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Self-intersection of the relative dualizing sheaf on modular curves $X_1(N)$

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Résumé. Soit N un entier naturel impair sans facteur carré ayant au moins deux diviseurs relativement premiers et supérieurs ou égaux à 4. Le théorème principal de cet article est une formule asymptotique exclusivement en termes de N pour l'auto-intersection arithmétique du dualisant relatif des courbes modulaires $X_1(N)/\mathbb{Q}$. Nous en déduisons une formule asymptotique pour la hauteur stable de Faltings de la Jacobienne $J_1(N)/\mathbb{Q}$ de $X_1(N)/\mathbb{Q}$ ainsi qu'une version effective de la conjecture de Bogomolov pour $X_1(N)/\mathbb{Q}$ pour N suffisamment grand.

Abstract. Let N be an odd and squarefree positive integer divisible by at least two relative prime integers bigger or equal than 4. Our main theorem is an asymptotic formula solely in terms of N for the stable arithmetic self-intersection number of the relative dualizing sheaf for modular curves $X_1(N)/\mathbb{Q}$. From our main theorem we obtain an asymptotic formula for the stable Faltings height of the Jacobian $J_1(N)/\mathbb{Q}$ of $X_1(N)/\mathbb{Q}$, and, for sufficiently large N, an effective version of Bogomolov's conjecture for $X_1(N)/\mathbb{Q}$.

1. Introduction

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathcal{X}/\mathcal{O}_K$ be the minimal regular model of a smooth projective curve X/K of genus $g_X > 0$. We call $\mathcal{X}/\mathcal{O}_K$ an arithmetic surface. In [2], S. J. Arakelov introduced an intersection theory for metrized invertible sheaves on $\mathcal{X}/\mathcal{O}_K$. G. Faltings established in his work [9] many fundamental results to the theory of Arakelov. Within this framework we may attach two important invariants to the curve X/K: Let $\overline{\omega}_{\mathcal{X}/\mathcal{O}_K}$ be the relative dualizing sheaf on $\mathcal{X}/\mathcal{O}_K$ equipped with the Arakelov metric. The first invariant is the stable arithmetic self-intersection number $\frac{1}{[K:\mathbb{Q}]}\overline{\omega}^2_{\mathcal{X}/\mathcal{O}_K}$ which is independent of the field K as long as X/K has semistable reduction over $\mathcal{O}_K$. The second invariant is the arithmetic degree of the direct image of $\overline{\omega}_{\mathcal{X}/\mathcal{O}_K}$ which is, in other words, the stable Faltings height of the Jacobian $\mathrm{Jac}(X)/K$ of X/K.

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The arithmetic significance of the stable arithmetic self-intersection number was given in [29] by showing that its strict positivity is equivalent to Bogomolov's conjecture (finally proven by E. Ullmo after partial results by Burnol, Szpiro, and Zhang). Recall that this conjecture claims that the set of algebraic points of the curve X/K embedded into its Jacobian are discretely distributed with respect to the "Néron-Tate topology" supposed that the genus of the curve is bigger than one. The second invariant is particularly interesting in the situation of modular curves. E.g., the stable Faltings height of the Jacobian of the modular curve $X_1(N)/\mathbb{Q}$ plays an important role in [8].

The only cases so far in which the stable arithmetic self-intersection number of the relative dualizing sheaf is known are arithmetic surfaces, where the generic fiber is a curve of genus one (see [9]), a curve of genus two (see [4]), or a modular curve $X_0(N)$, N squarefree and $2, 3 \nmid N$, (see [1], [23]). More recently, upper bounds in the cases of modular curves $X(N)$ and Fermat curves were found (see [5]). There are expressions for the stable Faltings height in the first two cases (see [4] and [9]). In the case of the modular curves $X_0(N)$, the stable Faltings height is asymptotically determined in [17]. Asymptotics in the case of the modular curve $X_1(N)$ are already given in [8] (cf. remark 8.3).

1.1. *Arakelov theory on arithmetic surfaces.* Let $\mathcal{X}/\mathcal{O}_K$ be the minimal regular model of a smooth projective curve X/K of genus $g_X > 0$. Let D be a divisor on $\mathcal{X}$ and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(D)$ the corresponding line bundle on $\mathcal{X}/\mathcal{O}_K$. For every embedding $\sigma : K \longrightarrow \mathbb{C}$ we equip the induced line bundle L_σ on the compact Riemann surface $X_\sigma(\mathbb{C})$, $X_\sigma := X \times_\sigma \mathrm{Spec}(\mathbb{C})$, with the unique admissible metric (Arakelov metric) with respect to the canonical volume form μ_{can} (cf. [28], p. 332). A line bundle $\mathcal{L}$ equipped with these metrics for all embeddings σ will be denoted by $\overline{\mathcal{L}}$. For the relative dualizing sheaf $\overline{\omega}_{\mathcal{X}/\mathcal{O}_K}$ the Arakelov metric has the following interpretation: the residual maps

$$\Omega^1_{X_\sigma(\mathbb{C})} \otimes \mathcal{O}_{X_\sigma(\mathbb{C})}(P)|_P \longrightarrow \mathbb{C}$$

are isometries for all points $P \in X_\sigma(\mathbb{C})$ and all embeddings $\sigma : K \longrightarrow \mathbb{C}$, where $\mathcal{O}_{X_\sigma(\mathbb{C})}(P)$ is equipped with the Arakelov metric and $\mathbb{C}$ with the standard hermitian metric (see [28], p. 333).

The intersection product of two metrized line bundles $\overline{\mathcal{L}} = \overline{\mathcal{O}}_{\mathcal{X}}(P)$ and $\overline{\mathcal{M}} = \overline{\mathcal{O}}_{\mathcal{X}}(Q)$, P, Q two horizontal prime divisors on $\mathcal{X}$ with no common component and induced points P_σ, Q_σ on $X_\sigma(\mathbb{C})$, is given by

$$\overline{\mathcal{L}} \cdot \overline{\mathcal{M}} = (P, Q)_{\mathrm{fin}} - \sum_{\sigma : K \to \mathbb{C}} g^\sigma_{\mathrm{can}}(P_\sigma, Q_\sigma),$$

where $(P,Q)_{\text{fin}}$ is their local intersection product on $\mathcal{X}/\mathcal{O}_K$ (see [28], p. 332) and g^{σ}_{can} is the canonical Green's function on $X_\sigma(\mathbb{C})\times X_\sigma(\mathbb{C})\setminus\Delta_{X_\sigma(\mathbb{C})}$, denoting by $\Delta_{X_\sigma(\mathbb{C})}$ the diagonal, (for the definition see section 3.2).

1.2. *Main results.* Let $X_1(N)/\mathbb{Q}$ be the smooth projective algebraic curve over $\mathbb{Q}$ that classifies elliptic curves equipped with a point of exact order N. Let N be an odd and squarefree integer of the form $N=N'qr>0$ with q and r relative prime integers satisfying $q,r\geq 4$. Then, the minimal regular model $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ is semistable (cf. proposition 7.3) and the genus of $X_1(N)/\mathbb{Q}$ denoted by g_N satisfies $g_N>0$. With the notation $\overline{\omega}^2_N=\dfrac{1}{[K:\mathbb{Q}]}\overline{\omega}^2_{\mathcal{X}_1(N)/\mathcal{O}_K}$ our main theorem (cf. theorem 7.7) is the following:

Theorem. *Let N be an odd and squarefree integer of the form $N=N'qr>0$ with q and r relative prime integers satisfying $q,r\geq 4$. Then, we have*

$$\overline{\omega}^2_N=3g_N\log(N)+o(g_N\log(N)).$$

Our first arithmetic application is the following asymptotic formula for the stable Faltings height $h_{\text{Fal}}(J_1(N))$ of the Jacobian $J_1(N)/\mathbb{Q}$ of the modular curves $X_1(N)/\mathbb{Q}$ (cf. theorem 8.2).

Theorem. *Let N be an odd and squarefree integer of the form $N=N'qr>0$ with q and r relative prime integers satisfying $q,r\geq 4$. Then, we have*

$$h_{\text{Fal}}(J_1(N))=\frac{g_N}{4}\log(N)+o(g_N\log(N)).$$

We also obtain an asymptotic formula for the admissible self-intersection number of the relative dualizing sheaf $\overline{\omega}^2_{a,N}$ in the sense of the theory of Zhang in [33]. From this we can deduce, for large N, an effective version of Bogomolov's conjecture for the modular curve $X_1(N)/\mathbb{Q}$: Let h_{NT} be the Néron-Tate height on the Jacobian $J_1(N)/\mathbb{Q}$, and let $\varphi_D: X_1(N)/\mathbb{Q}\longrightarrow J_1(N)/\mathbb{Q}$ be the embedding of the modular curve $X_1(N)/\mathbb{Q}$ into its Jacobian with respect to a divisor $D\in\operatorname{Div}(X_1(N))$ of degree one. With this notation we prove the following (cf. theorem 8.7):

Theorem. *Let N be an odd and squarefree positive integer of the form $N=N'qr$ with q and r relative prime integers satisfying $q,r\geq 4$. Then, for any $\varepsilon>0$, there is a sufficiently large N such that the set of algebraic points*

$$\left\{x\in X_1(N)(\overline{\mathbb{Q}})\,\middle|\,h_{\text{NT}}(\varphi_D(x))<\left(\frac{3}{4}-\varepsilon\right)\log(N)\right\}$$

is finite, and the set of algebraic points

$$\left\{x\in X_1(N)(\overline{\mathbb{Q}})\,\middle|\,h_{\text{NT}}(\varphi_D(x))\leq\left(\frac{3}{2}+\varepsilon\right)\log(N)\right\}$$

is infinite, if the class $[\mathcal{O}_{X_1(N)}(K_{X_1(N)} - (2g_N - 2)D)]$ *is a torsion element in* $J_1(N)/\mathbb{Q}$*, where* $K_{X_1(N)}$ *is the canonical divisor on* $X_1(N)/\mathbb{Q}$*.*

1.3. *Outline.* The main structure of this article derives from proposition 7.6 providing us with the formula

$$\overline{\omega}_N^2 = 4g_N(g_N-1)g_{\mathrm{can}}(0,\infty) + \frac{1}{\varphi(N)}\frac{g_N+1}{g_N-1}\,(V_0,V_\infty)_{\mathrm{fin}}\,,$$

where V_0, V_∞ are explicit vertical divisors on $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$. To achieve our main theorem, we first compute the analytic part $4g_N(g_N-1)g_{\mathrm{can}}(0,\infty)$ following the strategy of [1] using results of [16]. Afterwards we determine the geometric part $\frac{1}{\varphi(N)}\frac{g_N+1}{g_N-1}\,(V_0,V_\infty)_{\mathrm{fin}}$ of the stable arithmetic self-intersection number $\overline{\omega}_N^2$.

In section two we recall some basic facts of the (compactified) modular curves and present the spectral expansion of the automorphic kernels of weight 0 and 2. We conclude this section by observing that the arithmetic average

$$F(z) := \frac{1}{g_N}\sum_{j=1}^{g_N} y^2|f_j(z)|^2 \qquad\qquad (z = x+iy \in \mathbb{H})$$

of an orthonormal basis $\{f_j\}_{j=1}^{g_N}$ of holomorphic cusp forms of weight 2 with respect to the congruence subgroup $\Gamma_1(N)$ appears in the spectral expansions of the automorphic kernels mentioned above. In section three we obtain a formula which connects $g_{\mathrm{can}}(0,\infty)$ with the constant term C_F in the Laurent expansion at $s=1$ of the Rankin-Selberg transform $R_F(s)$, $s \in \mathbb{C}$, of the function $F(z)$. In section four we determine the Rankin-Selberg transform of the hyperbolic part of the automorphic kernels. In section five we analyze the Rankin-Selberg transform of the parabolic part of the automorphic kernels and their spectral expansions. Subsequently we obtain a first expression for C_F (cf. remark 3.9). In section six we may finally determine the analytic part purely in terms of N. In section seven we obtain our main theorem after having computed the geometric part of the stable arithmetic self-intersection number $\overline{\omega}_N^2$. In section eight we deduce the arithmetic applications mentioned above. In the appendix we finally study an Epstein zeta function that appears in section four.

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2. Background Material

Let us collect some basic material of modular curves $X(\Gamma_1(N))$ and their spectral theory. Our main references are [6], [13], and [26, 27]. The spectral theory in [13] and [26, 27] is formulated in a more general framework, namely for Fuchsian subgroups of the first kind. In order to keep this exposition short, we restrict the discussion to the congruence subgroup $\Gamma_1(N)$.

2.1. *The upper half-plane.* Let $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid \mathrm{Im}(z) = y > 0\}$ be the upper half-plane equipped with the hyperbolic metric

$$\mathrm{d}s^2_{\mathrm{hyp}}(z) := \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{2.1}$$

giving $\mathbb{H}$ the structure of a 2-dimensional Riemannian manifold of constant negative curvature equal to -1. The hyperbolic metric (2.1) induces the distance function ρ on $\mathbb{H}$ defined by $\cosh\big(\rho(z,w)\big) = 1 + 2u(z,w)$, where

$$u(z,w) = \frac{\mid z - w \mid^2}{4\mathrm{Im}(z)\mathrm{Im}(w)} \qquad (z, w \in \mathbb{H}), \tag{2.2}$$

and the hyperbolic volume form on $\mathbb{H}$ given by $\mu_{\mathrm{hyp}}(z) := \frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$.

2.2. *Modular curves* $X(\Gamma_1(N))$. Let $N \geq 1$ be a positive integer and $\Gamma_1(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ the congruence subgroup defined by

$$\Gamma_1(N) := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod N,\, c \equiv 0 \mod N\right\}$$

acting by fractional linear transformation $z \mapsto \gamma z := \frac{az+b}{cz+d}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$, on the upper half-plane $\mathbb{H}$. The quotient space $\Gamma_1(N)\backslash\mathbb{H}$ is denoted by $Y(\Gamma_1(N))$. Letting $\Gamma_1(N)$ act on the projective line $\mathbb{P}^1_{\mathbb{Q}}$ by $\gamma(s:t) := (as+bt : cs+dt)$, $(s:t) \in \mathbb{P}^1_{\mathbb{Q}}$, the (compactified) modular curve $X(\Gamma_1(N))$ associated to $\Gamma_1(N)$ is defined as the quotient space

$$X(\Gamma_1(N)) := \Gamma_1(N)\backslash(\mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}),$$

which can be endowed with a natural topology making the quotient space into a compact Riemann surface (see [6], chap. 2.4).
The finite set $P_{\Gamma_1(N)} := X(\Gamma_1(N))\backslash Y(\Gamma_1(N))$ is called the set of (inequivalent) cusps of $\Gamma_1(N)$ and is represented by the set of elements (a,c) of $(\mathbb{Z}/N\mathbb{Z})^2$ of order N modulo the equivalence relation $(a,c) \equiv (a',c')$ if and only if $(a',c') = (a+nc, c)$ for some $n \in \mathbb{Z}/N\mathbb{Z}$. Moreover, the hyperbolic volume form μ_{hyp} descends to a volume form on $X(\Gamma_1(N))$ (see [6], p. 181) which we will denote again by μ_{hyp}. For the moduli interpretation of the modular curve $X(\Gamma_1(N))$ we refer the reader to section 7.

2.3. *Genus and volume formula.* We assume $N \geq 5$ to avoid elliptic fixed points and to have uniform formulas for the following quantities. Let g_N be the genus and v_N the hyperbolic volume of $X(\Gamma_1(N))$, then we have

$$g_N = 1 + \frac{1}{24}\varphi(N)N\prod_{p|N}\left(1+\frac{1}{p}\right) - \frac{1}{4}\sum_{d|N}\varphi(d)\varphi(N/d) \tag{2.3}$$

and

$$v_N = \frac{\pi}{6}\varphi(N)N\prod_{p|N}\left(1+\frac{1}{p}\right), \tag{2.4}$$

where $\varphi(\,\cdot\,)$ is Euler's phi function (see [6], theorem 3.1.1, p. 68 and formula (5.15), p. 182). In particular, we have $g_N \geq 1$ for $N = 11$ or $N \geq 13$.

2.4. *The Hilbert space* $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)$. Recall that a function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called automorphic of weight k, $k \in \mathbb{N}$, with respect to $\Gamma_1(N)$, if it satisfies $f(\gamma z) = j_{k,\gamma}(z)f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_1(N)$, where $j_{k,\gamma}(z) := \frac{(cz+d)^k}{|cz+d|^k}$. We denote by $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)$ the Hilbert space of square-integrable automorphic functions of weight k with respect to $\Gamma_1(N)$ with scalar product given by

$$\langle f, g\rangle := \int_{\Gamma_1(N)\backslash\mathbb{H}} f(z)\overline{g}(z)\mu_{\mathrm{hyp}}(z) \qquad \Big(f, g \in L^2(\Gamma_1(N)\backslash\mathbb{H}, k)\Big).$$

The hyperbolic Laplacian of weight k

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}, \tag{2.5}$$

acting as a non-negative self-adjoint operator on $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)$ (in fact, it is the unique self-adjoint extension of Δ_k acting on the subspace of smooth and compactly supported automorphic functions of weight k; see [26], pp. 309–310), gives rise to the spectral decomposition

$$L^2(\Gamma_1(N)\backslash\mathbb{H}, k) = L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_0 \oplus L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_{\mathrm{r}} \oplus L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_{\mathrm{c}};$$

here $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_0$ is the space of cusp forms of weight k, i.e., of automorphic functions of weight k with vanishing 0-th Fourier coefficients in its Fourier expansions with respect to the diverse cusps of $\Gamma_1(N)$, which belongs to the discrete part of the spectrum, $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_{\mathrm{r}}$ is the discrete part of $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_0^{\perp}$, given by residues of Eisenstein series, and $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)_{\mathrm{c}}$ forms the continuous part of the spectrum, given by integrals of Eisenstein series.

2.5. *Eisenstein series and spectral expansion.* Let $\mathfrak{a} \in P_{\Gamma_1(N)}$ be a cusp of $\Gamma_1(N)$ and $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{R})$ a scaling matrix of $\mathfrak{a}$, i.e., $\sigma_\mathfrak{a}\infty = \mathfrak{a}$ and

$$\sigma_\mathfrak{a}^{-1}\Gamma_1(N)_\mathfrak{a}\sigma_\mathfrak{a} \cong \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \,|\, m \in \mathbb{Z} \right\}$$

for $\Gamma_1(N)_\mathfrak{a}$ the stabilizer group of the cusp $\mathfrak{a}$. The Eisenstein series for the cusp $\mathfrak{a}$ of weight k with respect to $\Gamma_1(N)$ is defined by (cf. [27], p. 291)

$$E_{\mathfrak{a},k}(z,s) := \sum_{\gamma \in \Gamma_1(N)_\mathfrak{a} \backslash \Gamma_1(N)} \mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^s j_{k,\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-1} \qquad (s \in \mathbb{C}, \mathrm{Re}(s) > 1).$$

The Eisenstein series $E_{\mathfrak{a},k}(z,s)$ defines an holomorphic function for $\mathrm{Re}(s) > 1$ and is in this range an automorphic function of weight k in the z-variable. Moreover, it possesses a meromorphic continuation to the whole s-plane. The meromorphically continued Eisenstein serie $E_{\mathfrak{a},k}(z,s)$ is holomorphic for $\mathrm{Re}(s) = \frac{1}{2}$, and the poles at s with $\mathrm{Re}(s) > \frac{1}{2}$ lie all in the interval $(1/2, 1]$ (cf. [27], Satz 10.3, Satz 10.4, and Satz 11.2).
If $\{u_j\}_{j=0}^\infty$ is an orthonormal basis of the discrete part of $L^2(\Gamma_1(N)\backslash\mathbb{H}, k)$, i.e., $\Delta_k u_j = \lambda_j u_j$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2, \ldots$, then every $f \in L^2(\Gamma_1(N)\backslash\mathbb{H}, k)$ has the spectral expansion

$$\text{(2.6)} \qquad f(z) = \sum_{j=0}^\infty \langle f, u_j \rangle u_j(z) + \sum_{\mathfrak{a} \in P_{\Gamma_1(N)}} \frac{1}{4\pi} \int_{-\infty}^{+\infty} \langle f, E_{\mathfrak{a},k}(\cdot, \tfrac{1}{2} + ir) \rangle E_{\mathfrak{a},k}(z, \tfrac{1}{2} + ir) \mathrm{d}r,$$

which converges in the norm topology. If furthermore, f is smooth and bounded, then the sums in (2.6) are uniformly convergent on compacta of $\mathbb{H}$ (see [27], Satz 7.2, Satz 12.2, and Satz 12.3).

2.6. *Shifting operator and eigenspaces of* $\lambda_0 = 0$. Noting that Δ_0 and Δ_2 have the same eigenvalues (see [26], lemma 3.2), we define $L^2_{\lambda_j}(\Gamma_1(N)\backslash\mathbb{H}, k)$, $k = 0, 2$, to be the eigenspace corresponding to the eigenvalue λ_j. The differential operator (loc. cit. denoted by K_0)

$$\text{(2.7)} \qquad \Lambda_0 := iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} : L^2_{\lambda_j}(\Gamma_1(N)\backslash\mathbb{H}, 0) \longrightarrow L^2_{\lambda_j}(\Gamma_1(N)\backslash\mathbb{H}, 2)$$

induces for $\lambda_j \neq 0$ a bijection satisfying

$$\text{(2.8)} \qquad \langle \Lambda_0(f), \Lambda_0(g) \rangle = \lambda_j \langle f, g \rangle \qquad (f, g \in L^2_{\lambda_j}(\Gamma_1(N)\backslash\mathbb{H}, 0)).$$

(see [26], lemma 6.1). For $\lambda_j = 0$ and $k = 0$, we have that $L^2_0(\Gamma_1(N)\backslash\mathbb{H}, 0)$ is one dimensional generated by the only residue of the Eisenstein series at

$s=1$ given by v_N^{-1} (see [13], theorem 11.3). For $\lambda_j = 0$ and $k=2$, we have an isometry

$$L_0^2(\Gamma_1(N)\backslash\mathbb{H},2) \longrightarrow S_2(\Gamma_1(N)) \tag{2.9}$$

by sending $f \mapsto y^{-1}f$(see [26], Satz 6.3), where $S_2(\Gamma_1(N))$ denotes the space of holomorphic cusp forms of weight 2 for $\Gamma_1(N)$. For later purposes we mention (see [27], p. 292, equation (10.8))

$$\Lambda_0\big(E_{\mathfrak{a},0}(z,s)\big) = sE_{\mathfrak{a},2}(z,s). \tag{2.10}$$

2.7. *Automorphic kernels of weight* 0 *and* 2. Let $h:\mathbb{R}\longrightarrow\mathbb{C}$ be a test function, i.e., an even function satisfying for an $A\in\mathbb{R}, A>1$:

(i) $h(r)$ can be extended holomorphically to the strip $|\mathrm{Im}(r)| < \frac{A}{2}$

(ii) $h(r) \ll (|r|+1)^{-2-\frac{A-1}{2}}$ for $|\mathrm{Im}(r)| < \frac{A}{2}$.

The inverse Selberg transform k_0 of h of weight 0 is given by the following three equations (see [13], p. 32):

$$\begin{aligned} g(w) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\exp(-iwr)\mathrm{d}r && (w\in\mathbb{R}),\\ q\big(e^v+e^{-v}-2\big) &= g(v) && (v\in\mathbb{R}),\\ k_0(u) &= -\frac{1}{\pi}\int_{-\infty}^{+\infty} q'(u+v^2)\mathrm{d}v && (u\geq 0). \end{aligned}$$

The inverse Selberg transform k_2 of h of weight 2 is given in the similar way with the only change in the last step (see [11], p. 402 and p. 455):

$$k_2(u) := -\frac{1}{\pi}\int_{-\infty}^{+\infty} q'(u+v^2)\frac{\sqrt{u+4+v^2}-v}{\sqrt{u+4+v^2}+v}\mathrm{d}v \qquad (u\geq 0).$$

The automorphic kernel of weight 0 with respect to $\Gamma_1(N)$ on $\mathbb{H}\times\mathbb{H}$ is defined by

$$K_0(z,w) := \sum_{\gamma\in\Gamma_1(N)} k_0\big(u(z,\gamma w)\big), \tag{2.11}$$

which is an automorphic function of weight 0 in the z-variable; here the function u is defined as in (2.2). Similarly, the automorphic kernel of weight 2 with respect to $\Gamma_1(N)$ on $\mathbb{H}\times\mathbb{H}$ is defined by

$$K_2(z,w) := \sum_{\gamma\in\Gamma_1(N)} k_2\big(u(z,\gamma w)\big)\,\frac{\gamma w-\overline{z}}{z-\gamma\overline{w}}\,j_{2,\gamma}(w), \tag{2.12}$$

which is an automorphic function of weight 2 in the z-variable.

2.8. *Spectral expansion of automorphic kernels.* From the orthogonal projections $\langle K_0(z,w),u_j(z)\rangle = h_0(r_j)\overline{u}_j(w)$ and $\langle K_0(z,w),E_{\mathfrak{a},0}(z,\frac{1}{2}+ir)\rangle = h_0(r)\overline{E}_{\mathfrak{a},0}(w,\frac{1}{2}+ir)$ (see [13], theorem 7.4), using the convention to write an eigenvalue λ_j as $\lambda_j = \frac{1}{4}+r_j^2$ with $r_j \in \mathbb{C}$, we find from (2.6) the following spectral expansion

$$K_0(z,w) = \frac{h(\frac{i}{2})}{v_N} + \sum_{j=1}^{\infty} h(r_j)u_j(z)\overline{u}_j(w) +$$

$$\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}} \int_{-\infty}^{+\infty} h_0(r)E_{\mathfrak{a},0}(z,\tfrac{1}{2}+ir)\overline{E}_{\mathfrak{a},0}(w,\tfrac{1}{2}+ir)\mathrm{d}r. \tag{2.13}$$

Noting that $\langle K_2(z,w), u_j(z)\rangle = h_2(r_j)\overline{u}_j(w)$ and $\langle K_2(z,w), E_{\mathfrak{a},2}(z,\frac{1}{2}+ir)\rangle = h_2(r)\overline{E}_{\mathfrak{a},2}(w,\frac{1}{2}+ir)$ (cf. [1], lemma 3.1.1), we find from (2.6) and observations (2.8) and (2.9) the following spectral expansion

$$K_2(z,w) = h(\tfrac{i}{2})\sum_{j=1}^{g_N} \mathrm{Im}(z)f_j(z)\mathrm{Im}(w)\overline{f}_j(w) + \sum_{j=1}^{\infty}\frac{h(r_j)}{\lambda_j}\Lambda_0(u_j)(z)\overline{\Lambda_0(u_j)}(w) +$$

$$\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}} \int_{-\infty}^{+\infty} h(r)E_{\mathfrak{a},2}(z,\tfrac{1}{2}+ir)\overline{E}_{\mathfrak{a},2}(w,\tfrac{1}{2}+ir)\mathrm{d}r, \tag{2.14}$$

where $\{f_1,\ldots,f_{g_N}\}$ is an orthonormal basis of $S_2(\Gamma_1(N))$.

2.9. Let $h(t,r) := \exp\left(-t\left(\frac{1}{4}+r^2\right)\right)$ be the test function with parameter $t \in \mathbb{R}_{>0}$. The inverse Selberg transform of $h(t,r)$ of weight 0 defines the function $k_0(t,u)$ for $u \geq 0$; for a fixed $\gamma \in \Gamma_1(N)$, we set

$$K_{0,\gamma}(t,z) := k_0\big(t, u(z,\gamma z)\big) \qquad (z \in \mathbb{H}).$$

Similarly, the inverse Selberg transform of $h(t,r)$ of weight 2 defines the function $k_2(t,u)$ for $u \geq 0$; for a fixed $\gamma \in \Gamma_1(N)$, we set

$$K_{2,\gamma}(t,z) := k_2\big(t, u(z,\gamma z)\big)\frac{\gamma z - \overline{z}}{z - \gamma\overline{z}}\, j_{2,\gamma}(z) \qquad (z \in \mathbb{H}).$$

2.10. Lemma. *Let be $l \in \mathbb{Z}$. With the above notation the following series*

$$\sum_{\substack{\gamma\in\Gamma_1(N)\\ \mathrm{tr}(\gamma)=l}} K_{0,\gamma}(t,z) \qquad \textit{and} \qquad \sum_{\substack{\gamma\in\Gamma_1(N)\\ \mathrm{tr}(\gamma)=l}} K_{2,\gamma}(t,z) \qquad (z \in \mathbb{H})$$

are automorphic functions of weight 0 *with respect to* $\Gamma_1(N)$.

Proof. This follows from the fact that $K_{k,\gamma}(t,\delta z) = K_{k,\delta^{-1}\gamma\delta}(t,z)$ for $k = 0,2$ and any $\delta = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_1(N)$. □

2.11. *Notation.* We define for $t > 0$, $k = 0,2$, and $l \in \mathbb{Z}$ with $|l| > 2$

$$H_{k,l}(t,z) := \sum_{\substack{\gamma\in\Gamma_1(N)\\ \mathrm{tr}(\gamma)=l}} K_{k,\gamma}(t,z), \qquad H_k(t,z) := \sum_{\substack{l\in\mathbb{Z}\\ |l|>2}} H_{k,l}(t,z), \tag{2.15}$$

(2.16)
$$P_k(t,z) := \sum_{\substack{\gamma\in\Gamma_1(N)\\ |\mathrm{tr}(\gamma)|=2}} K_{k,\gamma}(t,z),$$

(2.17)
$$C_k(t,z) := -\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{-\infty}^{+\infty} h(t,r)|E_{\mathfrak{a},k}(z,\tfrac{1}{2}+ir)|^2\mathrm{d}r - \frac{2-k}{2}\frac{1}{v_N},$$

and

(2.18)
$$D_0(t,z) := \sum_{j=1}^{\infty} h(t,r_j)|u_j(z)|^2,\ \ D_2(t,z) := \sum_{j=1}^{\infty}\frac{h(t,r_j)}{\lambda_j}|\Lambda_0(u_j)(z)|^2.$$

2.12. *Basic formula.* We assume that $N=11$ or $N\geq 13$. We define the $\Gamma_1(N)$-invariant functions (cf. lemma 2.10)

(2.19)
$$H(t,z) := H_2(t,z) - H_0(t,z) \qquad P(t,z) := P_2(t,z) - P_0(t,z)$$
$$C(t,z) := C_2(t,z) - C_0(t,z) \qquad D(t,z) := D_2(t,z) - D_0(t,z),$$

such that observations (2.13) and (2.14), taking the difference of $K_2(t,z)$ and $K_0(t,z)$, imply

(2.20)
$$g_N F(z) + D(t,z) = H(t,z) + P(t,z) + C(t,z),$$

where

(2.21)
$$F(z) := \frac{1}{g_N}\sum_{j=1}^{g_N} y^2|f_j(z)|^2$$

with $\{f_j\}_{j=1}^{g_N}$ an orthonormal basis of $S_2\big(\Gamma_1(N)\big)$. Note that we have $h(t,\frac{i}{2})=1$ and that there is no elliptic contribution, i.e., there is no $\gamma\in\Gamma_1(N)$ with $|\mathrm{tr}(\gamma)|<2$.

3. Green's function on cusps

In this section we recall the definition of the canonical Green's function and derive a formula for its evaluation on cusps essentially in terms of the function defined in (2.21). Our formula follows from previous work of A. Abbes and E. Ullmo in [1] and J. Jorgenson and J. Kramer in [16]. In the sequel we assume $g_N\geq 1$, i.e., that $N=11$ or $N\geq 13$.

3.1. *Canonical volume.* Let $S_2\big(\Gamma_1(N)\big)$ be the space of holomorphic cusp forms of weight 2 with respect to $\Gamma_1(N)$ equipped with the Petersson inner product

$$\langle f,g\rangle_{\mathrm{Pet},2} := \int_{X(\Gamma_1(N))} f(z)\overline{g}(z)\mathrm{Im}(z)^2\mu_{\mathrm{hyp}}(z) \qquad \big(f,g\in S_2\big(\Gamma_1(N)\big)\big).$$

Choosing an orthonormal basis $\{f_1,\ldots,f_{g_N}\}$ of $S_2(\Gamma_1(N))$, the canonical volume form on $X(\Gamma_1(N))$ is given by

$$\mu_{\mathrm{can}}(z) := \frac{i}{2g_N}\sum_{j=1}^{g_N}|f_j(z)|^2\mathrm{d}z\wedge\mathrm{d}\overline{z} = F(z)\mu_{\mathrm{hyp}}(z),$$

where $F(z)$ is defined by (2.21). Note that this volume form becomes under the ismomophism $S(\Gamma_1(N))\cong H^0(X(\Gamma_1(N)),\Omega^1_{X(\Gamma_1(N))})$ given by $f(z)\mapsto f(z)\mathrm{d}z$ the one considered in [2].

3.2. *Canonical Green's function.* The canonical Green's function g_{can} is the unique smooth function on $X(\Gamma_1(N))\times X(\Gamma_1(N))\setminus\Delta_{X(\Gamma_1(N))}$, denoting by $\Delta_{X(\Gamma_1(N))}$ the diagonal, which satisfies:

(i) $\frac{1}{\pi i}\frac{\partial^2}{\partial z\partial\overline{z}}g_{\mathrm{can}}(z,w)+\delta_w(z)=\mu_{\mathrm{can}}(z)$,

(ii) $\displaystyle\int_{X(\Gamma_1(N))}g_{\mathrm{can}}(z,w)\mu_{\mathrm{can}}(z)=0 \qquad \forall w\in X(\Gamma_1(N))$,

where $\delta_w(z)$ is the Dirac delta distribution.

3.3. To state the formula for the canonical Green's function on cusps from [1], we recall that, for $\mathfrak{a},\mathfrak{b}\in P_{\Gamma_1(N)}$ cusps of $\Gamma_1(N)$, the Eisenstein series $E_{\mathfrak{a},0}(z,s)$ of weight zero with respect to $\Gamma_1(N)$ admits at the cusp $\mathfrak{b}$ the Fourier expansion $E_{\mathfrak{a},0}(\sigma_{\mathfrak{b}}z,s)=\sum_{n\in\mathbb{Z}}a_n(y,s;\mathfrak{ab})\exp(2\pi inx)$ with

$$a_0(y,s;\mathfrak{ab})=\delta_{\mathfrak{ab}}y^s+\varphi_{\mathfrak{ab}}(s)y^{1-s},$$

where

$$\varphi_{\mathfrak{ab}}(s)=\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{1}{N^s}\sum_{c\in\mathbb{N}_{>0}}\frac{b_{\mathfrak{ab}}(c)}{c^{2s}} \tag{3.1}$$

with

$$b_{\mathfrak{ab}}(c)=\#\left\{\begin{pmatrix}\star & \star\\ c & \star\end{pmatrix}\in g_{\mathfrak{a}}^{-1}\Gamma_1(N)_{\mathfrak{a}}\,g_{\mathfrak{a}}\backslash g_{\mathfrak{a}}^{-1}\Gamma_1(N)g_{\mathfrak{b}}/g_{\mathfrak{b}}^{-1}\Gamma_1(N)_{\mathfrak{b}}\,g_{\mathfrak{b}}\right\}; \tag{3.2}$$

here $g_{\mathfrak{a}}$ and $g_{\mathfrak{b}}$ denote elements of $\mathrm{SL}_2(\mathbb{Z})$ mapping the standard cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$ with represantative $(1:0)\in\mathbb{P}^1_{\mathbb{Q}}$ to the cusps $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Note that $\varphi_{\mathfrak{ab}}(s)$ is a meromorphic function with a simple pole at $s=1$ and residue v_N^{-1} (see [13]).

3.4. Proposition. *Let N satisfy $N=11$ or $N\geq 13$. Let $X(\Gamma_1(N))$ be the modular curve associated to the congruence subgroup $\Gamma_1(N)$. Then, we* *have for two different cusps* $\mathfrak{a}, \mathfrak{b} \in P_{\Gamma_1(N)}$

$$g_{\mathrm{can}}(\mathfrak{a}, \mathfrak{b}) =$$

$$-2\pi \lim_{s\to 1} \left(\varphi_{\mathfrak{a}\mathfrak{b}}(s) - \frac{1}{v_N} \frac{1}{s-1} \right) + 2\pi \lim_{s\to 1} \left(\int_{\Gamma_1(N)\backslash\mathbb{H}} F(z) E_{\mathfrak{a},0}(z,s) \mu_{\mathrm{hyp}}(z) + \int_{\Gamma_1(N)\backslash\mathbb{H}} F(w) E_{\mathfrak{b},0}(w,s) \mu_{\mathrm{hyp}}(w) - \frac{2}{v_N} \frac{1}{s-1} \right) + O\left(\frac{1}{g_N} \right),$$

where the error term is independent of the cusps $\mathfrak{a}$ *and* $\mathfrak{b}$*.*

Proof. This follows from proposition E in [1] in combination with the bound on the hyperbolic Green's function in [16], lemma 3.7 and proposition 4.7 with the universal constants for $\Gamma_1(N)$ given in lemma 5.3 (c) and lemma 5.9. □

3.5. Lemma. *Let* N *satisfy* $N = 11$ *or* $N \geq 13$*. Let* $X(\Gamma_1(N))$ *be the modular curve associated to the congruence subgroup* $\Gamma_1(N)$ *and* $0, \infty_d$*,* $d \in (\mathbb{Z}/N\mathbb{Z})^\times$*, the cusps having representatives* $(0:1)$ *and* $(d:0)$ *in* $\mathbb{P}^1_{\mathbb{Q}}$*, respectively. With the notation* $\infty = \infty_1$*, we have*

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} F(z) E_{0,0}(z,s) \mu_{\mathrm{hyp}}(z) = \int_{\Gamma_1(N)\backslash\mathbb{H}} F(z) E_{\infty,0}(z,s) \mu_{\mathrm{hyp}}(z) \tag{3.3}$$

and

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} F(z) E_{\infty_d,0}(z,s) \mu_{\mathrm{hyp}}(z) = \int_{\Gamma_1(N)\backslash\mathbb{H}} F(z) E_{\infty,0}(z,s) \mu_{\mathrm{hyp}}(z). \tag{3.4}$$

Proof. We choose $\sigma_0^{-1} = \frac{1}{\sqrt{N}} W_N \in \mathrm{SL}_2(\mathbb{R})$ with $W_N = \left(\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right)$ the Atkin-Lehner involution and $\sigma_d^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c \equiv 0 \mod N$. Then, σ_0 and σ_d are scaling matrices of the cusps 0 and ∞_d, respectively, i.e., we have $\sigma_0 \infty = 0$ as well as $\sigma_0^{-1} \Gamma_1(N) \sigma_0 = \Gamma_1(N)$ and $\sigma_0^{-1} \Gamma_1(N)_0 \sigma_0 = \Gamma_1(N)_\infty$ and the same for σ_d. Hence it follows from the definitions that $E_{0,0}(z,s) = E_{\infty,0}(\sigma_0^{-1} z, s)$ and $E_{\infty_d,0}(z,s) = E_{\infty,0}(\sigma_d^{-1} z, s)$. Therefore, it suffices for the proof of equations (3.3) and (3.4) to show $F(z) = F(\sigma_0^{-1} z)$ and $F(z) = F(\sigma_d^{-1} z)$, which we do now starting with the first equality.
The Atkin-Lehner involution W_N acts on the space $S_2(\Gamma_1(N))$ of holomorphic cusp forms of weight 2 with respect to $\Gamma_1(N)$ by

$$f|_{W_N}(z) := \det(W_N)(-Nz)^{-2} f(W_N z) = N(Nz)^{-2} f(W_N z),$$

$f \in S_2(\Gamma_1(N))$, and we have $|f\,|_{W_N}|_{W_N}\,| = |f|$ for $f \in S_2(\Gamma_1(N))$ (see [3], proposition 1.1), from which we can deduce that $\{f_j|_{W_N}\}_{j=1}^{g_N}$ remains an orthonormal basis, and so

$$\sum_{j=1}^{g_N} |f_j(z)|^2 = \sum_{j=1}^{g_N} |f_j|_{W_N}(z)|^2. \tag{3.5}$$

Using equation (3.5) we calculate

$$F(\sigma_0^{-1}z) = \frac{1}{g_N}\sum_{j=1}^{g_N} \mathrm{Im}(W_N z)^2 |f_j(W_N z)|^2 = \frac{1}{g_N}\frac{N^2}{|Nz|^4}\sum_{j=1}^{g_N} y^2 |f_j(W_N z)|^2$$
$$= \frac{1}{g_N}\sum_{j=1}^{g_N} y^2 |f_j|_{W_N}(z)|^2 = \frac{1}{g_N}\sum_{j=1}^{g_N} y^2|f_j(z)|^2 = F(z).$$

This proves the first equality. For the second equality, note that the space $S_2(\Gamma_1(N))$ of cusp forms of weight 2 decomposes in

$$S_2(\Gamma_1(N)) = \bigoplus_{\epsilon} S_2(\Gamma_1(N), \epsilon),$$

where ϵ runs through all Dirichlet characters mod N (see [6]). Thereby, we define

$$S_2(\Gamma_1(N), \epsilon) := \{f \in S_2(\Gamma_1(N)) \mid f|_2\langle d\rangle = \epsilon(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times\}$$

denoting by $\langle d\rangle$ the diamond operator given by $f|_2\langle d\rangle(z) := \frac{1}{(cz+d)^2} f(\gamma z)$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d' \end{smallmatrix}\right) \in \Gamma_0(N)$ with $d' \equiv d \mod N$. But since $f_j|_2\langle d\rangle = \epsilon(d) f_j$ for some Dirichlet character ϵ, the set $\{f_j|_2\langle d\rangle\}_{j=1}^{g_N}$ remains an orthonormal basis of $S_2(\Gamma_1(N))$. Further, we have $\mathrm{Im}(\sigma_d^{-1}z)^2|f_j(\sigma_d^{-1}z)|^2 = \mathrm{Im}(z)^2|f_j(z)|^2$ showing $F(\sigma_d^{-1}z) = F(z)$. This completes the proof of the lemma. □

3.6. Lemma. *Let N satisfy $N = 11$ or $N \geq 13$. Let $X(\Gamma_1(N))$ be the modular curve associated to the congruence subgroup $\Gamma_1(N)$ and $0, \infty_d$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ the cusps having representatives $(0:1)$ and $(d:0)$ in $\mathbb{P}^1_{\mathbb{Q}}$, respectively. With the notation $\infty = \infty_1$, we then have that $\varphi_{0\infty_d}(s) = \varphi_{0\infty}(s)$ holds for all $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.*

Proof. Noting that $g_0^{-1}\Gamma_1(N)_\infty\, g_0 = \langle\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)\rangle$ and $g_d^{-1}\Gamma_1(N)_\infty\, g_d = \langle\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ with $g_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $g_d = \left(\begin{smallmatrix} a & b \\ N & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have to show by formula (3.1) that the number of elements of the sets

$$S_d(c) := \left\{ \begin{pmatrix} \alpha & \beta \\ c & \delta \end{pmatrix} \in \left\langle \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \right\rangle \Big\backslash g_0^{-1}\Gamma_1(N) g_d \Big/ \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \right\}$$

is independent of d. To this end, we consider the map $\psi : S_d(c) \longrightarrow (\mathbb{Z}/c\mathbb{Z})^\times$ induced by mapping $\left(\begin{smallmatrix} \alpha & \beta \\ c & \delta \end{smallmatrix}\right) \mapsto \delta \mod c$. Since $(c, \delta) = 1$ and the right action by $\langle\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ changes δ only by $\mod c$, the map ψ is well-defined. We now show that ψ is bijective starting from showing that ψ is injective. This will prove the lemma.

Let $\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ c & \delta_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \alpha_2 & \beta_2 \\ c & \delta_2 \end{smallmatrix}\right)$ be two representatives of elements of $S_d(c)$ such that $\delta_1 \equiv \delta_2 \mod c$, i.e., $\delta_2 = \delta_1 + nc$ for some $n \in \mathbb{Z}$. By the right action of $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ on $\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ c & \delta_1 \end{smallmatrix}\right)$, we obtain $\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ c & \delta_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_1 & \star \\ c & \delta_2 \end{smallmatrix}\right)$. From this we conclude

$\alpha_1\delta_1 \equiv \alpha_2\delta_2 \mod c$, i.e., $c|(\alpha_2-\alpha_1)$. Furthermore, being elements of $S_d(c)$, we have $N|\alpha_1$ and $N|\alpha_2$; since $(c,N)=1$, we find that $cN|(\alpha_2-\alpha_1)$, which shows $\alpha_2 = \alpha_1 + mcN$ for some $m \in \mathbb{Z}$. By the left action of $\left(\begin{smallmatrix}1 & mN\\ 0 & 1\end{smallmatrix}\right)$ on $\left(\begin{smallmatrix}\alpha_1 & \star\\ c & \delta_2\end{smallmatrix}\right)$, we obtain $\left(\begin{smallmatrix}1 & mN\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_1 & \star\\ c & \delta_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}\alpha_2 & \star\\ c & \delta_2\end{smallmatrix}\right)$, which proves the injectivity of ψ.

We now show the surjectivity of ψ. Let $\delta \mod c$ be given in $(\mathbb{Z}/c\mathbb{Z})^\times$; we let $\delta \in \mathbb{Z}$ be a representative satisfying $(c,\delta)=1$. We have to find $\alpha, \beta \in \mathbb{Z}$ such that the representative $\left(\begin{smallmatrix}\alpha & \beta\\ c & \delta\end{smallmatrix}\right)$ satisfies $\alpha \equiv 0 \mod N$, $\beta \equiv d \mod N$, and $\alpha\delta - \beta c = 1$, which, in fact, implies $g_0\left(\begin{smallmatrix}\alpha & \beta\\ c & \delta\end{smallmatrix}\right)g_d^{-1} \in \Gamma_1(N)$, as desired. The first and second condition forces us to choose α and β of the form $\alpha = xN$ and $\beta = d + yN$ with $x, y \in \mathbb{Z}$; we have to verify that there are $x, y \in \mathbb{Z}$ such that also the third condition is satisfied. The three conditions imply $-dc \equiv 1 \mod N$; hence, we find $v \in \mathbb{Z}$ with $vN - dc = 1$. Now, since $(c,\delta)=1$, there are $x, y \in \mathbb{Z}$ such that $xd - yc = v$. With this choice for $x, y \in \mathbb{Z}$, we find $\alpha\delta - \beta c = 1$. This completes the proof of the lemma. □

3.7. *Rankin-Selberg transform.* Let f be a $\Gamma_1(N)$-invariant function of rapid decay at the cusp $\mathfrak{a}$, i.e., the 0-th Fourier coefficient $a_0(y;\mathfrak{a})$ of the Fourier expansion $f(\sigma_\mathfrak{a} z) = \sum_{n\in\mathbb{Z}} a_n(y;\mathfrak{a})\exp(2\pi i n x)$ of f at the cusp $\mathfrak{a}$ satisfies $a_0(y;\mathfrak{a}) = O(y^{-M})$ for all $M > 0$ as $y \to \infty$. Then, the Rankin-Selberg transform $R_{f,\mathfrak{a}}(s)$ of f at the cusp $\mathfrak{a}$ is defined by

$$R_{f,\mathfrak{a}}(s) := \int_{\Gamma_1(N)\backslash\mathbb{H}} f(\sigma_\mathfrak{a} z)E_{\mathfrak{a},0}(z,s)\mu_{\mathrm{hyp}}(z) = \int_0^{+\infty} a_0(y;\mathfrak{a})y^{s-2}\mathrm{d}y$$

for $\mathrm{Re}(s) > 1$. The Rankin-Selberg transform $R_{f,\mathfrak{a}}(s)$ of f at the cusp $\mathfrak{a}$ can be continued meromorphically to the whole s-plane and has simple poles at $s = 0, 1$ with residue at $s = 1$ given by

$$\mathrm{res}_{s=1}\big(R_{f,\mathfrak{a}}(s)\big) = \frac{1}{v_N}\int_{\Gamma_1(N)\backslash\mathbb{H}} f(z)\mu_{\mathrm{hyp}}(z).$$

(see, e.g., [10], p. 9). Applying the Rankin-Selberg transform to the function $F(z)$ defined in (2.21), which is of rapid decay at all cusps, we have the Laurent expansion at $s = 1$, writing $R_F(s) := R_{F,\infty}(s)$,

$$R_F(s) = \frac{1}{v_N}\frac{1}{s-1} + C_F + O(s-1), \tag{3.6}$$

denoting by C_F the constant term of this expansion.

3.8. Proposition. *Let N satisfy $N = 11$ or $N \geq 13$. Let $X\big(\Gamma_1(N)\big)$ be the modular curve associated to the congruence subgroup $\Gamma_1(N)$ and $0, \infty_d$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, the cusps having representatives $(0:1)$ and $(d:0)$, respectively. Then, we have*

$$g_{\mathrm{can}}(0,\infty_d) = 4\pi C_F - 2\pi\lim_{s\to1}\left(\varphi_{0\infty}(s) - \frac{1}{v_N}\frac{1}{s-1}\right) + O\left(\frac{1}{g_N}\right),$$

where the error term is independent of d.

Proof. This follows from proposition 3.4, lemma 3.5, and lemma 3.6, using the notation of 3.7. □

3.9. Remark. In the next section we follow a strategy to determine the constant C_F which was carried out by A. Abbes and E. Ullmo for the modular curve $X_0(N)$, N squarefree and not divisible by 2 and 3 in [1] based on ideas of D. Zagier's proof of the Selberg trace formula in [31]: we will compute the Rankin-Selberg transforms, denoted by

$$R_H(t,s),\ \ R_P(t,s),\ \ R_C(t,s),\ \ R_D(t,s) \tag{3.7}$$

of all terms displayed in (2.19) and determine their constant terms in their Laurent expansions at $s = 1$. Letting t tend to infinity, the contribution of the discrete part $R_D(t,s)$ will vanish, and so we obtain the constant C_F by formula (2.20).
It might be interesting to look at the problem to determine C_F from the adelic point of view starting with [14].

4. Contribution of Rankin-Selberg: hyperbolic part

In this section we calculate the contribution of the Rankin-Selberg transform $R_H(t,s)$ in terms of the Selberg zeta function.

4.1. Rankin-Selberg of hyperbolic part. We begin with calculating the Rankin-Selberg transforms of the $\Gamma_1(N)$-invariant functions (see (2.15) at the end of section 3.5)

$$H_{k,l}(t,z) = \sum_{\substack{\gamma\in\Gamma_1(N)\\ \mathrm{tr}(\gamma)=l}} K_{k,\gamma}(t,z) \qquad (t>0; k=0,2)$$

for $l \in \mathbb{Z}$ with $|l| > 2$.
We first note that the Rankin-Selberg transforms of these functions exist for $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < 1 + A$ and A as in 2.7. This can be shown mutatis mutandis as in [1], proposition 3.2.1: one can reduce the question to show

$$\sum_{|l|>2}\int_2^{+\infty}\int_{-1/2}^{1/2}\sum_{\substack{\gamma\in\Gamma_1(N)\\ \mathrm{tr}(\gamma)=l}} |K_{0,\gamma}(t,z)|y^{\mathrm{Re}(s)-2}\mathrm{d}x\mathrm{d}y < \infty$$

for $1 < \mathrm{Re}(s) < 1 + A$, and the claim follows then from [1], lemma 3.2.1.

4.1.1. Elements of $\Gamma_1(N)$ give rise to quadratic forms. Let us briefly discuss this link. Therefore, we use the convention to write $[a,b,c]$ for an (integral binary) quadratic form $q(X,Y) = aX^2 + bXY + cY^2 \in \mathbb{Z}[X,Y]$. Let Q_l be the set of quadratic forms with discriminant $\operatorname{disc}(q) = l^2 - 4$. The modular group $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_1(1)$ acts on the set of quadratic forms Q_l via

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) \times Q_l &\longrightarrow Q_l \\ (\delta, q) &\mapsto q \circ \delta, \end{aligned}$$

where $(q \circ \delta)(X, Y) := q\left((X, Y)\delta^t\right)$ with δ^t the transpose of δ. For $q = [a, b, c] \in Q_l$ and $\delta = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the quadratic form $q \circ \delta$ is explicitly given by

$$q \circ \delta = [q(x, z), b(xt + yz) + 2(axy + czt), q(y, t)]. \tag{4.1}$$

4.1.2. Definition. For a positive integer N and $l \in \mathbb{Z}$ with $|l| > 2$, we define

$$Q_l(N) := \{q = [aN, bN, c] \mid a, b, c \in \mathbb{Z}; \operatorname{disc}(q) = l^2 - 4\} \subseteq Q_l$$

and

$$\Gamma_{1,l}(N) = \{\gamma \in \Gamma_1(N) \mid \operatorname{tr}(\gamma) = l\} \subseteq \Gamma_1(N).$$

4.1.3. Let N be a positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$. We have a map $\psi : \Gamma_{1,l}(N) \longrightarrow Q_l(N)$ defined by

$$\gamma = \begin{pmatrix} 1 + aN & b \\ cN & 1 + dN \end{pmatrix} \mapsto q_\gamma := [cN, (d - a)N, -b]. \tag{4.2}$$

Supposed that N is odd, the map ψ defines a bijection between the sets $\Gamma_{1,l}(N)$ and $Q_l(N)$ as one verifies that the map $\psi' : Q_l(N) \longrightarrow \Gamma_{1,l}(N)$ given by

$$q = [aN, bN, c] \mapsto \gamma_q := \begin{pmatrix} \frac{l - bN}{2} & -c \\ aN & \frac{l + bN}{2} \end{pmatrix}$$

is well-defined. Once this is shown one easily verifies that the two maps ψ and ψ' are inverse to each other, which establishes the claimed bijection.

4.1.4. The congruence subgroup $\Gamma_1(N)$ acts on the sets $Q_l(N)$ and $\Gamma_{1,l}(N)$ as follows: The action of $\Gamma_1(N)$ on $Q_l(N)$ is given by

$$\begin{aligned} \Gamma_1(N) \times Q_l(N) &\longrightarrow Q_l(N) \\ (\delta, q) &\mapsto q \circ \delta, \end{aligned} \tag{4.3}$$

where equation (4.1) shows that this action is well-defined; the action of $\Gamma_1(N)$ on $\Gamma_{1,l}(N)$ is given by conjugation

$$\begin{aligned} \Gamma_1(N) \times \Gamma_{1,l}(N) &\longrightarrow \Gamma_{1,l}(N) \\ (\delta, \gamma) &\mapsto \delta \cdot \gamma := \delta^{-1}\gamma\delta. \end{aligned} \tag{4.4}$$

Let N be an odd positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$. Then the diagram

$$\begin{array}{ccc} \Gamma_1(N) \times \Gamma_{1,l}(N) & \longrightarrow & \Gamma_{1,l}(N) \\ \downarrow{\scriptstyle \mathrm{id}\times\psi} & & \downarrow{\scriptstyle \psi} \\ \Gamma_1(N) \times Q_l(N) & \longrightarrow & Q_l(N) \end{array}$$

commutes, where the horizontal maps are the group actions (4.3), (4.4), and the vertical map ψ is given by (4.2). In particular, we have a bijection

$$\Gamma_{1,l}(N)/\Gamma_1(N) \cong Q_l(N)/\Gamma_1(N).$$

4.1.5. Let N be an odd positive integer. For $0 < u < N$ with $(u, N) = 1$, we set

$$M_u(N) := \left\{(m,n) \in \mathbb{Z}^2 \,|\, (m,n) \equiv (0,u) \mod N\right\}.$$

Note that the congruence subgroup $\Gamma_1(N)$ acts on $M_u(N)$ by

$$\begin{aligned} \Gamma_1(N) \times M_u(N) &\longrightarrow M_u(N) \\ (\delta, (m,n)) &\mapsto (m,n)\delta. \end{aligned}$$

Furthermore, for $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$, we set $\Delta_l^u(N) := \Gamma_{1,l}(N) \times M_u(N)$ and define

$$\Delta_l^{u\pm}(N) := \left\{\left(\gamma, (m,n)\right) \in \Delta_l^u(N) \,|\, q_\gamma(n,-m) \gtrless 0\right\}.$$

Note that $\Delta_l^u(N) = \Delta_l^{u+}(N) \,\dot\cup\, \Delta_l^{u-}(N)$, and that there is an action of the congruence subgroup $\Gamma_1(N)$ on $\Delta_l^u(N)$ defined by

$$\begin{aligned} \Gamma_1(N) \times \Delta_l^u(N) &\longrightarrow \Delta_l^u(N) \\ \left(\delta, \left(\gamma, (m,n)\right)\right) &\mapsto \delta \cdot \left(\gamma, (m,n)\right) := \left(\delta^{-1}\gamma\delta, (m,n)\delta\right), \end{aligned}$$

which preserves the subsets $\Delta_l^{u+}(N), \Delta_l^{u-}(N) \subseteq \Delta_l^u(N)$. Observing that

$$(q \circ \delta) \cdot ((m,n)\delta) = q \cdot (m,n)$$

using the notation $q \cdot (m,n) := q(n,-m)$ as in [30], we make the following

4.1.6. Definition. Let N be an odd positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$. For $0 < u < N$ with $(u,N) = 1$, we define for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, the zeta function

$$\zeta_{u,N}(s,l) = \sum_{(\gamma,(m,n)) \in \Delta_l^{u+}(N)/\Gamma_1(N)} \frac{1}{q_\gamma(n,-m)^s}. \tag{4.5}$$

We show in the appendix that this zeta function is well-defined for $\mathrm{Re}(s) > 1$. Finally, we denote for any $l \in \mathbb{Z}$ by γ_l the matrix $\gamma_l := \begin{pmatrix} \frac{l}{2} & \frac{l^2}{4}-1 \\ 1 & \frac{l}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

4.1.7. Proposition. *Let N be an odd positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$. For $s \in \mathbb{C}$ satisfying $1 < \operatorname{Re}(s) < 1 + A$, where A is as in 2.7, and $t > 0$, we have*

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} H_{k,l}(t,z) E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) = I_{k,l}(t,s) \sum_{\substack{0<u<N \\ (u,N)=1}} c_u(s)\zeta_{u,N}(s,l),$$

where we set

$$I_{k,l}(t,s) := \int_{\mathbb{H}} K_{k,\gamma_l}(t,z) y^s \mu_{\mathrm{hyp}}(z) + \int_{\mathbb{H}} K_{k,\gamma_{-l}}(t,z) y^s \mu_{\mathrm{hyp}}(z) \qquad (k = 0, 2)$$

and $c_u(s) := \sum_{\substack{d>0 \\ du \equiv 1 \bmod N}} \frac{\mu(d)}{d^{2s}}$ *denoting by $\mu(d)$ the Moebius function.*

Proof. First note that

$$E_{\infty,0}(z,s) = \sum_{\Gamma_1(N)_\infty\backslash\Gamma_1(N)} \operatorname{Im}(\gamma z)^s = \sum_{\left(\begin{smallmatrix} * & * \\ m & n \end{smallmatrix}\right)\in\Gamma_1(N)} \frac{y^s}{|mz+n|^{2s}}$$
$$= \sum_{\substack{(m,n)\equiv(0,1) \mod N \\ (m,n)=1}} \frac{y^s}{|mz+n|^{2s}}.$$

In order to write the Eisenstein series without the coprimality condition, we define $\zeta_N(s) := \sum_{\substack{d>0 \\ (d,N)=1}} \frac{1}{d^s}$, $\operatorname{Re}(s) > 1$, and observe that

$$\zeta_N(2s) \sum_{\substack{(m,n)\equiv(0,1) \bmod N \\ (m,n)=1}} \frac{y^s}{|mz+n|^{2s}} = \sum_{\substack{d>0 \\ (d,N)=1}} \sum_{\substack{(m,n)\equiv(0,1) \bmod N \\ (m,n)=1}} \frac{y^s}{|dmz+dn|^{2s}} =$$
$$\sum_{\substack{d>0 \\ (d,N)=1}} \sum_{\substack{(m',n')\equiv(0,d) \bmod N \\ (m',n')=d}} \frac{y^s}{|m'z+n'|^{2s}} = \sum_{\substack{0<u<N \\ (u,N)=1}} \sum_{\substack{(m',n')\in\mathbb{Z}^2 \\ (m',n')\equiv(0,u) \bmod N}} \frac{y^s}{|m'z+n'|^{2s}}.$$

Multiplying by $\zeta_N(2s)^{-1} = \sum_{\substack{d>0 \\ (d,N)=1}} \frac{\mu(d)}{d^{2s}}$ and writing (m,n) instead of (m',n') yields

$$E_{\infty,0}(z,s) = \sum_{\substack{0<u<N \\ (u,N)=1}} c_u(s) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\equiv(0,u) \bmod N}} \frac{y^s}{|mz+n|^{2s}}.$$

Now, since $\Gamma_1(N)$ acts freely on $\Delta_l^u(N)$ and $\Delta_l^u(N) = \Delta_l^{u+}(N) \,\dot\cup\, \Delta_l^{u-}(N)$, we obtain

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} H_{k,l}(t,z) E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) = \tag{4.6}$$

$$\sum_{\substack{0<u<N \\ (u,N)=1}} c_u(s) \sum_{(\gamma,(m,n))\in\Delta_l^u(N)/\Gamma_1(N)} \int_{\mathbb{H}} K_{k,\gamma}(t,z) \frac{y^s}{|mz+n|^{2s}} \mu_{\mathrm{hyp}}(z) =$$

$$\sum_{\substack{0<u<N \\ (u,N)=1}} c_u(s) \Bigg(\sum_{(\gamma,(m,n))\in\Delta_l^{u+}(N)/\Gamma_1(N)} \int_{\mathbb{H}} K_{k,\gamma}(t,z) \frac{y^s}{|mz+n|^{2s}} \mu_{\mathrm{hyp}}(z) +$$

$$\sum_{(\gamma,(m,n))\in\Delta_l^{u-}(N)/\Gamma_1(N)} \int_{\mathbb{H}} K_{k,\gamma}(t,z) \frac{y^s}{|mz+n|^{2s}} \mu_{\mathrm{hyp}}(z) \Bigg).$$

For $(\gamma,(m,n)) \in \Delta_l^{u+}(N)$ with $\gamma = \begin{pmatrix} 1+aN & b \\ cN & 1+dN \end{pmatrix}$, we define the matrix

$$M := \frac{1}{q_\gamma(n,-m)^{\frac{1}{2}}} \begin{pmatrix} n & -(d-a)N\frac{n}{2} - bm \\ -m & cNn - (d-a)N\frac{m}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

One computes $\frac{\mathrm{Im}(Mz)^s}{|mMz+n|^{2s}} = \frac{y^s}{q_\gamma(n,-m)^s}$ (see [30], p. 127) and an elementary calculation proves the equality $M^{-1}\gamma M = \gamma_l$. Recalling that $K_{k,\gamma}(t,\delta z) = K_{k,\delta^{-1}\gamma\delta}(t,z)$ $(k=0,2)$ for any $\delta \in \mathrm{SL}_2(\mathbb{R})$, we obtain by the change of variable $z \mapsto Mz$

$$\int_{\mathbb{H}} K_{k,\gamma}(t,z) \frac{y^s}{|mz+n|^{2s}} \mu_{\mathrm{hyp}}(z) = \frac{1}{q_\gamma(n,-m)^s} \int_{\mathbb{H}} K_{k,\gamma_l}(t,z) y^s \mu_{\mathrm{hyp}}(z),$$

and hence

$$\sum_{\Delta_l^{u+}(N)/\Gamma_1(N)} \int_{\mathbb{H}} K_{k,\gamma}(t,z) \frac{y^s}{|mz+n|^{2s}} \mu_{\mathrm{hyp}}(z) = \tag{4.7}$$

$$\zeta_{u,N}(s,l) \int_{\mathbb{H}} K_{k,\gamma_l}(t,z) y^s \mu_{\mathrm{hyp}}(z).$$

For $(\gamma,(m,n)) \in \Delta_l^{u-}(N)$ we define the matrix

$$M' := \frac{1}{q_{-\gamma}(n,-m)^{\frac{1}{2}}} \begin{pmatrix} n & (d-a)N\frac{n}{2} + bm \\ -m & -cNn + (d-a)N\frac{m}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and one verifies again $\frac{\mathrm{Im}(M'z)^s}{|mM'z+n|^{2s}} = \frac{y^s}{q_{-\gamma}(n,-m)^s}$ as well as $M'^{-1}(-\gamma)M' = \gamma_{-l}$. Since we have $K_{k,\gamma}(t,z) = K_{k,-\gamma}(t,z)$, the change of variable $z \mapsto M'z$

implies

$$\int_{\mathbb{H}} K_{k,\gamma}(t,z)\frac{y^s}{|mz+n|^{2s}}\mu_{\mathrm{hyp}}(z) = \int_{\mathbb{H}} K_{k,-\gamma}(t,z)\frac{y^s}{|mz+n|^{2s}}\mu_{\mathrm{hyp}}(z)$$
$$= \frac{1}{q_{-\gamma}(n,-m)^s}\int_{\mathbb{H}} K_{k,\gamma_{-l}}(t,z)y^s\mu_{\mathrm{hyp}}(z).$$

Observing that $q_{-\gamma} = q_{\gamma^{-1}}$ and $q_{\gamma^{-1}} = -q_\gamma$, we find

$$\sum_{(\gamma,(m,n))\in\Delta_l^{u-}(N)/\Gamma_1(N)} \int_{\mathbb{H}} K_{k,\gamma}(t,z)\frac{y^s}{|mz+n|^{2s}}\mu_{\mathrm{hyp}}(z) = \tag{4.8}$$
$$\sum_{(\gamma,(m,n))\in\Delta_l^{u-}(N)/\Gamma_1(N)} \frac{1}{q_{\gamma^{-1}}(n,-m)^s}\int_{\mathbb{H}} K_{k,\gamma_{-l}}(t,z)y^s\mu_{\mathrm{hyp}}(z) =$$
$$\sum_{(\gamma,(m,n))\in\Delta_l^{u+}(N)/\Gamma_1(N)} \frac{1}{q_{\gamma}(n,-m)^s}\int_{\mathbb{H}} K_{k,\gamma_{-l}}(t,z)y^s\mu_{\mathrm{hyp}}(z) =$$
$$\zeta_{u,N}(s,l)\int_{\mathbb{H}} K_{k,\gamma_{-l}}(t,z)y^s\mu_{\mathrm{hyp}}(z).$$

Equation (4.6) together with equations (4.7) and (4.8) proves the proposition. □

4.2. Hyperbolic contribution and Selberg zeta function. In this section we compute the constant term in the Laurent expansion at $s = 1$ of the hyperbolic contribution $R_H(t,s)$, i.e., by proposition 4.1.7, of

$$R_H(t,s) = \sum_{\substack{|l|>2 \\ l\equiv 2 \mod N}} (I_{2,l}(t,s) - I_{0,l}(t,s)) \sum_{\substack{0<u<N \\ (u,N)=1}} c_u(s)\zeta_{u,N}(s,l) \qquad (t>0).$$

4.2.1. Let N be an odd positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$. We denote by $h_l(N)$ the cardinality of the set $Q_l(N)/\Gamma_1(N)$. The finiteness of $h_l(N)$ follows from the finiteness of the class number $h_l(1)$ of properly equivalent quadratic forms with discriminant $l^2 - 4$.
Note that for a quadratic form $q = [aN, bN, c] \in Q_l(N)$ the stabilizer $\Gamma_1(N)_q = \mathrm{SL}_2(\mathbb{Z})_q \cap \Gamma_1(N)$ of $q \in Q_l(N)$ is an infinite cyclic group generated by $\alpha_q := \alpha_0^k$, where α_0 is the generator of $\mathrm{SL}_2(\mathbb{Z})_q$ and k is the least positive integer such that $\alpha_0^k \in \Gamma_1(N)$. We denote by ε_q the eigenvalue of α_q with $\varepsilon_q^2 > 1$ which is, in fact, the k-th power of the fundamental unit ε_0 in the real quadratic field $\mathbb{Q}(\sqrt{l^2-4})$. In particular, ε_q is independent of the choice q in $Q_l(N)$. Let us also mention that observation 4.1.4 implies $\Gamma_1(N)_q = Z(\gamma_q)$, where $Z(\gamma_q)$ is the centralizer of γ_q in $\Gamma_1(N)$.

4.2.2. Proposition. *Let N be an odd positive integer, $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$, and $0 < u < N$ with $(u,N) = 1$. Then, the zeta function $\zeta_{u,N}(s,l)$ defines for $\mathrm{Re}(s) > 1$ a holomorphic function and has a* *meromorphic continuation to the whole complex plane with a simple pole at* $s = 1$ *with residue*

$$\mathrm{res}_{s=1}\zeta_{u,N}(s,l) = \sum_{q\in Q_l(N)/\Gamma_1(N)} \frac{2}{N^2\sqrt{l^2-4}}\log(\varepsilon_q).$$

We prove this proposition in the appendix which is a slight variant of a result of E. Landau in [20].

4.2.3. Corollary. *Let* N *be an odd positive integer and* $l \in \mathbb{Z}$ *with* $|l| > 2$ *and* $l \equiv 2 \mod N$*. Then the series*

$$\sum_{\substack{0<u<N\\(u,N)=1}} c_u(s)\zeta_{u,N}(s,l)$$

has a meromorphic continuation to the half plane $\mathrm{Re}(s) > 1/2$ *with a simple pole at* $s = 1$ *and residue*

$$\mathrm{res}_{s=1}\sum_{\substack{0<u<N\\(u,N)=1}} c_u(s)\zeta_{u,N}(s,l) = \frac{2}{\pi v_N}\sum_{q\in Q_l(N)/\Gamma_1(N)}\frac{\log(\varepsilon_q)}{\sqrt{l^2-4}}.$$

Proof. Note that $c_u(s)$ with $0 < u < N$ and $(u,N) = 1$ is holomorphic for $\mathrm{Re}(s) > 1/2$. Therefore, by proposition 4.2.2, we have a meromorphic continuation of $\sum_{\substack{0<u<N\\(u,N)=1}} c_u(s)\zeta_{u,N}(s,l)$ to the half plane $\mathrm{Re}(s) > 1/2$ and

(4.9)

$$\mathrm{res}_{s=1}\sum_{\substack{0<u<N\\(u,N)=1}} c_u(s)\zeta_{u,N}(s,l) = \sum_{\substack{0<u<N\\(u,N)=1}}\frac{c_u(1)}{N^2}\sum_{q\in Q_l(N)/\Gamma_1(N)}\frac{2}{\sqrt{l^2-4}}\log(\varepsilon_q).$$

We have

(4.10)

$$\sum_{\substack{0<u<N\\(u,N)=1}} c_u(1) = \sum_{\substack{d>0\\(d,N)=1}}\frac{\mu(d)}{d^2} = \frac{1}{\zeta_N(2)} = \frac{1}{\zeta(2)}\prod_{p|N}\frac{1}{1-\frac{1}{p^2}} = \frac{6}{\pi^2}\prod_{p|N}\frac{1}{1-\frac{1}{p^2}}.$$

Plugging equation (4.10) in (4.9) and taking the formula (2.4) into account, the claim of the corollary follows immediately. □

Recall that we defined in section 3.3 the test function $h(t,r)$ to be the function $h(t,r) = \exp\left(-t\left(\frac{1}{4}+r^2\right)\right)$ $(t > 0, r \in \mathbb{R})$ with Fourier transform given by

$$g(t,w) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{t}{4}-\frac{w^2}{4t}\right) \qquad (t > 0, w \in \mathbb{R}). \tag{4.11}$$

4.2.4. Lemma. *For any $t > 0$, we have the Laurent expansion*

$$I_{2,l}(t,s) - I_{0,l}(t,s) = (s-1)A_l(t) + O_t\Big((s-1)^2\Big) \qquad (s \to 1),$$

where

$$A_l(t) := -\frac{\pi}{2n_l} + \frac{1}{4}\int_{-\infty}^{+\infty} \frac{h(t,r)}{\frac{1}{4}+r^2}\exp\big(-2ir\log(n_l)\big)\mathrm{d}r$$

with $n_l := \frac{l+\sqrt{l^2-4}}{2}$ and n_l^{-1} the eigenvalues of γ_l. Furthermore, we have

$$|A_l(t)| \leq \frac{\pi}{2(\log(n_l))^2} n_l^{-\log(n_l)/t}.$$

Proof. For the first statement see [1], proposition 3.2.2. For the second statement, we first observe that

(4.12)

$$A_l(t) = \frac{-\pi}{2n_l} + \frac{1}{4}\int_{-\infty}^{+\infty} \frac{h(t,r)}{\frac{1}{4}+r^2}\exp(-2ir\log(n_l))\mathrm{d}r = \frac{-\pi}{2}\int_0^t g(\xi, 2\log(n_l))\mathrm{d}\xi.$$

By definition, we have

$$\frac{1}{2}\int_0^t g(\xi, 2\log(n_l))\mathrm{d}\xi = \frac{1}{2}\int_0^t \frac{1}{\sqrt{4\pi\xi}}\exp\Big(-\frac{\xi}{4} - \frac{(\log(n_l))^2}{\xi}\Big)\,\mathrm{d}\xi \qquad \in \mathbb{R}_{>0},$$

and the change of variable $x := \frac{1}{\xi}$ yields then

$$A_l(t) = \frac{-\pi}{2}\int_{1/t}^{+\infty} \frac{\sqrt{x}}{\sqrt{4\pi}x^2}\exp\Big(-\frac{1}{4x} - (\log(n_l))^2 x\Big)\,\mathrm{d}x.$$

Since $\frac{\sqrt{x}}{\sqrt{4\pi}x^2}\exp\Big(-\frac{1}{4x}\Big) < 1$ for $x > 0$, we find

$$0 \geq A_l(t) \geq \frac{-\pi}{2}\int_{1/t}^{+\infty}\exp\Big(-(\log(n_l))^2 x\Big)\,\mathrm{d}x = \frac{-\pi}{2(\log(n_l))^2} n_l^{-\log(n_l)/t},$$

from which the claimed bound follows. □

4.2.5. Proposition. *For any $t > 0$, the function $R_H(t,s)$ is holomorphic at $s = 1$. Furthermore, we have with $n_l = \frac{l+\sqrt{l^2-4}}{2}$*

$$R_H(t,1) = \frac{1}{\pi v_N} \sum_{\substack{|l|>2 \\ l \equiv 2 \mod N}} \sum_{q \in Q_l(N)/\Gamma_1(N)} \frac{2}{\sqrt{l^2-4}} \log(\varepsilon_q) \times$$
$$\left(-\frac{\pi}{2n_l} + \frac{1}{4}\int_{-\infty}^{+\infty} \frac{h(t,r)}{\frac{1}{4}+r^2}\exp\big(-2ir\log(n_l)\big)\mathrm{d}r\right).$$

Proof. It suffices to prove that $R_H(t,s)$ is bounded at $s=1$. From corollary 4.2.3 and lemma 4.2.4 we find the inequality

$$\left|(I_{2,l}(t,1)-I_{0,l}(t,1))\sum_{\substack{0<u<N,\\(u,N)=1}} c_u(1)\zeta_{N,u}(1,l)\right| \le \frac{n_l^{-\log(n_l)/t}}{2(\log(n_l))^2}\frac{2h_l(N)\log(\varepsilon_q)}{v_N\sqrt{l^2-4}}.$$

We first note that $h_l(N)$ can be bounded by the classical class number $h_l(1)$ by $h_l(N)\le h_l(1)\,[\mathrm{SL}_2(\mathbb{Z}):\Gamma_1(N)]$ and that $\varepsilon_0^k=\varepsilon_q$ holds for some $0\le k\le\varphi(N)$. Since by Siegel's theorem (see [32], p. 85)

$$\sum_{0<D<l^2-4}\frac{h_D\log(\varepsilon_0)}{\sqrt{D}}=O(l^2)\qquad(l\to\infty),$$

we obtain

(4.13)

$$\left|\sum_{\substack{|l|>2\\ l\equiv 2 \mod N}}(I_{2,l}(t,1)-I_{0,l}(t,1))\sum_{\substack{0<u<N\\(u,N)=1}} c_u(1)\zeta_{N,u}(1,l)\right| \le C_N\sum_{|l|>2}\frac{l^2 n_l^{-\log(n_l)/t}}{2(\log(n_l))^2}$$

with C_N some constant depending solely on N. As for $l>\exp(3t)$ we have

$$\frac{l^2 n_l^{-\log(n_l)/t}}{2(\log(n_l))^2}<l^{-1-\varepsilon}$$

with $\varepsilon>0$ small enough, the series on the right hand side of inequality (4.13) converges, which proves the holomorphicity of $R_H(t,s)$ at $s=1$. The claimed value of $R_H(t,s)$ at $s=1$ follows now from corollary 4.2.3 and lemma 4.2.4. □

4.2.6. Definition. Let $\gamma\in\Gamma_1(N)$ be a hyperbolic element, i.e., $|\mathrm{tr}(\gamma)|>2$. The *norm* $N(\gamma)$ *of* γ is defined by $N(\gamma):=v^2$, where v is the eigenvalue of γ with $v^2>1$. We denote by γ_0 the generator of the centralizer $Z(\gamma)$ of γ, and call γ *primitive* if $\gamma=\gamma_0$ holds.

4.2.7. The Selberg zeta function $Z_{\Gamma_1(N)}(s)$ associated to $\Gamma_1(N)$ is defined via the Euler product expansion

$$Z_{\Gamma_1(N)}(s):=\prod_{[\gamma]\in H(\Gamma_1(N))}Z_\gamma(s)\qquad(\mathrm{Re}(s)>1),$$

where $H\big(\Gamma_1(N)\big)$ denotes the set of primitive conjugacy classes of hyperbolic elements in $\Gamma_1(N)$ and the local factors $Z_\gamma(s)$ are given by

$$Z_\gamma(s):=\prod_{n=0}^{\infty}\left(1-N(\gamma)^{-(s+n)}\right).$$

The Selberg zeta function $Z_{\Gamma_1(N)}(s)$ is known to have a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation (cf. [13], section 10.8).

4.2.8. The hyperbolic contribution $\Theta_{\Gamma_1(N)}(t)$ in the Selberg trace formula (cf. [13], theorem 10.2) is given by

$$\Theta_{\Gamma_1(N)}(t) = \sum_{\substack{[\gamma] \\ \text{hyperbolic}}} \frac{\log\left(N(\gamma_0)\right)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g(t, \log\left(N(\gamma)\right)).$$

For $\gamma \in \Gamma_{1,l}(N)$, one easily verifies, using the fact $\operatorname{tr}(\gamma) = N(\gamma)^{1/2} + N(\gamma)^{-1/2}$, the formulas

$$N(\gamma)^{1/2} - N(\gamma)^{-1/2} = \sqrt{l^2-4} \quad \text{and} \quad N(\gamma) = \left(\frac{l + \sqrt{l^2-4}}{2}\right)^2 = n_l^2.$$

Further, note that the primitive element γ_0 associated to the hyperbolic conjugacy class $[\gamma]$ generates the centralizer $Z(\gamma) = Z(\gamma_0)$ (see [13], p. 137), and hence equals the generator α_{q_γ} of the stabilizer $\Gamma_1(N)_{q_\gamma}$ of q_γ. Therefore, we have

$$\Theta_{\Gamma_1(N)}(t) = \sum_{\substack{|l|>2 \\ l\equiv 2 \mod N}} \sum_{q\in Q_l(N)/\Gamma_1(N)} \frac{2\log\left(\varepsilon_q\right)}{\sqrt{l^2-4}} g(t, 2\log(n_l)).$$

For any $t > 0$, we have by formula (4.12)

$$R_H(t,1) = -\frac{1}{2v_N}\int_0^t \Theta_{\Gamma_1(N)}(\xi)\mathrm{d}\xi. \tag{4.14}$$

4.2.9. Proposition. *The integral $\int_0^t \Theta_{\Gamma_1(N)}(\xi)\mathrm{d}\xi$ is asymptotically equivalent to t for $t \to \infty$, i.e.,*

$$\int_0^t \Theta_{\Gamma_1(N)}(\xi)\mathrm{d}\xi \sim t$$

holds. Furthermore, we have

$$\int_0^{+\infty} \left(\Theta_{\Gamma_1(N)}(t) - 1\right)\mathrm{d}t = \lim_{s\to 1}\left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1}\right) - 1.$$

Proof. Having McKean's formula (see [22], p. 239)

$$\frac{1}{2s-1}\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} = \int_0^{+\infty} \exp(-s(s-1)t)\Theta_{\Gamma_1(N)}(t)\mathrm{d}t,$$

the proof is exactly the same as the proof in [1], proposition 3.3.3, for $\Gamma_0(N)$. □

4.2.10. Corollary. *The hyperbolic contribution $R_H(t,1)$ for $t \to \infty$ is given by*

$$R_H(t,1) = -\frac{t}{2v_N} - \frac{1}{2v_N} \lim_{s\to 1} \left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1} \right) + \frac{1}{2v_N} + o(1).$$

Proof. This follows now immediately from observation (4.14) and proposition 4.2.9. □

5. Contribution of Rankin-Selberg: spectral and parabolic part

In this section we determine the contribution of the Rankin-Selberg transforms of the remaining $\Gamma_1(N)$-invariant functions $P_k(t,s)$, $C_k(t,s)$, and $D(t,s)$ $(k=0,2)$ of formula (2.20). The contribution of $P_k(t,s)$ can be taken from [1] which we cite at the end of the second section of this chapter.
First observe that we have (for suitable s)

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} P_k(t,z) E_{\infty,0}(z,s) \mu_{\mathrm{hyp}}(z) = \int_0^{+\infty} p_k(t,y) y^{s-2} \mathrm{d}y \qquad (k=0,2),$$

and

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} C_k(t,z) E_{\infty,0}(z,s) \mu_{\mathrm{hyp}}(z) = \int_0^{+\infty} c_k(t,y) y^{s-2} \mathrm{d}y \qquad (k=0,2),$$

where $p_k(t,y)$ and $c_k(t,y)$ are the 0-th Fourier coefficients of $P_k(t,z)$ and $C_k(t,z)$ with respect to the cusp ∞, respectively. Proceeding as in [1] we provide a further decomposition of the sums $p_k(t,y)+c_k(t,y)$, $k=0,2$. Recall that the 0-th Fourier coefficient of the Eisenstein series $E_{\mathfrak{a},0}(z,s)$, $\mathfrak{a} \in P_{\Gamma_1(N)}$, is given by

$$a_0(y,s;\mathfrak{a}\infty,0) = \delta_{\mathfrak{a}\infty} y^s + \varphi_{\mathfrak{a}\infty}(s) y^{1-s},$$

where $\varphi_{\mathfrak{a}\infty}(s)$ is defined as in (3.1). Then, for $s=\frac{1}{2}+ir \in \mathbb{C}$, we have

$$\text{(5.1)} \qquad \sum_{\mathfrak{a}\in P_{\Gamma_1(N)}} \left| a_0\left(y, \tfrac{1}{2}+ir; \mathfrak{a}\infty, 0\right) \right|^2 =$$
$$2y + \varphi_{\infty\infty}\left(\tfrac{1}{2}-ir\right) y^{1+2ir} + \varphi_{\infty\infty}\left(\tfrac{1}{2}+ir\right) y^{1-2ir}$$

since the scattering matrix $\Phi(s) = \left(\varphi_{\mathfrak{a}\mathfrak{b}}(s)\right)_{\mathfrak{a},\mathfrak{b}\in P_{\Gamma_1(N)}}$ is unitary for $\mathrm{Re}(s) = \frac{1}{2}$ (see [13], theorem 6.6). By observation (2.10), we find

$$\left(\tfrac{1}{2}+ir\right) a_0\left(y, \tfrac{1}{2}+ir; \mathfrak{a}\infty, 2\right) = \Lambda_0(a_0\left(y, \tfrac{1}{2}+ir; \mathfrak{a}\infty, 0\right)) =$$
$$\left(\tfrac{1}{2}+ir\right) \delta_{\mathfrak{a}\infty} y^{\frac{1}{2}+ir} + \varphi_{\mathfrak{a}\infty}\left(\tfrac{1}{2}+ir\right)\left(\tfrac{1}{2}-ir\right) y^{\frac{1}{2}-ir},$$

which implies

$$(5.2)\quad \sum_{\mathfrak{a}\in P_{\Gamma_1(N)}} \left|a_0\left(y,\tfrac{1}{2}+ir;\mathfrak{a}\infty,2\right)\right|^2 = 2y+\varphi_{\infty\infty}\left(\tfrac{1}{2}-ir\right)\frac{\frac{1}{2}+ir}{\frac{1}{2}-ir}y^{1+2ir}+ \varphi_{\infty\infty}\left(\tfrac{1}{2}+ir\right)\frac{\frac{1}{2}-ir}{\frac{1}{2}+ir}y^{1-2ir}.$$

Let us define for $k=0,2$ the functions

$$p_{k,1}(t,y) := \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{\substack{\gamma\in\Gamma_1(N)\\ |\mathrm{tr}(\gamma)|=2,\gamma\notin\Gamma_1(N)_\infty}} K_{k,\gamma}(t,x+iy)\mathrm{d}x,$$

$$p_{k,2}(t,y) := \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{\gamma\in\Gamma_1(N)_\infty} K_{k,\gamma}(t,x+iy)\mathrm{d}x - \frac{y}{2\pi}\int_{-\infty}^{+\infty} h(t,r)\mathrm{d}r$$

and

$$c_{k,1}(t,y) := -\frac{y}{2\pi}\int_{-\infty}^{+\infty} h(t,r)\varphi_{\infty\infty}\left(\tfrac{1}{2}-ir\right)\left(\frac{\frac{1}{2}+ir}{\frac{1}{2}-ir}\right)^{\frac{k}{2}} y^{2ir}\mathrm{d}r - \frac{2-k}{2}\frac{1}{v_N},$$

$$c_{k,2}(t,y) := -\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}} \int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\infty}^{+\infty} h(t,r)\left|\widetilde{E}_{\mathfrak{a},k}\left(x+iy,\tfrac{1}{2}+ir\right)\right|^2 \mathrm{d}r\,\mathrm{d}x,$$

where $\widetilde{E}_{\mathfrak{a},k}\left(x+iy,\frac{1}{2}+ir\right) := E_{\mathfrak{a},k}\left(x+iy,\frac{1}{2}+ir\right) - a_0\left(y,\frac{1}{2}+ir;\mathfrak{a}\infty,k\right)$. Using the above notation, observations (5.1) and (5.2) allow us to write

$$p_k(t,y)+c_k(t,y) = p_{k,1}(t,y)+p_{k,2}(t,y)+c_{k,1}(t,y)+c_{k,2}(t,y) \quad (k=0,2).$$

Defining the integrals

$$R_{P_{k,j}}(t,s) := \int_0^{+\infty} p_{k,j}(t,y)y^{s-2}\mathrm{d}y \qquad (k=0,2;j=1,2).$$

$$R_{C_{k,j}}(t,s) := \int_0^{+\infty} c_{k,j}(t,y)y^{s-2}\mathrm{d}y \qquad (k=0,2;j=1,2)$$

we find

$$(5.3)\quad R_P(t,s) = \left(R_{P_{2,1}}(t,s)-R_{P_{0,1}}(t,s)\right)+\left(R_{P_{2,2}}(t,s)-R_{P_{0,2}}(t,s)\right)$$

and

$$(5.4)\quad R_C(t,s) = \left(R_{C_{2,1}}(t,s)-R_{C_{0,1}}(t,s)\right)+\left(R_{C_{2,2}}(t,s)-R_{C_{0,2}}(t,s)\right).$$

5.1. Rankin-Selberg of spectral part.

5.1.1. Lemma. *For $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < A$, where A is as in 2.7, and $t > 0$, we have*

$$R_{C_{0,1}}(t,s) = -\frac{1}{2} h\left(t, \tfrac{is}{2}\right) \varphi_{\infty\infty}\left(\tfrac{1+s}{2}\right)$$

and

$$R_{C_{2,1}}(t,s) = -\frac{1}{2} h\left(t, \tfrac{is}{2}\right) \varphi_{\infty\infty}\left(\tfrac{1+s}{2}\right) \frac{1-s}{1+s}.$$

Proof. The proof is similar to the proof in [1], lemma 3.2.17 for the congruence subgroup $\Gamma_0(N)$. Since loc. cit. there is no proof in the case $k = 0$, we give for the convenience of the reader a proof here, and refer to [1] for the case $k = 2$.

The idea is to apply Mellin's inversion theorem. To this end, we consider, for a fixed $t > 0$, the function $h(t,r)\varphi_{\infty\infty}\left(\frac{1}{2} - ir\right) y^{2ir}$ as a function in the complex variable r. This function is holomorphic in the strip $0 < \mathrm{Im}(r) < \frac{A}{2}$ except for a simple pole at $r = \frac{i}{2}$, as $\varphi_{\infty\infty}\left(\frac{1}{2} - ir\right)$ has a simple pole there with residue $\frac{i}{v_N}$. Furthermore, this function tends to 0 for $\mathrm{Re}(r) \to \infty$ by the fact that $\varphi_{\infty\infty}\left(\frac{1}{2} - ir\right)$ is uniformly bounded by [12], theorem 12.9. Now, for $c \in \mathbb{R}$ such that $\frac{1}{2} < c < \frac{A}{2}$, the residue theorem implies

$$\begin{aligned}
&-\frac{y}{2\pi}\int_{-\infty}^{+\infty} h(t,r)\varphi_{\infty\infty}\left(\tfrac{1}{2} - ir\right) y^{2ir}\mathrm{d}r = \\
&-\frac{y}{2\pi}\int_{-\infty+ic}^{+\infty+ic} h(t,r)\varphi_{\infty\infty}\left(\tfrac{1}{2} - ir\right) y^{2ir}\mathrm{d}r - ih\left(t, \tfrac{i}{2}\right)\mathrm{res}_{r=\frac{i}{2}}\left(\varphi_{\infty\infty}\left(\tfrac{1}{2} - ir\right)\right) = \\
&-\frac{y}{2\pi}\int_{-\infty+ic}^{+\infty+ic} h(t,r)\varphi_{\infty\infty}\left(\tfrac{1}{2} - ir\right) y^{2ir}\mathrm{d}r + \frac{1}{v_N}.
\end{aligned}$$

Hence we have

$$c_{0,1}(t,y) = -\frac{y}{2\pi}\int_{-\infty+ic}^{+\infty+ic} h(t,r)\varphi_{\infty\infty}\left(\tfrac{1}{2} - ir\right) y^{2ir}\mathrm{d}r.$$

If we change the variable by $s = -2ir$, we obtain

$$c_{0,1}(t,y) = -\frac{y}{4\pi i}\int_{2c-i\infty}^{2c+i\infty} h\left(t, \tfrac{is}{2}\right)\varphi_{\infty\infty}\left(\tfrac{1+s}{2}\right) y^{-s}\mathrm{d}s,$$

and from the inverse Mellin transform we deduce the claimed equation

$$R_{C_{0,1}}(t,s) = \int_0^{+\infty} c_{0,1}(t,y) y^{s-2}\mathrm{d}y = -\frac{1}{2} h\left(t, \tfrac{is}{2}\right)\varphi_{\infty\infty}\left(\tfrac{1+s}{2}\right).$$

□

5.1.2. Let $f(z)$ be an automorphic function of weight 0 with respect to $\Gamma_1(N)$ with eigenvalue λ and of rapid decay at the cusp ∞. Then, we have

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} |\Lambda_0(f(z))|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) = \left(\lambda + \frac{s(s-1)}{2}\right)\int_{\Gamma_1(N)\backslash\mathbb{H}} |f(z)|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z).$$

The proof is analogous to the proof of [1], lemma 3.2.18, where this claim is formulated for the congruence subgroup $\Gamma_0(N)$. Namely, the Rankin-Selberg method implies as in [1]

$$\int_{\Gamma_1(N)\backslash\mathbb{H}} |\Lambda_0 f(z)|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) = -\frac{1}{2}\int_{\Gamma_1(N)\backslash\mathbb{H}} \Delta_0\left(|f(z)|^2\right) E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) + \lambda\int_{\Gamma_1(N)\backslash\mathbb{H}} |f(z)|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z). \tag{5.5}$$

Since $s(1-s)$ is the eigenvalue of $E_{\infty,0}(z,s)$, Green's second identity implies

$$-\frac{1}{2}\int_{\Gamma_1(N)\backslash\mathbb{H}} \Delta_0\left(|f(z)|^2\right) E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z) = \frac{s(s-1)}{2}\int_{\Gamma_1(N)\backslash\mathbb{H}} |f(z)|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z). \tag{5.6}$$

Plugging equation (5.6) in equation (5.5) the claim follows.

5.1.3. Lemma. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $t > 0$ the integrals*

$$R_{C_{k,2}}(t,s) = \int_0^{+\infty} c_{k,2}(t,y)y^{s-2}\mathrm{d}y \qquad (k = 0,2)$$

exist and have a meromorphic continuation to the whole s-plane with a simple pole at $s = 1$. Furthermore, we have

$$R_{C_{2,2}}(t,s) = R_{C_{0,2}}(t,s) + \frac{s(s-1)}{2}\int_0^{+\infty} c^*_{0,2}(t,y)y^{s-2}\mathrm{d}y,$$

where $c^*_{0,2}(t,y)$ is defined as $c_{0,2}(t,y)$, but with the function $h(t,r)$ replaced by $h^*(t,r) := \dfrac{h(t,r)}{\frac{1}{4}+r^2}$.

Proof. Recall that we have by definition

$$\int_0^{+\infty} c_{k,2}(t,y)y^{s-2}\mathrm{d}y = -\frac{1}{4\pi}\int_0^{+\infty}\left(\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{-1/2}^{+1/2}\int_{-\infty}^{+\infty} h(t,r)\left|\widetilde{E}_{\mathfrak{a},k}\left(x+iy,\tfrac{1}{2}+ir\right)\right|^2\mathrm{d}r\mathrm{d}x\right)y^{s-2}\mathrm{d}y.$$

Since $\widetilde{E}_{\mathfrak{a},0}(x+iy,\frac{1}{2}+ir)$ and also $\widetilde{E}_{\mathfrak{a},2}(x+iy,\frac{1}{2}+ir)$ are of rapid decay as $y\to\infty$ by [27], lemma 10.2, we find that the expressions

$$\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{-1/2}^{1/2}\int_{-\infty}^{+\infty} h(t,r)\left|\widetilde{E}_{\mathfrak{a},k}\left(x+iy,\tfrac{1}{2}+ir\right)\right|^2 \mathrm{d}r\mathrm{d}x \qquad (k=0,2)$$

satisfy the growth condition of 3.7, and the first statement of the lemma follows.

For the second statement, note that we are allowed to interchange the integrals for $\mathrm{Re}(s)>1$. Hence we have for $k=0,2$

$$\int_0^{+\infty} c_{k,2}(t,y)y^{s-2}\mathrm{d}y = \tag{5.7}$$

$$-\frac{1}{4\pi}\int_{-\infty}^{+\infty}h(t,r)\left(\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{\Gamma_1(N)\backslash\mathbb{H}}\left|\widetilde{E}_{\mathfrak{a},k}\left(z,\tfrac{1}{2}+ir\right)\right|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z)\right)\mathrm{d}r.$$

Since we have

$$\Lambda_0\left(\widetilde{E}_{\mathfrak{a},0}\left(z,\tfrac{1}{2}+ir\right)\right)=\left(\tfrac{1}{2}+ir\right)\widetilde{E}_{\mathfrak{a},2}\left(z,\tfrac{1}{2}+ir\right), \tag{5.8}$$

and as $\widetilde{E}_{\mathfrak{a},0}\left(z,\frac{1}{2}+ir\right)$ is an eigenfunction of the hyperbolic Laplacian Δ_0 with eigenvalue $\frac{1}{4}+r^2$, which is of rapid decay at the cusp ∞, equation (5.8) and observation 5.1.2 imply

$$\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{\Gamma_1(N)\backslash\mathbb{H}}\left|\widetilde{E}_{\mathfrak{a},2}\left(z,\tfrac{1}{2}+ir\right)\right|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z)=$$

$$\left(1+\frac{s(s-1)}{2(\frac{1}{4}+r^2)}\right)\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{\Gamma_1(N)\backslash\mathbb{H}}\left|\widetilde{E}_{\mathfrak{a},0}\left(z,\tfrac{1}{2}+ir\right)\right|^2 E_{\infty,0}(z,s)\mu_{\mathrm{hyp}}(z).$$

Therefore, we have by equations (5.7)

$$\int_0^{+\infty}c_{2,2}(t,y)y^{s-2}\mathrm{d}y=\int_0^{+\infty}c_{0,2}(t,y)y^{s-2}\mathrm{d}y+\frac{s(s-1)}{2}\int_0^{+\infty}c_{0,2}^*(t,y)y^{s-2}\mathrm{d}y,$$

where

$$c_{0,2}^*(t,y):=-\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\int_{-\infty}^{+\infty}h^*(t,r)\left|\widetilde{E}_{\mathfrak{a},0}\left(x+iy,\tfrac{1}{2}+ir\right)\right|^2\mathrm{d}r\,\mathrm{d}x$$

with $h^*(t,r)=\dfrac{h(t,r)}{\frac{1}{4}+r^2}$. This completes the proof of the lemma. □

5.1.4. Lemma. *For $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ and $t>0$, we have*

$$R_D(t,s)=\frac{s(s-1)}{2}\sum_{j=1}^{\infty}\frac{h(t,r_j)}{\lambda_j}R_{|u_j|^2}(s).$$

Proof. Since the u_j's are cusp forms they are of rapid decay at all cusps and the Rankin-Selberg transforms exist for $\mathrm{Re}(s) > 1$. By observation 5.1.2 we find

$$R_{|\Lambda_0(u_j)|^2}(t,s) = \left(\lambda_j + \frac{s(s-1)}{2}\right) R_{|u_j|^2}(t,s)$$

which implies the claim of the lemma. □

5.2. Spectral and parabolic contribution.

5.2.1. Note that for an odd and squarefree positive integer N, we have by [12], p. 566, proposition 6.3

$$\varphi_{\infty\infty}(s) = 2\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)} N^{-2s} \prod_{p|N} \frac{1}{1-p^{-2s}}$$

where $\varphi_{\infty\infty}(s)$ is the function in the constant term of the 0-th Fourier expansion of the Eisenstein series $E_{\infty,0}(z,s)$. (In the notation loc. cit. we have to choose $x_1 = x_2 = 1$ and $A_1 = A_2 = 1$, which corresponds to the cusp ∞). One easily computes its Laurent expansion to be

$$\varphi_{\infty\infty}(s) = \frac{1}{v_N}\frac{1}{s-1} + \frac{1}{v_N}\left(2\gamma + \frac{a\pi}{6} - 2\sum_{p|N}\frac{p^2\log(p)}{p^2-1}\right) + O(s-1), \tag{5.9}$$

where γ is the Euler constant and a is the derivative of $\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\zeta(2s)}$ at $s=1$.

5.2.2. Proposition. *Let N be an odd and squarefree positive integer. For any $t > 0$, we have the following Laurent expansion in a neighbourhood of $s = 1$*

$$R_{C_{2,1}}(t,s) - R_{C_{0,1}}(t,s) =$$
$$\frac{1}{v_N}\frac{1}{s-1} + \frac{t}{2v_N} + \frac{1}{2v_N}\left(1 + 2\gamma + \frac{a\pi}{6} - 2\sum_{p|N}\frac{p^2\log(p)}{p^2-1}\right) + O(s-1).$$

Proof. By lemma 5.1.1, we have

$$R_{C_{2,1}}(t,s) = -\frac{1}{2}h\left(t,\tfrac{is}{2}\right)\varphi_{\infty\infty}(\tfrac{1+s}{2})\frac{1-s}{1+s},$$

which is holomorphic at $s = 1$. Then, since $h(t,\frac{i}{2}) = 1$, we obtain by observation (5.9) the Laurent expansion

$$R_{C_{2,1}}(t,s) = \frac{1}{2v_N} + O(s-1). \tag{5.10}$$

Again by lemma 5.1.1, we have

$$-R_{C_{0,1}}(t,s) = \frac{1}{2}h\left(t, \tfrac{is}{2}\right)\varphi_{\infty\infty}(\tfrac{1+s}{2}),$$

which has a pole of order one at $s=1$. Then, since

$$\frac{1}{2}h\left(t, \tfrac{is}{2}\right) = \frac{1}{2}\exp(-\,t\left(\tfrac{1}{4}+(\tfrac{is}{2})^2\right)) = \frac{1}{2}+\frac{1}{4}t(s-1)+O\left((s-1)^2\right),$$

we obtain again by (5.9) the Laurent expansion

$$-\,R_{C_{0,1}}(t,s) =$$
$$\frac{1}{v_N}\frac{1}{s-1}+\frac{t}{2v_N}+\frac{1}{2v_N}\left(2\gamma+\frac{a\pi}{6}-2\sum_{p|N}\frac{p^2}{p^2-1}\log p\right)+O(s-1)$$

which implies with the Laurent expansion (5.10) the claim of the proposition. □

5.2.3. Proposition. *For any $t>0$, we have in a neighbourhood of $s=1$ the following Laurent expansion*

$$R_{C_{2,2}}(t,s)-R_{C_{0,2}}(t,s) = C_4(t)+O(s-1),$$

where the constant $C_4(t)$ tends to zero as $t\to\infty$.

Proof. By lemma 5.1.3 we have

$$R_{C_{2,2}}(t,s)-R_{C_{0,2}}(t,s) = \frac{s(s-1)}{2}\int_0^{+\infty} c^*_{0,2}(t,y)y^{s-2}\mathrm{d}y,$$

where we set

$$c^*_{0,2}(t,y) = -\frac{1}{4\pi}\sum_{\mathfrak{a}\in P_{\Gamma_1(N)}}\int_{-1/2}^{1/2}\int_{-\infty}^{+\infty}\frac{h(t,r)}{\frac{1}{4}+r^2}\left|\widetilde{E}_{\mathfrak{a},0}(x+iy,\tfrac{1}{2}+ir)\right|^2\mathrm{d}r\,\mathrm{d}x.$$

By means of 3.7 the integral

$$\int_0^{+\infty} c^*_{0,2}(t,y)y^{s-2}\mathrm{d}y$$

has a meromorphic continuation to the whole s-plane with a simple pole at $s=1$. Therefore $R_{C_{2,2}}(t,s)-R_{C_{0,2}}(t,s)$ is holomorphic at $s=1$ for any $t>0$. Since we have chosen $h(t,r)=\exp(-\,t(\frac{1}{4}+r^2))$ the constant $C_4(t)$ in the Laurent expansion tends to 0 as $t\to\infty$, as we claimed. □

5.2.4. Proposition. *For any $t>0$, we have in a neighbourhood of $s=1$ the following Laurent expansion*

$$R_D(t,s) = \frac{1}{2v_N}\sum_{j=1}^{\infty}\frac{h(t,r_j)}{\lambda_j}+O(s-1). \tag{5.11}$$

Proof. By lemma 5.1.4 we have

$$R_D(t,s) = \frac{s(s-1)}{2} \sum_{j=1}^{\infty} \frac{h(t,r_j)}{\lambda_j} R_{|u_j|^2}(s).$$

We know from 3.7 that all $R_{|u_j|^2}(s)$ have a meromorphic continuation to the whole s-plane with a simple pole at $s=1$ with residue v_N^{-1}. Hence the summands

$$\frac{s(s-1)}{2} \frac{h(t,r_j)}{\lambda_j} R_{|u_j|^2}(s)$$

are holomorphic at $s=1$, and since the series $R_D(t,s)$ converges uniformly in a neighbourhood of $s=1$, the series is holomorphic at $s=1$. From this the Laurent expansion (5.11) follows immediately. □

We conclude this section by recalling some known results from [1]. Let N be a positive integer. We denote by $d(N)$ the number of positive divisors of N and by $\sigma_s(N) := \sum_{d|N} d^s$, $s \in \mathbb{C}$, the divisor sum of N.

5.2.5. Proposition. *For any $t>0$, we have in a neighborhood of $s=1$ the following Laurent expansion*

$$R_{P_{2,1}}(t,s) - R_{P_{0,1}}(t,s) =$$
$$\left(-\frac{1}{2} + \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{h(t,r)}{\frac{1}{4}+r^2} \mathrm{d}r\right) \frac{\varphi(N)d(N)}{v_N} \frac{1}{s-1} +$$
$$\frac{\varphi(N)d(N)C_1(t)}{v_N} - \frac{\varphi(N)}{v_N}\left(\frac{1}{2} - \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{h(t,r)}{\frac{1}{4}+r^2}\mathrm{d}r\right) \times$$
$$d(N)\left(3\gamma + \frac{a\pi}{6} - \sum_{p|N} \frac{2p+1}{p+1}\log(p) + \left(1 - \frac{1}{d(N)}\right)\sigma_{-1}(N)\right) + O(s-1),$$

where a is the derivative of $\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)\zeta(2s)}$ at $s=1$ and $C_1(t)$ is a function, which converges for $t \to \infty$.

Proof. Every element $\gamma \in \Gamma_0(N)$ with $|\mathrm{tr}(\gamma)| = 2$ lies already in $\Gamma_1(N)$ as we can write $\gamma = \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix}$ with $a^2 = -bc \equiv 0 \mod N$. Hence we can just apply [1], p. 57, equation (28), and p. 60, lemma 3.3.10. □

5.2.6. Proposition. *For any $t>0$, we have in a neighborhood of $s=1$ the following Laurent expansion*

$$R_{P_{2,2}}(t,s) - R_{P_{0,2}}(t,s) =$$
$$\left(\frac{1}{4\pi} + C_2(t)\right)\frac{1}{s-1} + \left(\frac{\Gamma'(2)+\gamma-\log(4\pi)}{4\pi} + \gamma C_2(t) + C_3(t)\right) + O(s-1),$$

where $C_2(t)$ and $C_3(t)$ are functions in t tending to 0 as $t \to \infty$ and γ denotes the Euler constant.

Proof. Also here, as every element $\gamma \in \Gamma_0(N)$ with $|\mathrm{tr}(\gamma)| = 2$ lies already in $\Gamma_1(N)$, we can apply [1], p. 57, equation (28) and p. 60 on the top. $\square$

6. Analytic part of $\overline{\omega}_N^2$

In this section we determine the analytic part of the stable arithmetic self-intersection number of the relative dualizing sheaf, i.e., we obtain an asymptotic formula in N for the Greens function evaluated at the cusps 0 and ∞_d, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, of $\Gamma_1(N)$.

6.1. Lemma. *For an odd and squarefree positive integer N, we have*

$$\lim_{s\to 1}\left(\varphi_{0\infty}(s) - \frac{1}{v_N}\frac{1}{s-1}\right) = \frac{1}{v_N}\left(2\gamma + \frac{a\pi}{6} + \sum_{p|N}\frac{-p^2+2p+1}{p^2-1}\log(p)\right),$$

where γ is the Euler constant and a is the derivative of $\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\zeta(2s)}$ at $s = 1$.

Proof. From [19], Satz 1, p. iv, we have for any j

$$v_0^j b_{0\infty}^{\Gamma_0(N)}(c) = \sum_{\ell=1}^{r_\infty} b_{0_j\infty_\ell}(c);$$

here v_0^j denotes the ramification index of the cusp 0_j of $\Gamma_1(N)$ lying over the cusp 0 of the congruence subgroup $\Gamma_0(N)$ contained in $\Gamma_1(N)$, r_∞ refers to the number of cusps ∞_ℓ lying over the cusp ∞ of $\Gamma_0(N)$, and, for $c \in \mathbb{N}$,

$$b_{0\infty}^{\Gamma_0(N)}(c) = \#\left\{\begin{pmatrix}\star & \star\\ c & \star\end{pmatrix} \in g_0^{-1}\Gamma_0(N)_0\, g_0\backslash g_0^{-1}\Gamma_0(N)g_\infty/g_\infty^{-1}\Gamma_0(N)_\infty\, g_\infty\right\}$$

with $g_0 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $g_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ mapping the standard cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$ to the cusps 0 and ∞, respectively. The quantity $b_{0_j\infty_\ell}(c)$ is defined by (3.2).

In the sequel we will choose for 0_j the cusp 0 of $\Gamma_1(N)$ as the cusp lying over the cusp 0 of $\Gamma_0(N)$; since we have $\Gamma_0(N)_0 = \Gamma_1(N)_0$, the cusp 0 of $\Gamma_1(N)$ is unramified over the cusp 0 of $\Gamma_0(N)$, which shows $v_0^j = 1$. Furthermore, as the group $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and since again $\Gamma_0(N)_\infty = \Gamma_1(N)_\infty$, all the cusps ∞_ℓ lying over the cusp ∞ of $\Gamma_0(N)$ are unramified, whence

$$r_\infty = [\Gamma_0(N) : \Gamma_1(N)] = \varphi(N).$$

Note that we have (in the obvious notation)

$$\mathrm{vol}_{\mathrm{hyp}}(\Gamma_1(N)\backslash\mathbb{H}) = [\Gamma_0(N) : \Gamma_1(N)]\mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})\,.$$

Then, the lemma follows immediately from lemma 3.6 and the Laurent expansion for odd and squarefree N (see [1], p. 67)

$$\varphi_{0\infty}^{\Gamma_0(N)}(s) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})}\frac{1}{s-1} +$$
$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(\Gamma_0(N)\backslash\mathbb{H})}\left(2\gamma + \frac{a\pi}{6} + \sum_{p|N}\frac{-p^2+2p+1}{p^2-1}\log(p)\right) + O(s-1).$$

□

6.2. Theorem. *Let N be an odd and squarefree positive integer. Then the constant term C_F in the Laurent expansion of the Rankin-Selberg transform $R_F(s)$ of F at $s=1$ is given by*

$$C_F = -\frac{1}{2g_N v_N}\lim_{s\to 1}\left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1}\right) +$$
$$\frac{1}{2g_N v_N}\left(2 + 2\gamma + \frac{a\pi}{6} - 2\sum_{p|N}\frac{p^2\log(p)}{p^2-1}\right) + \frac{\Gamma'(2)+\gamma-\log(4\pi)}{4\pi g_N} +$$
$$\frac{\varphi(N)d(N)}{2g_N v_N}\left(2C_1 - 3\gamma - \frac{a\pi}{6} + \sum_{p|N}\frac{2p+1}{p+1}\log(p) - \left(1-\frac{1}{d(N)}\right)\sigma_{-1}(N)\right),$$

where γ is the Euler constant, a is the derivative of $\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\zeta(2s)}$ at $s=1$, and C_1 is the limit $C_1 := \lim_{t\to\infty} C_1(t)$ (coming from proposition 5.2.5).

Proof. The idea of the proof is to use equation (2.20) and to determine the constant terms in the Laurent expansions at $s=1$ of the Rankin-Selberg transforms on the right-hand side as $t\to\infty$. This gives an expression of C_F, since by proposition 5.2.4, we have $\lim_{t\to\infty} R_D(t,1) = 0$ showing that on the left-hand side there is no contribution from the discrete part adding to C_F.

Now we determine the constants in the Laurent expansions of the Rankin-Selberg transforms on the right-hand side of equation (2.20) as $t\to\infty$ using the facts (5.3) and (5.4). From corollary 4.2.10 and proposition 5.2.2 we find

$$\lim_{t\to\infty}\lim_{s\to 1}\left(R_H(t,s) + \left(R_{C_{2,1}}(t,s) - R_{C_{0,1}}(t,s)\right) - \frac{1}{v_N}\frac{1}{s-1}\right) = \tag{6.1}$$
$$-\frac{1}{2v_N}\lim_{s\to 1}\left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1}\right) + \frac{1}{2v_N}\left(2 + 2\gamma + \frac{a\pi}{6} - 2\sum_{p|N}\frac{p^2\log(p)}{p^2-1}\right).$$

Denoting by R_1 the residue of $R_{P_{2,1}}(t,s) - R_{P_{0,1}}(t,s)$ at $s = 1$, we find from proposition 5.2.5

$$\lim_{t\to\infty}\lim_{s\to 1}\left(R_{P_{2,1}}(t,s) - R_{P_{0,1}}(t,s) - \frac{R_1}{s-1}\right) = \frac{\varphi(N)d(N)}{2v_N}\left(2C_1 - 3\gamma - \frac{a\pi}{6} + \sum_{p\mid N}\frac{2p+1}{p+1}\log(p) - \left(1-\frac{1}{d(N)}\right)\sigma_{-1}(N)\right), \tag{6.2}$$

where C_1 is the limit $C_1 := \lim_{t\to\infty} C_1(t)$. From proposition 5.2.6 we find

$$\lim_{t\to\infty}\lim_{s\to 1}\left(R_{P_{2,2}}(t,s) - R_{P_{0,2}}(t,s) - \left(\frac{1}{4\pi} + C_2(t)\right)\frac{1}{s-1}\right) = \frac{\Gamma'(2) + \gamma - \log(4\pi)}{4\pi}, \tag{6.3}$$

and proposition 5.2.3 implies

$$\lim_{t\to\infty}\lim_{s\to 1}\left(R_{C_{2,2}}(t,s) - R_{C_{0,2}}(t,s)\right) = 0. \tag{6.4}$$

Collecting all constants (6.1), (6.2), (6.3), and (6.4) and dividing them by g_N, proves the claim of the theorem. □

6.3. Corollary. *Let N be an odd and squarefree positive integer satisfying $N = 11$ or $N \geq 13$. Let $X(\Gamma_1(N)) \cong X_1(N)(\mathbb{C})$ be the modular curve with genus $g_N > 0$. Then for the cusps $0, \infty_d \in X_1(N)(\mathbb{C})$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have*

$$g_{\mathrm{can}}(0,\infty_d) = -\frac{2\pi}{g_N v_N}\lim_{s\to 1}\left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1}\right) + \frac{2\pi}{g_N v_N}\left(2 + 2\gamma + \frac{a\pi}{6} - 2\sum_{p\mid N}\frac{p^2\log(p)}{p^2-1}\right) + \frac{\Gamma'(2)+\gamma-\log(4\pi)}{g_N} + \frac{2\pi\varphi(N)d(N)}{g_N v_N}\left(2C_1 - 3\gamma - \frac{a\pi}{6} + \sum_{p\mid N}\frac{2p+1}{p+1}\log(p) - \left(1-\frac{1}{d(N)}\right)\sigma_{-1}(N)\right) - \frac{2\pi}{v_N}\left(2\gamma + \frac{a\pi}{6} + \sum_{p\mid N}\frac{-p^2+2p+1}{p^2-1}\log(p)\right) + O\left(\frac{1}{g_N}\right),$$

where γ is the Euler constant, a is the derivative of $\sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)\zeta(2s)}$ at $s = 1$ and C_1 is the constant from theorem 6.2.

Proof. By proposition 3.8, we have

$$g_{\mathrm{can}}(0,\infty_d) = 4\pi C_F - 2\pi\lim_{s\to 1}\left(\varphi_{0\infty}(s) - \frac{1}{v_N}\frac{1}{s-1}\right) + O\left(\frac{1}{g_N}\right).$$

Hence lemma 6.1 and theorem 6.2 imply the statement of the corollary. □

6.4. A bound for the constant term in the Laurent expansion of the logarithmic derivative of the Selberg zeta function at $s = 1$ is provided by a result of J. Jorgenson and J. Kramer in [15], p. 29, namely

$$\lim_{s\to 1}\left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)} - \frac{1}{s-1}\right) = O_\varepsilon(N^\varepsilon), \tag{6.5}$$

where the implied constant depends only on ε. There is the weaker bound $O_\varepsilon(N^{\frac{7}{8}+\varepsilon})$ due to P. Michel and E. Ullmo in [23], corollary 1.4.

6.5. Theorem. *Let N be an odd and squarefree positive integer satisfying $N = 11$ or $N \geq 13$. Let $X(\Gamma_1(N)) \cong X_1(N)(\mathbb{C})$ be the modular curve with genus $g_N > 0$. Then for the cusps $0, \infty_d \in X_1(N)(\mathbb{C})$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have*

$$4g_N(g_N-1)g_{\mathrm{can}}(0,\infty_d) = 2g_N\log(N) + o\big(g_N\log(N)\big).$$

Proof. From corollary 6.3, we deduce

(6.6)

$$4\,g_N(g_N-1)g_{\mathrm{can}}(0,\infty_d) = -\frac{8\pi(g_N-1)}{v_N}\lim_{s\to 1}\left(\frac{Z'_{\Gamma_1(N)}(s)}{Z_{\Gamma_1(N)}(s)} - \frac{1}{s-1}\right) +$$
$$\frac{8\pi(g_N-1)}{v_N}\left(2+2\gamma+\frac{a\pi}{6}-2\sum_{p\mid N}\frac{p^2\log(p)}{p^2-1}\right) + 4\,(g_N-1)\,\big(\Gamma'(2)+\gamma-$$
$$\log(4\pi)\big) + \frac{8\pi\varphi(N)d(N)(g_N-1)}{v_N}\Bigg(2C_1-3\gamma-\frac{a\pi}{6}+\sum_{p\mid N}\frac{2p+1}{p+1}\log(p)-\Big(1-$$
$$\frac{1}{d(N)}\Big)\sigma_{-1}(N)\Bigg) - \frac{8\pi g_N(g_N-1)}{v_N}\left(2\gamma+\frac{a\pi}{6}+\sum_{p\mid N}\frac{-p^2+2p+1}{p^2-1}\log(p)\right) + O(g_N).$$

The asymptotic $\frac{24(g_N-1)}{\prod_{p\mid N}(p^2-1)} = 1 + o(1)$ together with $v_N = \frac{\pi}{6}\prod_{p\mid N}(p^2-1)$ imply

$$\frac{8\pi\,(g_N-1)}{v_N} = \frac{48\,(g_N-1)}{\prod_{p\mid N}(p^2-1)} = 2 + o(1). \tag{6.7}$$

Hence, since we have

$$2\gamma + \frac{a\pi}{6} + \sum_{p\mid N}\frac{-p^2+2p+1}{p^2-1}\log(p) = \log(N) + O\big(\log\log(N)\big),$$

we find with observation (6.7)

$$\frac{8\pi g_N(g_N-1)}{v_N}\Big(2\gamma+\frac{a\pi}{6}+\sum_{p|N}\frac{-p^2+2p+1}{p^2-1}\log(p)\Big)=$$
$$2g_N\log(N)+o\big(g_N\log(N)\big).$$

Noting that all other summands on the right-hand side of (6.6) vanish in the little o-term (for the first summand we use the bound (6.5)) we obtain

$$4\,g_N(g_N-1)g_{\mathrm{can}}(0,\infty_d)=2g_N\log(N)+o\big(g_N\log(N)\big).$$

□

7. Geometric part of $\overline{\omega}_N^2$ and main theorem

7.1. Let $X_1(N)/\mathbb{Q}$ be the smooth projective algebraic curve over $\mathbb{Q}$ that classifies elliptic curves equipped with a point of exact order N. There exists a canonical analytic isomorphism

$$j:X\big(\Gamma_1(N)\big)=\Gamma_1(N)\backslash\big(\mathbb{H}\cup\mathbb{P}^1_{\mathbb{Q}}\big)\longrightarrow X_1(N)(\mathbb{C}),$$

and we say that a K-rational point $x\in X_1(N)(K)$ is a cusp if $x\in j(\Gamma_1(N)\backslash\mathbb{P}^1_{\mathbb{Q}})$. Note that the cusp 0 of $X_1(N)/\mathbb{Q}$ is $\mathbb{Q}$-rational and the cusp ∞ of $X_1(N)/\mathbb{Q}$ is $\mathbb{Q}(\zeta_N)$-rational (see [25], proposition 1).

7.2. Let $X_1(N)/\mathbb{Q}(\zeta_N)=X_1(N)\times_{\mathbb{Q}}\mathbb{Q}(\zeta_N)$ be the modular curve over the cyclotomic field $\mathbb{Q}(\zeta_N)$. Let $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ be the minimal regular model of $X_1(N)/\mathbb{Q}(\zeta_N)$, i.e., a regular, projective, and flat $\mathbb{Z}[\zeta_N]$-scheme with generic fiber isomorphic to $X_1(N)/\mathbb{Q}(\zeta_N)$. Under the assumption that $g_N\geq 1$, minimality means by Castelnuovo's criterion that the canonical divisor K_N corresponding to the relative dualizing sheaf is numerically effective, i.e., $K_N\cdot V\geq 0$ holds for every vertical prime divisor V of $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$.
Integral models of modular curves were intensively studied by many people. We collect in the following proposition some facts from [18].

7.3. Proposition. *Let N be a squarefree positive integer of the form $N=N^{'}qr$ with q and r two relative prime integers satisfying $q,r\geq 4$. The minimal regular model $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ has smooth fibers over prime ideals $\mathfrak{p}$ of $\mathbb{Z}[\zeta_N]$ with $\mathfrak{p}\nmid N$. For $\mathfrak{p}|N$ the fiber of $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ over $\mathfrak{p}$ is the union of two irreducible, smooth, and proper $k(\mathfrak{p})$-curves $C_{1,\mathfrak{p}}$ and $C_{2,\mathfrak{p}}$, $k(\mathfrak{p})$ the residue field at $\mathfrak{p}$, intersecting transversally in*

$$s_{\mathfrak{p}}:=\frac{p-1}{24}\cdot\frac{\varphi(N/p)N}{p}\prod_{q|\frac{N}{p}}\left(1+\frac{1}{q}\right)\tag{7.1}$$

$k(\mathfrak{p})$-rational points. Moreover, the curves $C_{1,\mathfrak{p}}$ and $C_{2,\mathfrak{p}}$ are isomorphic.

Proof. First we suppose that there is a regular model of $X_1(N)/\mathbb{Q}(\zeta_N)$ with fibers as described in the proposition. Then, the adjunction formula for arithmetic surfaces implies $K_N \cdot V = 2p_a(V) - 2 + s_\mathfrak{p} \geq 0$ for every vertical prime divisor V in the fiber over $\mathfrak{p}|N$, where $p_a(V)$ denotes the arithmetic genus of V. In the case of good reduction, we find $K_N \cdot V = 2g_N - 2 \geq 0$ as $N \geq 13$. Hence the regular model is minimal. As a minimal regular model is unique up to isomorphism, it suffices to find such a regular model.
Let $\overline{\mathfrak{M}}(\Gamma_1(N))/\mathbb{Z}[\zeta_N]$ be the compactified coarse moduli scheme of canonical balanced $\Gamma_1(N)$-structures as described in [18], chap. 9, which exists only if N is a positive integer of the form $N = N' qr$ with q and r two relative prime integers satisfying $q, r \geq 4$. It follows from the modular interpretation that $\mathfrak{M}(\Gamma_1(N))/\mathbb{Z}[\zeta_N]$ is a model for $X_1(N)/\mathbb{Q}(\zeta_N)$. That $\overline{\mathfrak{M}}(\Gamma_1(N))/\mathbb{Z}[\zeta_N]$ is a regular model having smooth fibers over $\mathfrak{p} \nmid N$ follows from [18], theorem 5.5.1, theorem 10.9.1, and the summarizing table on p. 305. For $\mathfrak{p}|N$ the fiber of $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ over $\mathfrak{p}$ is the union of two irreducible, smooth, and proper $k(\mathfrak{p})$-curves which are isomorphic and intersect transversally, follows from [18], theorem 13.11.4. The formula for the intersection number $s_\mathfrak{p}$ follows from [18], corollaries 5.5.3 and 12.9.4. □

7.4. Let $0, \infty \in X_1(N)\big(\mathbb{Q}(\zeta_N)\big)$ be the cusps with representatives $(0:1)$, $(1:0)$ in $\mathbb{P}^1_\mathbb{Q}$, respectively. We let H_0, H_∞ be the horizontal divisors obtained by taking the Zariski closure of $0, \infty$ in $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$, respectively. Note that for $m = 0, \infty$ there exists an open subscheme containing H_m, which is smooth over $\mathbb{Z}[\zeta_N]$ (see [18], theorem 10.9.1).

7.5. Proposition. *Let N be a squarefree positive integer of the form $N = N' qr$ with q and r two relative prime integers satisfying $q, r \geq 4$. Then, there exist vertical divisors $V_m \in \mathrm{Div}(\mathcal{X}_1(N))_\mathbb{Q}$ $(m = 0, \infty)$ satisfying*

(7.2)
$$\Big(\overline{\omega}_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]} \otimes \overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_m)^{\otimes -(2g_N-2)} \otimes \overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_m)\Big) \cdot \overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V) = 0$$

for all vertical divisors V of $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$. Furthermore, we have the following intersection numbers

$$(V_0, V_0)_{\mathrm{fin}} = (V_\infty, V_\infty)_{\mathrm{fin}} = -(V_0, V_\infty)_{\mathrm{fin}} = \\ -\varphi(N) \frac{24(g_N-1)^2}{\prod_{p|N}(p^2-1)} \sum_{p|N} \frac{p+1}{p-1} \log(p).$$

Proof. We start by considering a fiber over a closed point $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[\zeta_N])$ with $\mathfrak{p}|N$. The fiber consists by proposition 7.3 of the two irreducible components $C_{1,\mathfrak{p}}$ and $C_{2,\mathfrak{p}}$. The horizontal divisors H_0 and H_∞ intersect the fiber in a smooth $k(\mathfrak{p})$-rational point, and, by the cusp and component labeling in [18], p. 296, they do not intersect the same component. We denote by $C_{0,\mathfrak{p}}$ and $C_{\infty,\mathfrak{p}}$ the component intersected by H_0 and H_∞, respectively. Let us define the $\mathbb{Q}$-divisors

$$V_{0,\mathfrak{p}} := -\frac{g_N-1}{s_\mathfrak{p}}C_{0,\mathfrak{p}} \qquad \text{and} \qquad V_{\infty,\mathfrak{p}} := -\frac{g_N-1}{s_\mathfrak{p}}C_{\infty,\mathfrak{p}}.$$

Then, we claim that

$$V_0 := \sum_{\substack{\mathfrak{p}\in\mathrm{Spec}(\mathbb{Z}[\zeta_N])\\ \mathfrak{p}|N}} V_{0,\mathfrak{p}} \qquad \text{and} \qquad V_\infty := \sum_{\substack{\mathfrak{p}\in\mathrm{Spec}(\mathbb{Z}[\zeta_N])\\ \mathfrak{p}|N}} V_{\infty,\mathfrak{p}}$$

fulfill the conditions stated in (7.2). Noting that

$$K_N\cdot C_{0,\mathfrak{p}} = \deg\omega_{C_{0,\mathfrak{p}}/k(\mathfrak{p})} = \deg\omega_{C_{\infty,\mathfrak{p}}/k(\mathfrak{p})} = K_N\cdot C_{\infty,\mathfrak{p}}$$

and $K_N\cdot(C_{0,\mathfrak{p}}+C_{\infty,\mathfrak{p}}) = 2g_N-2$, we find $K_N\cdot C_{0,\mathfrak{p}} = K_N\cdot C_{\infty,\mathfrak{p}} = g_N-1$. We have to consider the following three cases:

(i) $V=V_\mathfrak{p}$ with $\mathfrak{p}\nmid N$. In this case we calculate, using the adjunction formula

$$\Big(\overline{\omega}_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_m)^{\otimes-(2g_N-2)}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_m)\Big)\cdot\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V) =$$
$$\big(2g_N-2-(2g_N-2)\big)\log\big(\sharp k(\mathfrak{p})\big)=0.$$

(ii) $V=C_{m,\mathfrak{p}}$ with $\mathfrak{p}|N$. In this case we calculate

$$\Big(\overline{\omega}_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_m)^{\otimes-(2g_N-2)}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_m)\Big)\cdot\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V) =$$
$$\big(g_N-1-(2g_N-2)+g_N-1\big)\log\big(\sharp k(\mathfrak{p})\big)=0.$$

(iii) $V=C_{n,\mathfrak{p}}$ with $\mathfrak{p}|N$, $n\in\{0,\infty\}$, and $n\neq m$. In this case we calculate

$$\Big(\overline{\omega}_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_m)^{\otimes-(2g_N-2)}\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_m)\Big)\cdot\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V) =$$
$$\big(g_N-1-(g_N-1)\big)\log\big(\sharp k(\mathfrak{p})\big)=0.$$

This proves the first part of the proposition. Now, proposition 7.3 implies

$$(V_0,V_0)_{\mathrm{fin}} = (V_\infty,V_\infty)_{\mathrm{fin}} = -(V_0,V_\infty)_{\mathrm{fin}} = -\sum_{\mathfrak{p}|N}\frac{(g_N-1)^2}{s_\mathfrak{p}^2}C_{0,\mathfrak{p}}\cdot C_{\infty,\mathfrak{p}} =$$
$$-\sum_{\mathfrak{p}|N}\frac{(g_N-1)^2\log\big(\sharp k(\mathfrak{p})\big)}{s_\mathfrak{p}} = -24(g_N-1)^2\sum_{p|N}\frac{1}{p-1}\sum_{\mathfrak{p}|p}\frac{\log\big(\sharp k(\mathfrak{p})\big)}{\prod_{q|N/p}(q^2-1)},$$

which proves the proposition noting that $\sum_{\mathfrak{p}|p}\log\big(\sharp k(\mathfrak{p})\big) = \varphi(N/p)\log(p)$

□

7.6. Proposition. *Let N be a squarefree positive integer of the form $N = N^{'}qr$ with q and r two relative prime integers satisfying $q,r\geq 4$. Let $0,\infty$*

be the cusps of $X(\Gamma_1(N))$ *with representatives* $(0:1)$, $(1:0)$ *in* $\mathbb{P}^1_{\mathbb{Q}}$, *respectively, and let* V_0, V_∞ *be the vertical divisors of proposition 7.5. Then, we have*

$$\overline{\omega}_N^2 = 4g_N(g_N-1)g_{\mathrm{can}}(0,\infty) + \frac{1}{\varphi(N)}\frac{g_N+1}{g_N-1}\,(V_0,V_\infty)_{\mathrm{fin}}\,. \tag{7.3}$$

Proof. Formula (7.3) is analogous to the corresponding formula in proposition D of [1] and, in the smooth case, the formula given in [29], p. 241. We first claim that a multiple of the line bundle

$$\Big(\omega_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]}\otimes\mathcal{O}_{\mathcal{X}_1(N)}(H_0)^{\otimes-(2g_N-2)}\otimes\mathcal{O}_{\mathcal{X}_1(N)}(V_0)\Big)\big|_{X_1(N)}\in J_1(N)(\mathbb{Q}(\zeta_N))$$

has support in the cusps. Since $\Gamma_1(N)$ has no elliptic points for $N\geq 4$ (see [6], p. 107) the natural map $X(\Gamma_1(N))\longrightarrow\mathbb{P}^1_{\mathbb{C}}$ has constant ramification index at points over the elliptic point of order two in $\mathbb{P}^1_{\mathbb{C}}$ and has constant ramification index at points over the elliptic point of order 3 in $\mathbb{P}^1_{\mathbb{C}}$. Therefore, our first claim follows from [1], lemme 4.1.1. Hence a well-known theorem of Manin and Drinfeld (see [7]) says, that this line bundle is a torsion element in the Jacobian $J_1(N)/\mathbb{Q}(\zeta_N)$. Now, as condition (7.2) is satisfied, a theorem of Faltings and Hriljac (see [9], theorem 4), and the fact that the Néron-Tate height vanishes at torsion points, imply as in [1]

$$\begin{aligned}\overline{\omega}^2_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]} = &-2g_N(g_N-1)\Big(\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_0)^2+\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_\infty)^2\Big)+\\ &\frac{1}{2}\Big(\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_0)^2+\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_\infty)^2\Big)\,.\end{aligned} \tag{7.4}$$

Now we consider the admissible metrized line bundle

$$\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_\infty)\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_0)^{\otimes-1}\otimes\Big(\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_0)\otimes\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(V_\infty)^{\otimes-1}\Big)^{\otimes 1/(2g_N-2)}$$

which is orthogonal to all vertical divisors V of $\mathcal{X}_1(N)$ because of the conditions (7.2) and has the generic fiber with support in the cusps. Then, a similar argument as above shows

$$\begin{aligned}\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_0)^2+\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_\infty)^2 =&2\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_0)\cdot\overline{\mathcal{O}}_{\mathcal{X}_1(N)}(H_\infty)+\\ &\frac{(V_0,V_0)_{\mathrm{fin}}-2\,(V_0,V_\infty)_{\mathrm{fin}}+(V_\infty,V_\infty)_{\mathrm{fin}}}{(2g_N-2)^2}.\end{aligned} \tag{7.5}$$

Since by 7.4 the horizontal divisors H_0 and H_∞ do not intersect, lemma 7.5, substituting equation (7.5) into (7.4), implies

(7.6)

$$\overline{\omega}^2_{\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]} = 4g_N(g_N-1)\sum_{\sigma:\mathbb{Q}(\zeta_N)\to\mathbb{C}} g^\sigma_{\mathrm{can}}(0,\infty_\sigma)+\frac{g_N+1}{g_N-1}\,(V_0,V_\infty)_{\mathrm{fin}}\,.$$

Note that the modular curve $X_1(N)/\mathbb{Q}$ is defined over $\mathbb{Q}$, and hence the Riemann surfaces $X_1(N)_\sigma$ are in fact all equal to $X(\Gamma_1(N))$. Moreover, proposition 3.8 implies $g^{\sigma}_{\mathrm{can}}(0,\infty_\sigma) = g^{\sigma'}_{\mathrm{can}}(0,\infty_{\sigma'})$ for any two embeddings $\sigma,\sigma' : \mathbb{Q}(\zeta_N) \to \mathbb{C}$. Therefore, dividing both sides of equality (7.6) by $\varphi(N)$, we obtain formula (7.3). □

7.7. Theorem. *Let N be an odd and squarefree positive integer of the form $N = N^{'}qr$ with q and r two relative prime integers satisfying $q, r \geq 4$. Then, we have*

$$\overline{\omega}^2_N = 3g_N \log(N) + o\big(g_N \log(N)\big).$$

Proof. By proposition 7.6, using theorem 6.5 and proposition 7.5, we have

$$\overline{\omega}^2_N = 2g_N \log(N) + \frac{24(g_N+1)(g_N-1)}{\prod_{p|N}(p^2-1)} \sum_{p|N} \frac{p+1}{p-1}\log(p) + o\big(g_N \log(N)\big).$$

Noting that $\frac{24(g_N-1)}{\prod_{p|N}(p^2-1)} = 1 + o(1)$, the claimed asymptotic follows. □

8. Arithmetic applications

8.1. *Stable Faltings height.* Let $J_1(N)/\mathbb{Q}$ be the Jacobian variety of the modular curve $X_1(N)/\mathbb{Q}$ and let $h_{\mathrm{Fal}}(J_1(N))$ be the stable Faltings height of $J_1(N)/\mathbb{Q}$. The arithmetic Noether formula (see [24], theorem 2.5) implies

(8.1)
$$12h_{\mathrm{Fal}}(J_1(N)) = \overline{\omega}^2_N + \sum_{p|N} \frac{s_p}{p-1}\log(p) + \delta_{\mathrm{Fal}}\big(X\big(\Gamma_1(N)\big)\big) - 4g_N\log(2\pi),$$

where s_p is given by the formula (7.1) and $\delta_{\mathrm{Fal}}\big(X\big(\Gamma_1(N)\big)\big)$ denotes the Faltings's delta invariant of $X\big(\Gamma_1(N)\big)$ (for the definition see [9], theorem 1, or, for another approach due to J.-B. Bost, see [28]). In [17] it is proved (see loc. cit. theorem 5.3 and remark 5.8) that

$$\delta_{\mathrm{Fal}}\big(X\big(\Gamma_1(N)\big)\big) = O(g_N). \tag{8.2}$$

8.2. Theorem. *Let N be an odd and squarefree positive integer of the form $N = N^{'}qr$ with q and r two relative prime integers satisfying $q, r \geq 4$. Then, we have*

$$h_{\mathrm{Fal}}(J_1(N)) = \frac{g_N}{4}\log(N) + o\big(g_N \log(N)\big).$$

Proof. Noting that

$$\frac{1}{3}\sum_{p|N} \frac{s_p}{p-1}\log(p) = \frac{g_N}{3}\sum_{p|N}\frac{\log(p)}{p^2-1} + \frac{\varphi(N)d(N)}{12}\sum_{p|N}\frac{\log(p)}{p^2-1} + O\big(\log(N)\big)$$

and

$$\frac{g_N}{3}\sum_{p|N}\frac{\log(p)}{p^2-1} + \frac{\varphi(N)d(N)}{12}\sum_{p|N}\frac{\log(p)}{p^2-1} = o\big(g_N\log(N)\big),$$

we find

$$\frac{1}{3}\sum_{p|N}\frac{s_p}{p-1}\log(p) = o\big(g_N\log(N)\big). \tag{8.3}$$

Hence observations (8.1) and (8.2) together with theorem 7.7 imply statement of the theorem. ☐

8.3. Remark. In [8], p. 83, theorem 16.7, the authors found the bound

$$h_{\mathrm{Fal}}\big(J_1(pl)\big) = O\big((pl)^2\log(pl)\big),$$

where p and l are distinct primes with $l > 5$. In theorem 8.2 the leading term is explicit.
We note that theorem 8.2 might lead to a similar bound on the minimal number of congruences of modular forms with respect to $\Gamma_1(N)$ as in [17], remark 6.6, p. 37, in the case of $\Gamma_0(N)$.

8.4. *Admissible self-intersection number.* We assume that the reader is familiar with the theory of the admissible pairing in [33].

Let N be an odd and squarefree integer of the form $N = N^{'}qr$ with q and r relative prime integeres satisfying $q, r \geq 4$. The dual reduction graph $G_{\mathfrak{p}}$ of the fiber of $\mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ over the prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_N]$ with $\mathfrak{p} \nmid N$ consists by proposition 7.3 of two vertices which are connected by $s_{\mathfrak{p}} = \frac{p-1}{24}\cdot\frac{\varphi(N/p)N}{p}\prod_{q|\frac{N}{p}}\left(1+\frac{1}{q}\right)$ edges of length 1. Hereby the two vertices correpond to the irreducible components $C_{0,\mathfrak{p}}$ and $C_{\infty,\mathfrak{p}}$ over $\mathfrak{p}$. Note that the genera of the two components $C_{0,\mathfrak{p}}$ and $C_{\infty,\mathfrak{p}}$ are the same by proposition 7.3, which we denote by $g_{\mathfrak{p}}$. Then, we have $g_N = 2g_{\mathfrak{p}} + s_{\mathfrak{p}} - 1$ and the canonical divisor $K_{G_{\mathfrak{p}}}$ on $G_{\mathfrak{p}}$ (for the definition see [33], p. 175) is in this case given by

$$K_{G_{\mathfrak{p}}} = (g_N - 1)[0+\infty].$$

We set $a_{\mathfrak{p}} := \frac{s_{\mathfrak{p}}}{s_{\mathfrak{p}}-1}g_{\mathfrak{p}}$ and $l_{\mathfrak{p}} := 2a_{\mathfrak{p}} + s_{\mathfrak{p}}$, and the admissible measure with respect to $K_{G_{\mathfrak{p}}}$ on $G_{\mathfrak{p}}$ (for the definition see [33], theorem 3.2) is given by

$$\mu_{\mathfrak{p}} := \frac{a_{\mathfrak{p}}}{l_{\mathfrak{p}}}(\delta_0+\delta_\infty) + \frac{1}{l_{\mathfrak{p}}}\mathrm{d}x. \tag{8.4}$$

Recalling the notation from [1], we have for $n = s_{\mathfrak{p}}$ that the admissible Green's function $g_{\mu_{\mathfrak{p}}}^{G_{\mathfrak{p}}}$ with respect to $\mu_{\mathfrak{p}}$ of the graph $G_{\mathfrak{p}}$ is given by (cf. [1],

p. 65)

$$g_{\mu_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(x_j,0)=\frac{1}{2l_{\mathfrak{p}}}x_j^2-\frac{a_{\mathfrak{p}}+s_{\mathfrak{p}}}{s_{\mathfrak{p}}l_{\mathfrak{p}}}x_j-\frac{3a_{\mathfrak{p}}+2s_{\mathfrak{p}}}{6s_{\mathfrak{p}}l_{\mathfrak{p}}},$$

$$g_{\mu_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(x_j,\infty)=\frac{1}{2l_{\mathfrak{p}}}(1-x_j)^2-\frac{a_{\mathfrak{p}}+s_{\mathfrak{p}}}{s_{\mathfrak{p}}l_{\mathfrak{p}}}(1-x_j)-\frac{3a_{\mathfrak{p}}+2s_{\mathfrak{p}}}{6s_{\mathfrak{p}}l_{\mathfrak{p}}},$$

$$g_{\mu_{\mathfrak{p}}}^{G_{\mathfrak{p}}}(x_j,x_j)=\left(1-\frac{1}{s_{\mathfrak{p}}}-\frac{1}{l_{\mathfrak{p}}}\right)x_j(1-x_j)-\frac{3a_{\mathfrak{p}}+2s_{\mathfrak{p}}}{6s_{\mathfrak{p}}l_{\mathfrak{p}}}, \tag{8.5}$$

where $0\leq x_j\leq 1$ ($j=1,\ldots,s_{\mathfrak{p}}$) denotes the coordinate along an edge. Note that for prime ideals $\mathfrak{p},\mathfrak{p}'$ of $\mathbb{Z}[\zeta_N]$ with $\mathfrak{p}|p$ and $\mathfrak{p}'|p$, we have $G_{\mathfrak{p}}=G_{\mathfrak{p}'}$, and we simply write G_p for this graph; further, we write in this case also simply $s_p,a_p,l_p,K_{G_p},\mu_p$ for $s_{\mathfrak{p}},a_{\mathfrak{p}},l_{\mathfrak{p}},K_{G_{\mathfrak{p}}},\mu_{\mathfrak{p}}$, respectively. Let $\overline{\omega}_{a,N}$ be the admissible metrized relative dualizing sheaf of the curve $X_1(N)/\mathbb{Q}$ as defined in [33], p. 181 and p. 188.

8.5. Theorem. *Let N be an odd and squarefree integer of the form $N=N^{'}qr$ with q and r relative prime integeres satisfying $q,r\geq 4$. Then, we have*

$$\overline{\omega}_{a,N}^2=3g_N\log(N)+o(g_N\log(N)).$$

Proof. By [33], theorem 5.5, we have

$$\overline{\omega}_N^2-\overline{\omega}_{a,N}^2=\frac{1}{\varphi(N)}\sum_{\mathfrak{p}|N}r_{\mathfrak{p}}\log\left(\sharp k(\mathfrak{p})\right)=\sum_{p|N}\frac{r_p}{p-1}\log(p),$$

where

$$r_{\mathfrak{p}}:=r_p:=\int_{G_p}g_{\mu_p}^{G_p}(x,x)\big((2g_N-2)\mu_p(x)+\delta_{K_{G_p}}\big).$$

Now we calculate (as in [1], p. 65) the value r_p for the graph G_p with $p|N$. The definition of r_p and the formula for the admissible measure μ_p of (8.4) yield

$$r_p=-\frac{(g_N-1)(3g_N+s_p-1)}{3s_pg_N}+\frac{(g_N-1)^2}{3g_N^2}s_p-\frac{(2s_p-1)(g_N-1)^2}{s_pg_N^2}.$$

Noting that $s_p<g_N$ and $\frac{g_N}{s_p}=p+1+O(1)$ holds for $p|N$, we obtain

$$r_p=-(p+1)+\frac{s_p}{3}+O(1).$$

Therefore, we have

$$\sum_{p|N}\frac{r_p}{p-1}\log(p)=\sum_{p|N}\frac{s_p}{3(p-1)}\log(p)+O\big(\log(N)\big).$$

Then, theorem 7.7 and the asymptotic (8.3) imply

$$\overline{\omega}_{a,N}^2=3g_N\log(N)+o(g_N\log(N)).$$

This completes the proof of the theorem. □

8.6. *Effective Bogomolov.* Let X/K be a smooth projective and geometrically connected curve over a number field K of genus $g_X > 1$. For a divisor $D \in \mathrm{Div}(X)$ of degree 1, let

$$\varphi_D : X \longrightarrow \mathrm{Jac}(X)$$

be the embedding of the curve X/K into its Jacobian $\mathrm{Jac}(X)/K$ defined by the mapping $x \mapsto [\mathcal{O}_X(x - D)]$. Then there exists an $\varepsilon > 0$ such that the set of algebraic points

$$\left\{ x \in X(\overline{K}) \,|\, h_{\mathrm{NT}}(\varphi_D(x)) \leq \varepsilon \right\}$$

is finite, where h_{NT} denotes the Néron-Tate height on $\mathrm{Jac}(X)/K$ (Bogomolov's conjecture). The conjecture was proved by E. Ullmo in the case of curves and by S.-W. Zhang, more general, for any non-torsion subvariety X of an abelian variety A/K.

Due to L. Szpiro and S.-W. Zhang it is known (see [33], theorem 5.6) that for every $\varepsilon > 0$ the set of algebraic points

$$\left\{ x \in X(\overline{K}) \,|\, h_{\mathrm{NT}}(\varphi_D(x)) < \frac{\overline{\omega}_a^2}{4(g_X - 1)} - \varepsilon \right\} \tag{8.6}$$

is finite, and that the set of algebraic points

$$\left\{ x \in X(\overline{K}) \,|\, h_{\mathrm{NT}}(\varphi_D(x)) \leq \frac{\overline{\omega}_a^2}{2(g_X - 1)} \right\} \tag{8.7}$$

is infinite if $[\mathcal{O}_X(K_X - (2g_X - 2)D)]$ is a torsion element in $\mathrm{Jac}(X)/K$.

8.7. Theorem. *Let N be an odd and squarefree positive integer of the form $N = N'qr$ with q and r relative prime integers satisfying $q, r \geq 4$. Then, for any $\varepsilon > 0$, there is a sufficiently large N such that the set of algebraic points*

$$\left\{ x \in X_1(N)(\overline{\mathbb{Q}}) \,|\, h_{\mathrm{NT}}(\varphi_D(x)) < \left(\frac{3}{4} - \varepsilon\right) \log(N) \right\}$$

is finite, and the set of algebraic points

$$\left\{ x \in X_1(N)(\overline{\mathbb{Q}}) \,|\, h_{\mathrm{NT}}(\varphi_D(x)) \leq \left(\frac{3}{2} + \varepsilon\right) \log(N) \right\}$$

is infinite, if $[\mathcal{O}_{X_1(N)}(K_{X_1(N)} - (2g_N - 2)D)]$ is a torsion element in $J_1(N)/\mathbb{Q}$.

Proof. The first and second statement of the theorem follow immediately from the height bounds (8.6) and (8.7) in conjunction with theorem 8.5. □

8.8. Remark. If $D = [0]$ then $[\mathcal{O}_{X_1(N)}(K_{X_1(N)} - (2g_N - 2)D)]$ is a torsion element in $J_1(N)/\mathbb{Q}$, which gives an example for the second statement in theorem 8.7.

9. Appendix: Meromorphic Continuation

In this appendix we give a proof of proposition 4.2.2. Recall that for N an odd positive integer and $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$, as well as $0 < u < N$ with $(u, N) = 1$, we define the zeta function

$$\zeta_{u,N}(s,l) = \sum_{(\gamma,(m,n))\in\Delta_l^{u+}(N)/\Gamma_1(N)} \frac{1}{q_\gamma(n,-m)^s} \qquad (s \in \mathbb{C},\ \mathrm{Re}(s) > 1).$$

We will show that $\zeta_{u,N}(s,l)$ is well-defined and has a meromorphic continuation to the whole s-plane. Furthermore, we will determine its residue at $s = 1$.

Throughout this section we keep the above assumptions on N, l, and u.

9.1. Let $q = [aN, bN, c] \in Q_l(N)$ with $r := \gcd(aN, bN, c)$ the greatest common divisor of the coefficients of q. We define the set

$$M_u^q(N) := \{(m,n) \in \mathbb{Z}^2 \mid (m,n) \equiv (0,u) \mod N;\ q(n,-m) > 0\} \subseteq M_u(N).$$

Let us set $\theta := \frac{bN+\sqrt{l^2-4}}{2c}$ and $\overline{\theta} := \frac{bN-\sqrt{l^2-4}}{2c}$ such that

$$q(n,-m) = c(m-\theta n)(m-\overline{\theta}n). \tag{9.1}$$

Further, we set $(t_q, u_q) := (t_0, \frac{u_0}{r})$ with (t_0, u_0) the smallest positive solution of Pell's equation $X^2 - \frac{\mathrm{disc}(q)}{r^2}Y^2 = 4$ such that $\frac{t_q - bNu_q}{2} \equiv 1 \mod N$. Then, the generator α_q of $\Gamma_1(N)_q$ is explicitly given by (cf. [32], p. 63, Satz 2)

$$\alpha_q = \begin{pmatrix} \frac{t_q - bNu_q}{2} & -cu_q \\ aNu_q & \frac{t_q + bNu_q}{2} \end{pmatrix} \in \Gamma_1(N),$$

and the power ε_q of the fundamental unit ε_0 of 4.2.1 is of the form $\varepsilon_q = \frac{t_q + u_q\sqrt{l^2-4}}{2}$. Therefore, for $(m', n') = (m,n)\alpha_q^k \in M_u(N)$, $k \in \mathbb{Z}$, we find (cf. [32], p. 70)

$$m' - \theta n' = \varepsilon_q^k(m - \theta n) \qquad \text{and} \qquad m' - \overline{\theta}n' = \overline{\varepsilon}_q^k(m - \overline{\theta}n), \tag{9.2}$$

where $\overline{\varepsilon}_q := \frac{t_q - u_q\sqrt{l^2-4}}{2}$ is the conjugate of ε_q in the quadratic field $\mathbb{Q}(\sqrt{l^2-4})$.

As $\varepsilon_q\overline{\varepsilon}_q > 0$, equations (9.1) and (9.2) imply $q(n', -m') > 0$; this gives a well-defined action

$$\begin{aligned} \Gamma_1(N)_q \times M_u^q(N) &\longrightarrow M_u^q(N) \\ (\delta, (m,n)) &\mapsto (m,n)\delta. \end{aligned} \tag{9.3}$$

With the above notation and the isomorphy $\Gamma_{1,l}(N)/\Gamma_1(N) \cong Q_l(N)/\Gamma_1(N)$ in 4.1.4, we find

$$\zeta_{u,N}(s,l) = \sum_{q\in Q_l(N)/\Gamma_1(N)} \sum_{(m,n)\in M_u^q(N)/\Gamma_1(N)_q} \frac{1}{q(n,-m)^s}.$$

Finally, we set $E_q := \frac{t_q+bNu_q}{2cu_q}$ with (t_q,u_q) as above, and define the set

$$R_u^q(N) := \left\{(m,n)\in\mathbb{Z}^2 \mid (m,n)\equiv(0,v) \mod N; n>0, m\geq E_q n\right\}.$$

9.2. Remark. By observation (4.1) we may assume that the coefficients of some representative $q=[aN,bN,c]$ of an equivalence class of $Q_l(N)/\Gamma_1(N)$ satisfies $aN>0, bN<0$, and $c>0$. We will exploit this fact in the sequel.

9.3. Lemma. *Let N be an odd positive integer and $l\in\mathbb{Z}$ with $|l|>2$ and $l\equiv 2 \mod N$. Further, let $0<u<N$ with $(u,N)=1$ as well as $0<u'<N$ with $u'\equiv -u \mod N$. Then, we have for* $\mathrm{Re}(s)>1$

$$\zeta_{u,N}(s,l)=\sum_{q\in Q_l(N)/\Gamma_1(N)}\left(\sum_{(m,n)\in R_u^q(N)}\frac{1}{q(n,-m)^s}+\sum_{(m,n)\in R_{u'}^q(N)}\frac{1}{q(n,-m)^s}\right).$$

Proof. We prove that the zeta function is well-defined in 9.6. Let us define for $q=[aN,bN,c]\in Q_l(N)$ with $c>0$ (which we can assume) the sets

$$M_u^{q\pm}(N) := \{(m,n)\in M_u^q(N) \mid m-\theta n \gtrless 0\}.$$

This allows us to write $M_u^q(N)$ as a disjoint union $M_u^q(N)=M_u^{q+}(N)\,\dot\cup\, M_u^{q-}(N)$, which descends by observation (9.2) to

$$M_u^q(N)/\Gamma_1(N)_q = M_u^{q+}(N)/\Gamma_1(N)_q \,\dot\cup\, M_u^{q-}(N)/\Gamma_1(N)_q.$$

Now, we define a map $\varphi_u^+ : M_u^{q+}(N)/\Gamma_1(N)_q \longrightarrow R_u^q(N)$ as well as a map $\varphi_u^- : M_u^{q-}(N)/\Gamma_1(N)_q \longrightarrow R_{u'}^q(N)$ which will turn out to be bijections. This proves then the statement of the lemma.
To define the map φ_u^+, let $(m,n)\in M_u^{q+}(N)$. By observation (9.2) we find for $(m',n')=(m,n)\alpha_q^k$, $k\in\mathbb{Z}$, the equation (cf. [32], p. 70)

$$\frac{m'-\overline{\theta}n'}{m'-\theta n'}=\left(\frac{\overline{\varepsilon}_q}{\varepsilon_q}\right)^k\frac{m-\overline{\theta}n}{m-\theta n}=(\varepsilon_q)^{-2k}\frac{m-\overline{\theta}n}{m-\theta n}.$$

Hence in each orbit of $M_u^{q+}(N)$ by $\Gamma_1(N)_q$, there is an element (m',n') which satisfies $1<\frac{m'-\overline{\theta}n'}{m'-\theta n'}\leq\varepsilon_q^2$. We find

$$1<\frac{m'-\overline{\theta}n'}{m'-\theta n'}\Longleftrightarrow n'>0 \quad\text{and}\quad \frac{m'-\overline{\theta}n'}{m'-\theta n'}\leq\varepsilon_q^2\Longleftrightarrow m'\geq\frac{\varepsilon_q^2\theta-\overline{\theta}}{\varepsilon_q^2-1}n',$$

and a little computation shows $\frac{\varepsilon_q^2\theta-\overline{\theta}}{\varepsilon_q^2-1}=E_q$, from which we obtain a well-defined map $\varphi_u^+ : M_u^{q+}(N)/\Gamma_1(N)_q \longrightarrow R_u^q(N)$ given by $(m,n)\cdot\Gamma_1(N)_q\mapsto(m',n')$. Now, we define the map $(\varphi_u^+)' : R_u^q(N)\longrightarrow M_u^{q+}(N)/\Gamma_1(N)_q$ by $(m,n)\mapsto(m,n)\cdot\Gamma_1(N)_q$. It is straightforward that this map is well-defined. One easily verifies that the maps φ_u^+ and $(\varphi_u^+)'$ are inverse to each other, which proves that φ_u^+ is a bijection.

To define the map φ_u^-, we let $\varphi : M_u^{q-}(N)/\Gamma_1(N)_q \xrightarrow{\sim} M_{u'}^{q+}(N)/\Gamma_1(N)_q$ be the bijection induced by mapping $(m,n) \mapsto (-m,-n)$. Defining the map $\varphi_u^- := \varphi_{u'}^+ \circ \varphi$ gives rise to a bijection $\varphi_u^- : M_u^{q-}(N)/\Gamma_1(N)_q \longrightarrow R_{u'}^q(N)$. Since φ_u^+ and φ_u^- are bijections, the statement of the lemma follows. □

9.4. Let N be an odd positive integer, $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$, $0 < u < N$ with $(u,N) = 1$, and $q \in Q_l(N)$. Then, we define the theta series $\vartheta_{u,N}^q(t)$ by

$$\vartheta_{u,N}^q(t) := \sum_{(m,n)\in R_u^q(N)} \exp\big(-tq(n,-m)\big) \qquad (t \in \mathbb{R}_{>0}), \tag{9.4}$$

which is a little variant of the theta series studied by E. Landau in [20] defined as follows: Let $E \in \mathbb{R} \setminus \{0\}$ and L_E be the lattice defined by

$$L_E := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

with $\omega_1 = (1,0)$ and $\omega_2 := (E,1)$. Let $S_{E,P}$ be the truncated and shifted lattice defined by

$$S_{E,P} := \Big\{(x,y) \in \mathbb{R}^2 \,|\, (x,y) \in P + L_E; y > 0, x \geq Ey\Big\}.$$

and $P = (x_0, y_0) \in \mathbb{R}^2$ lying inside the parallelogram determined by the four points $(0,0), (1,0), (E,1)$, and $(E+1,1)$. Let $q = [a,b,c] \in Q_l(1)$ be a quadratic form with $a > 0, b > 0, c > 0$, and discriminant $D = l^2 - 4$. Then, E. Landau considers the theta series

$$\vartheta_{E,P}^q(t) := \sum_{(x,y)\in S_{E,P}} \exp\big(-t\,q(x,y)\big) \qquad (t \in \mathbb{R}_{>0}).$$

If E satisfies $E > \frac{-b+\sqrt{D}}{2a}$, then the theta series $\vartheta_{E,P}^q(t)$ converges for $t > 0$, and we have

$$\vartheta_{E,P}^q(t) = \frac{\alpha_{-2}}{t} + \frac{\alpha_{-1}}{t^{\frac{1}{2}}} + \alpha_0 + \alpha_1 t^{\frac{1}{2}} + \ldots + \alpha_k t^{\frac{k}{2}} + O_k\left(t^{\frac{k+1}{2}}\right) \tag{9.5}$$

as $t \to 0$ with $\alpha_j \in \mathbb{R}, k \geq -2$, where all appearing constants depend on the choice of E, P, and q. Furthermore, α_{-2} is given by

$$\alpha_{-2} = \frac{1}{2\sqrt{D}} \log \frac{2aE + b + \sqrt{D}}{2aE + b - \sqrt{D}} \tag{9.6}$$

(see [20], Hilfssatz 11; note that in [20] the quadratic forms are of the form $q(X,Y) = aX^2 + 2bXY + cY^2$ and the discriminant is defined by $\operatorname{disc}(q) = b^2 - ac$. This causes the factor of 2 appearing in the expression of α_{-2}).

9.5. Lemma. *Let N be an odd positive integer, $l \in \mathbb{Z}$ with $|l| > 2$ and $l \equiv 2 \mod N$, $0 < u < N$ with $(u,N) = 1$, and $q = [aN, bN, c] \in Q_l(N)$ a* *quadratic form with* $aN > 0$, $bN < 0, c > 0$. *Then, the theta series* $\vartheta^q_{u,N}(t)$ *converges for* $t > 0$, *and we have*

$$\vartheta^q_{u,N}(t) = \frac{\beta_{-2}}{t} + \frac{\beta_{-1}}{t^{\frac{1}{2}}} + \beta_0 + \beta_1 t^{\frac{1}{2}} + \ldots + \beta_k t^{\frac{k}{2}} + O_k\left(t^{\frac{k+1}{2}}\right) \tag{9.7}$$

as $t \to 0$ *with* $\beta_j \in \mathbb{R}, k \geq -2$, *where all appearing constants depend on the choice of* u, N, *and* q. *Furthermore,* β_{-2} *is given by*

$$\beta_{-2} = \frac{1}{N^2\sqrt{l^2-4}} \log(\varepsilon_q).$$

Proof. We show that $\vartheta^q_{u,N}(t)$ is a sum of Landau's theta series $\vartheta^q_{E,P}(t)$. Let be $L := cu_qN$. First note that the points $(m,n) \equiv (0,u) \mod N$ with $m \geq E_q n$ and $n > 0$ do lie in the interior of the parallelograms of width and height L of the cone defined by $y > 0$ and $x \geq E_q y$. Now, let be $\xi := \frac{x}{L}$ and $\eta := \frac{y}{L}$. Then, we obtain $c^2u_q^2$ points $P_h = (x_h, y_h)$, $h = 1, \ldots, c^2u_q^2$, in the parallelogram now of width and height 1 with respect to the coordinates ξ and η. We set

$$A := cL^2, \quad B := -bNL^2, \quad C := aNL^2$$

and define $q' := [A,B,C]$ such that $q(y,-x) = q'(\xi,\eta)$ and $\mathrm{disc}(q') = B^2 - 4AC = \mathrm{disc}(q)L^4$. By the definition of the theta series, we have

$$\vartheta^q_{u,N}(t) = \sum_{h=1}^{c^2u_q^2} \vartheta^{q'}_{E_q,P_h}(t).$$

Since

$$E_q = \frac{t_q + bNu_q}{2cu_q} > \frac{bN + \sqrt{l^2-4}}{2c} = \frac{-B + \sqrt{\mathrm{disc}(q')}}{2A},$$

we can apply Landau's result and obtain

$$\left|\sum_{h=1}^{c^2u_q^2}\left(\vartheta^{q'}_{E_q,P_h}(t) - \frac{\alpha_{-2,h}}{t} - \frac{\alpha_{-1,h}}{t^{\frac{1}{2}}} - \alpha_{0,h} - \alpha_{1,h}t^{\frac{1}{2}} - \ldots - \alpha_{n,h}t^{\frac{k}{2}}\right)\right| < Ct^{\frac{k+1}{2}}$$

with $C := \sum_{h=1}^{c^2u_q^2} C_h$, where the C_h are the implied constants of formula (9.5).

This implies with $\beta_j := \sum_{h=1}^{c^2u_q^2} \alpha_{j,h}$ the claimed growth behaviour (9.7) of $\vartheta^q_{u,N}(t)$ as $t \to 0$.

It remains to determine β_{-2}. By formula (9.6) we have

$$\begin{aligned}\beta_{-2} =&c^2u_q^2\frac{1}{2L^2\sqrt{l^2-4}}\log\frac{2cE_q-bN+\sqrt{l^2-4}}{2cE_q-bN-\sqrt{l^2-4}}\\ =&\frac{1}{2N^2\sqrt{l^2-4}}\log\frac{t_q+u_q\sqrt{l^2-4}}{t_q-u_q\sqrt{l^2-4}}\\ =&\frac{1}{N^2\sqrt{l^2-4}}\log\frac{t_q+u_q\sqrt{l^2-4}}{2}=\frac{1}{N^2\sqrt{l^2-4}}\log(\varepsilon_q).\end{aligned}$$

This finishes the proof of the lemma. □

9.6. *Proof of proposition 4.2.2.* Since $q(n,-m)>0$ holds for $(m,n)\in R_u^q(N)$, we can use the well-known integral representation

$$\sum_{(m,n)\in R_u^q(N)}\frac{1}{q(n,-m)^s}=\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\vartheta_{u,N}^q(t)\mathrm{d}t \qquad (\mathrm{Re}(s)>1). \tag{9.8}$$

Using the asymptotics (9.7), we find

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\vartheta_{u,N}^q(t)\mathrm{d}t =&\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\frac{\beta_{-2}}{t}+\frac{\beta_{-1}}{t^{\frac{1}{2}}}+\beta_0+\ldots+\beta_k t^{\frac{k}{2}}\right)\mathrm{d}t+\\ &\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}R_k(t)\mathrm{d}t+\frac{1}{\Gamma(s)}\int_1^{+\infty}t^{s-1}\vartheta_{u,N}^q(t)\mathrm{d}t\end{aligned} \tag{9.9}$$

with $R_k(t)=O_k(t^{\frac{k+1}{2}})$. For $\mathrm{Re}(s)>1$ the first term on the right hand side is well-defined and is equal to

$$\frac{1}{\Gamma(s)}\left(\frac{\beta_{-2}}{s-1}+\frac{\beta_{-1}}{s-1/2}+\frac{\beta_0}{s}+\frac{\beta_1}{s+1/2}+\ldots+\frac{\beta_k}{s+k/2}\right), \tag{9.10}$$

which extends to a meromorphic function on the whole complex plane. For $\mathrm{Re}(s)>\frac{-k+1}{2}$ with $k\geq 2$ the integral of second term on the right hand side converges absolutely and uniformly, since in this case we have

$$t^{s-1}R_k(t)=O(t) \qquad (t\to 0).$$

The third integral converges absolutely and uniformly for any $s\in\mathbb{C}$. So we can deduce from equations (9.8) and (9.9), applied to u and u', that $\zeta_{u,N}(s,l)$ is well-defined for $\mathrm{Re}(s)>1$ and has a meromorphic continuation to the half plane $\mathrm{Re}(s)>\frac{-k+1}{2}$ with $k\geq 2$. This finishes the proof of the first part of the proposition as k can be arbitrarily large.
It remains to calculate the residue of $\zeta_{u,N}(s,l)$ at $s=1$. We deduce from equation (9.10) that $\zeta_{u,N}(s,l)$ has a simple pole at $s=1$ with residue

$$\mathrm{res}_{s=1}\zeta_{u,N}(s,l)=2h_l(N)\beta_{-2}=\frac{2h_l(N)}{N^2\sqrt{l^2-4}}\log(\varepsilon_q)$$

as β_{-2} does not depend on u and u'. This proves the proposition 4.2.2.

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