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Stéphane VIGUIÉ

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Invariants and coinvariants of semilocal units modulo elliptic units

par STÉPHANE VIGUIÉ

RÉSUMÉ. Soient p un nombre premier, et k un corps quadratique imaginaire dans lequel p se décompose en deux idéaux maximaux \mathfrak{p} et $\bar{\mathfrak{p}}$. Soit k_∞ l'unique \mathbb{Z}_p -extension de k non ramifiée en dehors de \mathfrak{p} , et soit K_∞ une extension finie de k_∞ , abélienne sur k . Soit $\mathcal{U}_\infty/\mathcal{C}_\infty$ la limite projective du module des unités semi-locales principales modulo le module des unités elliptiques. Nous prouvons que les différents modules des invariants et des co-invariants de $\mathcal{U}_\infty/\mathcal{C}_\infty$ sont finis. Notre approche utilise les distributions et la fonction L p -adique, définie dans [5].

ABSTRACT. Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two primes \mathfrak{p} and $\bar{\mathfrak{p}}$. Let k_∞ be the unique \mathbb{Z}_p -extension of k which is unramified outside of \mathfrak{p} , and let K_∞ be a finite extension of k_∞ , abelian over k . Let $\mathcal{U}_\infty/\mathcal{C}_\infty$ be the projective limit of principal semi-local units modulo elliptic units. We prove that the various modules of invariants and coinvariants of $\mathcal{U}_\infty/\mathcal{C}_\infty$ are finite. Our approach uses distributions and the p -adic L-function, as defined in [5].

1. Introduction

Let p be a prime number, and let k be an imaginary quadratic number field in which p decomposes into two distinct primes \mathfrak{p} and $\bar{\mathfrak{p}}$. Let k_∞ be the unique \mathbb{Z}_p -extension of k which is unramified outside of \mathfrak{p} , and let K_∞ be a finite extension of k_∞ , abelian over k . Let G_∞ be the Galois group of K_∞/k . We choose a decomposition of G_∞ as a direct product of a finite group G (the torsion subgroup of G_∞) and a topological group Γ isomorphic to \mathbb{Z}_p , $G_\infty = G \times \Gamma$. For all $n \in \mathbb{N}$, let K_n be the field fixed by $\Gamma_n := \Gamma^{p^n}$, and let $G_n := \text{Gal}(K_n/k)$. Remark that there may be different choices for Γ , but when p^n is larger than the order of the p -part of G , the group Γ_n does not depend on the choice of Γ .

Let F/k be a finite abelian extension of k . We denote by \mathcal{O}_F the ring of integers of F . Then we write \mathcal{O}_F^\times for the group of global units of F , and C_F for the group of elliptic units of F (see section 3). We set $\mathcal{C}_F := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_F$.

For all prime ideal \mathfrak{q} of \mathcal{O}_F above \mathfrak{p} , we write $F_{\mathfrak{q}}$, $\mathcal{O}_{F_{\mathfrak{q}}}$, and $\mathcal{O}_{F_{\mathfrak{q}}}^{\times}$ respectively for the completion of F at \mathfrak{q} , the ring of integers of $F_{\mathfrak{q}}$, and the group of units of $\mathcal{O}_{F_{\mathfrak{q}}}$. Then we write \mathcal{U}_F for the pro- p -completion of $\prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$. The

injection $\mathcal{O}_F^{\times} \hookrightarrow \prod_{\mathfrak{q}|\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{q}}}^{\times}$ induces a canonical map $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F^{\times} \rightarrow \mathcal{U}_F$. The

Leopoldt conjecture, which is known to be true for abelian extensions of k , states that this map is injective. For all $n \in \mathbb{N}$, we write \mathcal{C}_n and \mathcal{U}_n for \mathcal{C}_{K_n} and \mathcal{U}_{K_n} respectively. We define $\mathcal{C}_{\infty} := \varprojlim \mathcal{C}_n$ and $\mathcal{U}_{\infty} := \varprojlim \mathcal{U}_n$ by taking projective limit under the norm maps. The injections $\mathcal{C}_n \hookrightarrow \mathcal{U}_n$ are norm compatible and taking the limit we obtain an injection $\mathcal{C}_{\infty} \hookrightarrow \mathcal{U}_{\infty}$.

For any profinite group \mathcal{G} , and any commutative ring R , we define the Iwasawa algebra

$$R[[\mathcal{G}]] := \varprojlim R[\mathcal{H}],$$

where the projective limit is over all finite quotients \mathcal{H} of \mathcal{G} . Then \mathcal{C}_{∞} and \mathcal{U}_{∞} are naturally $\mathbb{Z}_p[[G_{\infty}]]$ -modules. It is well known that they are finitely generated over $\mathbb{Z}_p[[\Gamma]]$. Moreover one can show that $\mathcal{U}_{\infty}/\mathcal{C}_{\infty}$ is torsion over $\mathbb{Z}_p[[\Gamma]]$ (see [17, Proposition 3.1]). Let us fix a topological generator γ of Γ , and set $T := \gamma - 1$. We denote by \mathbb{C}_p the completion of an algebraic closure of \mathbb{Q}_p . For any complete subfield L of \mathbb{C}_p , finitely ramified over \mathbb{Q}_p , we denote by \mathcal{O}_L the complete discrete valuation ring of integers of L . Then the ring $\mathcal{O}_L[[\Gamma]]$ is isomorphic to $\mathcal{O}_L[[T]]$. It is well known that $\mathcal{O}_L[[T]]$ is a noetherian, regular, local domain. We also recall that $\mathcal{O}_L[[T]]$ is a unique factorization domain. If \mathbf{u}_L is a uniformizer of \mathcal{O}_L , then the maximal ideal \mathfrak{M} of $\mathcal{O}_L[[T]]$ is generated by \mathbf{u}_L and T , and $\mathcal{O}_L[[T]]$ is a complete topological ring with respect to its \mathfrak{M} -adic topology. A morphism $f : M \rightarrow N$ between two finitely generated $\mathcal{O}_L[[T]]$ -module is called a pseudo-isomorphism if its kernel and its cokernel are finitely generated and torsion over \mathcal{O}_L . If a finitely generated $\mathcal{O}_L[[T]]$ -module M is given, then one may find elements P_1, \dots, P_r in $\mathcal{O}_L[[T]]$, irreducible in $\mathcal{O}_L[[T]]$, and nonnegative integers n_0, \dots, n_r , such that there is a pseudo-isomorphism

$$M \longrightarrow \mathcal{O}_L[[T]]^{n_0} \oplus \bigoplus_{i=1}^r \mathcal{O}_L[[T]] / (P_i^{n_i}).$$

Moreover, the integer n_0 and the ideals $(P_1^{n_1}), \dots, (P_r^{n_r})$, are uniquely determined by M . If $n_0 = 0$, then the ideal generated by $P_1^{n_1} \dots P_r^{n_r}$ is called the characteristic ideal of M , and is denoted by $\text{char}_{\mathcal{O}_L[[T]]}(M)$.

Let χ be an irreducible \mathbb{C}_p -character of G . Let $L(\chi) \subset \mathbb{C}_p$ be the abelian extension of L generated by the values of χ . The group G acts naturally on $L(\chi)$ if we set, for all $g \in G$ and all $x \in L(\chi)$, $g.x := \chi(g)x$. For any $\mathcal{O}_L[G]$ -module Y , we define the χ -quotient Y_{χ} by $Y_{\chi} := \mathcal{O}_{L(\chi)} \otimes_{\mathcal{O}_L[G]} Y$. If Y is an $\mathcal{O}_L[[G_{\infty}]]$ -module, then Y_{χ} is an $\mathcal{O}_{L(\chi)}[[T]]$ -module in a natural

way. Moreover if L contains a $[K_0 : k]$ -th primitive root of unity, then there is $(a, b) \in \mathbb{N}^2$ such that

$$(1.1) \quad \mathbf{u}_L^a \text{char}_{\mathcal{O}_L[[T]]}(M) = \mathbf{u}_L^b \prod_{\chi} \text{char}_{\mathcal{O}_L[[T]]}(M_{\chi}),$$

where the product is over all irreducible \mathbb{C}_p -character on G .

For any profinite group \mathcal{G} , any normal subgroup \mathcal{H} of \mathcal{G} and any $\mathcal{O}_L[[\mathcal{G}]]$ -module M , we denote by $M^{\mathcal{H}}$ the module of \mathcal{H} -invariants of M , that is to say the maximal submodule of M which is invariant under the action of \mathcal{H} . We denote by $M_{\mathcal{H}}$ the module of \mathcal{H} -coinvariants of M , which is the quotient of M by the closed submodule topologically generated by the elements $(h - 1)m$ with $h \in \mathcal{H}$ and $m \in M$.

In this article, we prove that for all $n \in \mathbb{N}$, the module of Γ_n -invariants and the module of Γ_n -coinvariants of $\mathcal{U}_{\infty}/\mathcal{C}_{\infty}$ are finite (see Theorem 6.1). It generalizes a part of a result of Coates-Wiles [4, Theorem 1], where this result is shown at the χ^i -parts, for $i \not\equiv 0$ modulo $p - 1$, and for χ the character giving the action of G on the \mathfrak{p} -torsion points of a certain elliptic curve. But the result of [4] is stated for non-exceptional primes p (in particular $p \notin \{2, 3\}$), and under the assumption that \mathcal{O}_k is principal. Here we prove the general case.

Moreover we would like to mention an application of Theorem 6.1 to the main conjecture of Iwasawa theory. For all $n \in \mathbb{N}$, we set $\mathcal{E}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{K_n}^{\times}$ and we denote by A_n the p -part of the class-group $\text{Cl}(\mathcal{O}_{K_n})$ of \mathcal{O}_{K_n} . We define $\mathcal{E}_{\infty} := \varprojlim \mathcal{E}_n$ and $A_{\infty} := \varprojlim A_n$, projective limits under the norm maps. A formulation of the (one variable) main conjecture says that $\text{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_{\infty}/\mathcal{C}_{\infty})_{\chi} = \text{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty, \chi})$, where $\mathbb{Z}_p(\chi)$ is the ring of integers of $\mathbb{Q}_p(\chi)$. It has been proved in many cases by the use of Euler systems. We refer the reader to the pioneering work of Rubin in [15, Theorem 4.1] and [16, Theorem 2], adapted to the cyclotomic case by Greither in [7, Theorem 3.2]. The method is now classical, applied by many authors, see [2, Theorem 3.1], [11] and [17]. However the result of Gillard [6] which implies the nullity of the μ -invariant of A_{∞} is stated for $p \notin \{2, 3\}$, and for $p \in \{2, 3\}$ we just obtain a divisibility relation

$$(1.2) \quad \text{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty, \chi}) \text{ divides } p^a \text{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_{\infty}/\mathcal{C}_{\infty})_{\chi},$$

for some $a \in \mathbb{N}$ (see [11] and [17]). Following the ideas of Belliard in [1], in [18] we deduce from Theorem 6.1 that for $p \in \{2, 3\}$ the $\mathbb{Z}_p[[T]]$ -modules $\mathcal{E}_{\infty}/\mathcal{C}_{\infty}$ and A_{∞} have the same Iwasawa's μ and λ invariants. This result, together with (1.2), implies that there is $(a, b) \in \mathbb{N}^2$ such that the following raw form of the main conjecture holds,

$$\mathbf{u}_{\chi}^a \text{char}_{\mathbb{Z}_p(\chi)[[T]]}(A_{\infty, \chi}) = \mathbf{u}_{\chi}^b \text{char}_{\mathbb{Z}_p(\chi)[[T]]}(\mathcal{E}_{\infty}/\mathcal{C}_{\infty})_{\chi},$$

where \mathbf{u}_{χ} is a uniformizer of $\mathbb{Z}_p(\chi)$.

2. Distributions.

In this section, let A be a commutative ring and let \mathcal{G} be a profinite group. We denote by $\mathfrak{X}(\mathcal{G})$ the set of compact-open subsets of \mathcal{G} . Remark that for any $X \in \mathfrak{X}(\mathcal{G})$, one can find a finite subset F of X , and an open normal subgroup \mathcal{H} of \mathcal{G} , such that $X = \bigcup_{x \in F} x\mathcal{H}$.

Definition 1. An A -distribution on \mathcal{G} is an application $\mu : \mathfrak{X}(\mathcal{G}) \rightarrow A$, such that for all $(X_1, X_2) \in \mathfrak{X}(\mathcal{G})^2$, if $X_1 \cap X_2 = \emptyset$, then

$$\mu(X_1 \cup X_2) = \mu(X_1) + \mu(X_2).$$

We denote by $\mathcal{M}(\mathcal{G}, A)$ the A -module of A -distributions on \mathcal{G} . Moreover for $X \in \mathfrak{X}(\mathcal{G})$ and $\mu \in \mathcal{M}(\mathcal{G}, A)$, we denote by $\mu|_X$ the A -distribution on \mathcal{G} defined by

$$\mu|_X : \mathfrak{X}(X) \longrightarrow A, \quad Y \longmapsto \mu(Y \cap X).$$

Let $\pi : \mathcal{G} \rightarrow \mathcal{G}'$ be a continuous map between two profinite groups. To any distribution $\mu \in \mathcal{M}(\mathcal{G}, A)$ we attach the unique A -distribution $\pi_*\mu$ on \mathcal{G}' , such that for all $X \in \mathfrak{X}(\mathcal{G}')$,

$$\pi_*\mu(X) = \mu(\pi^{-1}(X)).$$

For any $\sigma \in \mathcal{G}$, let us also denote by $\sigma_*\mu$ the unique A -distribution on \mathcal{G} , such that for all $X \in \mathfrak{X}(\mathcal{G})$,

$$\sigma_*\mu(X) = \mu(\sigma^{-1}X).$$

Assume moreover that π is an open (continuous) group morphism, such that $\text{Ker}(\pi)$ is finite. To any distribution $\mu' \in \mathcal{M}(\mathcal{G}', A)$ we attach the unique A -distribution $\pi^\sharp\mu'$ on \mathcal{G} , such that for all $g \in \mathcal{G}$, and all open subgroup \mathcal{H} of \mathcal{G} ,

$$(2.1) \quad \pi^\sharp\mu'(g\mathcal{H}) = \#(\mathcal{H} \cap \text{Ker}(\pi)) \mu'(\pi(g\mathcal{H})).$$

Then we have

$$(2.2) \quad \pi^\sharp\pi_*\mu = \sum_{\sigma \in \text{Ker}(\pi)} \sigma_*\mu \quad \text{and} \quad \pi_*\pi^\sharp\mu' = \#(\text{Ker}(\pi)) \mu'|_{\text{Im}(\pi)}.$$

For $(\alpha_1, \alpha_2) \in \mathcal{M}(\mathcal{G}, A)^2$, there is a unique A -distribution β on $\mathcal{G} \times \mathcal{G}$ such that for all $(X_1, X_2) \in \mathfrak{X}(\mathcal{G})^2$, $\beta(X_1 \times X_2) = \alpha_1(X_1)\alpha_2(X_2)$. Then the convolution product $\alpha_1\alpha_2$ of α_1 and α_2 is defined by $\alpha_1\alpha_2 := m_*\beta$, where $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, $(\sigma_1, \sigma_2) \mapsto \sigma_1\sigma_2$. Once equipped with the convolution product, $\mathcal{M}(\mathcal{G}, A)$ is an A -algebra. For any A -distribution μ on \mathcal{G} , let us denote by $\underline{\mu}$ the unique element in $A[[\mathcal{G}]]$ such that for all open normal

subgroup \mathcal{H} of \mathcal{G} , the image $\underline{\mu}_{\mathcal{H}}$ of $\underline{\mu}$ in $A[\mathcal{G}/\mathcal{H}]$ is given by

$$\underline{\mu}_{\mathcal{H}} = \sum_{g \in \mathcal{G}/\mathcal{H}} \mu(\tilde{g}\mathcal{H})g,$$

where for any $g \in \mathcal{G}/\mathcal{H}$, $\tilde{g} \in \mathcal{G}$ is an arbitrary preimage of g . Then we have a canonical isomorphism

$$\mathcal{M}(\mathcal{G}, A) \xrightarrow{\sim} A[[\mathcal{G}]], \quad \mu \longmapsto \underline{\mu},$$

and for any $\mu \in \mathcal{M}(\mathcal{G}, A)$ and any $\sigma \in \mathcal{G}$, we have $\sigma_*\mu = \sigma\underline{\mu}$. Also we mention that if $\tilde{\pi} : A[[\mathcal{G}]] \rightarrow A[[\mathcal{G}']]$ is the canonical morphism defined by π , then we have the following commutative squares,

$$\begin{array}{ccc} \mathcal{M}(\mathcal{G}, A) \xrightarrow{\sim} A[[\mathcal{G}]] & \text{and} & \mathcal{M}(\mathcal{G}, A) \xrightarrow{\sim} A[[\mathcal{G}]] & \Sigma h \\ \pi_* \downarrow & & \uparrow \pi^\# & \uparrow \\ \mathcal{M}(\mathcal{G}', A) \xrightarrow{\sim} A[[\mathcal{G}']] & & \mathcal{M}(\mathcal{G}', A) \xrightarrow{\sim} A[[\mathcal{G}']] & \downarrow g \end{array}$$

where for all $g \in \mathcal{G}'$, Σh is the sum over all $h \in \mathcal{G}$ such that $\pi(h) = g$.

Proposition 2.1. *Let $\pi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be an open continuous morphism of profinite groups, such that $\text{Ker}(\pi)$ is finite. The morphism $\pi^\# : \mathcal{M}(\mathcal{G}_2, A) \rightarrow \mathcal{M}(\mathcal{G}_1, A)$ is injective if and only if π is surjective. Moreover if $\#(\text{Ker}(\pi))$ is not a zero divisor in A , then the image of $\pi^\#$ is $\mathcal{M}(\mathcal{G}_1, A)^{\text{Ker}(\pi)}$.*

Proof. Let $\mu_2 \in \mathcal{M}(\mathcal{G}_2, A)$. From (2.1) it is straightforward to check that $\pi^\#\mu_2 = 0$ if and only if $\mu_2(X) = 0$ for all $X \in \mathfrak{X}(\text{Im}(\pi))$, and then we deduce that $\pi^\#$ is injective if and only if π is surjective. For any $\sigma \in \text{Ker}(\pi)$, any $g \in \mathcal{G}_1$, and any open subgroup \mathcal{H} of \mathcal{G}_1 , we have

$$\begin{aligned} \sigma_*\pi^\#\mu_2(g\mathcal{H}) &= \#(\mathcal{H} \cap \text{Ker}(\pi)) \mu_2(\pi(\sigma^{-1}g\mathcal{H})) \\ &= \#(\mathcal{H} \cap \text{Ker}(\pi)) \mu_2(\pi(g\mathcal{H})) \\ &= \pi^\#\mu_2(g\mathcal{H}), \end{aligned}$$

hence $\sigma_*\pi^\#\mu_2 = \pi^\#\mu_2$, and $\text{Im}(\pi^\#) \subseteq \mathcal{M}(\mathcal{G}_1, A)^{\text{Ker}(\pi)}$.

Now let $\mu_1 \in \mathcal{M}(\mathcal{G}_1, A)^{\text{Ker}(\pi)}$. Let \mathcal{H} be an open subgroup of $\text{Im}(\pi)$, and $g \in \mathcal{G}_1$. Let \mathcal{W} be an open normal subgroup of $\pi^{-1}(\mathcal{H})$ such that $\mathcal{W} \cap \text{Ker}(\pi)$ is trivial. Let R be a complete representative system of $\pi^{-1}(\mathcal{H})$ modulo $\mathcal{W}\text{Ker}(\pi)$. Then $(\sigma r)_{(\sigma, r) \in \text{Ker}(\pi) \times R}$ is a complete representative system of

$\pi^{-1}(\mathcal{H})$ modulo \mathcal{W} , and we have

$$\begin{aligned} \mu_1 \left(\pi^{-1}(\mathcal{H})g \right) &= \sum_{(\sigma,r) \in \text{Ker}(\pi) \times R} \mu_1(\sigma r \mathcal{W}g) \\ &= \sum_{(\sigma,r) \in \text{Ker}(\pi) \times R} \left(\sigma^{-1} \right)_* \mu_1(r \mathcal{W}g) \\ &= \sum_{(\sigma,r) \in \text{Ker}(\pi) \times R} \mu_1(r \mathcal{W}g) \\ &= \#(\text{Ker}(\pi)) \sum_{r \in R} \mu_1(r \mathcal{W}g). \end{aligned}$$

Hence $\pi_*\mu_1$ takes values in $\#(\text{Ker}(\pi))A$, and we deduce the equality $\mu_1 = \pi^\# \left(\#(\text{Ker}(\pi))^{-1} \pi_*\mu_1 \right)$ from (2.2). □

Now assume $A := \mathcal{O}_L$ for some complete subfield L of \mathbb{C}_p , finitely ramified over \mathbb{Q}_p . An A -distribution on \mathcal{G} is called a measure. Let $\mu \in \mathcal{M}(\mathcal{G}, A)$ be such a measure, and let V be a complete separated topological A -module, such that the open submodules of V form a neighborhood basis for 0. Let $\mathcal{C}(\mathcal{G}, V)$ be the A -module of continuous maps from \mathcal{G} to V , equipped with the uniform convergence topology. For any $X \in \mathfrak{X}(\mathcal{G})$, we denote by $1_X : \mathcal{G} \rightarrow A$ the map such that $1_X(x) = 1$ for $x \in X$ and $1_X(x) = 0$ for $x \in \mathcal{G} \setminus X$. Then there is a unique continuous A -linear map

$$\mathcal{C}(\mathcal{G}, V) \longrightarrow V, \quad f \longmapsto \int f(t).d\mu(t),$$

such that for all $X \in \mathfrak{X}(\mathcal{G})$ and all $v \in V$, $\int 1_X(t)v.d\mu(t) = \mu(X)v$ (see [9, Chapter 4, §1]). For $X \in \mathfrak{X}(\mathcal{G})$ and $f \in \mathcal{C}(\mathcal{G}, V)$, we write $\int_X f.d\mu$ for $\int 1_X f.d\mu$. Then for $\sigma \in \mathcal{G}$, we have

$$(2.3) \quad \int_X f(t).d\sigma_*\mu(t) = \int_{\sigma^{-1}X} f(\sigma t).d\mu(t),$$

the equality being obvious if f is locally constant, and then extended to all $f \in \mathcal{C}(\mathcal{G}, V)$ by continuity. Then for $\mu \in \mathcal{M}(\Gamma, A)$, we have

$$(2.4) \quad \underline{\mu} = \int (1 + T)^{\kappa(\sigma)}.d\mu(\sigma) \quad \text{in } A[[T]],$$

where $\kappa : \Gamma \rightarrow \mathbb{Z}_p$ is the unique isomorphism of profinite groups such that $\kappa(\gamma) = 1$. Moreover if we write $\mathfrak{m}_{\mathbb{C}_p}$ for the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, then for any $x \in \mathfrak{m}_{\mathbb{C}_p}$ we have

$$(2.5) \quad \underline{\mu}(x) = \int (1 + x)^{\kappa(\sigma)}.d\mu(\sigma) \quad \text{in } \mathbb{C}_p,$$

see [9, Chapter 4, §1, Theorem 1.2, p. 98].

3. Elliptic units.

For L and L' two \mathbb{Z} -lattices of \mathbb{C} such that $L \subseteq L'$ and $[L' : L]$ is prime to 6, we denote by $z \mapsto \psi(z; L, L')$ the elliptic function defined in [14]. For \mathfrak{m} a nonzero proper ideal of \mathcal{O}_k , and \mathfrak{a} a nonzero ideal of \mathcal{O}_k prime to $6\mathfrak{m}$, G. Robert proved that $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \in k(\mathfrak{m})$, where $k(\mathfrak{m})$ is the ray class field of k , modulo \mathfrak{m} . If $\varphi_{\mathfrak{m}}(1) \in k(\mathfrak{m})^\times$ is the Robert-Ramachandra invariant, as defined in [12, p. 15], or in [5, p. 55], we have by [13, Corollaire 1.3, (iii)]

$$(3.1) \quad \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{12m} = \varphi_{\mathfrak{m}}(1)^{N(\mathfrak{a}) - (\mathfrak{a}, k(\mathfrak{m})/k)},$$

where m is the positive generator of $\mathfrak{m} \cap \mathbb{Z}$, $N(\mathfrak{a}) := \#(\mathcal{O}_k/\mathfrak{a})$ and $(\mathfrak{a}, k(\mathfrak{m})/k)$ is the Artin automorphism of $k(\mathfrak{m})/k$ defined by \mathfrak{a} . Let $S(\mathfrak{m})$ be the set of maximal ideals of \mathcal{O}_k which divide \mathfrak{m} . Then $\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})$ and $\varphi_{\mathfrak{m}}(1)$ are units if and only if $|S(\mathfrak{m})| \geq 2$. More precisely, if we denote by $w_{\mathfrak{m}}$ the number of roots of unity of k which are congruent to 1 modulo \mathfrak{m} , and if we write w_k for the number of roots of unity in k , then by [13, (iv'), p. 21], we have

$$(3.2) \quad \varphi_{\mathfrak{m}}(1)\mathcal{O}_{k(\mathfrak{m})} = \begin{cases} (1) & \text{if } 2 \leq |S(\mathfrak{m})| \\ (\mathfrak{q})_{k(\mathfrak{m})}^{12mw_{\mathfrak{m}}/w_k} & \text{if } S(\mathfrak{m}) = \{\mathfrak{q}\}, \end{cases}$$

where $(\mathfrak{q})_{k(\mathfrak{m})}$ is the product of the prime ideals of $\mathcal{O}_{k(\mathfrak{m})}$ which lie above \mathfrak{q} . Moreover, if \mathfrak{a} is prime to $6\mathfrak{m}\mathfrak{q}$, then by [13, Corollaire 1.3, (ii-1)] we have

$$(3.3) \quad N_{k(\mathfrak{m}\mathfrak{q})/k(\mathfrak{m})} \left(\psi(1; \mathfrak{m}\mathfrak{q}, \mathfrak{a}^{-1}\mathfrak{m}\mathfrak{q}) \right)^{w_{\mathfrak{m}}w_{\mathfrak{m}\mathfrak{q}}^{-1}} = \begin{cases} \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m})^{1 - (\mathfrak{q}, k(\mathfrak{m})/k)^{-1}} & \text{if } \mathfrak{q} \nmid \mathfrak{m}, \\ \psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) & \text{if } \mathfrak{q} \mid \mathfrak{m}. \end{cases}$$

Definition 2. Let $F \subset \mathbb{C}$ be a finite abelian extension of k , and write $\mu(F)$ for the group of roots of unity in F . Let \mathfrak{m} be a nonzero proper ideal of \mathcal{O}_k . We define the $\mathbb{Z}[\text{Gal}(F/k)]$ -submodule $\Psi(F, \mathfrak{m})$ of F^\times , generated by the $w_{\mathfrak{m}}$ -roots of all $N_{k(\mathfrak{m})/k(\mathfrak{m}) \cap F} \left(\psi(1; \mathfrak{m}, \mathfrak{a}^{-1}\mathfrak{m}) \right)$, where \mathfrak{a} is any nonzero ideal of \mathcal{O}_k prime to $6\mathfrak{m}$. Also, we set $\Psi'(F, \mathfrak{m}) := \mathcal{O}_F^\times \cap \Psi(F, \mathfrak{m})$.

Then, we let C_F be the group generated by $\mu(F)$ and by all $\Psi'(F, \mathfrak{m})$, for any nonzero proper ideal \mathfrak{m} of \mathcal{O}_k .

Remark 1. Let \mathfrak{m} and \mathfrak{g} be two nonzero proper ideals of \mathcal{O}_k , such that the conductor of F/k divides \mathfrak{m} . Let us denote by $\mathfrak{g} \wedge \mathfrak{m}$ the gcd of \mathfrak{g} and \mathfrak{m} . If $\mathfrak{g} \wedge \mathfrak{m} = 1$, then $\Psi'(F, \mathfrak{g}) \subseteq C_F \cap \mathcal{O}_{k(1)}^\times$. Else by (3.3) we have $\Psi'(F, \mathfrak{g}) \subseteq \Psi'(F, \mathfrak{g} \wedge \mathfrak{m})$.

We define $\mathcal{C}_n := \mathbb{Z}_p \otimes_{\mathbb{Z}} C_{K_n}$, and $\mathcal{C}_\infty := \varprojlim (\mathcal{C}_n)$, projective limit under the norm maps. For any nonzero ideal \mathfrak{g} of \mathcal{O}_k , we define

$$\Psi(K_n, \mathfrak{gp}^\infty) := \bigcup_{i=1}^\infty \Psi(K_n, \mathfrak{gp}^i) \quad \text{and} \quad \Psi'(K_n, \mathfrak{gp}^\infty) := \bigcup_{i=1}^\infty \Psi'(K_n, \mathfrak{gp}^i).$$

Then the projective limits under the norm maps are denoted by

$$\begin{aligned} \overline{\Psi}(K_\infty, \mathfrak{gp}^\infty) &:= \varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi(K_n, \mathfrak{gp}^\infty)), \\ \overline{\Psi}'(K_\infty, \mathfrak{gp}^\infty) &:= \varprojlim (\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi'(K_n, \mathfrak{gp}^\infty)). \end{aligned}$$

Let us write \mathcal{I} for the set of nonzero ideals of \mathcal{O}_k which are prime to \mathfrak{p} . For $\mathfrak{g} \in \mathcal{I}$, we set $K_{\mathfrak{g},\infty} := k(\mathfrak{gp}^\infty) = \bigcup_{n \in \mathbb{N}} k(\mathfrak{gp}^n)$, and $G_{\mathfrak{g},\infty} := \text{Gal}(K_{\mathfrak{g},\infty}/k)$. Then we write $G_{\mathfrak{g}}$ for the torsion subgroup of $G_{\mathfrak{g},\infty}$. We denote by \mathcal{I}' the subset of \mathcal{I} containing all the $\mathfrak{g} \in \mathcal{I}$ such that $w_{\mathfrak{g}} = 1$. In the sequel, we fix once and for all $\mathfrak{f} \in \mathcal{I}'$ such that $K_\infty \subseteq K_{\mathfrak{f},\infty}$. We choose arbitrarily a subgroup of $G_{\mathfrak{f},\infty}$, isomorphic to \mathbb{Z}_p , such that its image in G_∞ is Γ . Then for any $\mathfrak{g} \in \mathcal{I}$ such that $\mathfrak{g}|\mathfrak{f}$, we have the decomposition $G_{\mathfrak{g},\infty} = G_{\mathfrak{g}} \times \Gamma$.

Remark 2. From Remark 1, \mathcal{C}_∞ is generated by all the $\overline{\Psi}'(K_\infty, \mathfrak{gp}^\infty)$, where $\mathfrak{g} \in \mathcal{I}$ is such that $\mathfrak{g}|\mathfrak{f}$.

From (3.3), for $\mathfrak{g} \in \mathcal{I}$ such that $\mathfrak{g}|\mathfrak{f}$, and for any nonzero ideal \mathfrak{a} of \mathcal{O}_k which is prime to $6\mathfrak{gp}$, there is a unique

$$\psi \langle \mathfrak{g}, \mathfrak{a} \rangle \in \overline{\Psi}(K_{\mathfrak{g},\infty}, \mathfrak{gp}^\infty)$$

such that for large enough $n \in \mathbb{N}$, the canonical image of $\psi \langle \mathfrak{g}, \mathfrak{a} \rangle$ in $\mathbb{Z}_p \otimes_{\mathbb{Z}} \Psi(k(\mathfrak{gp}^n), \mathfrak{gp}^\infty)$ is $1 \otimes \psi(1; \mathfrak{gp}^n, \mathfrak{a}^{-1}\mathfrak{gp}^n)$.

4. From semilocal units to measures.

Let $\mathbb{Q}_p^{\text{nr}} \subseteq \mathbb{C}_p$ be the maximal unramified algebraic extension of \mathbb{Q}_p , and let L be the completion of \mathbb{Q}_p^{nr} . We denote by $\mathcal{O}_{\mathfrak{f}}$ the ring $\mathcal{O}_L[\zeta]$, where ζ is any primitive $[K_{\mathfrak{f},0} : k]$ -th root of unity in \mathbb{C}_p . For all $(\mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{I}^2$ such that $\mathfrak{g}_1|\mathfrak{g}_2$, we denote by $\pi_{\mathfrak{g}_2, \mathfrak{g}_1} : G_{\mathfrak{g}_2, \infty} \rightarrow G_{\mathfrak{g}_1, \infty}$ the restriction map. We write $N_{\mathfrak{g}_2, \mathfrak{g}_1} : \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2, \infty} \rightarrow \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1, \infty}$ for the norm map and we write $v_{\mathfrak{g}_2, \mathfrak{g}_1} : \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1, \infty} \rightarrow \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2, \infty}$ for the canonical injection.

For all $\mathfrak{g} \in \mathcal{I}'$, de Shalit defined in [5, I.3.4, II.4.6, and II.4.7] an injective morphism of $\mathbb{Z}_p[[G_\infty]]$ -modules $i_{\mathfrak{g}}^0 : \mathcal{U}_{\mathfrak{g}, \infty} \rightarrow \mathcal{M}(G_{\mathfrak{g}, \infty}, \mathcal{O}_L)$, which we extend by linearity to a morphism $i_{\mathfrak{g}}$ from $\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}, \infty}$ to $\mathcal{M}(G_{\mathfrak{g}, \infty}, \mathcal{O}_{\mathfrak{f}})$.

Lemma 4.1. *There is a unique way to extend the family $(i_{\mathfrak{g}})_{\mathfrak{g} \in \mathcal{I}'}$ to \mathcal{I} such that for all $(\mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{I}^2$, if $\mathfrak{g}_1 | \mathfrak{g}_2$ then the following squares are commutative,*

$$(4.1) \quad \begin{array}{ccc} \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2, \infty} & \xrightarrow{i_{\mathfrak{g}_2}} & \mathcal{M}(G_{\mathfrak{g}_2, \infty}, \mathcal{O}_{\mathfrak{f}}) & \text{and} & \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1, \infty} & \xrightarrow{i_{\mathfrak{g}_1}} & \mathcal{M}(G_{\mathfrak{g}_1, \infty}, \mathcal{O}_{\mathfrak{f}}) \\ \downarrow N_{\mathfrak{g}_2, \mathfrak{g}_1} & & \downarrow (\pi_{\mathfrak{g}_2, \mathfrak{g}_1})_* & & \downarrow v_{\mathfrak{g}_2, \mathfrak{g}_1} & & \downarrow (\pi_{\mathfrak{g}_2, \mathfrak{g}_1})^{\sharp} \\ \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1, \infty} & \xrightarrow{i_{\mathfrak{g}_1}} & \mathcal{M}(G_{\mathfrak{g}_1, \infty}, \mathcal{O}_{\mathfrak{f}}) & & \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2, \infty} & \xrightarrow{i_{\mathfrak{g}_2}} & \mathcal{M}(G_{\mathfrak{g}_2, \infty}, \mathcal{O}_{\mathfrak{f}}) \end{array}$$

Proof. This was proved by de Shalit in the case $p \neq 2$ (see [5, III.1.2 and III.1.3]). Let $\mathfrak{g}_1 \in \mathcal{I} \setminus \mathcal{I}'$, and let $\mathfrak{g}_2 \in \mathcal{I}'$ be such that $\mathfrak{g}_1 | \mathfrak{g}_2$. When $p \neq 2$ de Shalit uses the surjectivity of $N_{\mathfrak{g}_2, \mathfrak{g}_1}$ in order to construct $i_{\mathfrak{g}_1}$. If $p = 2$, $N_{\mathfrak{g}_2, \mathfrak{g}_1}$ may not be surjective. However we have $\text{Im}(i_{\mathfrak{g}_2} \circ v_{\mathfrak{g}_2, \mathfrak{g}_1}) \subseteq \mathcal{M}(G_{\mathfrak{g}_2, \infty}, \mathcal{O}_{\mathfrak{f}})^{\text{Ker}(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})}$. But by Proposition 2.1, $(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})^{\sharp}$ is injective and $\text{Im}(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})^{\sharp} = \mathcal{M}(G_{\mathfrak{g}_2, \infty}, \mathcal{O}_{\mathfrak{f}})^{\text{Ker}(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})}$. Hence there is a unique map $i_{\mathfrak{g}_1}$ such that the right hand square of (4.1) is commutative. The rest of the proof is identical to [5]. □

Lemma 4.2. *For all $\mathfrak{g} \in \mathcal{I}$, $i_{\mathfrak{g}}$ is an injective pseudo-isomorphism of $\mathcal{O}_{\mathfrak{f}}[[T]]$ -modules.*

Proof. Let $\mathfrak{g}_1 \in \mathcal{I}$, and let $\mathfrak{g}_2 \in \mathcal{I}'$ be such that $\mathfrak{g}_1 | \mathfrak{g}_2$. Then $(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})^{\sharp}$, $v_{\mathfrak{g}_2, \mathfrak{g}_1}$, and $i_{\mathfrak{g}_2}$ are injective, and by (4.1) we deduce the injectivity of $i_{\mathfrak{g}_1}$.

By class field theory, one can show that for any prime \mathfrak{q} of $K_{\mathfrak{g}_2, \infty}$ above \mathfrak{p} , the number of p -power roots of unity in $(K_{\mathfrak{g}_2, n})_{\mathfrak{q}}$ is bounded independantly of n (see [17, Lemma 2.1]). Then it follows from [5, I.3.7, Theorem] that $i_{\mathfrak{g}_2}$ is a pseudo-isomorphism. Since $(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_2, \infty})^{\text{Ker}(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})} = \mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_{\mathfrak{g}_1, \infty}$ and since $(\pi_{\mathfrak{g}_2, \mathfrak{g}_1})^{\sharp}$ is injective, it follows from (4.1) that $\mathcal{M}(G_{\mathfrak{g}_1, \infty}, \mathcal{O}_{\mathfrak{f}}) / \text{Im}(i_{\mathfrak{g}_1})$ is a submodule of $\mathcal{M}(G_{\mathfrak{g}_2, \infty}, \mathcal{O}_{\mathfrak{f}}) / \text{Im}(i_{\mathfrak{g}_2})$, which is pseudo-nul since $i_{\mathfrak{g}_2}$ is a pseudo-isomorphism. □

An element of the total fraction ring of $\mathcal{M}(G_{\mathfrak{g}, \infty}, \mathcal{O}_L)$ is called an \mathcal{O}_L -pseudo-measure. For $\mathfrak{g} \in \mathcal{I}$, let $\mu(\mathfrak{g})$ be the \mathcal{O}_L -pseudo-measure on $G_{\mathfrak{g}, \infty}$ defined in [5, II.4.12, Theorem]. It is a measure if $\mathfrak{g} \neq (1)$, and $\alpha\mu(1)$ is a measure for all $\alpha \in \mathcal{J}_{(1)}$, where we write $\mathcal{J}_{(1)}$ for the augmentation ideal of $\mathcal{O}_{\mathfrak{f}}[[G_{(1), \infty}]]$. By definition of $\mu(\mathfrak{g})$, we have

$$(4.2) \quad i_{\mathfrak{g}}(\psi \langle \mathfrak{g}, \mathfrak{a} \rangle) = ((\mathfrak{a}, K_{\mathfrak{g}, \infty}/k)_* - N(\mathfrak{a})) \mu(\mathfrak{g}).$$

Moreover, for $(\mathfrak{g}_1, \mathfrak{g}_2) \in \mathcal{I}^2$ such that $\mathfrak{g}_1 | \mathfrak{g}_2$, we have

$$(4.3) \quad (\pi_{\mathfrak{g}_2, \mathfrak{g}_1})_* \mu(\mathfrak{g}_2) = \prod_{\substack{l \text{ prime of } \mathcal{O}_k \\ l | \mathfrak{g}_2 \text{ and } l \nmid \mathfrak{g}_1}} \left(1 - (l, K_{\mathfrak{g}_1, \infty}/k)_*^{-1}\right) \mu(\mathfrak{g}_1).$$

Lemma 4.3. For $\mathfrak{g} \in \mathcal{I}$, we denote by $\mu_{p^\infty}(K_{\mathfrak{g},\infty})$ the group of p -power roots of unity in $K_{\mathfrak{g},\infty}$. Then we have

$$i_{\mathfrak{g}} \left(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \overline{\Psi}'(K_{\mathfrak{g},\infty}, \mathfrak{g}\mathfrak{p}^\infty) \right) = \mathcal{J}_{\mathfrak{g}}\mu(\mathfrak{g}),$$

where $\mathcal{J}_{\mathfrak{g}}$ is the annihilator of the $\mathcal{O}_{\mathfrak{f}}[[G_{\mathfrak{g},\infty}]]$ -module $\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mu_{p^\infty}(K_{\mathfrak{g},\infty})$ if $\mathfrak{g} \neq (1)$, and where $\mathcal{J}_{(1)}$ is the augmentation ideal of $\mathcal{O}_{\mathfrak{f}}[[G_{(1),\infty}]]$.

Proof. We refer the reader to [5, III.1.4]. □

5. Generation of the characteristic ideal.

For any $\mathfrak{g} \in \mathcal{I}$ such that $\mathfrak{g}|f$, and any irreducible (\mathbb{C} or \mathbb{C}_p) character χ of $G_{\mathfrak{g}}$, let $\mathfrak{f}_\chi \in \mathcal{I}$ be such that the conductor of χ is $\mathfrak{f}_\chi \mathfrak{p}^n$ for some $n \in \mathbb{N}$. Then χ defines a character on $G_{\mathfrak{f}_\chi}$, which we denote by χ_0 . We have

$$\mathcal{O}_{\mathfrak{f}}[[G_{\mathfrak{g},\infty}]]_\chi \simeq \mathcal{O}_{\mathfrak{f}}[[\Gamma]] \quad \text{and} \quad \mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_{\mathfrak{f}})_\chi \simeq \mathcal{M}(\Gamma, \mathcal{O}_{\mathfrak{f}}),$$

where the isomorphisms are induced by the following maps,

$$\tilde{\chi} : \mathcal{O}_{\mathfrak{f}}[[G_{\mathfrak{g},\infty}]] \rightarrow \mathcal{O}_{\mathfrak{f}}[[\Gamma]] \quad \text{and} \quad \chi' : \mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_{\mathfrak{f}}) \rightarrow \mathcal{M}(\Gamma, \mathcal{O}_{\mathfrak{f}}),$$

such that for any $(g, \sigma) \in G_{\mathfrak{g}} \times \Gamma$, $\tilde{\chi}(\sigma g) = \chi(g)\sigma$, and such that for any $\mu \in \mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_{\mathfrak{f}})$, $\chi'(\mu) = \tilde{\chi}(\underline{\mu})$. Moreover, remark that we have

$$(5.1) \quad \chi'(\mu) = \chi'_0 \left(\left(\pi_{\mathfrak{g},\mathfrak{f}_\chi} \right)_* \mu \right) \quad \text{for all } \mu \in \mathcal{M}(G_{\mathfrak{g},\infty}, \mathcal{O}_{\mathfrak{f}}),$$

and

$$(5.2) \quad \chi' \circ (\pi_{\mathfrak{g},\mathfrak{h}})^\sharp = 0 \quad \text{for all } \mathfrak{h} \in \mathcal{I} \text{ such that } \mathfrak{h} \neq \mathfrak{f}_\chi \text{ and } \mathfrak{h}|f_\chi.$$

For any finite group \mathcal{G} , any irreducible \mathbb{C}_p -character χ of \mathcal{G} , and any morphism $f : M \rightarrow N$ of $\mathcal{O}_{\mathfrak{f}}[\mathcal{G}]$ -modules, we denote by $f_\chi : M_\chi \rightarrow N_\chi$ the morphism defined by f . For any $x \in M$, we write x_χ for the canonical image of x in M_χ .

Lemma 5.1. Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g}|f$. Let $\chi \neq 1$ be an irreducible \mathbb{C}_p -character of $G_{\mathfrak{g}}$. Then

$$(i_{\mathfrak{g}})_\chi \left(\left(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}_{\mathfrak{g},\infty} \right)_\chi \right) \subseteq (i_{\mathfrak{f}_\chi})_{\chi_0} \left(\left(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \overline{\Psi}'(K_{\mathfrak{f}_\chi,\infty}, \mathfrak{f}_\chi \mathfrak{p}^\infty) \right)_{\chi_0} \right),$$

and the quotient is a pseudo-null $\mathcal{O}_{\mathfrak{f}}[[T]]$ -module.

Proof. Let $\mathfrak{h} \in \mathcal{I}$ be such that $\mathfrak{h}|g$, and let $x \in \overline{\Psi}'(K_{\mathfrak{g},\infty}, \mathfrak{h}\mathfrak{p}^\infty)$. From Remark 1, there is $y \in \overline{\Psi}'(K_{\mathfrak{h} \wedge \mathfrak{f}_\chi, \infty}, (\mathfrak{h} \wedge \mathfrak{f}_\chi) \mathfrak{p}^\infty)$ such that $N_{\mathfrak{g},\mathfrak{f}_\chi}(x) =$

$v_{\mathfrak{f}_\chi, \mathfrak{h} \wedge \mathfrak{f}_\chi}(y)$. From (5.1), and then from (4.1), one has

$$\begin{aligned}
 (i_{\mathfrak{g}})_\chi(x_\chi) &= \chi' \circ i_{\mathfrak{g}}(x) = \chi'_0 \circ \left(\pi_{\mathfrak{g}, \mathfrak{f}_\chi}\right)_* \circ i_{\mathfrak{g}}(x) \\
 &= \chi'_0 \circ i_{\mathfrak{f}_\chi} \circ N_{\mathfrak{g}, \mathfrak{f}_\chi}(x) \\
 &= \chi'_0 \circ i_{\mathfrak{f}_\chi} \circ v_{\mathfrak{f}_\chi, \mathfrak{h} \wedge \mathfrak{f}_\chi}(y) \\
 (5.3) \qquad &= \chi'_0 \circ \left(\pi_{\mathfrak{f}_\chi, \mathfrak{h} \wedge \mathfrak{f}_\chi}\right)^\sharp \circ i_{\mathfrak{h} \wedge \mathfrak{f}_\chi}(y).
 \end{aligned}$$

From (5.2) and (5.3), we deduce $(i_{\mathfrak{g}})_\chi(x_\chi) = 0$ if $\mathfrak{f}_\chi \nmid \mathfrak{h}$, and $(i_{\mathfrak{g}})_\chi(x_\chi) = \chi'_0 \circ i_{\mathfrak{f}_\chi}(y) = (i_{\mathfrak{f}_\chi})_{\chi_0}(y_{\chi_0})$ if $\mathfrak{f}_\chi | \mathfrak{h}$. By Remark 2, this states the inclusion $\mathcal{B} \subseteq \mathcal{A}$, where we set

$$\mathcal{A} := (i_{\mathfrak{f}_\chi})_{\chi_0} \left(\left(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \overline{\Psi}'(K_{\mathfrak{f}_\chi, \infty}, \mathfrak{f}_\chi \mathfrak{p}^\infty) \right)_{\chi_0} \right)$$

and

$$\mathcal{B} := (i_{\mathfrak{g}})_\chi \left(\left(\mathcal{O}_{\mathfrak{f}} \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}_{\mathfrak{g}, \infty} \right)_\chi \right).$$

Let $m := [k(\mathfrak{gp}^\infty) : k(\mathfrak{f}_\chi \mathfrak{p}^\infty)]$, and let $x \in \overline{\Psi}'(K_{\mathfrak{f}_\chi, \infty}, \mathfrak{f}_\chi \mathfrak{p}^\infty)$. Then $mx = N_{\mathfrak{g}, \mathfrak{f}_\chi} \circ v_{\mathfrak{g}, \mathfrak{f}_\chi}(x)$, and from (4.1) and (5.1), we obtain

$$\begin{aligned}
 m (i_{\mathfrak{f}_\chi})_{\chi_0}(x_{\chi_0}) &= \chi'_0 \circ i_{\mathfrak{f}_\chi} \circ N_{\mathfrak{g}, \mathfrak{f}_\chi} \circ v_{\mathfrak{g}, \mathfrak{f}_\chi}(x) \\
 &= \chi'_0 \circ \left(\pi_{\mathfrak{g}, \mathfrak{f}_\chi}\right)_* \circ i_{\mathfrak{g}} \circ v_{\mathfrak{g}, \mathfrak{f}_\chi}(x) \\
 &= \chi' \circ i_{\mathfrak{g}} \circ v_{\mathfrak{g}, \mathfrak{f}_\chi}(x) \\
 &= (i_{\mathfrak{g}})_\chi(v_{\mathfrak{g}, \mathfrak{f}_\chi}(x)_\chi),
 \end{aligned}$$

and we deduce that m annihilates \mathcal{A}/\mathcal{B} . Let $\alpha := \prod_{\substack{\mathfrak{l} \text{ prime of } \mathcal{O}_k \\ \mathfrak{l} | \mathfrak{g} \text{ and } \mathfrak{l} \nmid \mathfrak{f}_\chi}} (1 - \tilde{\chi}_0(\sigma_{\mathfrak{l}}^{-1}))$,

where $\sigma_{\mathfrak{l}}$ is the Fröbenius of \mathfrak{l} in $K_{\mathfrak{f}_\chi, \infty}/k$. Let $x \in \overline{\Psi}'(K_{\mathfrak{f}_\chi, \infty}, \mathfrak{f}_\chi \mathfrak{p}^\infty)$. From (3.3), there is $y \in \overline{\Psi}'(K_{\mathfrak{g}, \infty}, \mathfrak{gp}^\infty)$ such that $\alpha x = N_{\mathfrak{g}, \mathfrak{f}_\chi}(y)$. Then by (4.1) and (5.1), we have

$$\begin{aligned}
 \alpha (i_{\mathfrak{f}_\chi})_{\chi_0}(x_{\chi_0}) &= \chi'_0 \circ i_{\mathfrak{f}_\chi} \circ N_{\mathfrak{g}, \mathfrak{f}_\chi}(y) \\
 &= \chi'_0 \circ \left(\pi_{\mathfrak{g}, \mathfrak{f}_\chi}\right)_* \circ i_{\mathfrak{g}}(y) \\
 &= \chi' \circ i_{\mathfrak{g}}(y) \\
 &= (i_{\mathfrak{g}})_\chi(y_\chi).
 \end{aligned}$$

Hence α annihilates \mathcal{A}/\mathcal{B} . As a particular case, if there is no maximal ideal \mathfrak{l} of \mathcal{O}_k such that $\mathfrak{l} | \mathfrak{g}$ and $\mathfrak{l} \nmid \mathfrak{f}_\chi$, then $\alpha = 1$, $\mathcal{A} = \mathcal{B}$, and Lemma 5.1 is

proved in this case. Now assume that there is a maximal ideal \mathfrak{l} of \mathcal{O}_k such that $\mathfrak{l} \nmid \mathfrak{g}$ and $\mathfrak{l} \nmid \mathfrak{f}_\chi$. By class field theory, the decomposition group of \mathfrak{l} in $K_{\mathfrak{f}_\chi, \infty}/k$ has a finite index in $\text{Gal}(K_{\mathfrak{f}_\chi, \infty}/k)$. Hence $\sigma_{\mathfrak{l}} \notin G_{\mathfrak{f}_\chi}$, and there are a topological generator $\tilde{\gamma}$ of Γ , $n \in \mathbb{N}$, and $g \in G_{\mathfrak{f}_\chi}$ such that $\sigma_{\mathfrak{l}}^{-1} = g\tilde{\gamma}^{p^n}$. Then

$$(5.4) \quad 1 - \tilde{\chi}_0(\sigma_{\mathfrak{l}}^{-1}) = 1 - \chi_0(g)\tilde{\gamma}^{p^n} = 1 - \chi_0(g) \sum_{i=0}^{p^n} \binom{p^n}{i} \tilde{T}^i,$$

where $\tilde{T} := \tilde{\gamma} - 1$. Since m and $\chi_0(g)$ are coprime, and since $-\chi_0(g)$ is the coefficient of \tilde{T}^{p^n} in the decomposition (5.4), we deduce that m and $1 - \tilde{\chi}_0(\sigma_{\mathfrak{l}}^{-1})$ are coprime. Then m and α are coprime, and annihilate \mathcal{A}/\mathcal{B} , so that Lemma 5.1 follows. \square

Lemma 5.2. *Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g} \nmid \mathfrak{f}$. Let $\chi \neq 1$ be an irreducible \mathbb{C}_p -character of $G_{\mathfrak{g}}$.*

- (i) *If $p \neq 2$ or if $w_{\mathfrak{g}} = w_{\mathfrak{f}_\chi}$, then $\text{Im}(i_{\mathfrak{g}})_{\chi} = \text{Im}(i_{\mathfrak{f}_\chi})_{\chi_0}$.*
- (ii) *If $p = 2$, then $\text{Im}(i_{\mathfrak{g}})_{\chi} \subseteq \text{Im}(i_{\mathfrak{f}_\chi})_{\chi_0}$, and the quotient is annihilated by 2.*

Proof. For $x \in \mathcal{U}_{\mathfrak{g}, \infty}$, by (5.1) and (4.1), we have

$$(5.5) \quad (i_{\mathfrak{g}})_{\chi}(x_{\chi}) = \chi'_0 \circ (\pi_{\mathfrak{g}, \mathfrak{f}_\chi})_* \circ i_{\mathfrak{g}}(x) = \chi'_0 \circ i_{\mathfrak{f}_\chi} \circ N_{\mathfrak{g}, \mathfrak{f}_\chi}(x) \\ = (i_{\mathfrak{f}_\chi})_{\chi_0}(N_{\mathfrak{g}, \mathfrak{f}_\chi}(x)_{\chi_0}).$$

We deduce $\text{Im}(i_{\mathfrak{g}})_{\chi} \subseteq \text{Im}(i_{\mathfrak{f}_\chi})_{\chi_0}$. For n large enough, the ramification index of the primes above \mathfrak{p} in $K_{\mathfrak{g}, n}/K_{\mathfrak{f}_\chi, n}$ is $w_{\mathfrak{f}_\chi} w_{\mathfrak{g}}^{-1}$. If $p \neq 2$, then $w_{\mathfrak{f}_\chi} w_{\mathfrak{g}}^{-1}$ is prime to p . Hence in case (i), $K_{\mathfrak{g}, n}/K_{\mathfrak{f}_\chi, n}$ is tamely ramified. Then $N_{\mathfrak{g}, \mathfrak{f}_\chi}$ is a surjection from $\mathcal{U}_{\mathfrak{g}, \infty}$ onto $\mathcal{U}_{\mathfrak{f}_\chi, \infty}$, and we deduce $\text{Im}(i_{\mathfrak{g}})_{\chi} \supseteq \text{Im}(i_{\mathfrak{f}_\chi})_{\chi_0}$ from (5.5). If $p = 2$, $\mathcal{U}_{\mathfrak{f}_\chi, \infty}/N_{\mathfrak{g}, \mathfrak{f}_\chi}(\mathcal{U}_{\mathfrak{g}, \infty})$ is annihilated by $w_{\mathfrak{f}_\chi} w_{\mathfrak{g}}^{-1}$ which is 1 or 2, and we deduce (ii) from (5.5). \square

For $p \neq 2$, Theorems 5.1 and 5.2 below were already proved by de Shalit in [5, III.1.10].

Theorem 5.1. *Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g} \nmid \mathfrak{f}$. Let u be a uniformizer of $\mathcal{O}_{\mathfrak{f}}$. Let $\chi \neq 1$ be an irreducible \mathbb{C}_p -character of $G_{\mathfrak{g}}$.*

- (i) *If $p \neq 2$ or if $w_{\mathfrak{g}} = w_{\mathfrak{f}_\chi}$, then $\text{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}(\mathcal{O}_{\mathfrak{f}} \hat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g}, \infty}/\mathcal{C}_{\mathfrak{g}, \infty}))_{\chi}$ is generated by $\tilde{\chi}_0(\mu(\mathfrak{f}_\chi))$.*
- (ii) *If $p = 2$, then the ideal $\text{char}_{\mathcal{O}_{\mathfrak{f}}[[T]]}(\mathcal{O}_{\mathfrak{f}} \hat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g}, \infty}/\mathcal{C}_{\mathfrak{g}, \infty}))_{\chi}$ is generated by $u^{-m_{\chi}} \tilde{\chi}_0(\mu(\mathfrak{f}_\chi))$, for some $m_{\chi} \in \mathbb{N}$.*

(In case $f_\chi = (1)$, we have expanded $\tilde{\chi}_0$ to the total fraction ring of $\mathcal{O}_f[[G_{(1),\infty}]]$ and to the fraction field of $\mathcal{O}_f[[\Gamma]]$. We still have $\tilde{\chi}_0(\underline{\mu}(1)) \in \mathcal{O}_f[[\Gamma]]$.)

Proof. Let us set $\tilde{\mathcal{C}}_{\mathfrak{g}} := (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}_{\mathfrak{g},\infty})_{\chi}$. We have the tautological exact sequence below,

$$(5.6) \quad 0 \rightarrow \text{Im}(i_{\mathfrak{g}})_{\chi} / (i_{\mathfrak{g}})_{\chi}(\tilde{\mathcal{C}}_{\mathfrak{g}}) \rightarrow \text{Im}(i_{f_{\chi}})_{\chi_0} / (i_{\mathfrak{g}})_{\chi}(\tilde{\mathcal{C}}_{\mathfrak{g}}) \rightarrow \text{Im}(i_{f_{\chi}})_{\chi_0} / \text{Im}(i_{\mathfrak{g}})_{\chi} \rightarrow 0.$$

From Lemma 5.2, we deduce the existence of $m_{\chi} \in \mathbb{N}$ such that

$$(5.7) \quad \text{char}_{\mathcal{O}_f[[T]]} \left(\text{Im}(i_{f_{\chi}})_{\chi_0} / \text{Im}(i_{\mathfrak{g}})_{\chi} \right) = (\mathbf{u}^{m_{\chi}}),$$

with $m_{\chi} = 0$ in case (i). Since $\text{Im}(i_{\mathfrak{g}})_{\chi} / (i_{\mathfrak{g}})_{\chi}(\tilde{\mathcal{C}}_{\mathfrak{g}}) \simeq (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_{\chi}$, from (5.6) and (5.7), we deduce that

$$(5.8) \quad \text{char}_{\mathcal{O}_f[[T]]} \left(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty})_{\chi} \right) = \mathbf{u}^{-m_{\chi}} \text{char}_{\mathcal{O}_f[[T]]} \left(\text{Im}(i_{f_{\chi}})_{\chi_0} / (i_{\mathfrak{g}})_{\chi}(\tilde{\mathcal{C}}_{\mathfrak{g}}) \right).$$

We set $\tilde{\Psi} := \left(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Psi}'(K_{f_{\chi},\infty}, f_{\chi} \mathfrak{p}^{\infty}) \right)_{\chi_0}$. From (5.8) and Lemma 5.1, we deduce

$$(5.9) \quad \text{char}_{\mathcal{O}_f[[T]]} \left(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty})_{\chi} \right) = \mathbf{u}^{-m_{\chi}} \text{char}_{\mathcal{O}_f[[T]]} \left(\text{Im}(i_{f_{\chi}})_{\chi_0} / (i_{f_{\chi}})_{\chi_0}(\tilde{\Psi}) \right).$$

Since $\text{Im}(i_{f_{\chi}})_{\chi_0} / (i_{f_{\chi}})_{\chi_0}(\tilde{\Psi}) \simeq \left(\text{Im}(i_{f_{\chi}}) / (i_{f_{\chi}}) \right)_{\chi_0}(\tilde{\Psi})$ and since $i_{f_{\chi}}$ is a pseudo-isomorphism, we deduce from (5.9) and Lemma 4.3 that

$$(5.10) \quad \begin{aligned} \text{char}_{\mathcal{O}_f[[T]]} \left(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty})_{\chi} \right) &= \mathbf{u}^{-m_{\chi}} \text{char}_{\mathcal{O}_f[[T]]} \left(\text{Im}(i_{f_{\chi}}) / (i_{f_{\chi}}) \right)_{\chi_0}(\tilde{\Psi}) \\ &= \mathbf{u}^{-m_{\chi}} \text{char}_{\mathcal{O}_f[[T]]} \left(\mathcal{M}(G_{f_{\chi},\infty}, \mathcal{O}_f) / \mathcal{J}_{f_{\chi}} \mu(f_{\chi}) \right)_{\chi_0} \\ &= \mathbf{u}^{-m_{\chi}} \text{char}_{\mathcal{O}_f[[T]]} \left(\mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0 \left(\mathcal{J}_{f_{\chi}} \mu(f_{\chi}) \right) \right). \end{aligned}$$

First we assume that $f_{\chi} \neq (1)$. Then $\chi'_0(\mu(f_{\chi})) \mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0(\mathcal{J}_{f_{\chi}} \mu(f_{\chi}))$ is isomorphic to $\left(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} \mu_{p^{\infty}}(K_{f_{\chi},\infty}) \right)_{\chi_0}$, hence pseudo-nul since $\mu_{p^{\infty}}(K_{f_{\chi},\infty})$

is finite. Then from (5.10) we deduce

$$\begin{aligned} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_\chi &= \mathbf{u}^{-m_\chi} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0(\mu(\mathfrak{f}_\chi)) \mathcal{M}(\Gamma, \mathcal{O}_f)) \\ &= \mathbf{u}^{-m_\chi} \tilde{\chi}_0(\underline{\mu(\mathfrak{f}_\chi)}) \mathcal{O}_f[[T]], \end{aligned}$$

and Theorem 5.1 follows in this case. Now assume $\mathfrak{f}_\chi = (1)$. Then we expand χ'_0 to the total fraction ring of $\mathcal{M}(G_{(1),\infty}, \mathcal{O}_f)$ and to the fraction field of $\mathcal{M}(\Gamma, \mathcal{O}_f)$. There is $\sigma \in G_{\mathfrak{g}}$ such that $\chi(\sigma) \neq 1$. Then

$$\chi'_0(\mu(1)) \mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0(\mathcal{J}_{(1)}\mu(1))$$

is pseudo-nul, annihilated by $1 - \chi(\sigma)$ and T . Since we have

$$\chi'_0(\mathcal{J}_{(1)}\mu(1)) \subseteq \mathcal{M}(\Gamma, \mathcal{O}_f),$$

we deduce the inclusion $\chi'_0(\mu(1)) \mathcal{M}(\Gamma, \mathcal{O}_f) \subseteq \mathcal{M}(\Gamma, \mathcal{O}_f)$ and from (5.10) we obtain

$$\begin{aligned} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_\chi &= \mathbf{u}^{-m_\chi} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0(\mu(1)) \mathcal{M}(\Gamma, \mathcal{O}_f)). \end{aligned}$$

(i) and (ii) follow immediately in this case. □

Theorem 5.2. *Let $\mathfrak{g} \in \mathcal{I}$ be such that $\mathfrak{g}|\mathfrak{f}$. Let χ be the trivial character on $G_{\mathfrak{g}}$.*

(i) *If $p \neq 2$ or if $w_{\mathfrak{g}} = |\mu(k)|$, then $\text{char}_{\mathcal{O}_f[[T]]} (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_\chi$ is generated by $\tilde{\chi}_0(\underline{T\mu(1)})$.*

(ii) *If $p = 2$, then the ideal $\text{char}_{\mathcal{O}_f[[T]]} (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_\chi$ is generated by $\mathbf{u}^{-m_\chi} \tilde{\chi}_0(\underline{T\mu(1)})$, for some $m_\chi \in \mathbb{N}$.*

Proof. As in the proof of Theorem 5.1, we have

$$(5.11) \quad \begin{aligned} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{U}_{\mathfrak{g},\infty} / \mathcal{C}_{\mathfrak{g},\infty}))_\chi &= \mathbf{u}^{-m_\chi} \text{char}_{\mathcal{O}_f[[T]]} (\mathcal{M}(\Gamma, \mathcal{O}_f) / \chi'_0(\mathcal{J}_{(1)}\mu(1))), \end{aligned}$$

where $m_\chi \in \mathbb{N}$ is zero in case (i). But $\chi'_0(\mathcal{J}_{(1)}\mu(1)) = \chi'_0(\underline{T\mu(1)}) \mathcal{M}(\Gamma, \mathcal{O}_f)$, and the theorem follows. □

6. Finiteness of invariants and coinvariants.

For any $\mathfrak{h} \in \mathcal{I}$, we write $L_{p,\mathfrak{h}}$ for the p -adic L-function of k with modulus \mathfrak{h} , as defined in [5, II.4.16]. It is the map defined on the set of all continuous

group morphisms ξ from $\text{Gal}(K_{\mathfrak{h},\infty}/k)$ to \mathbb{C}_p^\times (with $\xi \neq 1$ if $\mathfrak{h} = (1)$), such that

$$(6.1) \quad L_{p,\mathfrak{h}}(\xi) = \int \xi(\sigma)^{-1} \cdot d\mu(\mathfrak{h})(\sigma).$$

Let $n \in \mathbb{N}$, and let χ be an irreducible \mathbb{C}_p -character on $\text{Gal}(k(\mathfrak{h}\mathfrak{p}^n)/k)$ (with $\chi \neq 1$ if $\mathfrak{h} = (1)$). We write F_χ for the subfield of $k(\mathfrak{h}\mathfrak{p}^n)$ fixed by $\text{Ker}(\chi)$, and we write χ_{pr} for the character on $\text{Gal}(F_\chi/k)$ defined by χ . By inflation we can consider χ as a group morphism $\text{Gal}(K_{\mathfrak{h},\infty}/k) \rightarrow \mathbb{C}_p^\times$, so that the notation $L_{p,\mathfrak{h}}(\chi)$ makes sense. As in [5, II.5.2], if $n > 0$ we set

$$(6.2) \quad L_{p,\mathfrak{h}\mathfrak{p}^n}(\chi) := \begin{cases} (1 - \chi_{\text{pr}}(\mathfrak{p}, F_\chi/k)) L_{p,\mathfrak{h}}(\chi) & \text{if } \mathfrak{p} \text{ is unramified in } F_\chi, \\ L_{p,\mathfrak{h}}(\chi) & \text{if } \mathfrak{p} \text{ is ramified in } F_\chi. \end{cases}$$

Lemma 6.1. *Let $\mathfrak{g} \notin \{(0), (1)\}$ be an ideal of \mathcal{O}_k , and let χ be an irreducible \mathbb{C}_p -character on $\text{Gal}(k(\mathfrak{g})/k)$. If $\chi \neq 1$ and if none of the prime ideals dividing \mathfrak{g} are totally split in F_χ/k , then $L_{p,\mathfrak{g}}(\chi) \neq 0$. If $\chi = 1$, if \mathfrak{g} is a power of a prime ideal, and if $\mathfrak{p} \nmid \mathfrak{g}$, then $L_{p,\mathfrak{g}}(\chi) \neq 0$.*

Proof. We set $H := \text{Gal}(k(\mathfrak{g})/k)$. For all maximal ideal \mathfrak{r} of $\mathcal{O}_{k(\mathfrak{g})}$, let us denote by $v_{\mathfrak{r}}$ the normalized valuation at \mathfrak{r} . Let $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ be such that $v_{\mathfrak{r}}(\varphi_{\mathfrak{g}}(1)) = 0$ for all maximal ideal \mathfrak{r} of $\mathcal{O}_{k(\mathfrak{g})}$ not lying above \mathfrak{q} . Let $U \subset k(\mathfrak{g})^\times$ be the subgroup of all the numbers $x \in k(\mathfrak{g})^\times$ verifying the two following conditions,

- $v_{\mathfrak{r}}(x) = 0$ for all maximal ideal \mathfrak{r} of $\mathcal{O}_{k(\mathfrak{g})}$ not lying above \mathfrak{q} ,
- $v_{\mathfrak{r}}(x) = v_{\mathfrak{s}}(x)$ for all maximal ideals \mathfrak{r} and \mathfrak{s} of $\mathcal{O}_{k(\mathfrak{g})}$ above \mathfrak{q} .

Using Dirichlet’s theorem and the product formula, we see that $\mathbb{Q} \otimes_{\mathbb{Z}} U \simeq \mathbb{Q}[H]$. Hence we can fix $u \in U$ such that $\mathbb{Q} \otimes_{\mathbb{Z}} U$ is freely generated by $1 \otimes u$ over $\mathbb{Q}[H]$. Let us fix an embedding $\iota_p : k^{\text{alg}} \hookrightarrow \mathbb{C}_p$. We define the morphism of $k^{\text{alg}}[H]$ -modules below,

$$\ell_p : k^{\text{alg}} \otimes_{\mathbb{Z}} U \rightarrow \mathbb{C}_p[H], \quad a \otimes x \mapsto \iota_p(a) \sum_{\sigma \in H} \log_p(\iota_p(x^\sigma)) \sigma^{-1},$$

where \log_p is the p -adic logarithm, as defined in [8, §4]. Let us show that ℓ_p is injective on $k^{\text{alg}} \otimes_{\mathbb{Z}} U$. We assume that it is not injective, and a contradiction will arise. There is an irreducible \mathbb{C}_p -character ξ of H such that $e_\xi \ell_p(1 \otimes u) = 0$, and then the family $(\log_p(\iota_p(u^\sigma)))_{\sigma \in H}$ is not linearly independent over $\iota_p(k^{\text{alg}})$. By a theorem of Brumer [3, Theorem 1], we deduce that there are integers $\lambda_\sigma \in \mathbb{Z}$, $\sigma \in H$, with $\lambda_{\sigma_0} \neq 0$ for some $\sigma_0 \in H$, such that

$$\log_p \left(\iota_p \left(\prod_{\sigma \in H} u^{\lambda_\sigma \sigma} \right) \right) = \sum_{\sigma \in H} \lambda_\sigma \log_p(\iota_p(u^\sigma)) = 0.$$

It is well known that $\text{Ker}(\log_p)$ is generated by the roots of powers of p , hence $\prod_{\sigma \in H} u^{\lambda_\sigma}$ is a root of unity. Then we must have $\lambda_\sigma = 0$ for all $\sigma \in H$,

which contradicts $\lambda_{\sigma_0} \neq 0$. Thus we have verified the injectivity of ℓ_p . Now assume $L_{p,\mathfrak{g}}(\chi) = 0$. From the p -adic version of the Kronecker limit formula [5, II.5.2, Theorem], we deduce that $e_{\chi^{-1}}\ell_p(1 \otimes \varphi_{\mathfrak{g}}(1)) = 0$ in $\mathbb{C}_p[H]$. Then

$$(6.3) \quad e_{\chi^{-1}}(1 \otimes \varphi_{\mathfrak{g}}(1)) = 0 \quad \text{in} \quad k^{\text{alg}} \otimes_{\mathbb{Z}} U,$$

where χ is identified to a group morphism $H \rightarrow k^{\text{alg}}$ via ι_p . If $\chi \neq 1$, then from [12, Théorème 10] we deduce the existence of a maximal ideal \mathfrak{r} of \mathcal{O}_k , unramified in F_χ/k , such that $\mathfrak{r}|\mathfrak{g}$, and such that $\chi_{\text{pr}}(\mathfrak{r}, F_\chi/k) = 1$ (hence totally split in F_χ/k). If $\chi = 1$, from (6.3) we deduce $N_{k(\mathfrak{g})/k}(\varphi_{\mathfrak{g}}(1)) \in \mu(k)$. Then \mathfrak{g} must be divisible by at least two distinct prime ideals in virtue of (3.2). □

Theorem 6.1. *For all $n \in \mathbb{N}$, the module of Γ_n -invariants and the module of Γ_n -coinvariants of $\mathcal{U}_\infty/\mathcal{C}_\infty$ are finite.*

Proof. By [10, p. 254, Exercise 3], it is sufficient to verify that

$$(6.4) \quad \text{char}_{\mathbb{Z}_p[[T]]}(\mathcal{U}_\infty/\mathcal{C}_\infty) \text{ is prime to } \left((1+T)^{p^n} - 1 \right) \text{ in } \mathbb{Z}_p[[T]], \text{ for all } n \in \mathbb{N}.$$

For n large enough, $K_{f,n}/K_n$ is tamely ramified if $p \neq 2$, and if $p = 2$ the ramification index is 1 or 2. Hence we deduce that the cokernel of the norm maps $\mathcal{U}_{f,\infty} \rightarrow \mathcal{U}_\infty$ and $\mathcal{U}_{f,\infty}/\mathcal{C}_{f,\infty} \rightarrow \mathcal{U}_\infty/\mathcal{C}_\infty$ are annihilated by 2. Then we have

$$(6.5) \quad \text{char}_{\mathbb{Z}_p[[T]]}(\mathcal{U}_\infty/\mathcal{C}_\infty) \text{ divides } 2^a \text{char}_{\mathbb{Z}_p[[T]]}(\mathcal{U}_{f,\infty}/\mathcal{C}_{f,\infty}),$$

for some $a \in \mathbb{N}$. By (6.5), we are reduced to prove (6.4) in the case $K_\infty = K_{f,\infty}$. Then by (1.1), in order to verify (6.4) we only have to show that the ideal $\text{char}_{\mathcal{O}_f[[T]]}(\mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{U}_\infty / \mathcal{O}_f \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{C}_\infty)_\chi$ is prime to $\left((1+T)^{p^n} - 1 \right)$ in $\mathcal{O}_f[[T]]$, for all $n \in \mathbb{N}$, and all irreducible \mathbb{C}_p -character χ on G_f . Let χ be such a character, and let $\zeta \in \mu_{p^\infty}(\mathbb{C}_p)$. We choose a maximal ideal ℓ of \mathcal{O}_k , prime to \mathfrak{fp} , such that $\chi_{\text{pr}}(\ell, F_\chi/k) \neq 1$ if $\chi \neq 1$, and such that ℓ is not totally split in k_1 (the subfield of k_∞ fixed by Γ^p) if $\chi = 1$. By Theorem 5.1 and Theorem 5.2, it suffices to prove $\tilde{\chi}_0 \left(\left(1 - \sigma_\ell^{-1} \right) \underline{\mu}(f_\chi) \right) |_{T=\zeta^{-1}} \neq 0$, where $\sigma_\ell := (\ell, K_{f,\infty}/k)$. By (4.3) and by (2.5), we have

$$(6.6) \quad \begin{aligned} \tilde{\chi}_0 \left(\left(1 - \sigma_\ell^{-1} \right) \underline{\mu}(f_\chi) \right) |_{T=\zeta^{-1}} &= \tilde{\chi}_0 \left(\tilde{\pi}_{f_\chi \ell, f_\chi} \left(\underline{\mu}(f_\chi \ell) \right) \right) |_{T=\zeta^{-1}} \\ &= \int_{\Gamma} \zeta^{\kappa(\sigma)}. d\chi'_{f_\chi \ell} \left(\underline{\mu}(f_\chi \ell) \right) (\sigma), \end{aligned}$$

where $\chi_{f_\chi \ell}$ is the character on $G_{f_\chi \ell}$ defined by χ_0 , and where $\kappa : \Gamma \rightarrow \mathbb{Z}_p$ is the unique morphism of topological groups such that $\kappa(\gamma) = 1$. From (6.6)

and (2.3) we deduce

$$\begin{aligned}
 \tilde{\chi}_0 \left(\left(1 - \sigma_\ell^{-1} \right) \underline{\mu}(\mathfrak{f}_\chi) \right) |_{T=\zeta-1} &= \sum_{g \in G_{\mathfrak{f}_\chi \ell}} \chi_{\mathfrak{f}_\chi \ell}(g) \int_{\Gamma} \zeta^{\kappa(\sigma)} \cdot d \left(g^{-1} \right)_* \mu(\mathfrak{f}_\chi \ell)(\sigma). \\
 &= \sum_{g \in G_{\mathfrak{f}_\chi \ell}} \chi_{\mathfrak{f}_\chi \ell}(g) \int_{g\Gamma} \zeta^{\kappa(g^{-1}\sigma)} \cdot d\mu(\mathfrak{f}_\chi \ell)(\sigma) \\
 (6.7) \qquad \qquad \qquad &= \int_{G_{\mathfrak{f}_\chi \ell, \infty}} \zeta^{\kappa(g_\sigma^{-1}\sigma)} \chi_{\mathfrak{f}_\chi \ell}(g_\sigma) \cdot d\mu(\mathfrak{f}_\chi \ell)(\sigma),
 \end{aligned}$$

where for any $\sigma \in G_{\mathfrak{f}_\chi \ell, \infty}$, g_σ is the image of σ through the projection $G_{\mathfrak{f}_\chi \ell, \infty} \rightarrow G_{\mathfrak{f}_\chi \ell}$. We define $\xi : G_{\mathfrak{f}_\chi \ell, \infty} \rightarrow \mathbb{C}_p^\times$, $\sigma \mapsto \zeta^{\kappa(g_\sigma^{-1}\sigma)} \chi_{\mathfrak{f}_\chi \ell}(g_\sigma)$. Then ξ is a group morphism, and if $n \in \mathbb{N}$ is such that $\zeta^{p^n} = 1$, then ξ defines an irreducible \mathbb{C}_p -character on $G_{\mathfrak{f}_\chi \ell, n} := \text{Gal}(K_{\mathfrak{f}_\chi \ell, n}/k)$. Let \mathfrak{g} be the conductor of F_ξ . Since the restriction of ξ to $G_{\mathfrak{f}_\chi \ell} \hookrightarrow G_{\mathfrak{f}_\chi \ell, n}$ is $\chi_{\mathfrak{f}_\chi \ell}$, we deduce that there is $m \in \mathbb{N}$ such that $\mathfrak{g} = \mathfrak{f}_\chi \mathfrak{p}^m$, and from (6.2) we deduce that

$$(6.8) \qquad \qquad \qquad L_{p, \mathfrak{g}\ell}(\xi^{-1}) = L_{p, \mathfrak{f}_\chi \ell}(\xi^{-1}).$$

Then from (6.7) and (6.1) we deduce

$$\begin{aligned}
 (6.9) \quad \left(1 - \tilde{\chi}_0 \left(\sigma_\ell^{-1} \right) \right) \tilde{\chi}_0 \left(\underline{\mu}(\mathfrak{f}_\chi) \right) |_{T=\zeta-1} \\
 = \int_{G_{\mathfrak{f}_\chi \ell, \infty}} \xi(\sigma) \cdot d\mu(\mathfrak{f}_\chi \ell)(\sigma) = L_{p, \mathfrak{f}_\chi \ell}(\xi^{-1}).
 \end{aligned}$$

If $\chi \neq 1$, then $\chi_{\text{pr}}(\ell, F_\chi/k) \neq 1$ implies that ℓ is not totally split in F_ξ/k . If $\chi = 1$ and $\zeta \neq 1$, then $k_1 \subseteq F_\xi$ and ℓ is not totally split in F_ξ/k . If $\chi = 1$ and $\zeta = 1$, then $\xi = 1$ and $\mathfrak{g} = (1)$. From (6.9), (6.8), and Lemma 6.1, we deduce

$$\left(1 - \tilde{\chi}_0 \left(\sigma_\ell^{-1} \right) \right) \tilde{\chi}_0 \left(\underline{\mu}(\mathfrak{f}_\chi) \right) |_{T=\zeta-1} = L_{p, \mathfrak{g}\ell}(\xi^{-1}) \neq 0.$$

□

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Stéphane VIGUIÉ

Université de Franche-Comté

16 route de Gray

25030 Besançon cedex, France

E-mail: stephane.viguie@univ-fcomte.fr

URL: <http://www.math.u-bordeaux.fr/A2X/>