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par Oleg N. GERMAN et Nikolay G. MOSHCHEVITIN

Résumé. Nous démontrons un résultat sur l'existence des formes linéaires de type Diophantien donné.

Abstract. We prove a result on the existence of linear forms of a given Diophantine type.

1. Approximation to irrational numbers

Let α be an irrational number. The Hurwitz theorem says that the inequality

$$\|q\alpha\| < \frac{1}{\sqrt{5}q},$$

where $\|\cdot\|$ denotes the distance to the nearest integer, has infinitely many solutions in integer q. Moreover, there is a countable set of numbers α for which this inequality is exact, that is for every positive ε the inequality

$$\|q\alpha\| < \left(\frac{1}{\sqrt{5}} - \varepsilon\right)\frac{1}{q}$$

admits only a finite number of solutions in integer q.

The numbers λ under the condition that there is an $\alpha = \alpha(\lambda)$ for which one has

$$\lambda = \liminf_{q\to+\infty} q||q\alpha||$$

form *the Lagrange spectrum.* It is a well-known fact that the Lagrange spectrum has a discrete part

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \ldots,$$

and the maximal λ for which there are continuously many $\alpha(\lambda)$ is $\lambda = 1/3$. It is also well-known that the Lagrange spectrum contains an interval $[0; \lambda^*]$. This interval is known as *Hall's ray* as M. Hall [7] was the first to prove that $\lambda^* > 0$. These and many other results concerning the Lagrange spectrum can be found in the book [5].

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Furthermore, V. Jarnik (see [8], Satz 6) showed that for every decreasing function $\varphi(y) = o(y^{-1})$ there is an uncountable set of real numbers α satisfying the following conditions: the inequality

$$\|q\alpha\| < \varphi(q)$$

has infinitely many solutions but for any $\varepsilon > 0$ the stronger inequality

$$\|q\alpha\| < (1-\varepsilon)\varphi(q)$$

has only a finite number of solutions.

The results mentioned above use the theory of continued fractions.

Recently V. Beresnevich, H. Dickinson and S. Velani in [2] and Y. Bugeaud in [3], [4] obtained a precise metric version of Jarnik's result. For example, Y. Bugeaud [3] showed that for every decreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that the function $x \mapsto x^2\varphi(x)$ is non-increasing and the series

$$\sum_{x=1}^{\infty} x\varphi(x) \tag{1.1}$$

converges, the sets

$$\mathcal{K}(\varphi) = \left\{\alpha \in \mathbb{R} : \ \left|\alpha - \frac{p}{q}\right| < \varphi(q) \ \text{ for infinitely many rationals } \ \frac{p}{q}\right\}$$

and

$$\mathrm{EXACT}(\varphi) = \mathcal{K}(\varphi) \setminus \left(\bigcup_{\varepsilon>0} \mathcal{K}((1-\varepsilon)\varphi)\right)$$

have the same Hausdorff dimension. Moreover, Y. Bugeaud [3] proved that the sets $\mathcal{K}(\varphi)$ and $\mathrm{EXACT}(\varphi)$ have the same $\mathcal{H}^f$-Hausdorff measure for a certain choice of the dimension function $f : \mathbb{R}_+ \to \mathbb{R}_+$. A certain result in the case when the series (1.1) diverges was obtained by Y. Bugeaud in [4].

2. General result by Jarnik

Throughout the paper for each $\mathbf{x} = (x_1, \ldots, x_n)$ we denote by $|\mathbf{x}|$ the Euclidean norm

$$|\mathbf{x}| = |\mathbf{x}|_2 = (x_1^2 + \ldots + x_n^2)^{1/2}$$

and by $|\mathbf{x}|_\infty$ the sup-norm

$$|\mathbf{x}|_\infty = \max_{1\leqslant i\leqslant n} |x_i|.$$

We also denote by $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ the inner product of $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$. For a fixed $\boldsymbol{\alpha}$ we get a linear form $\langle \boldsymbol{\alpha}, \cdot \rangle$.

We formulate a general result from [9]. Consider a real matrix

$$\Theta = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,n} \\ \cdots & \cdots & \cdots \\ \theta_{m,1} & \cdots & \theta_{m,n} \end{pmatrix}.$$

Given a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ we say that a set of $n+m$ integers

$$x_1, \dots, x_n,\ y_1, \dots, y_m$$

is a *φ-approximation for* Θ if with $\boldsymbol{\theta}_i = (\theta_{i,1}, \dots, \theta_{i,n})$ we have

$$\begin{cases} |\langle \boldsymbol{\theta}_i, \mathbf{x}\rangle - y_i| < \varphi(|\mathbf{x}|_\infty), \quad i = 1, \dots m, \\ |\mathbf{x}|_\infty > 0. \end{cases}$$

V. Jarnik considered an arbitrary non-increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and an arbitrary function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, such that the following conditions are satisfied:

- $\lambda(x) \to 0, \quad x \to \infty$;
- the functions $\varphi(x) \cdot x^{1/k}, \quad k = 1, \dots, m, \quad \varphi(x) \cdot x^{1+\varepsilon}$ and $\varphi(x) \cdot x^{(n-1)/m}$ are monotone;
- the integral $\int_A^\infty x^{n-1}(\varphi(x))^m dx$ converges.

For such $\varphi(x)$ and $\lambda(x)$ he proved in [9][1] that there is an uncountable set of matrices Θ, each having infinitely many φ-approximations but not more than a finite collection of $\lambda\varphi$-approximations.

Another result by Jarnik (see [9], Théorème B) gives a more precise statement under stronger conditions on $\varphi(x)$. Namely, he considered an arbitrary function $\varphi(x)$ satisfying the following conditions:

- $\varphi(x) \cdot x \to 0, \quad x \to \infty$;
- the functions $\varphi(x) \cdot x$ and $\varphi(x) \cdot x^{(n-1)/m}$ are monotone;
- the integral $\int_A^\infty x^{n-1}(\varphi(x))^m dx$ converges;

and proved the existence of an uncountable set of matrices Θ, each having infinitely many φ-approximations but not more than a finite collection of $(1-\varepsilon)\varphi$-approximations, for any positive ε.

So we see that the additional condition

$$\varphi(x) = o(x^{-1}), \quad x \to \infty,$$

allows obtaining sharper results concerning systems of linear forms of a given Diophantine type.

Here we would like to note that a nice metric generalization of Jarnik's result was obtained by V. Beresnevich, H. Dickinson and S. Velani in [2]. There the authors deal with a general setting for Diophantine approximations for systems of linear forms and prove certain results on the "exact logarithmic" order of approximations.

In the next two sections we discuss some improvements of Jarnik's result in the cases $n = 1$, $m \geqslant 2$ (simultaneous approximations) and $m = 1$,

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[1] for the special case of simultaneous approximations ($n = 1$) see [8]

$n \geqslant 2$ (linear forms). Here we should note that there is an old problem, still unsolved, to generalize the result on the existence of Hall's ray mentioned in Section 1 to the cases of simultaneous approximations and linear forms (and even to the general case). This problem seems to be a difficult one.

3. Simultaneous approximations

In this section for convenience we put

$$\varphi(t) = \frac{\psi(t)}{t^{1/m}}.$$

In Jarnik's theorem discussed in Section 2 by certain reasons the monotonicity conditions may be omitted. This observation in the case of simultaneous approximations (see also [8], Satz 5) leads to the following

Theorem 3.1. *Let m be a positive integer. Given an arbitrary decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and an arbitrary function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, decreasing to zero, suppose that the integral*

$$\int_A^\infty \frac{(\psi(x))^m}{x} dx$$

converges. Then one can find an uncountable set of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, such that for every sufficiently large positive integer q one has

$$\max_{1 \leqslant i \leqslant m} \|q\alpha_i\| \geqslant \frac{\lambda(q)\psi(q)}{q^{1/m}},$$

but the inequality

$$\max_{1 \leqslant i \leqslant m} \|q\alpha_i\| \leqslant \frac{\psi(q)}{q^{1/m}}$$

has infinitely many solutions in positive integer q.

In [1] R. Akhunzhanov and N. Moshchevitin generalizing the approach from [10] proved the following

Theorem 3.2. *Let m be a positive integer. Then there are explicit positive constants A_m, B_m with the following property. Given an arbitrary non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(1) \leqslant A_m$, one can find an uncountable set of vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ such that for every positive integer q*

$$\max_{1 \leqslant i \leqslant m} \|q\alpha_i\| \geqslant \frac{\psi(q)}{q^{1/m}} \left(1 - B_m\psi(q)\right),$$

but the inequality

$$\max_{1 \leqslant i \leqslant m} \|q\alpha_i\| \leqslant \frac{\psi(q)}{q^{1/m}} \left(1 + B_m\psi(q)\right)$$

has infinitely many solutions in positive integer q.

Here we would like to note that in the paper [10] the author attributes to V. Jarnik a stronger result than he actually proved in [8], [9].

4. Linear forms

In this section we put

$$\varphi(t) = \frac{\psi(t)}{t^n}.$$

Jarnik's theorem discussed in Section 2 in the case of linear forms leads to the following

Theorem 4.1. *Let n be a positive integer. Given an arbitrary decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and an arbitrary function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, decreasing to zero, suppose that the integral*

$$\int_A^\infty \frac{\psi(x)}{x} dx$$

converges. Then there is an uncountable set of $\boldsymbol{\alpha} \in \mathbb{R}^n$, such that for all $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $|\mathbf{x}|$ sufficiently large one has

$$\|\langle \boldsymbol{\alpha}, \mathbf{x} \rangle\| \geqslant \frac{\lambda(|\mathbf{x}|)\psi(|\mathbf{x}|)}{|\mathbf{x}|_\infty^n},$$

but the inequality

$$\|\langle \boldsymbol{\alpha}, \mathbf{x} \rangle\| \leqslant \frac{\psi(|\mathbf{x}|)}{|\mathbf{x}|_\infty^n}$$

has infinitely many solutions in $\mathbf{x} \in \mathbb{Z}^n$.

We now formulate the main result of this paper. For simplicity we restrict ourselves to the case $n = 2$ and use the Euclidean norm, though we believe that a similar result should be valid for systems of linear forms and for arbitrary norms.

Theorem 4.2. *There are explicit positive constants A, B with the following property. Given an arbitrary non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(1) \leqslant A$, one can find an uncountable set of $\boldsymbol{\alpha} \in \mathbb{R}^2$, such that for all $\mathbf{x} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$*

$$\|\langle \boldsymbol{\alpha}, \mathbf{x} \rangle\| \geqslant \frac{\psi(|\mathbf{x}|)}{|\mathbf{x}|^2}(1 - B\psi(|\mathbf{x}|))$$

but the inequality

$$\|\langle \boldsymbol{\alpha}, \mathbf{x} \rangle\| \leqslant \frac{\psi(|\mathbf{x}|)}{|\mathbf{x}|^2}(1 + B\psi(|\mathbf{x}|))$$

has infinitely many solutions in $\mathbf{x} \in \mathbb{Z}^2$.

5. Best approximations

Definition 1. A point $\mathbf{m} \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$ is said to be a *best approximation* for $\langle \boldsymbol{\alpha}, \cdot \rangle$ if

$$\| \langle \boldsymbol{\alpha}, \mathbf{m} \rangle \| < \| \langle \boldsymbol{\alpha}, \mathbf{m}' \rangle \|$$

for every $\mathbf{m}' \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$, such that $|\mathbf{m}'| < |\mathbf{m}|$, and

$$\| \langle \boldsymbol{\alpha}, \mathbf{m} \rangle \| \leqslant \| \langle \boldsymbol{\alpha}, \mathbf{m}' \rangle \|$$

for every $\mathbf{m}' \in \mathbb{Z}^2 \backslash \{\pm \mathbf{m}\}$, such that $|\mathbf{m}'| = |\mathbf{m}|$.

The set of all the best approximations for $\langle \boldsymbol{\alpha}, \cdot \rangle$ is infinite if and only if the coordinates of $\boldsymbol{\alpha}$ are linearly independent with the unit over $\mathbb{Q}$. If this is the case, then for each possible absolute value there are exactly two best approximations, on which this absolute value is attained, and they differ only in the sign. Thus, we can order the set of all the best approximations for $\langle \boldsymbol{\alpha}, \cdot \rangle$ with respect to the absolute value and obtain a sequence $\{\pm \mathbf{m}_k\}_{k=1}^{\infty}$.

Set

$$\gamma = \frac{18}{9 - \sqrt{2}} \approx 2.373\,. \tag{5.1}$$

Theorem 4.2 is a corollary of the following, more precise, theorem, which is the main result of the paper. In this theorem the *whole* sequence of best approximations is concerned and *all* of them are required to be of a given order.

Theorem 5.1. *Given an arbitrary non-increasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(1) \leqslant (9\gamma)^{-1}$, *there is an* $\boldsymbol{\alpha} \in \mathbb{R}^2$, *such that all the best approximations* $\mathbf{m}_k$ *for* $\langle \boldsymbol{\alpha}, \cdot \rangle$ *satisfy the condition*

$$\psi(|\mathbf{m}_k|) - 4\gamma\psi(|\mathbf{m}_k|)^2 < \| \langle \boldsymbol{\alpha}, \mathbf{m}_k \rangle \| \cdot |\mathbf{m}_k|^2 \leqslant \psi(|\mathbf{m}_k|) + \gamma\psi(|\mathbf{m}_k|)^2. \tag{5.2}$$

Moreover, there is a continuum of such $\boldsymbol{\alpha}$.

We note that the technique used here to prove Theorem 5.1 is similar to the technique developed in [6].

6. Proof of Theorem 5.1

6.1. Description of the set of forms having a given point m as a best approximation. Given a primitive point $\mathbf{m} \in \mathbb{Z}^2$, the set of all $\boldsymbol{\alpha} \in \mathbb{R}^2$, such that $\mathbf{m}$ is a best approximation for $\langle \boldsymbol{\alpha}, \cdot \rangle$, is contained in the set

$$\mathfrak{S} = \bigcap_{\substack{\mathbf{m}' \in \mathbb{Z}^2 \backslash \{\mathbf{0}, \pm \mathbf{m}\} \\ |\mathbf{m}'| \leqslant |\mathbf{m}|}} \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, \| \langle \mathbf{m}, \mathbf{x} \rangle \| \leqslant \| \langle \mathbf{m}', \mathbf{x} \rangle \| \right\} \tag{6.1}$$

and contains its interior. One can easily see that each of the sets in the intersection (6.1) is simply a union of parallelograms. Besides that, one of the two diagonals of each of these parallelograms lies on an integer level of the form $\langle \mathbf{m}, \cdot \rangle$, and the union of such diagonals coincides with the union of all the integer levels of $\langle \mathbf{m}, \cdot \rangle$. Thus, all the connected components of $\operatorname{int} \mathfrak{S}$ are open convex polygons. None of these polygons can be too small. To see this we shall use the following

Lemma 6.1. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^2$ *and let* $\mathbf{b}$ *and* $\mathbf{c}$ *be linearly independent. Let also* $\boldsymbol{\alpha} \in \mathbb{R}^2$, $\langle \mathbf{b}, \boldsymbol{\alpha} \rangle \in \mathbb{Z}$, $\langle \mathbf{c}, \boldsymbol{\alpha} \rangle \in \mathbb{Z}$. *Then for every* $\lambda \in \mathbb{Z}$ *the (Euclidean) distance from* $\boldsymbol{\alpha}$ *to the line defined by the equation* $\langle \mathbf{a}, \mathbf{x} \rangle = \lambda$ *is an integer multiple of*

$$\frac{1}{|\mathbf{a}||\det(\mathbf{b}, \mathbf{c})|}\,.$$

Proof. The index of $\mathbb{Z}^2$ as a sublattice of the lattice, dual to $\operatorname{span}_{\mathbb{Z}}(\mathbf{b}, \mathbf{c})$, is equal to $|\det(\mathbf{b}, \mathbf{c})|$. Hence $\det(\mathbf{b}, \mathbf{c})\boldsymbol{\alpha} \in \mathbb{Z}^2$, and thus,

$$\langle \mathbf{a}, \boldsymbol{\alpha} \rangle \in \frac{\mathbb{Z}}{|\det(\mathbf{b}, \mathbf{c})|}\,.$$

It remains to notice that the Euclidean distance between two adjacent integer levels of the form $\langle \mathbf{a}, \mathbf{x} \rangle$ equals $1/|\mathbf{a}|$. □

Since not all the linear forms determining the boundary of a connected component of $\operatorname{int} \mathfrak{S}$ are necessarily integer, but some of them may have half–integer coefficients, the fact that none of those components can be too small is implied by the following obvious corollary to Lemma 6.1:

Corollary 6.1. *Let* $\mathbf{a} \in \frac{1}{2}\mathbb{Z}^2$, $\mathbf{b}, \mathbf{c} \in \mathbb{Z}^2$ *and let* $\mathbf{b}$ *and* $\mathbf{c}$ *be linearly independent. Let also* $\boldsymbol{\alpha} \in \mathbb{R}^2$, $\langle \mathbf{b}, \boldsymbol{\alpha} \rangle \in \mathbb{Z}$, $\langle \mathbf{c}, \boldsymbol{\alpha} \rangle \in \mathbb{Z}$. *Then for every* $\lambda \in \frac{1}{2}\mathbb{Z}$ *the (Euclidean) distance from* $\boldsymbol{\alpha}$ *to the line defined by the equation* $\langle \mathbf{a}, \mathbf{x} \rangle = \lambda$ *is an integer multiple of*

$$\frac{1}{2|\mathbf{a}||\det(\mathbf{b}, \mathbf{c})|}\,.$$

6.2. Basis change. The following statement supports one of the crucial steps in the proof of Theorem 5.1.

Lemma 6.2. *Suppose* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^2$ *and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^2$ *satisfy the relations* $\langle \boldsymbol{\alpha}, \mathbf{b} \rangle \in \mathbb{Z}$, $\langle \boldsymbol{\alpha}, \mathbf{c} \rangle = \langle \boldsymbol{\beta}, \mathbf{c} \rangle \in \mathbb{Z}$, *and* $\langle \boldsymbol{\beta}, \mathbf{a} \rangle$ *equals the nearest integer to* $\langle \boldsymbol{\alpha}, \mathbf{a} \rangle$ *(in case* $\langle \boldsymbol{\alpha}, \mathbf{a} \rangle = 1/2$ *one can take any of the two nearest integers). Suppose also that* $\mathbf{b}, \mathbf{c}$ *are linearly independent,* $\mathbf{a}, \mathbf{c}$ *are linearly independent,*

$$\{\mathbf{x} \in \mathbb{Z}^2 \mid \langle \boldsymbol{\alpha}, \mathbf{x} \rangle \in \mathbb{Z}\} = \operatorname{span}_{\mathbb{Z}}(\mathbf{b}, \mathbf{c}) \tag{6.2}$$

and

$$\|\langle \boldsymbol{\alpha}, \mathbf{a} \rangle\| = |\det(\mathbf{b}, \mathbf{c})|^{-1}. \tag{6.3}$$

Then

$$\{\mathbf{x} \in \mathbb{Z}^2 \mid \langle \boldsymbol{\beta}, \mathbf{x} \rangle \in \mathbb{Z}\} = \operatorname{span}_{\mathbb{Z}}(\mathbf{a}, \mathbf{c}) \tag{6.4}$$

and

$$\|\langle\boldsymbol{\beta},\mathbf{b}\rangle\| = |\det(\mathbf{a},\mathbf{c})|^{-1}. \tag{6.5}$$

Proof. Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, $\mathbf{c} = (c_1, c_2)$ and set $a_3 = -\langle\boldsymbol{\beta},\mathbf{a}\rangle$, $b_3 = -\langle\boldsymbol{\alpha},\mathbf{b}\rangle$, $c_3 = -\langle\boldsymbol{\alpha},\mathbf{c}\rangle = -\langle\boldsymbol{\beta},\mathbf{c}\rangle$.

Let us prove that the points

$$\overline{\mathbf{a}} = (a_1, a_2, a_3), \quad \overline{\mathbf{b}} = (b_1, b_2, b_3), \quad \overline{\mathbf{c}} = (c_1, c_2, c_3)$$

form a basis of $\mathbb{Z}^3$. It follows from (6.2) that all the integer points contained in the plane π_α spanned by $\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ belong to the lattice

$$\mathrm{span}_{\mathbb{Z}}(\overline{\mathbf{b}},\overline{\mathbf{c}}).$$

This means that $\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ can be supplemented to a basis of $\mathbb{Z}^3$. Hence $\mathbb{Z}^3$ splits into "layers" contained in two-dimensional planes parallel to π_α. Moreover, any two neighbouring planes cut a segment in the vertical axis of length $|\det(\mathbf{b},\mathbf{c})|^{-1}$. Now, using (6.3) and the fact that a_3 equals the nearest integer to $-\langle\boldsymbol{\alpha},\mathbf{a}\rangle$ we see that $\overline{\mathbf{a}}$ lies in a plane next to π_α. This shows that $\overline{\mathbf{a}},\overline{\mathbf{b}},\overline{\mathbf{c}}$ form a basis of $\mathbb{Z}^3$.

Thus, all the integer points of the plane π_β spanned by $\overline{\mathbf{a}}$ and $\overline{\mathbf{c}}$ are in

$$\mathrm{span}_{\mathbb{Z}}(\overline{\mathbf{a}},\overline{\mathbf{c}}),$$

which immediately implies (6.4). As before, $\mathbb{Z}^3$ splits into "layers" contained in two-dimensional planes parallel to π_β, such that any two neighbouring ones cut a segment in the vertical axis of length $|\det(\mathbf{a},\mathbf{c})|^{-1}$. Noticing that $\overline{\mathbf{b}}$ lies in a plane next to π_β we get (6.5). □

6.3. Induction lemma. To prove theorem 5.1 we shall construct a sequence of embedded two-dimensional "half-balls" $\{\Omega_k\}_{k=1}^{\infty}$ with their common point $\boldsymbol{\alpha}$ satisfying the statement of the corresponding theorem. We say that a set Ω is a *half-ball of radius R centered at a point* $\mathbf{x}$ if Ω is the intersection of a closed Euclidean ball of radius R centered at $\mathbf{x}$ and a closed half-plane with the supporting line containing $\mathbf{x}$.

The following lemma gives the induction step.

Lemma 6.3. *Let $k \in \mathbb{Z}_+$, $\psi_k, \psi_{k+1} \in \mathbb{R}_+$, $\psi_k, \psi_{k+1} \leqslant (9\gamma)^{-1}$, and let $\mathbf{m}_k, \mathbf{m}_{k+1} \in \mathbb{Z}^2$. Let $\Omega_k \subset \mathbb{R}^2$ be a half-ball of radius*

$$R_k = (2|\mathbf{m}_{k+1}||\det(\mathbf{m}_k,\mathbf{m}_{k+1})|)^{-1}$$

centred at $\boldsymbol{\alpha}_k$ with the line

$$\ell_k = \{\mathbf{x} \in \mathbb{R}^2 \mid \langle\mathbf{x},\mathbf{m}_k\rangle = \langle\boldsymbol{\alpha}_k,\mathbf{m}_k\rangle\}$$

supporting it. Suppose that the following conditions are satisfied:

1) $\langle\mathbf{m}_k,\mathbf{m}_{k+1}\rangle \leqslant 0$;

2) $\{\mathbf{x} \in \mathbb{Z}^2 \mid \langle\boldsymbol{\alpha}_k,\mathbf{x}\rangle \in \mathbb{Z}\} = \mathrm{span}_{\mathbb{Z}}(\mathbf{m}_k,\mathbf{m}_{k+1})$;

3) *for every* $\boldsymbol{\alpha} \in \Omega_k$ *and every* $\mathbf{m} \in \mathbb{Z}^2 \setminus \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$*, such that* $|\mathbf{m}| < |\mathbf{m}_{k+1}|$*, one has*

$$\|\langle \boldsymbol{\alpha}, \mathbf{m}_k \rangle\| < \|\langle \boldsymbol{\alpha}, \mathbf{m} \rangle\|;$$

4) $\gamma < \dfrac{|\det(\mathbf{m}_k, \mathbf{m}_{k+1})|}{|\mathbf{m}_k|^2} < 3\gamma$;

5) $(2\gamma\psi_k)^{-1/2} \leqslant \dfrac{|\mathbf{m}_{k+1}|}{|\mathbf{m}_k|} < (\gamma\psi_k)^{-1/2}$.

Then there is a point $\mathbf{m}_{k+2} \in \mathbb{Z}^2$*, linearly independent with* $\mathbf{m}_{k+1}$*, and a half-ball* $\Omega_{k+1} \subset \Omega_k$ *of radius*

$$R_{k+1} = (2|\mathbf{m}_{k+2}||\det(\mathbf{m}_{k+1}, \mathbf{m}_{k+2})|)^{-1}$$

centred at $\boldsymbol{\alpha}_{k+1}$ *with the line*

$$\ell_{k+1} = \{\mathbf{x} \in \mathbb{R}^2 \mid \langle \mathbf{x}, \mathbf{m}_{k+1} \rangle = \langle \boldsymbol{\alpha}_{k+1}, \mathbf{m}_{k+1} \rangle\}$$

supporting it which satisfy the following conditions:

1) $\langle \mathbf{m}_{k+1}, \mathbf{m}_{k+2} \rangle \leqslant 0$;

2) $\langle \boldsymbol{\alpha}_{k+1}, \mathbf{m}_{k+1} \rangle = \langle \boldsymbol{\alpha}_k, \mathbf{m}_{k+1} \rangle$ *and*

$$\{\mathbf{x} \in \mathbb{Z}^2 \mid \langle \boldsymbol{\alpha}_{k+1}, \mathbf{x} \rangle \in \mathbb{Z}\} = \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_{k+1}, \mathbf{m}_{k+2});$$

3) *for every* $\boldsymbol{\alpha} \in \Omega_{k+1}$ *and every* $\mathbf{m} \in \mathbb{Z}^2 \setminus \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_{k+1}, \mathbf{m}_{k+2})$*, such that* $|\mathbf{m}| < |\mathbf{m}_{k+2}|$*, one has*

$$\|\langle \boldsymbol{\alpha}, \mathbf{m}_{k+1} \rangle\| < \|\langle \boldsymbol{\alpha}, \mathbf{m} \rangle\|;$$

4) *for every* $\boldsymbol{\alpha} \in \Omega_{k+1}$ *and every* $\mathbf{m} \in \mathbb{Z}^2 \setminus \{\mathbf{0}, \pm\mathbf{m}_k\}$*, such that* $|\mathbf{m}| < |\mathbf{m}_{k+1}|$*, one has*

$$\|\langle \boldsymbol{\alpha}, \mathbf{m}_k \rangle\| < \|\langle \boldsymbol{\alpha}, \mathbf{m} \rangle\|;$$

5) $(2\gamma\psi_{k+1})^{-1/2} \leqslant \dfrac{|\mathbf{m}_{k+2}|}{|\mathbf{m}_{k+1}|} < (\gamma\psi_{k+1})^{-1/2}$.

6) $\psi_k^{-1} \leqslant \dfrac{|\det(\mathbf{m}_{k+1}, \mathbf{m}_{k+2})|}{|\mathbf{m}_k|^2} < \psi_k^{-1} + 3\gamma$.

Proof. Denote by $\tilde{\Omega}_k$ the ball, of which Ω_k is a half, and by $\tilde{\Omega}_{k+1}$ the corresponding ball for Ω_{k+1}, which is to be constructed.

Define δ_k as follows. Set $\delta_k = 1$ if for every $\mathbf{x} \in \Omega_k$

$$\langle \mathbf{x}, \mathbf{m}_k \rangle \leqslant \langle \boldsymbol{\alpha}_k, \mathbf{m}_k \rangle,$$

and set $\delta_k = -1$ if for every $\mathbf{x} \in \Omega_k$

$$\langle \mathbf{x}, \mathbf{m}_k \rangle \geqslant \langle \boldsymbol{\alpha}_k, \mathbf{m}_k \rangle.$$

Since ℓ_k supports Ω_k, δ_k is defined correctly. Hence

$$\delta_k \langle \mathbf{x}, \mathbf{m}_k \rangle \leqslant \delta_k \langle \boldsymbol{\alpha}_k, \mathbf{m}_k \rangle$$

for every $\mathbf{x} \in \Omega_k$.

There is exactly one integer point $\mathbf{w}$ in the parallelogram spanned by $\mathbf{m}_k$ and $\mathbf{m}_{k+1}$, such that the fractional part of $\langle\boldsymbol{\alpha}_k,\mathbf{w}\rangle$ is minimal and positive, i.e.

$$\{\langle\boldsymbol{\alpha}_k,\mathbf{w}\rangle\}=|\det(\mathbf{m}_k,\mathbf{m}_{k+1})|^{-1}.$$

Denote by $\mathbf{m}_{k+1}^{\perp}$ the integer point satisfying the conditions $\langle\mathbf{m}_{k+1}^{\perp},\mathbf{m}_{k+1}\rangle=0$, $|\mathbf{m}_{k+1}^{\perp}|=|\mathbf{m}_{k+1}|$ and $\langle\mathbf{m}_{k+1}^{\perp},\mathbf{m}_k\rangle>0$. Consider the point

$$\mathbf{v}=\left(\psi_k^{-1}\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}\right)\delta_k\mathbf{m}_{k+1}^{\perp}-\sqrt{(2\gamma\psi_{k+1})^{-1}-\left(\psi_k^{-1}\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}\right)^2}\,\mathbf{m}_{k+1}.$$

The subset of the affine lattice

$$\mathbf{w}+\mathrm{span}_{\mathbb{Z}}(\mathbf{m}_k,\mathbf{m}_{k+1})$$

consisting of points $\mathbf{x}$, such that the quantity $\langle\mathbf{x}-\mathbf{v},\delta_k\mathbf{m}_{k+1}^{\perp}\rangle$ is minimal and non-negative, lies on a line parallel to $\mathbf{m}_{k+1}$. Define $\mathbf{m}_{k+2}$ to be the point of this set, such that the quantity $\langle\mathbf{m}_{k+2}-\mathbf{v},-\mathbf{m}_{k+1}\rangle$ is minimal and non-negative. Notice that, due to the definition of the point $\mathbf{m}_{k+1}^{\perp}$, the sign of the coefficient λ_1 in the decomposition $\mathbf{m}_{k+2}=\lambda_1\mathbf{m}_k+\lambda_2\mathbf{m}_{k+1}$ is equal to that of δ_k, i.e.

$$\frac{\lambda_1}{|\lambda_1|}=\delta_k. \tag{6.6}$$

We shall use this fact when defining Ω_{k+1}. But now let us turn to the statements 5) and 6), for their proof involves neither Ω_{k+1}, nor α_{k+1}.

With $\mathbf{m}_{k+2}$ chosen as above the statement 5) follows from the inequalities

$$|\mathbf{v}|^2\leqslant|\mathbf{m}_{k+2}|^2<|\mathbf{v}|^2+|\mathbf{m}_{k+1}|^2+\left(\frac{\langle\mathbf{m}_k,\mathbf{m}_{k+1}^{\perp}\rangle}{|\mathbf{m}_{k+1}^{\perp}|}\right)^2<|\mathbf{v}|^2+\frac{|\mathbf{m}_{k+1}|^2}{2\gamma\psi_{k+1}}\,,$$

the latter being a consequence of the relation

$$\frac{\langle\mathbf{m}_k,\mathbf{m}_{k+1}^{\perp}\rangle}{|\mathbf{m}_{k+1}^{\perp}|^2}=\frac{|\det(\mathbf{m}_k,\mathbf{m}_{k+1})|}{|\mathbf{m}_{k+1}|^2}<6\gamma^2\psi_k$$

and the condition $\psi_k,\psi_{k+1}\leqslant(9\gamma)^{-1}$.

As for the statement 6), it follows from the inequalities

$$\begin{aligned}\psi_k^{-1}\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}\leqslant\frac{\langle\mathbf{m}_{k+2},\mathbf{m}_{k+1}^{\perp}\rangle}{|\mathbf{m}_{k+1}^{\perp}|^2}&<\psi_k^{-1}\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}+\frac{|\det(\mathbf{m}_k,\mathbf{m}_{k+1}|}{|\mathbf{m}_{k+1}|^2}\\&\leqslant\psi_k^{-1}\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}+\frac{3\gamma|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}\\&\leqslant(\psi_k^{-1}+3\gamma)\frac{|\mathbf{m}_k|^2}{|\mathbf{m}_{k+1}|^2}\,.\end{aligned}$$

Indeed, taking into account that

$$\frac{|\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|}{|\mathbf{m}_{k+1}|^2} = \frac{\langle \mathbf{m}_{k+2},\mathbf{m}_{k+1}^{\perp}\rangle}{|\mathbf{m}_{k+1}^{\perp}|^2}$$

we get

$$\psi_k^{-1} \leqslant \frac{|\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|}{|\mathbf{m}_k|^2} < \psi_k^{-1} + 3\gamma.$$

Now let us define $\boldsymbol{\alpha}_{k+1}$ and Ω_{k+1} and prove the rest of the statements. Let us define $\boldsymbol{\alpha}_{k+1}$ by the equalities

$$\langle \boldsymbol{\alpha}_{k+1},\mathbf{m}_{k+1}\rangle = \langle \boldsymbol{\alpha}_k,\mathbf{m}_{k+1}\rangle, \tag{6.7}$$

$$\langle \boldsymbol{\alpha}_{k+1},\mathbf{m}_{k+2}\rangle = [\langle \boldsymbol{\alpha}_k,\mathbf{m}_{k+2}\rangle]. \tag{6.8}$$

Note that, due to Lemma 6.2, for $\mathbf{m}_{k+2}$ and $\boldsymbol{\alpha}_{k+1}$ thus chosen the statement 2) of the Lemma holds. It also follows from Lemma 6.2 that the distance from $\boldsymbol{\alpha}_{k+1}$ to ℓ_k is equal to

$$(|\mathbf{m}_k||\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|)^{-1},$$

which in its turn implies that

$$|\boldsymbol{\alpha}_{k+1}-\boldsymbol{\alpha}_k| = \frac{|\mathbf{m}_{k+1}|}{|\det(\mathbf{m}_k,\mathbf{m}_{k+1})\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|}\,.$$

Hence

$$\begin{aligned}|\boldsymbol{\alpha}_{k+1}-\boldsymbol{\alpha}_k| + R_{k+1} &= \frac{2|\mathbf{m}_{k+1}|^2}{|\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|}R_k \\ &\quad + \frac{|\det(\mathbf{m}_k,\mathbf{m}_{k+1})|}{|\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|}\frac{|\mathbf{m}_{k+1}|}{|\mathbf{m}_{k+2}|}R_k \\ &< \frac{2}{\gamma}R_k + 3\gamma\psi_k\sqrt{2\gamma\psi_{k+1}}R_k \\ &< \ R_k.\end{aligned}$$

Here we have made use of the statements 5) and 6) we have already proved, the assumption 4), the condition $\psi_k,\psi_{k+1} \leqslant (9\gamma)^{-1}$ and the definition of γ. Thus,

$$\tilde{\Omega}_{k+1} \subset \tilde{\Omega}_k.$$

More than that, $\tilde{\Omega}_{k+1}$ is contained either in Ω_k, or in $\tilde{\Omega}_k\backslash\Omega_k$, since

$$R_{k+1} < (|\mathbf{m}_k||\det(\mathbf{m}_{k+1},\mathbf{m}_{k+2})|)^{-1}$$

and the righthand side of this inequality, as we have already noticed before, is the distance from $\boldsymbol{\alpha}_{k+1}$ to ℓ_k.

Let us prove that $\tilde{\Omega}_{k+1} \subset \Omega_k$. By (6.8),

$$\langle \boldsymbol{\alpha}_{k+1},\mathbf{m}_{k+2}\rangle < \langle \boldsymbol{\alpha}_k,\mathbf{m}_{k+2}\rangle,$$

so, if $\mathbf{m}_{k+2} = \lambda_1\mathbf{m}_k + \lambda_2\mathbf{m}_{k+1}$, then, due to (6.7),

$$\langle \boldsymbol{\alpha}_{k+1}, \lambda_1\mathbf{m}_k\rangle < \langle \boldsymbol{\alpha}_k, \lambda_1\mathbf{m}_k\rangle,$$

which, in view of (6.6), implies that

$$\delta_k\langle \boldsymbol{\alpha}_{k+1}, \mathbf{m}_k\rangle < \delta_k\langle \boldsymbol{\alpha}_k, \mathbf{m}_k\rangle.$$

This shows that

$$\tilde{\Omega}_{k+1} \subset \Omega_k.$$

To define Ω_{k+1} it remains to choose between the two parts of $\tilde{\Omega}_{k+1}$ separated by the line ℓ_{k+1}. Corollary 6.1 together with the definition of R_{k+1} implies that the statement 3) of the Lemma holds for every $\boldsymbol{\alpha} \in \tilde{\Omega}_{k+1}$, so we have to specify one of the two halves of $\tilde{\Omega}_{k+1}$ only to provide the statement 4).

Between the described two halves of $\tilde{\Omega}_{k+1}$ let us choose to be Ω_{k+1} the one that is closest to ℓ_k.

Let us prove the statement 4). Notice first that for each $\boldsymbol{\alpha} \in \Omega_k$ the linear form $\langle \boldsymbol{\alpha}, \cdot\rangle$ does not attain integer values at any point of the set $\mathbb{Z}^2 \setminus \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$ with absolute value not exceeding $|\mathbf{m}_{k+1}|$. At the same time for every $\mathbf{m} \in \pm\mathbf{w} + \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$ we have

$$\|\langle \boldsymbol{\alpha}_k, \mathbf{m}\rangle\| = |\det(\mathbf{m}_k, \mathbf{m}_{k+1})|^{-1}.$$

Hence, taking into account the assumptions 4) and 5), we see that for every $\boldsymbol{\alpha} \in \Omega_k$ and every $\mathbf{m} \in \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$, such that $|\mathbf{m}| \leqslant |\mathbf{m}_{k+1}|$, we have

$$\big|\langle \boldsymbol{\alpha}, \mathbf{m}\rangle - \langle \boldsymbol{\alpha}_k, \mathbf{m}\rangle\big| < 1/2.$$

This means that for any $\boldsymbol{\alpha} \in \Omega_k$, non-collinear with $\boldsymbol{\alpha}_k$, and any two points $\mathbf{m}', \mathbf{m}'' \in \operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$, such that $|\mathbf{m}'|, |\mathbf{m}''| \leqslant |\mathbf{m}_{k+1}|$, the quotient

$$\|\langle \boldsymbol{\alpha}, \mathbf{m}'\rangle\| \Big/ \|\langle \boldsymbol{\alpha}, \mathbf{m}''\rangle\|$$

is equal to the quotient of distances from the points $\mathbf{m}'$ and $\mathbf{m}''$ to the line

$$\big\{\mathbf{x} \in \mathbb{R}^2 \,\big|\, \langle \boldsymbol{\alpha}, \mathbf{x}\rangle = \langle \boldsymbol{\alpha}_k, \mathbf{x}\rangle\big\}. \tag{6.9}$$

Due to the choice of Ω_{k+1} and the assumption 1), for every $\boldsymbol{\alpha} \in \Omega_{k+1}$ the points $\pm\mathbf{m}_k$ are closer to the line (6.9) than any other non-zero point of the set $\operatorname{span}_{\mathbb{Z}}(\mathbf{m}_k, \mathbf{m}_{k+1})$ with the absolute value less than $|\mathbf{m}_{k+1}|$. This implies the statement 4). □

Let us describe now the base of induction. Set $\psi_1 = \psi(1)$ and

$$\mathbf{m}_1 = (1, 0), \quad \mathbf{m}_2 = \Big(\Big\lceil \sqrt{(2\gamma\psi_1)^{-1} - \gamma^2}\Big\rceil, 3\Big), \quad \boldsymbol{\alpha}_1 = (0, 3^{-1}).$$

It is easily verified that the assumptions 1), 2), 4), 5) of Lemma 6.3 are satisfied for these points. Setting

$$R_1 = (2|\mathbf{m}_2||\det(\mathbf{m}_1, \mathbf{m}_2)|)^{-1},$$

choosing as Ω_1 any of the two corresponding half-balls and taking into account Corollary 6.1, we see that all the assumptions of Lemma 6.3 are fulfilled. This gives the induction base.

For each $k = 2, 3, 4, \dots$ let us consequently set $\psi_k = \psi(|\mathbf{m}_k|)$ and apply Lemma 6.3. We can do so since the assumption 5) together with the statement 6) of Lemma 6.3 for a fixed $k \in \mathbb{Z}_+$ imply the assumption 4) with k substituted by $k+1$. Thus we get a sequence $\{\Omega_k\}_{k=1}^{\infty}$ of embedded half-balls with a common point $\boldsymbol{\alpha}$ and a sequence $\{\mathbf{m}_k\}_{k=1}^{\infty}$. It follows from the statement 4) of Lemma 6.3 that for each $k \in \mathbb{Z}_+$ the pair $\pm\mathbf{m}_k$ is the k-th pair of best approximations for $\langle\boldsymbol{\alpha}, \cdot\rangle$, whereas due to the statements 5) and 6) we have for each $k \in \mathbb{Z}_+$ the inequalities

$$\psi_k^{-1} \leqslant \frac{|\det(\mathbf{m}_{k+1}, \mathbf{m}_{k+2})|}{|\mathbf{m}_k|^2} < \psi_k^{-1} + 3\gamma < (\psi_k - 3\gamma\psi_k^2)^{-1}$$

and

$$R_{k+1}|\mathbf{m}_k|^3 \leqslant \Big(2(2\gamma\psi_{k+1})^{-1/2}(2\gamma\psi_k)^{-1/2}\psi_k^{-1}\Big)^{-1} \leqslant \gamma\psi_k^2.$$

Hence for all $\boldsymbol{\alpha} \in \Omega_{k+1}$

$$\psi_k - 4\gamma\psi_k^2 < \|\langle\boldsymbol{\alpha}, \mathbf{m}_k\rangle\| \cdot |\mathbf{m}_k|^2 \leqslant \psi_k + \gamma\psi_k^2,$$

which proves Theorem 5.1.

Remark 1. In a similar way we can apply Lemma 6.3 to prove a bit different fact. Namely, within the assumptions of Theorem 5.1 we can prove that there are continuously many forms $\langle\boldsymbol{\alpha}, \cdot\rangle$, such that their best approximations $\mathbf{m}_k$ satisfy the condition

$$\psi(k) - 4\gamma\psi(k)^2 < \|\langle\boldsymbol{\alpha}, \mathbf{m}_k\rangle\| \cdot |\mathbf{m}_k|^2 \leqslant \psi(k) + \gamma\psi(k)^2.$$

To this effect we just have to put $\psi_k = \psi(k)$ and repeat the arguments from the proof of Theorem 5.1.

Remark 2. It is not difficult to improve the constants in Theorem 5.1. However, the wish to make better constants, if realized, would make the proof look more cumbersome and blur the main ideas.

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