, provides the measure criterion for $m \geqslant 3$; neither Harman's nor Schmidt's result covers the $m = 2$ case. By adapting the arguments of Beresnevich and Velani in [7], we will show that no restrictions are necessary in the $m = 2$ case. In doing so, we are able to establish the following best possible statement.

Theorem 1. *Let* $\underline{\psi} : \mathbb{N} \to \mathbb{R}^+$, $m > 1$, $n \geqslant 1$. *Then*

$$\lambda\left(W(m,n;\underline{\psi})\right) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} q^{m-1}\psi_1(q)\cdots\psi_n(q) = \infty.$$

The corresponding convergence case follows once more upon application of Borel–Cantelli lemma and is free from any assumption on the choices of m, n and the approximating functions. Note also that the proof given here will not be valid for the $m = 1$ case, as one needs some more assumptions on the functions $\psi_1, \dots, \psi_n$ as shown by Harman.

For the sake of completeness we mention that a Hausdorff measure version of Theorem 1 can be straightforwardly established using the Mass Transference Principle of [4] along with the 'slicing' technique [5]. The slicing technique is broad ranging and has been successfully employed in various related settings for a similar purpose [8, 13, 14].

Our paper will be structured as follows. In Section 2, we reduce the proof of Theorem 1 to establishing the analogous statement for a certain subset of $W(m,n;\underline{\psi})$ and then to a 'quasi-independence on average' statement. In Section 3, we establish various key measure theoretic estimates and in doing so completes the proof of Theorem 1.

Notation. Throughout, the symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are comparable. We will denote by φ the Euler's well known totient function.

2. Preliminaries

Consider the set

$$\begin{aligned} W'(m,n;\underline{\psi}) := \{\mathbf{X} \in \mathbb{I}^{mn} : &\text{ system of inequalities (1.1) holds} \\ &\text{for infinitely many } (\mathbf{p},\mathbf{q}) \in \mathbb{Z}^n \times \mathbb{Z}^m \setminus \{\mathbf{0}\} \text{ with } \gcd(\mathbf{p},\mathbf{q}) = 1\}, \end{aligned}$$

where $\gcd(\mathbf{p},\mathbf{q})$ denote the greatest common divisor of $p_1, \dots, p_n$, $q_1, \dots, q_m$. The set $W'(m,n;\underline{\psi})$ differs from $W(m,n;\underline{\psi})$ only by the coprimeness condition imposed on $\mathbf{p}$ and $\mathbf{q}$, and so we clearly have that

$W'(m,n;\underline{\psi}) \subset W(m,n;\underline{\psi})$. In addition, there is no loss of generality in assuming that

$$\psi_i(q) < c \qquad \text{for all } q \in \mathbb{N}, i = 1, \ldots, n, \text{ and } c > 0\ . \tag{2.1}$$

To see this, suppose for the moment that this was not the case; i.e. for some i statement (2.1) is false. Let

$$\widehat{\psi} : q \to \widehat{\psi}(q) := \min\{c, \psi_i(q)\}\ .$$

It is easily verified that if $\sum q^{m-1}\psi_1(q)\cdots\psi_i(q)\cdots\psi_n(q)$ diverges, then $\sum q^{m-1}\psi_1(q)\cdots\widehat{\psi}(q)\cdots\psi_n(q)$ diverges. Furthermore, $W'(m,n;\psi_1,\ldots,\widehat{\psi},\ldots,\psi_m) \subset W'(m,n;\underline{\psi})$ and so it suffices to establish Theorem 1 for $\widehat{\psi}$ as defined above.

The limsup nature of the sets $W(m,n;\underline{\psi})$ and $W'(m,n;\underline{\psi})$ is vital for the measure theoretic investigations we shall perform below. As such, it will be useful to express them in a limsup form. For any point $\underline{\delta} := (\delta_1,\ldots,\delta_n) \in \mathbb{R}^n$ with $\delta_i > 0$ for $1 \le i \le n$ and for any $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, let

$$B(\mathbf{q},\underline{\delta}) := \{\mathbf{X} \in \mathbb{I}^{mn} : |q_1 x_{1i} + \ldots + q_m x_{mi} + p_i| < \delta_i \text{ for all } i = 1,\ldots,n \text{ and some } \mathbf{p} \in \mathbb{Z}^n\}.$$

Furthermore, let

$$B'(\mathbf{q},\underline{\delta}) := \{\mathbf{X} \in \mathbb{I}^{mn} : |q_1 x_{1i} + \ldots + q_m x_{mi} + p_i| < \delta_i \text{ for all } i = 1,\ldots,n \text{ and some } \mathbf{p} \in \mathbb{Z}^n \text{ with } \gcd(\mathbf{p},\mathbf{q}) = 1\}\,.$$

Once more, the set $B'(\mathbf{q},\underline{\delta})$ differs from $B(\mathbf{q},\underline{\delta})$ by only the coprimeness condition. It is easily verified that

$$W(m,n;\underline{\psi}) = \limsup_{|\mathbf{q}|\to\infty} B(\mathbf{q},\underline{\psi}(|\mathbf{q}|))$$

and

$$W'(m,n;\underline{\psi}) = \limsup_{|\mathbf{q}|\to\infty} B'(\mathbf{q},\underline{\psi}(|\mathbf{q}|)).$$

The following statement helps us to reduce the proof of Theorem 1 to showing that $W'(m,n;\underline{\psi})$ is of positive Lebesgue measure.

Lemma 2. *For any m, $n \geqslant 1$ and $\underline{\psi} : \mathbb{N} \to \mathbb{R}^+$,*

$$\lambda(W'(m,n;\underline{\psi})) > 0 \qquad \Longrightarrow \qquad \lambda(W'(m,n;\underline{\psi})) = 1.$$

The proof of Lemma 2 follows on combining Theorem 4 of [6] and Lemma 2.2 of [17] as described in [6]. It can also be proven using the "cross-fibering principle" described in [3], which allowed the authors to establish a Zero-One Law in the multiplicative setup. The technique is very general and can have a number of different applications. For the proof of Lemma 2 using cross-fibering principle we refer the reader to [19].

Now, in order to prove positive measure, we make use of the following lemma which is a generalisation of the divergent part of the Borel-Cantelli lemma, tailored to our needs.

Lemma 3. *Let $E_k \subset \mathbb{I}^{mn}$ be a sequence of measurable sets such that $\sum_{k=1}^{\infty} \lambda(E_k) = \infty$. Then*

$$\lambda(\limsup_{k\to\infty} E_k) \ \geqslant \ \limsup_{N\to\infty} \frac{\left(\sum_{s=1}^{N} \lambda(E_s)\right)^2}{\sum_{s,t=1}^{N} \lambda(E_s \cap E_t)} \quad .$$

3. Proof of Theorem 1

In view of Lemma 3, the desired statement $\lambda(W'(m,n;\underline{\psi})) > 0$ will follow upon showing that the sets $B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))$ are quasi-independent on average and that the sum of their measures diverges. Essentially, we shall prove the following statement, which we include for clarity and completeness.

Proposition 4 (Quasi-independence on average)**.** *Let $m > 1$, $n \geqslant 1$ and $\underline{\psi} : \mathbb{N} \to \mathbb{R}^+$ satisfy $\psi_i(q) < 1/2$ for all $q \in \mathbb{N}$ and all $i = 1, \ldots, n$ and $\sum_{q=1}^{\infty} q^{m-1}\psi_1(q)\cdots\psi_n(q) = \infty$. Then,*

$$\sum_{\mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \lambda\left(B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))\right) \ = \ \infty \ , \tag{3.1}$$

and there exists a constant $C > 1$ such that for N sufficiently large,

$$\begin{aligned}\sum_{\substack{|\mathbf{q}^{(1)}|\leqslant N\\ |\mathbf{q}^{(2)}|\leqslant N}} &\lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right)\\ &\leqslant C\Big(\sum_{|\mathbf{q}^{(1)}|\leqslant N} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|))\right)\Big)^2.\end{aligned} \tag{3.2}$$

We first estimates the measure of $B'(\mathbf{q}, \underline{\delta})$. Given $\underline{\delta} \in \mathbb{R}^n$ with $\delta_i > 0$ for every i, $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\mathbf{p} \in \mathbb{Z}^n$, let

$$B(\mathbf{q}, \mathbf{p}, \underline{\delta}) := \{\mathbf{X} \in \mathbb{I}^{mn} : |q_1 x_{1i} + \ldots + q_m x_{mi} + p_i| < \delta_i\}.$$

Our estimate is a consequence of the following Lemmas (5, 6 and 7) which are adapted from [7] to the current setup. The proofs are almost identical therefore we leave the details for the reader.

Lemma 5. *Let $m > 1$, $n \geqslant 1$ and let $\underline{\delta} \in (0, 1/2)^n$ and $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Then, for any $l | \gcd(\mathbf{q})$*

$$\sum_{\mathbf{p}\in\mathbb{Z}^n} \lambda\left(B(\mathbf{q}, l\mathbf{p}, \underline{\delta})\right) = \left(\frac{2}{l}\right)^n \delta_1 \cdots \delta_n.$$

Lemma 6. *Let $m > 1$, $n \geqslant 1$ and let $\underline{\delta} \in (0,1/2)^n$ and $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$. Then,*

$$\lambda\left(B'(\mathbf{q},\underline{\delta})\right) = 2^n \delta_1 \cdots \delta_n \prod_{p|d} (1-p^{-n})\,.$$

The product is over prime divisors p of $d := \gcd(\mathbf{q})$ and is defined to be one if $d = 1$.

The following is a consequence of examining the product term in Lemma 6 and provides us the estimate we want for the measure of $B'(\mathbf{q},\underline{\delta})$.

Lemma 7. *Let $m > 1$, $n \geqslant 1$ and let $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$, $d := \gcd(\mathbf{q})$ and $\underline{\delta} \in (0,1/2)^n$. If $n = 1$, then*

$$\lambda\left(B'(\mathbf{q},\delta_1)\right) = 2\delta_1\,\frac{\varphi(d)}{d}\,.$$

If $n > 1$, then

$$\frac{6}{\pi^2}\,2^n \delta_1 \cdots \delta_n \leqslant \lambda\left(B'(\mathbf{q},\underline{\delta})\right) \leqslant 2^n \delta_1 \cdots \delta_n\,. \tag{3.3}$$

We now turn our attention to estimating the measure of the pairwise intersection of the sets $B'(\mathbf{q},\underline{\delta})$ i.e., the intersection of two sets $B'(\mathbf{q}^{(1)},\underline{\delta}^{(1)})$ and $B'(\mathbf{q}^{(2)},\underline{\delta}^{(2)})$ for $\mathbf{q}^{(1)},\mathbf{q}^{(2)} \in \mathbb{Z}^m\setminus\{\mathbf{0}\}$ and $\underline{\delta}^{(1)},\underline{\delta}^{(2)} \in \mathbb{R}^n$ with $\delta_i^{(1)},\delta_i^{(2)} > 0\ \forall\ i$. Naturally, there are two possibilities to be discussed; the case when $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are parallel and the case when they are not parallel. In the latter case, the following lemma, which can be found in [18], provides the relevant result. For an alternative proof using torus geometry see [9, p. 83-86].

Lemma 8. *Let $m,\ n \geqslant 1$ and let $\mathbf{q}^{(1)},\mathbf{q}^{(2)} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $\underline{\delta}^{(1)} := (\delta_1^{(1)},\ldots,\delta_n^{(1)})$, $\underline{\delta}^{(2)} := (\delta_1^{(2)},\ldots,\delta_n^{(2)}) \in (0,1/2)^n$. Then,*

$$\lambda\left(B(\mathbf{q}^{(1)},\underline{\delta}^{(1)})\right) = 2^n \delta_1^{(1)} \cdots \delta_n^{(1)}$$

and

$$\lambda\left(B(\mathbf{q}^{(1)},\underline{\delta}^{(1)}) \cap B(\mathbf{q}^{(2)},\underline{\delta}^{(2)})\right) = \lambda\left(B(\mathbf{q}^{(1)},\underline{\delta}^{(1)})\right)\cdot\lambda\left(B(\mathbf{q}^{(2)},\underline{\delta}^{(2)})\right) \qquad \text{if}\quad \mathbf{q}^{(1)} \nparallel \mathbf{q}^{(2)}.$$

Here, the notation $\mathbf{q}^{(1)} \nparallel \mathbf{q}^{(2)}$ means that $\mathbf{q}^{(1)}$ is not parallel to $\mathbf{q}^{(2)}$. To deal with the case that $\mathbf{q}^{(1)} \parallel \mathbf{q}^{(2)}$, that is, $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are parallel, we prove the following statement.

Lemma 9. *Let $m > 1$, $n \geqslant 1$. There is a constant $C > 0$ such that for $\underline{\delta}^{(1)},\underline{\delta}^{(2)} \in (0,1/2)^n$ and $\mathbf{q}^{(1)},\mathbf{q}^{(2)} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ satisfying $\mathbf{q}^{(1)} \neq \pm\mathbf{q}^{(2)}$*

$$\lambda\left(B'(\mathbf{q}^{(1)},\underline{\delta}^{(1)}) \cap B'(\mathbf{q}^{(2)},\underline{\delta}^{(2)})\right) \leqslant C \prod_{i=1}^{n} \delta_i^{(1)}\delta_i^{(2)}\,. \tag{3.4}$$

Proof. In view of Lemma 8, we only need to deal with the situation that $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are parallel.
It can be verified via geometric considerations that the left hand side of (3.4) is the product of the measures of the intersection on each axis; that is

$$\lambda\left(B'(\mathbf{q}^{(1)},\underline{\delta}^{(1)})\cap B'(\mathbf{q}^{(2)},\underline{\delta}^{(2)})\right)=\prod_{i=1}^{n}\lambda\left(B'(\mathbf{q}^{(1)},\delta_i^{(1)})\cap B'(\mathbf{q}^{(2)},\delta_i^{(2)})\right).$$

Indeed, as the vectors $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are parallel, the sets $B'(\mathbf{q}^{(1)},\underline{\delta}^{(1)})$ and $B'(\mathbf{q}^{(2)},\underline{\delta}^{(2)})$ can be visualized as n-dimensional boxes, which are oriented the same way in the Euclidian space, and the measure of their intersection can be thought of as the n-dimensional volume of the intersection of these boxes. The upshot of this is that we can restrict our attention to the case $n=1$ and we will write δ for δ_1.
Since the statement of Theorem 1 was only previously unverified in the case $m=2$, we will provide the argument for this value of m. However, we stress that the same techniques are valid for $m>2$, but do require some more tedious calculations.
Let us consider the two sets of lines

$$q_1^{(1)}x_1+q_2^{(1)}x_2=-p_1 \quad\text{and}\quad q_1^{(2)}x_1+q_2^{(2)}x_2=-p_2 \quad\text{with}\quad p_1,p_2\in\mathbb{Z}.$$

The sets $B'(\mathbf{q}^{(1)},\delta^{(1)})$ and $B'(\mathbf{q}^{(2)},\delta^{(2)})$ correspond to $\frac{\delta^{(1)}}{|\mathbf{q}^{(1)}|_2}$–neighborhood of the first line and $\frac{\delta^{(2)}}{|\mathbf{q}^{(2)}|_2}$–neighborhood of the second line respectively. Where, $|\cdot|_2$ denotes the standard Euclidean norm. Denote by $0<\gamma\leqslant\pi$ the angle between these lines and the positive direction of the x_1–axis. The aim is to estimate the measure of the intersection of the neighborhoods of these lines.

Suppose that $0<\gamma<\pi/4$. For the other values of γ the argument will be similar. For the sake of convenience we will rotate each line, including the boundaries of $\delta^{(i)}$-neighborhoods, clockwise by the angle γ around the point of its intersection with the x_2–axis (when $\pi/4<\gamma<\pi/2$ we rotate the lines anti-clockwise and proceed similarly). This procedure will remove the $q_1^{(i)}$ coordinates from our inequalities at the cost of altering the measure of the neighborhoods we are working with. The sets $B'(\mathbf{q}^{(1)},\delta^{(1)})$ and $B'(\mathbf{q}^{(2)},\delta^{(2)})$ become

$$S_1=S(\mathbf{q}^{(1)},\delta^{(1)})=\{X\in\mathbb{I}^2:|x_2q_2^{(1)}-p_1|<\frac{\delta^{(1)}}{\cos\gamma}$$
$$\text{for some } p_1\in\mathbb{Z},\ \gcd(p_1,q_1^{(1)},q_2^{(1)})=1\}$$

and

$$S_2 = S(\mathbf{q}^{(2)}, \delta^{(2)}) = \{X \in \mathbb{I}^2 : |x_2 q_2^{(2)} - p_2| < \frac{\delta^{(2)}}{\cos\gamma}$$
$$\text{for some } p_2 \in \mathbb{Z},\ \gcd(p_2, q_1^{(2)}, q_2^{(2)}) = 1\},$$

respectively.
Furthermore, $\lambda\left(B'(\mathbf{q}^{(1)}, \delta^{(1)}) \cap B'(\mathbf{q}^{(2)}, \delta^{(2)})\right) = \lambda\left(S_1 \cap S_2\right)$.

This measure can be estimated as the product of the number of points, which are sufficiently close to each other, and the measure of intersecting $(\delta^{(i)}/q_2^{(i)}\cos\gamma)$-neighborhoods at each point, i.e.

$$\lambda\left(S_1 \cap S_2\right) \leqslant \frac{1}{\cos\gamma} \min\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\} \cdot N,$$

where N is the number of pairs p_1, p_2 for which the following conditions hold for given $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$:

$$\begin{gathered} 0 \leqslant p_1 < q_2^{(1)},\ 0 \leqslant p_2 < q_2^{(2)}, \\ \gcd(p_1, q_1^{(1)}, q_2^{(1)}) = 1,\ \gcd(p_2, q_1^{(2)}, q_2^{(2)}) = 1 : \\ \left|\frac{p_1}{q_2^{(1)}} - \frac{p_2}{q_2^{(2)}}\right| < \frac{2}{\cos\gamma} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\}. \end{gathered}$$

This condition is equivalent to

$$\begin{gathered} 0 \leqslant p_1 < q_2^{(1)},\ 0 \leqslant p_2 < q_2^{(2)}, \\ \gcd(p_1, q_1^{(1)}, q_2^{(1)}) = 1,\ \gcd(p_2, q_1^{(2)}, q_2^{(2)}) = 1 : \\ \left|p_1 \cdot q_2^{(2)} - p_2 \cdot q_2^{(1)}\right| < \frac{2q_2^{(1)}q_2^{(2)}}{\cos\gamma} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\}. \end{gathered} \tag{3.5}$$

Note that $|p_1 \cdot q_2^{(2)} - p_2 \cdot q_2^{(1)}|$ is non-zero as otherwise the coprimeness condition would be contravened. To see this, suppose to the contrary that

$$|p_1 \cdot q_2^{(2)} - p_2 \cdot q_2^{(1)}| = 0. \tag{3.6}$$

Note that as the vectors $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are parallel, it is possible to choose a vector $\mathbf{q}^*$ such that $\mathbf{q}^{(1)} = k\mathbf{q}^*$ for some $k \in \mathbb{Z}$ and $\mathbf{q}^{(2)} = l\mathbf{q}^*$ for some $l \in \mathbb{Z}$ with $\gcd(k, l) = 1$. Neither k nor l can be equal to 1, as if that happens it means that one of the vectors $\mathbf{q}^{(i)}$ is a multiple of the other one, say, $\mathbf{q}^{(2)} = l\mathbf{q}^{(1)}$ and (3.6) only holds when $p_2 = lp_1$, which contradicts the assumption of coprimeness of $q_1^{(2)}, q_2^{(2)}$ and p_2. Now, (3.6) trivially holds if $p_1 = p_2 = 0$. In this case both

$$\gcd(q_1^{(1)}, q_2^{(1)}) = \gcd(kq_1^*, kq_2^*) = k \neq 1$$

and

$$\gcd(q_1^{(2)}, q_2^{(2)}) = \gcd(lq_1^*, lq_2^*) = l \neq 1,$$

which contradicts the definition of B'. Therefore, suppose that $p_1 \neq 0$ (the proof will be the same for $p_2 \neq 0$). Then $p_1 q_2^{(2)} - p_2 q_2^{(1)} = 0$ if and only if $p_1 \cdot lq_2^* - p_2 \cdot kq_2^* = 0$. Taking the last expression modulo k we get

$$p_1 \cdot l \equiv 0 \mod k,$$

which is equivalent to

$$p_1 \equiv 0 \mod k$$

as k and l are coprime. This gives us $p_1 = kp^*$ with $p^* \neq 0$ and $\gcd(p_1, q_1^{(1)}, q_2^{(1)}) = \gcd(kp^*, kq_1^*, kq_2^*) = k \neq 1$, which again contradicts the coprimality condition. Therefore, there are no such values of p_1 and p_2 for which (3.6) holds.

With this in mind we see that the expression $|p_1 \cdot q_2^{(2)} - p_2 \cdot q_2^{(1)}|$ can take at most

$$\frac{2q_2^{(1)} q_2^{(2)}}{\gcd(q_2^{(1)}, q_2^{(2)}) \cos\gamma} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\}$$

integer values in (3.5) and each value can be obtained $\gcd(q_2^{(1)}, q_2^{(2)})$ times. This means that

$$N \leqslant \frac{2q_2^{(1)} q_2^{(2)}}{\cos\gamma} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\} \leqslant 4q_2^{(1)} q_2^{(2)} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\}$$

since $\cos\gamma > 1/\sqrt{2}$ (due to the choice of γ). Thus, the measure of the intersection

$$\lambda\left(S_1 \cap S_2\right) \ll q_2^{(1)} q_2^{(2)} \max\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\} \min\left\{\frac{\delta^{(1)}}{q_2^{(1)}}, \frac{\delta^{(2)}}{q_2^{(2)}}\right\} = \delta^{(1)} \cdot \delta^{(2)},$$

and for $n > 1$

$$\lambda\left(B'(\mathbf{q}^{(1)}, \underline{\delta}^{(1)}) \cap B'(\mathbf{q}^{(2)}, \underline{\delta}^{(2)})\right) \leqslant C \prod_{i=1}^{n} \delta_i^{(1)} \delta_i^{(2)},$$

as required. □

In view of the fact that $B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|)) \subseteq B(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))$, to complete the proof of Theorem 1 it remains to establish Proposition 4. The following two lemmas enable us to accomplish this.

Lemma 10. *Let $m > 1$, $n \geqslant 1$ and $\psi_i(Q) < 1/2$ for all $Q \in \mathbb{N}$, $i = 1, \ldots, n$. Then with $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and $N \in \mathbb{N}$,*

$$\sum_{|\mathbf{q}| \leqslant N} \lambda\left(B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))\right) \asymp \sum_{Q=1}^{N} Q^{m-1} \psi_1(Q) \cdots \psi_n(Q)\ . \tag{3.7}$$

Proof. The proof splits into two cases: $n > 1$ and $n = 1$. We begin by considering the easy case $n > 1$. By (3.3) and the fact that the number of integer points $\mathbf{q} \in \mathbb{Z}^m$ with $|\mathbf{q}| = Q$ is comparable to Q^{m-1} (see [18, p. 39]), we have that

$$\begin{aligned}\sum_{\mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\},\ |\mathbf{q}|\leqslant N} \lambda\left(B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))\right) &\asymp \sum_{\mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\},\ |\mathbf{q}|\leqslant N} \psi_1(|\mathbf{q}|)\cdots\psi_n(|\mathbf{q}|) \\ &\asymp \sum_{Q=1}^{N}\sum_{|\mathbf{q}|=Q} \psi_1(|\mathbf{q}|)\cdots\psi_n(|\mathbf{q}|) \\ &\asymp \sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)\,.\end{aligned}$$

This establishes (3.7) in the case $n > 1$. The case $n = 1$ is very similar to the corresponding proof in [7] and therefore omitted. ☐

A clear implication of Lemma 10 is that $\sum_{\mathbf{q}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \lambda\left(B'(\mathbf{q}, \underline{\psi}(|\mathbf{q}|))\right) = \infty$; in other words, statement (3.1) holds subject to the conditions of Proposition 4. The truth of inequality (3.2) is a consequence of the following lemma.

Lemma 11. *Let $m > 1$, $n \geqslant 1$, $\psi_i(Q) < 1/2$ for all $Q \in \mathbb{N}$ and $\sum Q^{m-1}\psi_1(Q)\cdots\psi_n(Q) = \infty$. Then with $\mathbf{q}^{(1)}, \mathbf{q}^{(2)} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ and N sufficiently large,*

$$\begin{aligned}\sum_{|\mathbf{q}^{(1)}|\leqslant N,\ |\mathbf{q}^{(2)}|\leqslant N} &\lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right) \\ &\ll \Big(\sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)\Big)^2\,.\end{aligned} \tag{3.8}$$

Proof. We can express the left hand sum of (3.8) as

$$\sum_{|\mathbf{q}^{(1)}|\leqslant N,\ |\mathbf{q}^{(2)}|\leqslant N} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right) = M_1 + M_2,$$

where

$$M_1 = \sum_{\substack{|\mathbf{q}^{(1)}|\leqslant N,\ |\mathbf{q}^{(2)}|\leqslant N \\ \mathbf{q}^{(2)}=\pm\mathbf{q}^{(1)}}} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right)$$

and

$$M_2 = \sum_{\substack{|\mathbf{q}^{(1)}|\leqslant N,\ |\mathbf{q}^{(2)}|\leqslant N \\ \mathbf{q}^{(2)}\neq\pm\mathbf{q}^{(1)}}} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right).$$

We first deal with the case M_1. Since the sum $\sum Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)$ diverges, there exists a positive integer N_0 such that $\sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q) > 1$ for all $N > N_0$. Then, by Lemma 10 it follows that for $N > N_0$

$$\begin{aligned} M_1 = 2\sum_{|\mathbf{q}^{(1)}|\leqslant N} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|))\right) &\ll \sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q) \\ &< \left(\sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)\right)^2 . \end{aligned}$$

Now we need to obtain a similar estimate for M_2. In view of Lemma 9, it follows that

$$\begin{aligned} M_2 &= \sum_{Q=1}^{N}\sum_{l=1}^{N} \sum_{\substack{|\mathbf{q}^{(1)}|=Q,\ |\mathbf{q}^{(2)}|=l \\ \mathbf{q}^{(2)}\neq\pm\mathbf{q}^{(1)}}} \lambda\left(B'(\mathbf{q}^{(1)}, \underline{\psi}(|\mathbf{q}^{(1)}|)) \cap B'(\mathbf{q}^{(2)}, \underline{\psi}(|\mathbf{q}^{(2)}|))\right) \\ &\ll \sum_{Q=1}^{N}\sum_{l=1}^{N} \sum_{\substack{|\mathbf{q}^{(1)}|=Q, \\ |\mathbf{q}^{(2)}|=l}} \psi_1(|\mathbf{q}^{(1)}|)\cdots\psi_n(|\mathbf{q}^{(1)}|)\cdot\psi_1(|\mathbf{q}^{(2)}|)\cdots\psi_n(|\mathbf{q}^{(2)}|) \\ &= \sum_{Q=1}^{N}\sum_{l=1}^{N} \psi_1(Q)\cdots\psi_n(Q)\cdot\psi_1(l)\cdots\psi_n(l) \sum_{|\mathbf{q}^{(1)}|=Q} 1 \sum_{|\mathbf{q}^{(2)}|=l} 1 \\ &\ll \sum_{Q=1}^{N}\sum_{l=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)\cdot l^{m-1}\psi_1(l)\cdots\psi_n(l) \\ &\ll \left(\sum_{Q=1}^{N} Q^{m-1}\psi_1(Q)\cdots\psi_n(Q)\right)^2 . \end{aligned}$$

This completes the proof of Lemma 11 and hence the proof of Theorem 1. $\square$

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