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The divisor problem for binary cubic forms

par Tim BROWNING

Résumé. Nous étudions l'ordre moyen du nombre de diviseurs des valeurs de certaines formes binaires cubiques qui ne sont pas irréductibles sur $\mathbb{Q}$ et discutons quelques applications.

Abstract. We investigate the average order of the divisor function at values of binary cubic forms that are reducible over $\mathbb{Q}$ and discuss some applications.

1. Introduction

This paper is motivated by the well-known problem of studying the average order of the divisor function $\tau(n) = \sum_{d|n} 1$, as it ranges over the values taken by polynomials. Our focus is upon the case of binary forms $C \in \mathbb{Z}[x_1, x_2]$ of degree 3, the treatment of degree 1 or 2 being trivial.

We wish to understand the behaviour of the sum

$$T(X; C) = \sum_{x_1, x_2 \leqslant X} \tau(C(x_1, x_2)),$$

as $X \to \infty$. The hardest case is when C is irreducible over $\mathbb{Q}$ with non-zero discriminant, a situation first handled by Greaves [7]. He establishes the existence of constants $c_0, c_1 \in \mathbb{R}$, with $c_0 > 0$, such that

$$T(X; C) = c_0 X^2 \log X + c_1 X^2 + O_{\varepsilon, C}(X^{2 - \frac{1}{14} + \varepsilon}),$$

for any $\varepsilon > 0$. Here, as throughout our work, any dependence in the implied constant will be indicated explicitly by an appropriate subscript. This was later improved by Daniel [4], who improved $2 - \frac{1}{14} + \varepsilon$ to $2 - \frac{1}{8} + \varepsilon$. Daniel also achieves asymptotic information about the sum associated to irreducible binary forms of degree 4, which is at the limit of what is currently possible.

Our aim is to investigate the corresponding sums $T(X) = T(X; L_1 L_2 L_3)$ when C is assumed to factorise as a product of linearly independent linear forms $L_1, L_2, L_3 \in \mathbb{Z}[x_1, x_2]$. In doing so we will gain a respectable improvement in the quality of the error term apparent in the work of Greaves and Daniel. The following result will be established in §4.

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Theorem 1. *For any $\varepsilon > 0$ there exist constants $c_0, \dots, c_3 \in \mathbb{R}$, with $c_0 > 0$, such that*

$$T(X) = \sum_{i=0}^{3} c_i X^2 (\log X)^{3-i} + O_{\varepsilon, L_1, L_2, L_3}(X^{2-\frac{1}{4}+\varepsilon}).$$

Our proof draws heavily on a series of joint papers of the author with la Bretèche [2, 3]. These involve an analysis of the more exacting situation wherein $\tau(L_1L_2L_3)$ is replaced by $r(L_1L_2L_3L_4)$ or $\tau(L_1L_2Q)$, for an irreducible binary quadratic form Q.

One of the motivations for studying the divisor problem for binary forms is the relative lack of progress for the divisor problem associated to polynomials in a single variable. It follows from work of Ingham [8] that

$$\sum_{n \leqslant X} \tau(n)\tau(n+h) \sim \frac{6}{\pi^2}\sigma_{-1}(h) X (\log X)^2$$

as $X \to \infty$, for given $h \in \mathbb{N}$. Exploiting connections with Kloosterman sums, Estermann [6] obtained a cleaner asymptotic expansion with a reasonable degree of uniformity in h. Several authors have since revisited this problem achieving asymptotic formulae with h in an increasingly large range compared to X. The best results in the literature are due to Duke, Friedlander and Iwaniec [5] and to Motohashi [9].

A successful analysis of the sum

$$T_h(X) = \sum_{n \leqslant X} \tau(n-h)\tau(n)\tau(n+h),$$

has not yet been forthcoming for a single positive integer h. It is conjectured that $T_h(X) \sim c_h X (\log X)^3$ as $X \to \infty$, for a suitable constant $c_h > 0$. A straightforward heuristic analysis based on the underlying Diophantine equations suggests that one should take

$$c_h = \frac{11}{8} f(h) \prod_p \Big(1 - \frac{1}{p}\Big)^2 \Big(1 + \frac{2}{p}\Big), \tag{1.1}$$

where f is given multiplicatively by $f(1) = 1$ and

$$f(p^\nu) = \begin{cases} \dfrac{1+\frac{4}{p}+\frac{1}{p^2}-\frac{3\nu+4}{p^{\nu+1}}-\frac{4}{p^{\nu+2}}+\frac{3\nu+2}{p^{\nu+3}}}{(1+\frac{2}{p})(1-\frac{1}{p})}, & \text{if } p > 2, \\ \frac{52}{11} - \frac{41+15\nu}{11\times 2^\nu}, & \text{if } p = 2, \end{cases} \tag{1.2}$$

for $\nu \geqslant 1$. In the following result we provide some support for this expectation.

Theorem 2. *Let $\varepsilon > 0$ and let $H \geqslant X^{\frac{3}{4}+\varepsilon}$. Then we have*

$$\sum_{h \leqslant H} (T_h(X) - c_h X (\log X)^3) = o(HX(\log X)^3).$$

This result will be established in §5, where we will see that $HX(\log X)^3$ represents the true order of magnitude of the two sums on the left hand side. It would be interesting to reduce the lower bound for H assumed in this result.

Throughout our work it will be convenient to reserve i, j for generic distinct indices from the set $\{1, 2, 3\}$. For any $\mathbf{h} \in \mathbb{N}^3$, we let

$$\Lambda(\mathbf{h}) = \{\mathbf{x} \in \mathbb{Z}^2 : h_i \mid L_i(\mathbf{x})\}, \tag{1.3}$$

$$\varrho(\mathbf{h}) = \#(\Lambda(\mathbf{h}) \cap [0, h_1h_2h_3)^2). \tag{1.4}$$

It is clear that $\Lambda(\mathbf{h})$ defines an integer sublattice of rank 2. In what follows let $\mathcal{R}$ always denote a compact subset of $\mathbb{R}^2$ whose boundary is a piecewise continuously differentiable closed curve with length

$$\partial(\mathcal{R}) \ll \sup_{\mathbf{x}\in\mathcal{R}} \max\{|x_1|, |x_2|\}.$$

This is in contrast to our earlier investigations [2, 3], where a hypothesis of this sort is automatically satisfied by working with closed convex subsets of $\mathbb{R}^2$. Let $\mathbf{d}, \mathbf{D} \in \mathbb{N}^3$ with $d_i \mid D_i$. We shall procure Theorems 1 and 2 through an analysis of the auxiliary sum

$$S(X; \mathbf{d}, \mathbf{D}) = \sum_{\mathbf{x}\in\Lambda(\mathbf{D})\cap X\mathcal{R}} \tau\Big(\frac{L_1(\mathbf{x})}{d_1}\Big)\tau\Big(\frac{L_2(\mathbf{x})}{d_2}\Big)\tau\Big(\frac{L_3(\mathbf{x})}{d_3}\Big), \tag{1.5}$$

where $X\mathcal{R} = \{X\mathbf{x} : \mathbf{x} \in \mathcal{R}\}$. We will also assume that $L_i(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathcal{R}$.

Before revealing our estimate for $S(X; \mathbf{d}, \mathbf{D})$ we will first need to introduce some more notation. We write

$$L_\infty = L_\infty(L_1, L_2, L_3) = \max\{\|L_1\|, \|L_2\|, \|L_3\|\}, \tag{1.6}$$

where $\|L_i\|$ denotes the maximum modulus of the coefficients of L_i. We will set

$$r_\infty = r_\infty(\mathcal{R}) = \sup_{\mathbf{x}\in\mathcal{R}} \max\{|x_1|, |x_2|\}, \tag{1.7}$$

$$r' = r'(L_1, L_2, L_3, \mathcal{R}) = \max_{1\leqslant i\leqslant 3} \sup_{\mathbf{x}\in\mathcal{R}} |L_i(\mathbf{x})|. \tag{1.8}$$

These are positive real numbers. Furthermore, let $D = D_1D_2D_3$ and let $\delta(\mathbf{D}) \in \mathbb{N}$ denote the largest $\delta \in \mathbb{N}$ for which $\Lambda(\mathbf{D}) \subseteq \{\mathbf{x} \in \mathbb{Z}^2 : \delta \mid \mathbf{x}\}$. Bearing this notation in mind we will establish the following result in §2 and §3.

Theorem 3. *Let $\varepsilon > 0$ and let $\theta \in (\frac{1}{4}, 1)$. Assume that $r'X^{1-\theta} \geqslant 1$. Then there exists a polynomial $P \in \mathbb{R}[x]$ of degree 3 such that*

$$\begin{aligned} S(X; \mathbf{d}, \mathbf{D}) = {} & \operatorname{vol}(\mathcal{R})X^2P(\log X) \\ & + O_\varepsilon\Big(\frac{D^\varepsilon L_\infty^{2+\varepsilon} r_\infty^\varepsilon}{\delta(\mathbf{D})}(r_\infty r'^{\frac{3}{4}} + r_\infty^2)X^{\frac{7}{4}+\varepsilon}\Big), \end{aligned}$$

where the coefficients of P have size $O_\varepsilon(D^\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon(1+r'^{-1})^\varepsilon(\det \mathsf{\Lambda}(\mathbf{D}))^{-1})$. Moreover, the leading coefficient of P is $C=\prod_p \sigma_p(\mathbf{d},\mathbf{D})$, with

$$\sigma_p(\mathbf{d},\mathbf{D})=\Big(1-\frac{1}{p}\Big)^3\sum_{\boldsymbol{\nu}\in\mathbb{Z}_{\geqslant 0}^3}\frac{\varrho(p^{N_1},p^{N_2},p^{N_3})}{p^{2N_1+2N_2+2N_3}} \tag{1.9}$$

and $N_i=\max\{v_p(D_i),\nu_i+v_p(d_i)\}$.

While the study of the above sums is interesting in its own right, it turns out that there are useful connections to conjectures of Manin and his collaborators [1] concerning the growth rate of rational points on Fano varieties. Consider for example the bilinear hypersurface

$$W_s: \quad x_0y_0+\cdots+x_sy_s=0$$

in $\mathbb{P}^s\times\mathbb{P}^s$. This defines a flag variety and it can be embedded in $\mathbb{P}^{s(s+2)}$ via the Segre embedding ϕ. Let $U_s\subset W_s$ be the open subset on which $x_iy_j\neq 0$ for $0\leqslant i,j\leqslant n$. If $H:\mathbb{P}^{s(s+2)}(\mathbb{Q})\to\mathbb{R}_{>0}$ is the usual exponential height then we wish to analyse the counting function

$$\begin{aligned} N(B)&=\#\{v\in U_s(\mathbb{Q}): H(\phi(v))\leqslant B\}\\ &=\frac{1}{4}\#\{(\mathbf{x},\mathbf{y})\in\mathbb{Z}_*^{s+1}\times\mathbb{Z}_*^{s+1}:\ \max|x_iy_j|^s\leqslant B,\ \mathbf{x}.\mathbf{y}=0\},\end{aligned}$$

as $B\to\infty$, where $\mathbb{Z}_*^k$ denotes the set of primitive vectors in $\mathbb{Z}^k$ with non-zero components. It follows from work of Robbiani [10] that there is a constant $c_s>0$ such that $N(B)\sim c_sB\log B$, for $s\geqslant 3$, which thereby confirms the Manin conjecture in this case. This is established using the Hardy–Littlewood circle method. Spencer [11] has given a substantially shorter treatment, which also handles $s=2$. By casting the problem in terms of a restricted divisor sum in §6, we will modify the proof of Theorem 3 to provide an independent proof of Spencer's result in the case $s=2$.

Theorem 4. *For $s=2$ we have $N(B)=cB\log B+O(B)$, with*

$$c=\frac{12}{\zeta(2)^2}\prod_p\Big(1+\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p}+\frac{1}{p^2}\Big).$$

2. Theorem 3: special case

Our proof follows the well-trodden paths of [2, §4] and [3, §§5,6]. We will begin by establishing a version of Theorem 3 when $d_i=D_i=1$. Let us write $S(X)$ for the sum in this special case. In §3 we shall establish the general case by reducing the situation to this case via a linear change of variables.

Recall that the linear forms under consideration are not necessarily primitive. We therefore fix integers ℓ_i such that L_i^* is a primitive linear form, with

$$L_i = \ell_i L_i^*. \tag{2.1}$$

It will be convenient to define the least common multiple

$$L_* = [\ell_1, \ell_2, \ell_3]. \tag{2.2}$$

Let $\varepsilon > 0$ and assume that $r' X^{1-\psi} \geqslant 1$ for some parameter $\psi \in (0,1)$. Throughout our work we will follow common practice and allow the small parameter $\varepsilon > 0$ to take different values at different parts of the argument, so that $x^\varepsilon \log x \ll_\varepsilon x^\varepsilon$, for example. In this section we will show that there exists a polynomial $P \in \mathbb{R}[x]$ of degree 3 such that

$$\begin{aligned} S(X) = {} & \operatorname{vol}(\mathcal{R}) X^2 P(\log X) \\ & + O_\varepsilon\big(L_\infty^\varepsilon r_\infty^\varepsilon r'^{\frac{3}{4}} (r_\infty + L_*^{\frac{1}{2}} \operatorname{vol}(\mathcal{R})^{\frac{1}{2}}) X^{\frac{7}{4}+\varepsilon}, \end{aligned} \tag{2.3}$$

where the leading coefficient of P is $\prod_p \sigma_p$, with

$$\sigma_p = \Big(1 - \frac{1}{p}\Big)^3 \sum_{\boldsymbol{\nu} \in \mathbb{Z}_{\geqslant 0}^3} \frac{\varrho(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{p^{2\nu_1 + 2\nu_2 + 2\nu_3}}. \tag{2.4}$$

Moreover, the coefficients of P have modulus $O_\varepsilon(L_\infty^\varepsilon r_\infty^\varepsilon (1 + r'^{-1})^\varepsilon)$.

As a first step we deduce from the trivial bound for the divisor function the estimate

$$S(X) \ll_\varepsilon L_\infty^\varepsilon r_\infty^{2+\varepsilon} X^{2+\varepsilon}. \tag{2.5}$$

We will also need to record the inequalities

$$\frac{r'}{2L_\infty} \leqslant r_\infty \leqslant 2r' L_\infty, \quad \operatorname{vol}(\mathcal{R}) \leqslant 4r_\infty^2. \tag{2.6}$$

The lower bounds for r_∞ and $4r_\infty^2$ are trivial. To see the remaining bound we suppose that $L_i(\mathbf{x}) = a_i x_1 + b_i x_2$. Let $\Delta_{i,j} = a_i b_j - a_j b_i$ denote the resultant of L_i, L_j. By hypothesis $\Delta_{i,j}$ is a non-zero integer. We have

$$x_1 = \frac{b_j L_i(\mathbf{x}) - b_i L_j(\mathbf{x})}{\Delta_{i,j}}, \quad x_2 = \frac{a_i L_j(\mathbf{x}) - a_j L_i(\mathbf{x})}{\Delta_{i,j}},$$

for any i, j. It therefore follows that $r_\infty \leqslant 2r' L_\infty$, as required for (2.6).

The technical tool underpinning the proof of (2.3) is an appropriate "level of distribution" result. Recall the definitions (1.3) and (1.4). The following is a trivial modification of the proofs of [2, Lemma 3] and [4, Lemma 3.2].

Lemma 2.1. *Let $\varepsilon > 0$. Let $X \geqslant 1$, $Q_i \geqslant 2$ and $Q = Q_1Q_2Q_3$. Then there exists an absolute constant $A > 0$ such that*

$$\sum_{\substack{\mathbf{d}\in\mathbb{N}^3\\ d_i\leqslant Q_i}} \Big|\#(\Lambda(\mathbf{d}) \cap X\mathcal{R}_{\mathbf{d}}) - \frac{\mathrm{vol}(X\mathcal{R}_{\mathbf{d}})\varrho(\mathbf{d})}{(d_1d_2d_3)^2}\Big|$$
$$\ll_\varepsilon L_\infty^\varepsilon(MX(\sqrt{Q} + \max Q_i) + Q)(\log Q)^A,$$

where $\mathcal{R}_{\mathbf{d}} \subseteq \mathcal{R}$ is any compact set depending on $\mathbf{d}$ whose boundary is a piecewise continuously differentiable closed curve of length at most M.

Recall the definition of r' from (1.8). In what follows it will be convenient to set

$$X' = r'X.$$

For any $1 \leqslant i \leqslant 3$ and $\mathbf{x} \in X\mathcal{R}$ we have

$$\begin{aligned}\tau(L_i(\mathbf{x})) &= \sum_{\substack{d\mid L_i(\mathbf{x})\\ d\leqslant\sqrt{X'}}} 1 + \sum_{\substack{d\mid L_i(\mathbf{x})\\ d>\sqrt{X'}}} 1\\ &= \sum_{\substack{d\mid L_i(\mathbf{x})\\ d\leqslant\sqrt{X'}}} 1 + \sum_{\substack{e\mid L_i(\mathbf{x})\\ e\sqrt{X'}<L_i(\mathbf{x})}} 1 \qquad (2.7)\\ &= \tau_+(L_i(\mathbf{x})) + \tau_-(L_i(\mathbf{x})),\end{aligned}$$

say. In this way we may produce a decomposition into 8 subsums

$$S(X) = \sum S_{\pm,\pm,\pm}(X), \qquad (2.8)$$

where

$$S_{\pm,\pm,\pm}(X) = \sum_{\mathbf{x}\in\mathbb{Z}^2\cap X\mathcal{R}} \tau_\pm(L_1(\mathbf{x}))\tau_\pm(L_2(\mathbf{x}))\tau_\pm(L_3(\mathbf{x})).$$

Each sum $S_{\pm,\pm,\pm}(X)$ is handled in the same way. Let us treat the sum $S_{+,+,-}(X)$, which is typical.

On noting that $L_i(\mathbf{x}) \leqslant X'$ for any $\mathbf{x} \in X\mathcal{R}$ we deduce that

$$S_{+,+,-}(X) = \sum_{d_1,d_2,d_3\leqslant\sqrt{X'}} \#(\Lambda(\mathbf{d}) \cap \mathcal{S}_{\mathbf{d}}),$$

where $\mathcal{S}_{\mathbf{d}}$ is the set of $\mathbf{x} \in X\mathcal{R}$ for which $d_3\sqrt{X'} < L_3(\mathbf{x})$. To estimate this sum we apply Lemma 2.1 with $Q_1 = Q_2 = Q_3 = \sqrt{X'}$. This gives

$$S_{+,+,-}(X) - \sum_{d_1,d_2,d_3\leqslant\sqrt{X'}} \frac{\varrho(\mathbf{d})\,\mathrm{vol}(\mathcal{S}_{\mathbf{d}})}{(d_1d_2d_3)^2}$$
$$\ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon(r_\infty r'^{\frac{3}{4}}X^{\frac{7}{4}+\varepsilon} + r'^{\frac{3}{2}}X^{\frac{3}{2}+\varepsilon}),$$

since $\partial(\mathcal{R}) \ll r_\infty$. If $r'^{\frac{3}{4}} \leqslant r_\infty X^{\frac{1}{4}}$ then this error term is satisfactory for (2.3). Alternatively, if $r'^{\frac{3}{4}} > r_\infty X^{\frac{1}{4}}$, then the conclusion follows from (2.5) instead. It remains to analyse the main term, the starting point for which is an analysis of the sum

$$M(\mathbf{T}) = \sum_{d_i \leqslant T_i} \frac{\varrho(\mathbf{d})}{(d_1 d_2 d_3)^2}, \tag{2.9}$$

for $T_1, T_2, T_3 \geqslant 1$. We will establish the following result.

Lemma 2.2. *Let $\varepsilon > 0$ and $T = T_1 T_2 T_3$. Then there exist $c, c_{i,j}, c_k, c_0 \in \mathbb{R}$, with modulus $O_\varepsilon(L_\infty^\varepsilon)$, such that*

$$\begin{aligned} M(\mathbf{T}) = c \prod_{i=1}^{3} \log T_i + \sum_{1 \leqslant i < j \leqslant 3} c_{i,j} (\log T_i)(\log T_j) + \sum_{1 \leqslant k \leqslant 3} c_k \log T_k + c_0 \\ + O_\varepsilon(L_\infty^\varepsilon L_*^{\frac{1}{2}} \min\{T_1, T_2, T_3\}^{-\frac{1}{2}} T^\varepsilon), \end{aligned}$$

where L_* is given by (2.2) *and*

$$c = \prod_p \sigma_p. \tag{2.10}$$

Before proving this result we first show how it leads to (2.3). For ease of notation we write $d_3 = z$ and $f(z) = \mathrm{vol}(\mathcal{S}_{\mathbf{d}})$. Let $Q = \sqrt{X'}$. Since $f(Q) = 0$, it follows from partial summation that

$$\sum_{d_1, d_2, d_3 \leqslant Q} \frac{\varrho(\mathbf{d}) \, \mathrm{vol}(\mathcal{S}_{\mathbf{d}})}{(d_1 d_2 d_3)^2} = -\int_1^Q f'(z) M(Q, Q, z) \mathrm{d}z,$$

in the notation of (2.9). An application of Lemma 2.2 reveals that there exist constants $c, a_1, \ldots, a_5 \ll_\varepsilon L_\infty^\varepsilon$ such that

$$\begin{aligned} M(Q, Q, z) = c(\log Q)^2(\log z) + a_1(\log Q)^2 + a_2(\log Q)(\log z) \\ + a_3 \log Q + a_4 \log z + a_5 + O_\varepsilon(L_\infty^\varepsilon L_*^{\frac{1}{2}} z^{-\frac{1}{2}} Q^\varepsilon), \end{aligned}$$

with c given by (2.10). However we claim that

$$f'(z) \ll \mathrm{vol}(\mathcal{R})^{\frac{1}{2}} Q X.$$

To see this we suppose that $L_3(\mathbf{x}) = a_3 x_1 + b_3 x_2$ with $|a_3| \geqslant |b_3| > 0$. Then

$$\begin{aligned} -f'(z) &= \lim_{\Delta \to 0} \Delta^{-1} \mathrm{vol}\{\mathbf{x} \in X\mathcal{R} : zQ < L_3(\mathbf{x}) \leqslant (z + \Delta)Q\} \\ &= \lim_{\Delta \to 0} \Delta^{-1} \mathrm{vol}\left\{\left(y_1, \frac{y_2 + zQ - a_3 y_1}{b_3}\right) \in X\mathcal{R} : 0 < y_2 \leqslant \Delta Q\right\} \\ &\ll Q \, \mathrm{vol}(X\mathcal{R})^{\frac{1}{2}}, \end{aligned}$$

on making the change of variables $y_1 = x_1$ and $y_2 = L_3(\mathbf{x}) - zQ$. This therefore establishes the claim and we see that the error term contributes

$$\begin{aligned}
&\ll_\varepsilon L_\infty^\varepsilon L_*^{\frac12} Q^\varepsilon \int_1^Q |f'(z)| z^{-\frac12} \mathrm{d}z \\
&\ll_\varepsilon L_\infty^\varepsilon L_*^{\frac12} \operatorname{vol}(\mathcal{R})^{\frac12} Q^{\frac32+\varepsilon} X \\
&\ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon L_*^{\frac12} \operatorname{vol}(\mathcal{R})^{\frac12} r'^{\frac34} X^{\frac74+\varepsilon}.
\end{aligned}$$

Moreover, we have

$$\int_1^Q f'(z)\mathrm{d}z = -f(1) = -X^2 \operatorname{vol}(\mathcal{R}) + O(r_\infty Q X),$$

and

$$\begin{aligned}
\int_1^Q (\log z) f'(z) \mathrm{d}z &= -\int_1^Q \frac{f(z)}{z}\mathrm{d}z \\
&= -\int_{\mathbf{x}\in X\mathcal{R}} \int_{1<z<Q^{-1}L_3(\mathbf{x})} \frac{\mathrm{d}z\mathrm{d}\mathbf{x}}{z} \\
&= -\int_{\mathbf{x}\in X\mathcal{R}} \big(\log L_3(\mathbf{x}) - \log Q\big)\mathrm{d}\mathbf{x} \\
&= -X^2 \operatorname{vol}(\mathcal{R}) \log Q + bX^2,
\end{aligned}$$

for a constant $b \ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon \operatorname{vol}(\mathcal{R})(1 + r'^{-1})^\varepsilon$. Putting everything together we conclude

$$\begin{aligned}
\sum_{d_1,d_2,d_3 \leqslant Q} \frac{\varrho(\mathbf{d}) \operatorname{vol}(\mathcal{S}_\mathbf{d})}{(d_1 d_2 d_3)^2} = {}& 2^{-3} \operatorname{vol}(\mathcal{R}) X^2 P(\log X) \\
&+ O_\varepsilon\big(L_\infty^\varepsilon r_\infty r'^{\frac12} X^{\frac32+\varepsilon}\big) \\
&+ O_\varepsilon\big(L_\infty^\varepsilon r_\infty^\varepsilon L_*^{\frac12} \operatorname{vol}(\mathcal{R})^{\frac12} r'^{\frac34} X^{\frac74+\varepsilon}\big),
\end{aligned}$$

for a suitable polynomial $P \in \mathbb{R}[x]$ of degree 3 with leading coefficient $\prod_p \sigma_p$ and all coefficients having modulus $O_\varepsilon(L_\infty^\varepsilon r_\infty^\varepsilon (1 + r'^{-1})^\varepsilon)$. The error terms in this expression are satisfactory for (2.3). Once taken in conjunction with the analogous estimates for the remaining 7 sums in (2.8), this therefore completes the proof of (2.3).

We may now turn to the proof of Lemma 2.2, which rests upon an explicit investigation of the function $\varrho(\mathbf{d})$. Now it follows from the Chinese remainder theorem that there is a multiplicativity property

$$\varrho(g_1h_1, g_2h_2, g_3h_3) = \varrho(g_1, g_2, g_3)\varrho(h_1, h_2, h_3),$$

whenever $\gcd(g_1g_2g_3, h_1h_2h_3) = 1$. Recall that $\Delta_{i,j}$ is used to denote the resultant of L_i, L_j and set

$$\Delta = |\Delta_{1,2}\Delta_{1,3}\Delta_{2,3}| \neq 0.$$

Recall the definition of ℓ_i and L_i^* from (2.1). The following result collects together some information about the behaviour of $\varrho(\mathbf{d})$ at prime powers.

Lemma 2.3. *Let p be a prime. Suppose that* $\min\{e_i, \nu_p(\ell_i)\} = 0$. *Then we have*

$$\varrho(p^{e_1}, 1, 1) = p^{e_1}, \quad \varrho(1, p^{e_2}, 1) = p^{e_2}, \quad \varrho(1, 1, p^{e_3}) = p^{e_3}.$$

Next suppose that $0 \leqslant e_i \leqslant e_j \leqslant e_k$ *for a permutation* $\{i, j, k\}$ *of* $\{1, 2, 3\}$. *Then we have*

$$\varrho(p^{e_1}, p^{e_2}, p^{e_3}) \begin{cases} = p^{2e_i+e_j+e_k}, & \text{if } p \nmid \Delta, \\ \leqslant p^{2e_i+e_j+e_k+\min\{e_j, v_p(\Delta)\}+\min\{e_k, v_p(\ell_k)\}}, & \text{if } p \mid \Delta. \end{cases}$$

Proof. The first part of the lemma is obvious. To see the second part we suppose without loss of generality that $e_1 \leqslant e_2 \leqslant e_3$.

When $p \nmid \Delta$ the conditions $p^{e_i} \mid L_i(\mathbf{x})$ in $\varrho(p^{e_1}, p^{e_2}, p^{e_3})$ are equivalent to $p^{e_2} \mid \mathbf{x}$ and $p^{e_3} \mid L_3(\mathbf{x})$. Thus we conclude that

$$\varrho(p^{e_1}, p^{e_2}, p^{e_3}) = \#\{\mathbf{x} \,(\mathrm{mod}\, p^{e_1+e_3}) : p^{e_3-e_2} \mid L_3(\mathbf{x})\} = p^{2e_1+e_2+e_3},$$

as required.

Turning to the case $p \mid \Delta$, we begin with the inequalities

$$\begin{aligned} \varrho(p^{e_1}, p^{e_2}, p^{e_3}) &\leqslant p^{2e_1} \varrho(1, p^{e_2}, p^{e_3}) \\ &\leqslant p^{2e_1} \#\{\mathbf{x} \,(\mathrm{mod}\, p^{e_2+e_3}) : p^{e_2} \mid \Delta_{2,3}\mathbf{x}, \ p^{e_3} \mid L_3(\mathbf{x})\}. \end{aligned}$$

Let us write $\delta = v_p(\Delta_{2,3})$ and $\lambda = v_p(\ell_3)$ for short. In particular it is clear that $\delta \geqslant \lambda$. In this way we deduce that $\varrho(p^{e_1}, p^{e_2}, p^{e_3})$ is at most

$$p^{2e_1} \#\{\mathbf{x} \,(\mathrm{mod}\, p^{e_2+e_3}) : p^{\max\{e_2-\delta, 0\}} \mid \mathbf{x}, \ p^{\max\{e_3-\lambda, 0\}} \mid L_3^*(\mathbf{x})\}.$$

Suppose first that $e_2 \geqslant \delta$. Then $0 \leqslant e_2 - \delta \leqslant e_3 - \lambda$ and it follows that

$$\begin{aligned} \varrho(p^{e_1}, p^{e_2}, p^{e_3}) &\leqslant p^{2e_1} \#\{\mathbf{x} \,(\mathrm{mod}\, p^{e_3+\delta}) : p^{e_3-\lambda} \mid p^{e_2-\delta} L_3^*(\mathbf{x})\} \\ &= p^{2e_1} \cdot p^{e_3+\delta} \cdot p^{e_2+\lambda} \\ &= p^{2e_1+e_2+e_3+\delta+\lambda}, \end{aligned}$$

since L_3^* is primitive. Alternatively, if $e_2 < \delta$, we deduce that

$$\begin{aligned} \varrho(p^{e_1}, p^{e_2}, p^{e_3}) &\leqslant p^{2e_1} \#\{\mathbf{x} \,(\mathrm{mod}\, p^{e_2+e_3}) : p^{\max\{e_3-\lambda, 0\}} \mid L_3^*(\mathbf{x})\} \\ &= p^{2e_1+2e_2+e_3+\min\{e_3, \lambda\}}. \end{aligned}$$

Taking together these two estimates completes the proof of the lemma. $\square$

We now have the tools with which to tackle the proof of Lemma 2.2. We will argue using Dirichlet convolution, as in [3, Lemma 4]. Let

$$f(\mathbf{d}) = \frac{\varrho(\mathbf{d})}{d_1 d_2 d_3}$$

and let $h : \mathbb{N}^3 \to \mathbb{N}$ be chosen so that $f(\mathbf{d}) = (1 * h)(\mathbf{d})$, where $1(\mathbf{d}) = 1$ for all $\mathbf{d} \in \mathbb{N}^3$. We then have

$$h(\mathbf{d}) = (\mu * f)(\mathbf{d}),$$

where $\mu(\mathbf{d}) = \mu(d_1)\mu(d_2)\mu(d_3)$. The following result is the key technical estimate in our analysis of $M(\mathbf{T})$.

Lemma 2.4. *For any $\varepsilon > 0$ and any $\delta_1, \delta_2, \delta_3 \geqslant 0$ such that $\delta_1+\delta_2+\delta_3 < 1$, we have*

$$\sum_{\mathbf{k}\in\mathbb{N}^3} \frac{|h(\mathbf{k})|}{k_1^{1-\delta_1}k_2^{1-\delta_2}k_3^{1-\delta_3}} \ll_{\delta_1,\delta_2,\delta_3,\varepsilon} L_\infty^\varepsilon L_*^{\delta_1+\delta_2+\delta_3},$$

where L_* is given by (2.2)*.*

Proof. On noting that $k_1^{\delta_1}k_2^{\delta_2}k_3^{\delta_3} \leqslant k_1^{\delta_\Sigma} + k_2^{\delta_\Sigma} + k_3^{\delta_\Sigma}$, with $\delta_\Sigma = \delta_1 + \delta_2 + \delta_3$, it clearly suffices to establish the lemma in the special case $\delta_2 = \delta_3 = 0$ and $0 \leqslant \delta_1 < 1$.

Using the multiplicativity of h, our task is to estimate the Euler product

$$P = \prod_p \Big(1 + \sum_{\substack{\nu_i\geqslant 0\\ \boldsymbol{\nu}\neq\mathbf{0}}} \frac{|h(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})|p^{\nu_1\delta_1}}{p^{\nu_1+\nu_2+\nu_3}}\Big) = \prod_p P_p,$$

say. Now for any prime p, we deduce that

$$|h(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})| = |(\mu * f)(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})| \leqslant (1 * f)(p^{\nu_1}, p^{\nu_2}, p^{\nu_3}), \tag{2.11}$$

whence

$$P_p \leqslant 1 + \sum_{\substack{\alpha_i,\beta_i\geqslant 0\\ \boldsymbol{\alpha}+\boldsymbol{\beta}\neq\mathbf{0}}} \frac{p^{\alpha_1\delta_1}}{p^{\alpha_1+\alpha_2+\alpha_3}} \cdot \frac{f(p^{\beta_1}, p^{\beta_2}, p^{\beta_3})p^{\beta_1\delta_1}}{p^{\beta_1+\beta_2+\beta_3}}.$$

We may conclude that the contribution to the above sum from $\boldsymbol{\alpha}, \boldsymbol{\beta}$ such that $\boldsymbol{\beta} = \mathbf{0}$ is $O(p^{-1+\delta_1})$.

Suppose now that $\boldsymbol{\beta} \neq \mathbf{0}$, with $\beta_i \leqslant \beta_j \leqslant \beta_k$ for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ such that $\beta_k \geqslant 1$. Then Lemma 2.3 implies that

$$\frac{f(p^{\beta_1}, p^{\beta_2}, p^{\beta_3})p^{\beta_1\delta_1}}{p^{\beta_1+\beta_2+\beta_3}} \leqslant p^{\beta_1\delta_1} \cdot \frac{p^{\min\{\beta_j,v_p(\Delta)\}+\min\{\beta_k,\lambda_k\}}}{p^{\beta_j+\beta_k}},$$

where we have written $\lambda_k = v_p(\ell_k)$ for short. Summing this contribution over $\boldsymbol{\beta} \neq \mathbf{0}$ we therefore arrive at the contribution

$$\begin{aligned}
&\leqslant \sum_{1\leqslant k\leqslant 3} \sum_{\max\{\beta_1,\beta_2,\beta_3\}=\beta_k\geqslant 1} p^{\beta_1\delta_1} \cdot p^{\min\{\beta_k,\lambda_k\}-\beta_k} \\
&\ll \sum_{1\leqslant k\leqslant 3} \sum_{\beta_k\geqslant 1} p^{\beta_k(\delta_1-1)+\min\{\beta_k,\lambda_k\}} \\
&\ll p^{\max\{\lambda_1,\lambda_2,\lambda_3\}\delta_1}.
\end{aligned}$$

It now follows that

$$\prod_{p\mid\Delta} P_p \leqslant \prod_{p\mid\Delta}\Big(1+O(p^{-1+\delta_1})+O(p^{\max\{\lambda_1,\lambda_2,\lambda_3\}\delta_1})\Big) \ll_\varepsilon L_\infty^\varepsilon L_*^{\delta_1},$$

where L_* is given by (2.2). This is satisfactory for the lemma.

Turning to the contribution from $p\nmid\Delta$, it is a simple matter to conclude that

$$\varrho(p^{\nu_1},p^{\nu_2},p^{\nu_3};L_1,L_2,L_3)=\varrho(p^{\nu_1},p^{\nu_2},p^{\nu_3};L_1^*,L_2^*,L_3^*).$$

Hence Lemma 2.3 yields $h(p^\nu,1,1)=h(1,p^\nu,1)=h(1,1,p^\nu)=0$ if $\nu\geqslant 1$ and $p\nmid\Delta$, since then $f(p^\nu,1,1)=f(1,p^\nu,1)=f(1,1,p^\nu)=1$. Moreover, we deduce from Lemma 2.3 and (2.11) that for $p\nmid\Delta$ we have

$$\begin{aligned}|h(p^{\nu_1},p^{\nu_2},p^{\nu_3})| &\leqslant (1+\nu_1)(1+\nu_2)(1+\nu_3)\sum_{0\leqslant n_i\leqslant\nu_i} f(p^{n_1},p^{n_2},p^{n_3})\\ &\leqslant (1+\nu_1)^2(1+\nu_2)^2(1+\nu_3)^2\max_{0\leqslant n_i\leqslant\nu_i} f(p^{n_1},p^{n_2},p^{n_3})\\ &= (1+\nu_1)^2(1+\nu_2)^2(1+\nu_3)^2 p^{\min\{\nu_1,\nu_2,\nu_3\}}.\end{aligned}$$

Thus

$$\prod_{p\nmid\Delta} P_p=\prod_{p\nmid\Delta}\Big(1+\sum_{\boldsymbol{\nu}}\frac{(1+\nu_1)^2(1+\nu_2)^2(1+\nu_3)^2p^{\min\{\nu_1,\nu_2,\nu_3\}}}{p^{\nu_1(1-\delta_1)+\nu_2+\nu_3}}\Big),$$

where the sum over $\boldsymbol{\nu}$ is over all $\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}$ such that $\nu_1+\nu_2+\nu_3\geqslant 2$, with at least two of the variables being non-zero. The overall contribution to the sum arising from precisely two variables being non-zero is clearly $O(p^{-2})$. Likewise, we see that the contribution from all three variables being non-zero is $O(p^{-2+\delta_1})$. It therefore follows that

$$\prod_{p\nmid\Delta} P_p=\prod_{p\nmid\Delta}\Big(1+O(p^{-2+\delta_1})\Big)\ll_{\delta_1,\varepsilon} L_\infty^\varepsilon,$$

since $\delta_1<1$. This completes the proof of the lemma. □

We are now ready to complete the proof of Lemma 2.2. On recalling the definition (2.9), we see that

$$M(\mathbf{T})=\sum_{d_i\leqslant T_i}\frac{f(\mathbf{d})}{d_1d_2d_3}=\sum_{d_i\leqslant T_i}\frac{(1*h)(\mathbf{d})}{d_1d_2d_3}=\sum_{k_i\leqslant T_i}\frac{h(\mathbf{k})}{k_1k_2k_3}\sum_{e_i\leqslant\frac{T_i}{k_i}}\frac{1}{e_1e_2e_3}.$$

Now the inner sum is estimated as

$$\prod_{i=1}^{3}\Big(\log T_i-\log k_i+\gamma+O(k_i^{\frac12}T_i^{-\frac12})\Big).$$

The main term in this estimate is equal to

$$\prod_{i=1}^{3} \log T_i + R(\log T_1, \log T_2, \log T_3),$$

for quadratic $R \in \mathbb{R}[x,y,z]$ with coefficients bounded by $\ll_\varepsilon (k_1k_2k_3)^\varepsilon$ and no non-zero coefficients of x^2, y^2 or z^2. The error term is seen to be $\ll_\varepsilon T^\varepsilon \max\{k_i T_i^{-1}\}^{\frac{1}{2}}$, with $T = T_1T_2T_3$. We may therefore apply Lemma 2.4 to obtain an overall error of

$$\ll_\varepsilon L_\infty^\varepsilon L_*^{\frac{1}{2}} \min\{T_i\}^{-\frac{1}{2}} T^\varepsilon, \tag{2.12}$$

where L_* is given by (2.2).

Our next step is to show that the sums involving $\mathbf{k}$ can be extended to infinity with negligible error. If $a \ll_\varepsilon (k_1k_2k_3)^\varepsilon$ is any of the coefficients in our cubic polynomial main term, then for $j \in \{1,2,3\}$ Rankin's trick yields

$$\sum_{\substack{\mathbf{k}\in\mathbb{N}^3\\ k_j>T_j}} \frac{|h(\mathbf{k})||a|}{k_1k_2k_3} \ll_\varepsilon \sum_{\substack{\mathbf{k}\in\mathbb{N}^3\\ k_j>T_j}} \frac{|h(\mathbf{k})|}{(k_1k_2k_3)^{1-\varepsilon}} < \frac{1}{T_j^{\frac{1}{2}}} \sum_{\mathbf{k}\in\mathbb{N}^3} \frac{|h(\mathbf{k})| k_j^{\frac{1}{2}}}{(k_1k_2k_3)^{1-\varepsilon}},$$

which Lemma 2.4 reveals is bounded by (2.12). We have therefore arrived at the asymptotic formula for $M(\mathbf{T})$ in Lemma 2.2, with coefficients of size $O_\varepsilon(L_\infty^\varepsilon)$, as follows from Lemma 2.4. Moreover, the leading coefficient takes the shape

$$\sum_{\mathbf{k}\in\mathbb{N}^3} \frac{h(\mathbf{k})}{k_1k_2k_3} = \sum_{\mathbf{k}\in\mathbb{N}^3} \frac{(\mu * f)(\mathbf{k})}{k_1k_2k_3} = \prod_p \sigma_p,$$

in the notation of (2.4). This therefore concludes the proof of Lemma 2.2.

3. Theorem 3: general case

Let $\mathbf{d}, \mathbf{D} \in \mathbb{N}^3$, with $d_i \mid D_i$, and assume that $r'X^{1-\theta} \geqslant 1$ for $\theta \in (\frac{1}{4}, 1)$. In estimating $S(X;\mathbf{d},\mathbf{D})$, our goal is to replace the summation over $\mathsf{\Lambda}(\mathbf{D})$ by a summation over $\mathbb{Z}^2$, in order to relate it to the sum $S(X)$ that we studied in the previous section. We begin by recording the upper bound

$$S(X;\mathbf{d},\mathbf{D}) \ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon \Big(\frac{\mathrm{vol}(\mathcal{R}) X^{2+\varepsilon}}{\det \mathsf{\Lambda}(\mathbf{D})} + r_\infty X^{1+\varepsilon} \Big). \tag{3.1}$$

This follows immediately on taking the trivial estimate for the divisor function and applying standard lattice point counting results.

Given any basis $\mathbf{e}_1, \mathbf{e}_2$ for $\mathsf{\Lambda}(\mathbf{D})$, let $M_i(\mathbf{v})$ be the linear form obtained from $d_i^{-1}L_i(\mathbf{x})$ via the change of variables $\mathbf{x} \mapsto v_1\mathbf{e}_1 + v_2\mathbf{e}_2$. By choosing $\mathbf{e}_1, \mathbf{e}_2$ to be a minimal basis, we may further assume that

$$1 \leqslant |\mathbf{e_1}| \leqslant |\mathbf{e}_2|, \quad |\mathbf{e}_1||\mathbf{e}_2| \ll \det \mathsf{\Lambda}(\mathbf{D}), \tag{3.2}$$

where $|\mathbf{z}| = \max|z_i|$ for $\mathbf{z} \in \mathbb{R}^2$. Write $\mathbf{M}$ for the matrix formed from $\mathbf{e}_1, \mathbf{e}_2$. Carrying out this change of variables, we obtain

$$S(X; \mathbf{d}, \mathbf{D}) = \sum_{\mathbf{v} \in \mathbb{Z}^2 \cap X\mathcal{R}_{\mathbf{M}}} \tau(M_1(\mathbf{v}))\tau(M_2(\mathbf{v}))\tau(M_3(\mathbf{v})),$$

where $\mathcal{R}_{\mathbf{M}} = \{\mathbf{M}^{-1}\mathbf{z} : \mathbf{z} \in \mathcal{R}\}$. Note that $M_i(\mathbf{v}) > 0$ for every $\mathbf{v}$ in the summation. Moreover, the M_i will be linearly independent linear forms defined over $\mathbb{Z}$ and $\partial(\mathcal{R}_{\mathbf{M}}) \ll r_\infty(\mathcal{R}_{\mathbf{M}})$ in the notation of (1.7), where $\partial(\mathcal{R}_{\mathbf{M}})$ is the length of the boundary of $\mathcal{R}_{\mathbf{M}}$.

We now wish to estimate this quantity. In view of (3.2) and the fact that $\det \mathsf{\Lambda}(\mathbf{D}) = [\mathbb{Z}^2 : \mathsf{\Lambda}(\mathbf{D})]$ divides $D = D_1D_2D_3$, it is clear that

$$L_\infty(M_1, M_2, M_3) \leqslant DL_\infty(L_1, L_2, L_3) = DL_\infty,$$

in the notation of (1.6). In a similar fashion, recalling the definitions (1.7) and (1.8), we observe that

$$r_\infty(\mathcal{R}_{\mathbf{M}}) \ll \frac{|\mathbf{e}_1||\mathbf{e}_2|}{|\det \mathbf{M}|} r_\infty(\mathcal{R}) \ll r_\infty(\mathcal{R}) = r_\infty$$

and $r'(M_1, M_2, M_3, \mathcal{R}_{\mathbf{M}}) \leqslant \min\{d_1, d_2, d_3\}^{-1} r'(L_1, L_2, L_3, \mathcal{R}) \leqslant r'$.

Note that $r_\infty X \leqslant r_\infty r'^{\frac{3}{4}} X^{\frac{7}{4}}$, by our hypothesis on r'. Moreover, since

$$\det \mathsf{\Lambda}(\mathbf{D}) = \frac{D^2}{\varrho(\mathbf{D})},$$

Lemma 2.3 yields $\det \mathsf{\Lambda}(\mathbf{D}) \gg d_k \gcd(d_k, \ell_k)^{-1}$ for any $1 \leqslant k \leqslant 3$. Suppose for the moment that $d_k = \max\{d_i\} > X^{\frac{1}{4}}$. Then an application of (2.6) and (3.1) easily reveals that

$$\begin{aligned} S(X; \mathbf{d}, \mathbf{D}) &\ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon \Big(\frac{r_\infty^2 X^{2+\varepsilon} \gcd(d_k, \ell_k)}{d_k} + r_\infty X^{1+\varepsilon} \Big) \\ &\ll_\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon (r_\infty r'^{\frac{3}{4}} + L_\infty^{\frac{1}{2}} L_*^{\frac{1}{2}} r_\infty^2) X^{\frac{7}{4}+\varepsilon}, \end{aligned} \tag{3.3}$$

where ℓ_k is defined in (2.1) and L_* by (2.2). Alternatively, if $\max\{d_i\} \leqslant X^{\frac{1}{4}}$ then for any $\psi > 0$ we have

$$r'(M_1, M_2, M_3, \mathcal{R}_{\mathbf{M}}) X^{1-\psi} \geqslant r' X^{\frac{3}{4}-\psi} \geqslant r' X^{1-\theta} \geqslant 1,$$

provided that $\psi \leqslant \theta - \frac{1}{4}$. Taking $\psi = \theta - \frac{1}{4} \in (0, \frac{3}{4})$ all the hypotheses are therefore met for an application of (2.3).

To facilitate this application we note that $\mathrm{vol}(\mathcal{R}_{\mathbf{M}}) = |\det \mathbf{M}|^{-1} \mathrm{vol}(\mathcal{R})$. Moreover, if m_i denotes the greatest common divisor of the coefficients of M_i then $m_i \mid \ell_i \det \mathsf{\Lambda}(\mathbf{D})$. Hence we have

$$L_*(M_1, M_2, M_3) = [m_1, m_2, m_3] \leqslant [\ell_1, \ell_2, \ell_3] \det \mathsf{\Lambda}(\mathbf{D}) = L_* \det \mathsf{\Lambda}(\mathbf{D}),$$

from which it is clear that

$$L_*(M_1,M_2,M_3)^{\frac{1}{2}}\operatorname{vol}(\mathcal{R}_{\mathbf{M}})^{\frac{1}{2}} \leqslant L_*^{\frac{1}{2}}\operatorname{vol}(\mathcal{R})^{\frac{1}{2}} \leqslant 2L_*^{\frac{1}{2}}r_\infty,$$

by (2.6). Finally we recall that $r'(M_1,M_2,M_3,\mathcal{R}_{\mathbf{M}}) \geqslant (\max\{d_i\})^{-1}r'$. Collecting all of this together, it now follows from (2.3) and (3.3) that

$$S(X;\mathbf{d},\mathbf{D}) = \frac{\operatorname{vol}(\mathcal{R})}{\det \mathsf{\Lambda}(\mathbf{D})}X^2P(\log X) + O_\varepsilon(\mathcal{E}),$$

where the leading coefficient of P is $\prod_p \sigma_p^*$ and σ_p^* is defined as for σ_p in (2.4), but with $\varrho(\mathbf{h};L_1,L_2,L_3)$ replaced by $\varrho(\mathbf{h};M_1,M_2,M_3)$, and

$$\mathcal{E} = D^\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon (L_*^{\frac{1}{2}}r_\infty r'^{\frac{3}{4}} + L_\infty^{\frac{1}{2}}L_*^{\frac{1}{2}}r_\infty^2)X^{\frac{7}{4}+\varepsilon}.$$

Furthermore, the coefficients of P are all $O_\varepsilon(D^\varepsilon L_\infty^\varepsilon r_\infty^\varepsilon(1+r'^{-1})^\varepsilon)$ in modulus, so that the coefficients of the polynomial appearing in Theorem 3 have the size claimed there. Following the calculations in [2, §6] one finds that

$$\frac{1}{\det \mathsf{\Lambda}(\mathbf{D})}\prod_p \sigma_p^* = \prod_p \sigma_p(\mathbf{d},\mathbf{D}),$$

in the notation of (1.9).

Let us write $S(X;\mathbf{d},\mathbf{D}) = S(X;\mathbf{d},\mathbf{D};L_1,L_2,L_3,\mathcal{R})$ in (1.5) in order to stress the various dependencies. Recall the notation $\delta = \delta(\mathbf{D})$ that was introduced prior to the statement of Theorem 3. In order to obtain the factor δ^{-1} in the error term $\mathcal{E}$ we simply observe that

$$S(X;\mathbf{d},\mathbf{D};L_1,L_2,L_3,\mathcal{R}) = S(X;\mathbf{d},\mathbf{D};\delta L_1,\delta L_2,\delta L_3,\delta^{-1}\mathcal{R}).$$

According to (1.7) and (1.8), the value of r' is left unchanged and r_∞ should be divided by δ. However, L_∞ is replaced by δL_∞ and L_* becomes δL_*. On noting that $L_* \leqslant \ell_1\ell_2\ell_3 \leqslant L_\infty^3$, we easily conclude that the new error term is as in Theorem 3. Finally the constants obtained as factors of $X^2(\log X)^i$ in the main term must be the same since they are independent of X. This therefore concludes the proof of Theorem 3.

4. Treatment of $T(X)$

In this section we establish Theorem 1. For convenience we will assume that the coefficients of L_1,L_2,L_3 are all positive so that $L_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in [0,1]^2$. The general case is readily handled by breaking the sum over $\mathbf{x}$ into regions on which the sign of each $L_i(\mathbf{x})$ is fixed. In order to transfigure $T(X)$ into the sort of sum defined in (1.5), we will follow the opening steps of the argument in [3, §7]. This hinges upon the formula

$$\tau(n_1n_2n_3) = \sum_{\substack{\mathbf{e}\in\mathbb{N}^3\\ e_ie_j|n_k}} \frac{\mu(e_1e_2)\mu(e_3)}{2^{\omega(\gcd(e_1,n_1))+\omega(\gcd(e_2,n_2))}}\tau\Big(\frac{n_1}{e_2e_3}\Big)\tau\Big(\frac{n_2}{e_1e_3}\Big)\tau\Big(\frac{n_3}{e_1e_2}\Big),$$

which is established in [3, Lemma 10] and is valid for any $\mathbf{n} \in \mathbb{N}^3$. In this way we deduce that

$$T(X) = \sum_{\mathbf{e}\in\mathbb{N}^3} \mu(e_1e_2)\mu(e_3) \sum_{\substack{\mathbf{k}=(k_1,k_2,k_1',k_2')\in\mathbb{N}^4 \\ k_ik_i'|e_i}} \frac{\mu(k_1')\mu(k_2')}{2^{\omega(k_1)+\omega(k_2)}} T_{\mathbf{e},\mathbf{k}}(X),$$

with

$$T_{\mathbf{e},\mathbf{k}}(X) = \sum_{\mathbf{x}\in\mathsf{\Lambda}\cap[0,X]^2} \tau\Big(\frac{L_1(\mathbf{x})}{e_2e_3}\Big)\tau\Big(\frac{L_2(\mathbf{x})}{e_1e_3}\Big)\tau\Big(\frac{L_3(\mathbf{x})}{e_1e_2}\Big)$$

and $\mathsf{\Lambda} = \mathsf{\Lambda}([e_2e_3, k_1k_1'], [e_1e_3, k_2k_2'], e_1e_2)$ given by (1.3). Under the conditions $k_ik_i' \mid e_i$ and $|\mu(e_1e_2)| = |\mu(e_3)| = 1$, we clearly have

$$\mathsf{\Lambda} = \mathsf{\Lambda}([e_2e_3, k], [e_1e_3, k], e_1e_2),$$

with $k = k_1k_1'k_2k_2'$. Thus $T_{\mathbf{e},\mathbf{k}}(X)$ depends only on $k \mid e_1e_2$. Noting that $T_{\mathbf{e},\mathbf{k}}(X) = 0$ unless $|\mathbf{e}| \leqslant X$, and

$$\sum_{\substack{\mathbf{k}=(k_1,k_2,k_1',k_2')\in\mathbb{N}^4 \\ k_ik_i'=\gcd(k,e_i)}} \frac{\mu(k_1')\mu(k_2')}{2^{\omega(k_1)+\omega(k_2)}} = \frac{\mu(\gcd(k,e_1))\mu(\gcd(k,e_2))}{2^{\omega(\gcd(k,e_1))+\omega(\gcd(k,e_2))}} = \frac{\mu(k)}{2^{\omega(k)}},$$

we may therefore write

$$T(X) = \sum_{|\mathbf{e}|\leqslant X} \mu(e_1e_2)\mu(e_3) \sum_{k|e_1e_2} \frac{\mu(k)}{2^{\omega(k)}} T_{\mathbf{e},k}(X), \tag{4.1}$$

with $T_{\mathbf{e},k}(X) = S(X, \mathbf{d}, \mathbf{D})$ in the notation of (1.5) and

$$\mathbf{d} = (e_2e_3, e_1e_3, e_1e_2), \quad \mathbf{D} = ([e_2e_3, k], [e_1e_3, k], e_1e_2).$$

For the rest of this section we will allow all of our implied constants to depend upon ε and L_1, L_2, L_3. In particular we may clearly assume that $r_\infty = 1$, $L_\infty \ll 1$ and $1 \leqslant r' \ll 1$. Now let $\delta = \delta(\mathbf{D})$ be the quantity defined in the buildup to Theorem 3. A little thought reveals that

$$\delta \geqslant [e_1', e_2', e_3', k'] \gg [e_1e_2, e_3],$$

since e_1e_2 is square-free, where $e_i' = \frac{e_i}{\gcd(e_i,\Delta_{j,k})}$ and $k' = \frac{k}{\gcd(k,\Delta_{1,2})}$ and we recall that $\Delta_{j,k}$ is the resultant of L_j, L_k.

In view of the inequality $|\mathbf{e}| \leqslant X$, we conclude from Theorem 3 that

$$T_{\mathbf{e},k}(X) = X^2P(\log X) + O([e_1e_2, e_3]^{-1}X^{\frac{7}{4}+\varepsilon}),$$

for a cubic polynomial P with coefficients of size $\ll (e_1e_2e_3)^\varepsilon[e_1e_2, e_3]^{-2}$, since we have $\det \mathsf{\Lambda}(\mathbf{D}) \geqslant \delta^2$. The overall contribution from the error term, once inserted into (4.1), is

$$\begin{aligned}
&\ll X^{\frac{7}{4}+\varepsilon} \sum_{|\mathbf{e}|\leqslant X} \frac{|\mu(e_1e_2)\mu(e_3)|}{[e_1e_2, e_3]}\\
&\leqslant X^{\frac{7}{4}+\varepsilon} \sum_{|\mathbf{e}|\leqslant X} \frac{\gcd(e_1e_2, e_3)}{e_1e_2e_3}\\
&= X^{\frac{7}{4}+\varepsilon} \sum_{e_1,e_2\leqslant X} \frac{1}{e_1e_2} \sum_{h|e_1e_2} h \sum_{\substack{e_3\leqslant X\\ h|e_3}} \frac{1}{e_3}\\
&\ll X^{\frac{7}{4}+\varepsilon}.
\end{aligned}$$

This is clearly satisfactory from the point of view of Theorem 1. Similarly we deduce that the overall error produced by extending the summation over $\mathbf{e}$ to infinity is

$$\begin{aligned}
&\ll X^{2+\varepsilon} \sum_{|\mathbf{e}|>X} \frac{|\mu(e_1e_2)\mu(e_3)|(e_1e_2e_3)^{\varepsilon}}{[e_1e_2, e_3]^2}\\
&\ll X^{\frac{7}{4}+\varepsilon} \sum_{\mathbf{e}\in\mathbb{N}^3} \frac{\gcd(e_1e_2, e_3)^2|\mathbf{e}|^{\frac{1}{4}}}{(e_1e_2e_3)^2}\\
&\ll X^{\frac{7}{4}+\varepsilon}.
\end{aligned}$$

This therefore concludes the proof of Theorem 1.

5. Divisor problem on average

In this section we prove Theorem 2. We begin by writing

$$\sum_{h\leqslant H} \big(T_h(X) - c_h X(\log X)^3\big) = \Sigma_1 - \Sigma_2,$$

say, where c_h is given by (1.1). The following result deals with the second term.

Lemma 5.1. *Let $H \geqslant 1$. Then we have*

$$\Sigma_2 = cXH(\log X)^3 + O\big(XH^{\frac{1}{2}}(\log X)^3\big),$$

where

$$c = \frac{4}{3} \prod_{p>2} \Big(1+\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p}+\frac{1}{p^2}\Big). \tag{5.1}$$

Proof. We have $\Sigma_2 = c_1X(\log X)^3 S(H)$, where c_1 is given by taking $h = 1$ in (1.1), and $S(H) = \sum_{h\leqslant H} f(h)$, with f given multiplicatively by (1.2).

Using the equality $f=(f*\mu)*1$ and the trivial estimate $[x]=x+O(x^{\frac{1}{2}})$, we see that

$$S(H)=\sum_{d=1}^{\infty}(f*\mu)(d)\Big[\frac{H}{d}\Big]=H\sum_{d=1}^{\infty}\frac{(f*\mu)(d)}{d}+O\Big(H^{\frac{1}{2}}\sum_{d=1}^{\infty}\frac{|(f*\mu)(d)|}{d^{\frac{1}{2}}}\Big),$$

provided that the error term is convergent.

For $k\geqslant 1$ we have $(f*\mu)(p^k)=f(p^k)-f(p^{k-1})$. Hence we calculate

$$(f*\mu)(p^k)=\begin{cases}\frac{1}{p^k}\cdot\frac{1+3k-\frac{3k}{p}-\frac{3+3k}{p^2}+\frac{3k+2}{p^3}}{(1+\frac{2}{p})(1-\frac{1}{p})^2}, & \text{if } p>2,\\ \frac{1}{2^k}\cdot(1+\frac{15k}{11}), & \text{if } p=2,\end{cases}$$

for $k\geqslant 2$, and

$$(f*\mu)(p)=\begin{cases}\frac{1}{p}\cdot\frac{4+\frac{5}{p}}{1+\frac{2}{p}}, & \text{if } p>2,\\ \frac{13}{11}, & \text{if } p=2.\end{cases}$$

In particular it is clear that $|(f*\mu)(p^k)|\ll kp^{-k}$, whence

$$\sum_{d=1}^{\infty}\frac{|(f*\mu)(d)|}{d^{\frac{1}{2}}}\ll_\varepsilon\sum_{d=1}^{\infty}d^{-\frac{3}{2}+\varepsilon}\ll 1,$$

for $\varepsilon<\frac{1}{2}$. It follows that $S(H)=c_1'H+O(H^{\frac{1}{2}})$, where

$$\begin{aligned}c_1'&=\prod_p\sum_{k\geqslant 0}\frac{(f*\mu)(p^k)}{p^k}\\&=\frac{64}{33}\prod_{p>2}\Big(1+\frac{2}{p}\Big)^{-1}\Big(1-\frac{1}{p}\Big)^{-2}\Big(1+\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p}+\frac{1}{p^2}\Big).\end{aligned}$$

We conclude the proof of the lemma by noting that $c_1c_1'=c$. □

It would be easy to replace the exponent $\frac{1}{2}$ of H by any positive number, but this would not yield an overall improvement of Theorem 2. We now proceed to an analysis of the sum

$$\Sigma_1=\sum_{h\leqslant H}T_h(X)=\sum_{\substack{h\leqslant H\\ n\leqslant X}}\tau(n-h)\tau(n)\tau(n+h),$$

in which we follow the convention that $\tau(-n)=\tau(n)$. This corresponds to a sum of the type considered in (1.5), with $d_i=D_i=1$ and

$$L_1(\mathbf{x})=x_1-x_2,\quad L_2(\mathbf{x})=x_1,\quad L_3(\mathbf{x})=x_1+x_2.$$

The difference is that we are now summing over a lopsided region.

Lemma 5.2. *Let $H\geqslant 1$ and let $\varepsilon>0$. Then we have*

$$\Sigma_1=cXH(\log X)^3+O_\varepsilon(XH(\log X)^2+X^{\frac{1}{2}+\varepsilon}H+X^{\frac{7}{4}+\varepsilon}),$$

where c is given by (5.1).

Proof. Tracing through the proof of (2.3) one is led to consider 8 sums

$$\Sigma_1^{\pm,\pm,\pm} = \sum_{\substack{h\leqslant H\\ n\leqslant X}} \tau_\pm(n-h)\tau_\pm(n)\tau_\pm(n+h),$$

with $X' = 2X$ in the construction (2.7) of $\tau_\pm$. Arguing as before we examine a typical sum

$$\Sigma_1^{+,+,-} = \sum_{d_1,d_2,d_3\leqslant\sqrt{2X}} \#(\Lambda(\mathbf{d})\cap\mathcal{R}_{\mathbf{d}}(X,H)),$$

where $\mathcal{R}_{\mathbf{d}}(X,H) = \{\mathbf{x}\in(0,X]\times(0,H] : d_3\sqrt{2X} < L_3(\mathbf{x})\}$. An entirely analogous version of Lemma 2.1 for our lopsided region readily leads to the conclusion that

$$\Sigma_1^{+,+,-} = \sum_{d_1,d_2,d_3\leqslant\sqrt{2X}} \frac{\varrho(\mathbf{d})\operatorname{vol}(\mathcal{R}_{\mathbf{d}}(X,H))}{(d_1d_2d_3)^2} + O_\varepsilon\big(X^{\frac12+\varepsilon}H + X^{\frac74+\varepsilon}\big).$$

Combining Lemma 2.2 with partial summation, as previously, we conclude that

$$\sum_{d_1,d_2,d_3\leqslant\sqrt{2X}} \frac{\varrho(\mathbf{d})\operatorname{vol}(\mathcal{R}_{\mathbf{d}}(X,H))}{(d_1d_2d_3)^2} = XH(\log X)^3\prod_p \sigma_p + O_\varepsilon(XH(\log X)^2 + X^{\frac74+\varepsilon}),$$

with σ_p given by (2.4). This gives the lemma with $c = \prod_p \sigma_p$.

It remains to show that c matches up with (5.1). For any $\mathbf{a}\in\mathbb{Z}^3_{\geqslant 0}$ let $m(\mathbf{a}) = \max_{i\neq j}\{a_i+a_j\}$. For $z\in\mathbb{C}$ such that $|z|<1$ we claim that

$$S(z) = \sum_{\nu_1,\nu_2,\nu_3\geqslant 0} z^{m(\boldsymbol{\nu})} = \frac{1+z+z^2}{(1-z)^2(1-z^2)}. \tag{5.2}$$

But this follows easily from the observation

$$S(z) = 1 + 3\sum_{\substack{\nu_1=\nu_2=0\\ \nu_3\geqslant 1}} z^{\nu_3} + 3\sum_{\substack{\nu_1=0\\ \nu_2,\nu_3\geqslant 1}} z^{\nu_2+\nu_3} + z^2 S(z).$$

The linear forms arising in our analysis have resultants $\Delta_{1,2}=1, \Delta_{1,3}=2$ and $\Delta_{2,3}=1$. Moreover, $\ell_1=\ell_2=\ell_3=1$ in the notation of (2.1). Suppose that $p>2$ and write $z=\frac1p$. Then it follows from Lemma 2.3 that

$$\sum_{\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}} \frac{\varrho(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{p^{2\nu_1+2\nu_2+2\nu_3}} = S(z) = \frac{1+z+z^2}{(1-z)^2(1-z^2)}.$$

If $p=2$, it will be necessary to revisit the proof of Lemma 2.3. To begin with it is clear that $\varrho(2^{\nu_1},2^{\nu_2},2^{\nu_3}) = 2^{\nu_1+\nu_2+\nu_3+\min\{\nu_i\}}$ if $\min\{\nu_1,\nu_3\}\leqslant\nu_2$.

If $\nu_2 < \nu_i \leqslant \nu_j$ for some permutation $\{i,j\}$ of $\{1,3\}$ then

$$\begin{aligned}\varrho(2^{\nu_1},2^{\nu_2},2^{\nu_3}) &= \#\{\mathbf{x} \,(\mathrm{mod}\, 2^{\nu_1+\nu_2+\nu_3}) : 2^{\nu_i} \mid \Delta_{1,3}\mathbf{x},\ 2^{\nu_j} \mid L_j(\mathbf{x})\}\\ &= 2^{\nu_1+2\nu_2+\nu_3+1}.\end{aligned}$$

Writing $z = \frac{1}{2}$ we obtain

$$\begin{aligned}\sum_{\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}} \frac{\varrho(2^{\nu_1},2^{\nu_2},2^{\nu_3})}{2^{2\nu_1+2\nu_2+2\nu_3}} &= \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}\\ \min\{\nu_1,\nu_3\}\leqslant\nu_2}} z^{m(\boldsymbol{\nu})} + \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}\\ \min\{\nu_1,\nu_3\}>\nu_2}} z^{m(\boldsymbol{\nu})-1}\\ &= S(z) + \sum_{\substack{\boldsymbol{\nu}\in\mathbb{Z}^3_{\geqslant 0}\\ \min\{\nu_1,\nu_3\}>\nu_2}} z^{\nu_1+\nu_3-1}(1-z)\\ &= \frac{1+z+z^2}{(1-z)^2(1-z^2)} + \frac{z}{(1-z)^2(1+z)}.\end{aligned}$$

Hence, (2.4) becomes

$$\sigma_p = \begin{cases}(1+\frac{1}{p})^{-1}(1+\frac{1}{p}+\frac{1}{p^2}), & \text{if } p>2,\\ \frac{4}{3}, & \text{if } p=2,\end{cases}$$

as required to complete the proof of the lemma. □

Once combined, Lemmas 5.1 and 5.2 yield

$$\Sigma_1 - \Sigma_2 \ll_\varepsilon XH^{\frac{1}{2}}(\log X)^3 + XH(\log X)^2 + X^{\frac{1}{2}+\varepsilon}H + X^{\frac{7}{4}+\varepsilon}.$$

This is $o(XH(\log X)^3)$ for $H \geqslant X^{\frac{3}{4}+\varepsilon}$, as claimed in Theorem 2.

6. Bilinear hypersurfaces

In this section we establish Theorem 4, for which we begin by studying the counting function

$$N_0(X) = \#\{(\mathbf{u},\mathbf{v}) \in (\mathbb{Z}\setminus\{0\})^6 : |\mathbf{v}| \leqslant v_0 \leqslant X^{\frac{1}{2}},\ |\mathbf{u}| \leqslant v_0^{-1}X,\ \mathbf{u}.\mathbf{v}=0\},$$

for large X, where $|\mathbf{x}| = \max\{|x_0|,|x_1|,|x_2|\}$ for any $\mathbf{x}=(x_0,x_1,x_2)\in\mathbb{R}^3$. The overall contribution from vectors with $|v_1| = v_0$ is

$$\begin{aligned}&\ll \sum_{|v_2|\leqslant v_0\leqslant X^{\frac{1}{2}}} \#\{\mathbf{u}\in\mathbb{Z}^3 :\ |\mathbf{u}| \leqslant v_0^{-1}X,\ u_0v_0+u_1v_0+u_2v_2=0\}\\ &\ll \sum_{|v_2|\leqslant v_0\leqslant X^{\frac{1}{2}}} \frac{X^2}{v_0^3} \ll X^2,\end{aligned}$$

as can be seen using the geometry of numbers. Similarly there is a contribution of $O(X^2)$ to $N_0(X)$ from vectors for which $|v_2| = v_0$. Thus we may conclude that

$$N_0(X) = 2^3N_1(X) + O(X^2),$$

where $N_1(X)$ is the contribution to $N_0(X)$ from vectors with $0 < v_1, v_2 < v_0$ and $u_2 > 0$, with the equation $\mathbf{u}.\mathbf{v} = 0$ replaced by $u_0v_0 + u_1v_1 = u_2v_2$.

Define the region

$$V = \Big\{\boldsymbol{\alpha} \in [0,1]^6 : \alpha_2, \alpha_3 < \alpha_1 \leqslant \frac{1}{2},\ \alpha_1 + \alpha_5 - \alpha_2 \leqslant 1,\ \alpha_1 + \alpha_6 - \alpha_3 \leqslant 1\Big\},$$

and set

$$L_1(\mathbf{x}) = x_1, \quad L_2(\mathbf{x}) = x_2, \quad L_3(\mathbf{x}) = x_1 + x_2.$$

We will work with the region $\mathcal{R} = \{\mathbf{x} \in [-1,1]^2 : x_1x_2 \neq 0,\ x_1 + x_2 > 0\}$. Then we clearly have $N_1(X) = R(X)$, with

$$R(X) = \sum_{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{R}} \#\Big\{\mathbf{e} \in \mathbb{N}^3 : e_i \mid L_i(\mathbf{x}),\ (\boldsymbol{\epsilon}, \boldsymbol{\xi}) \in V\Big\},$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3), \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and

$$\epsilon_i = \frac{\log e_i}{\log X}, \quad \xi_i = \frac{\log |L_i(\mathbf{x})|}{\log X}.$$

Note that for $V = [0,1]^6$ this sum coincides with (1.5) for $d_i = D_i = 1$. We establish an asymptotic formula for $R(X)$ along the lines of the proof of Theorem 3. We will need to arrange things so that we are only considering small divisors in the summand. It is easy to see that the overall contribution to the sum from $\mathbf{e}$ such that $e_j^2 = L_j(\mathbf{x})$ for some $j \in \{1,2,3\}$ is

$$\ll_\varepsilon X^\varepsilon \sum_{e_j \leqslant \sqrt{X}} \#\{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{R} :\ L_j(\mathbf{x}) = e_j^2\} \ll_\varepsilon X^{\frac{3}{2}+\varepsilon}.$$

It follows that we may write

$$R(X) = \sum_{\mathbf{m} \in \{\pm 1\}^3} R^{(\mathbf{m})}(X) + O_\varepsilon(X^{\frac{3}{2}+\varepsilon}), \tag{6.1}$$

where $R^{(\mathbf{m})}(X)$ is the contribution from $m_ie_i \leqslant m_i\sqrt{|L_i(\mathbf{x})|}$.

We indicate how to handle $R^{(1,1,-1)}(X) = R^{+,+,-}(X)$, say, which is typical. Writing $L_3(\mathbf{x}) = e_3f_3$, we see that $f_3 \leqslant \sqrt{L_3(\mathbf{x})}$ and

$$\epsilon_3 = \frac{\log(f_3^{-1}L_3(\mathbf{x}))}{\log X} = \xi_3 - \frac{\log f_3}{\log X}.$$

On relabelling the variables we may therefore write

$$R^{+,+,-}(X) = \sum_{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathcal{R}} \#\left\{\mathbf{e} \in \mathbb{N}^3 : \begin{array}{l} e_i \mid L_i(\mathbf{x}),\ e_i \leqslant \sqrt{|L_i(\mathbf{x})|} \\ (\boldsymbol{\epsilon}, \boldsymbol{\xi}) \in V^{+,+,-} \end{array}\right\},$$

where

$$V^{+,+,-} = \{(\boldsymbol{\epsilon}, \boldsymbol{\xi}) \in \mathbb{R}^6 : (\epsilon_1, \epsilon_2, \xi_3 - \epsilon_3, \boldsymbol{\xi}) \in V\}.$$

Interchanging the order of summation we obtain

$$R^{+,+,-}(X) = \sum_{\mathbf{e}\in\mathbb{N}^3} \#\left\{\mathbf{x} \in \Lambda(\mathbf{e}) \cap X\mathcal{R} : \boldsymbol{\xi} \in V^{+,+,-}(\mathbf{e})\right\},$$

where $\boldsymbol{\xi} \in V^{+,+,-}(\mathbf{e})$ if and only if $(\boldsymbol{\epsilon}, \boldsymbol{\xi}) \in V^{+,+,-}$ and $2\epsilon_i \leqslant \xi_i$.

On verifying that the underlying region is a union of two convex regions, an application of Lemma 2.1 yields

$$R^{+,+,-}(X) = \sum_{\mathbf{e}\in\mathbb{N}^3} \frac{\mathrm{vol}\{\mathbf{x} \in X\mathcal{R} : \boldsymbol{\xi} \in V^{+,+,-}(\mathbf{e})\}\varrho(\mathbf{e})}{(e_1e_2e_3)^2} + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}).$$

Lemma 2.3 implies that

$$\frac{\varrho(\mathbf{e})}{e_1e_2e_3} = \gcd(e_1, e_2, e_3) = f(\mathbf{e}),$$

say, whence

$$R^{+,+,-}(X) = \int_{\mathbf{x}\in X\mathcal{R}} \sum_{\substack{\mathbf{e}\in\mathbb{N}^3 \\ 2\epsilon_i\leqslant\xi_i}} \frac{\chi_V(\epsilon_1, \epsilon_2, \xi_3 - \epsilon_3, \boldsymbol{\xi})f(\mathbf{e})}{e_1e_2e_3}\mathrm{d}\mathbf{x} + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}),$$

where χ_V is the characteristic function of the set V. We now write $f = h * 1$ as a convolution, for a multiplicative arithmetic function h. Opening it up gives

$$R^{+,+,-}(X) = \sum_{\mathbf{k}\in\mathbb{N}^3} \frac{h(\mathbf{k})}{k_1k_2k_3} \int_{\mathbf{x}\in X\mathcal{R}} M(X)\mathrm{d}\mathbf{x} + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}), \tag{6.2}$$

where for $\kappa_i = \frac{\log k_i}{\log X}$ we set

$$M(X) = \sum_{\substack{\mathbf{e}\in\mathbb{N}^3 \\ 2\epsilon_i+2\kappa_i\leqslant\xi_i}} \frac{\chi_V(\epsilon_1 + \kappa_1, \epsilon_2 + \kappa_2, \xi_3 - \epsilon_3 - \kappa_3, \boldsymbol{\xi})}{e_1e_2e_3}.$$

The estimation of $M(X)$ will depend intimately on the set V. Indeed we wish to show that $\int M(X)\mathrm{d}\mathbf{x}$ has order $X^2 \log X$, whereas taking $V = [0, 1]^6$ leads to a sum with order $X^2(\log X)^3$.

Writing out the definition of the set V we see that

$$M(X) = \sum_{\substack{e_1\in\mathbb{N} \\ 0\leqslant\epsilon_1+\kappa_1\leqslant\frac{1}{2} \\ 2\epsilon_1+2\kappa_1\leqslant\xi_1}} \frac{1}{e_1} \sum_{\substack{e_2\in\mathbb{N} \\ 0\leqslant\epsilon_2+\kappa_2<\epsilon_1+\kappa_1 \\ \epsilon_1+\kappa_1+\xi_2\leqslant1+\epsilon_2+\kappa_2 \\ 2\epsilon_2+2\kappa_2\leqslant\xi_2}} \frac{1}{e_2} \sum_{\substack{e_3\in\mathbb{N} \\ \xi_3<\epsilon_1+\kappa_1+\epsilon_3+\kappa_3\leqslant1 \\ 2\epsilon_3+2\kappa_3\leqslant\xi_3}} \frac{1}{e_3},$$

where $\epsilon_i = \frac{\log e_i}{\log X}$, $\kappa_i = \frac{\log k_i}{\log X}$ and $\xi_i = \frac{\log |L_i(\mathbf{x})|}{\log X}$. Further thought shows that the outer sum over e_1 can actually be taken over e_1 such that

$$\frac{\xi_3}{2} < \epsilon_1 + \kappa_1 \leqslant \min\left\{\frac{1}{2}, \frac{\xi_1}{2}, 1 - \frac{\xi_2}{2}\right\}.$$

The inner sums over e_2, e_3 can be approximated simultaneously by integrals, giving

$$\Big(\log X\int_{\max\{0,\epsilon_1+\kappa_1+\xi_2-1\}}^{\min\{\epsilon_1+\kappa_1,\frac{\xi_2}{2}\}}\mathrm{d}\tau_2+O(1)\Big)\Big(\log X\int_{\max\{0,\xi_3-\epsilon_1-\kappa_1\}}^{\min\{1-\epsilon_1-\kappa_1,\frac{\xi_3}{2}\}}\mathrm{d}\tau_3+O(1)\Big),$$

after an obvious change of variables. We see that the overall contribution to $M(X)$ from the error terms is

$$\begin{aligned}&\ll \log X\int_{\frac{\xi_3}{2}}^{\frac{\xi_1}{2}}\Big(1+\log X\int_{\xi_2+\tau_1-1}^{\tau_1}\mathrm{d}\tau_2+\log X\int_{\xi_3-\tau_1}^{\frac{\xi_3}{2}}\mathrm{d}\tau_3\Big)\mathrm{d}\tau_1\\ &=(I_1+I_2+I_3)\log X,\end{aligned}$$

say. Let $\mathcal{I}_i$ denote the integral of $I_i\log X$ over $\mathbf{x}\in X\mathcal{R}$. We see that

$$\mathcal{I}_1\leqslant\frac{1}{2}\int_{\{\mathbf{x}\in X\mathcal{R}:\ x_1+x_2<|x_1|\}}\big(\log|x_1|-\log(x_1+x_2)\big)\mathrm{d}\mathbf{x}\ll X^2.$$

Next we note that

$$\begin{aligned}\mathcal{I}_2&\ll(\log X)^2\int_{(\tau_1,\tau_2)\in[0,\frac12]^2}\int_{\{\mathbf{x}\in X\mathcal{R}:\ \xi_3\leqslant 2\tau_1,\ \xi_2\leqslant 1+\tau_2-\tau_1,\ x_2>0\}}\mathrm{d}\mathbf{x}\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &\leqslant(\log X)^4\int_{(\tau_1,\tau_2)\in[0,\frac12]^2}\int_{-\infty}^{2\tau_1}\int_{-\infty}^{1+\tau_2-\tau_1}X^{u+v}\mathrm{d}u\mathrm{d}v\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &=(\log X)^2\int_{(\tau_1,\tau_2)\in[0,\frac12]^2}X^{1+\tau_1+\tau_2}\mathrm{d}\tau_1\mathrm{d}\tau_2\ll X^2,\end{aligned}$$

and likewise,

$$\begin{aligned}\mathcal{I}_3&\ll(\log X)^2\int_{(\tau_1,\tau_3)\in[0,\frac12]^2}\int_{\{\mathbf{x}\in X\mathcal{R}:\ \xi_2\leqslant 1,\ \xi_3\leqslant\tau_1+\tau_3,\ x_2>0\}}\mathrm{d}\mathbf{x}\mathrm{d}\tau_1\mathrm{d}\tau_3\\ &\leqslant(\log X)^4\int_{(\tau_1,\tau_3)\in[0,\frac12]^2}\int_{-\infty}^{1}\int_{-\infty}^{\tau_1+\tau_3}X^{u+v}\mathrm{d}u\mathrm{d}v\mathrm{d}\tau_1\mathrm{d}\tau_3\ll X^2.\end{aligned}$$

Interchanging the sum over e_1 with the integrals over τ_2,τ_3 one uses the same sort of argument to show that the final summation can be approximated by an integral.

This therefore leads to the conclusion that

$$\int_{\mathbf{x}\in X\mathcal{R}}M(X)\mathrm{d}\mathbf{x}=(\log X)^3\int_{\mathbf{x}\in X\mathcal{R}}\int_{\substack{2\tau_1\leqslant\xi_1\\ 2\tau_2\leqslant\xi_2\\ 2\tau_3>\xi_3}}\chi_V(\boldsymbol{\tau},\boldsymbol{\xi})\mathrm{d}\boldsymbol{\tau}\mathrm{d}\mathbf{x}+O(X^2),$$

after an obvious change of variables. We insert this into (6.2) and then, on assuming analogous formulae for all the sums $R^{\pm,\pm,\pm}(X)$, we sum over all of the various permutations of $\mathbf{m}$ in (6.1). This gives

$$R(X)=c_0I(X)+O(X^2),$$

where

$$c_0 = \sum_{\mathbf{k}\in\mathbb{N}^3} \frac{h(\mathbf{k})}{k_1k_2k_3}, \quad I(X) = (\log X)^3 \int_{\mathbf{x}\in X\mathcal{R}} \int_{\boldsymbol{\tau}\in\mathbb{R}^3} \chi_V(\boldsymbol{\tau},\boldsymbol{\xi})\mathrm{d}\boldsymbol{\tau}\mathrm{d}\mathbf{x}.$$

Recalling (5.2) we easily deduce that

$$\begin{aligned}
c_0 &= \sum_{\mathbf{a}\in\mathbb{N}^3} \frac{\mu(a_1)\mu(a_2)\mu(a_3)}{a_1a_2a_3} \sum_{\mathbf{b}\in\mathbb{N}^3} \frac{\gcd(b_1,b_2,b_3)}{b_1b_2b_3} \\
&= \prod_p \Big(1-\frac{1}{p}\Big)^3 S\Big(\frac{1}{p}\Big) \\
&= \prod_p \Big(1+\frac{1}{p}\Big)^{-1} \Big(1+\frac{1}{p}+\frac{1}{p^2}\Big).
\end{aligned}$$

It remains to analyse the term

$$\begin{aligned}
I(X) &= (\log X)^3 \operatorname{vol}\left\{ (\mathbf{x},\boldsymbol{\tau}) \in \mathbb{R}^2\times[0,1]^3 : \begin{array}{l} x_1+x_2>0,\ |x_1|\leqslant X, \\ \tau_2,\tau_3<\tau_1\leqslant \frac{1}{2}, \\ \frac{\log|x_2|}{\log X} \leqslant 1+\tau_2-\tau_1, \\ \frac{\log x_1+x_2}{\log X} \leqslant 1+\tau_3-\tau_1 \end{array} \right\} \\
&= I^{+,+}(X) + I^{-,+}(X) + I^{+,-}(X),
\end{aligned}$$

where $I^{+,+}(X)$ (resp. $I^{-,+}(X)$, $I^{+,-}(X)$) is the contribution from $\mathbf{x},\boldsymbol{\tau}$ such that $x_1>0$ and $x_2>0$ (resp. $x_1<0$ and $x_2>0$, $x_1>0$ and $x_2<0$). In the first integral it is clear that $x_1<x_1+x_2\leqslant X$ so that the condition $|x_1|\leqslant X$ is implied by the others. Likewise, in the second volume integral we will have $x_2>|x_1|$ and so the condition $|x_1|\leqslant X$ is implied by the inequalities involving x_2. An obvious change of variables readily leads to the conclusion that $I^{+,+}(X)+I^{-,+}(X)$ is

$$\begin{aligned}
&= (\log X)^5 \int_{\{\boldsymbol{\tau}\in[0,\frac{1}{2}]^3:\tau_2,\tau_3<\tau_1\}} \int_{-\infty}^{1+\tau_3-\tau_1} \int_{-\infty}^{1+\tau_2-\tau_1} X^{u+v}\mathrm{d}u\mathrm{d}v\mathrm{d}\boldsymbol{\tau} \\
&= X^2(\log X)^3 \int_{\{\boldsymbol{\tau}\in[0,\frac{1}{2}]^3:\tau_2,\tau_3<\tau_1\}} X^{\tau_2+\tau_3-2\tau_1}\mathrm{d}\boldsymbol{\tau} \\
&= \frac{1}{2}X^2\log X + O(X^2).
\end{aligned}$$

The final integral $I^{+,-}(X)$ can be written as in the first line of the above, but with the added constraint that $X^u+X^v\leqslant X$ in the inner integration over u,v. For large X this constraint can be dropped with acceptable error, which thereby leads to the companion estimate

$$I^{+,-}(X) = \frac{1}{2}X^2\log X + O(X^2).$$

Putting everything together we have therefore shown that

$$N_0(X) = 2^3 N_1(X) + O(X^2) = 8c_0 X^2 \log X + O(X^2),$$

with c_0 given above. Running through the reduction steps in [11, §5] rapidly leads from this asymptotic formula to the statement of Theorem 4.

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