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A note on the Hermite–Rankin constant

par Kazuomi SAWATANI, Takao WATANABE et Kenji OKUDA

To the memory of Anne-Marie Bergé

Résumé. Nous généralisons l'inégalité de Poor et Yuen au cas des constantes $\gamma_{n,k}$ d'Hermite-Rankin et $\gamma'_{n,k}$ de Bergé–Martinet. En outre, nous donnons les valeurs exactes de certaines constantes d'Hermite-Rankin et de Bergé–Martinet de petite dimension en appliquant certaines inégalités démontrées par Bergé et Martinet aux valeurs explicites de γ'_5, γ'_7, $\gamma_{4,2}$ et γ_n, $n \leq 8$.

Abstract. We generalize Poor and Yuen's inequality to the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$. Moreover, we determine explicit values of some low-dimensional Hermite–Rankin and Bergé–Martinet constants by applying Rankin's inequality and some inequalities proven by Bergé and Martinet to explicit values of γ'_5, γ'_7, $\gamma_{4,2}$ and γ_n $(n \leqq 8)$.

1. Introduction

In the recent paper [PY3], Poor and Yuen proved an inequality among the Hermite constant γ_n, the Bergé–Martinet constant γ'_n and the constant c_n defined from the dyadic trace. They also studied the condition of the equality $(\gamma'_n)^2 = n/c_n$ and applied this to determine low dimensional Bergé–Martinet's constants γ'_5, γ'_6 and γ'_7.

In the first half of this paper, we generalize Poor and Yuen's inequality to the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$. In the second half, we show by using Rankin's inequality and some inequalities proven by Bergé and Martinet in [BM] that explicit values of γ'_5, γ'_7, $\gamma_{4,2}$ and γ_n $(n \leqq 8)$ lead us to explicit values of $\gamma_{6,2}$, $\gamma'_{6,2}$, $\gamma_{8,2}$, $\gamma'_{8,2}$, $\gamma_{8,3}$, $\gamma'_{8,3}$ and $\gamma_{8,4} = \gamma'_{8,4}$.

2. A generalization of Poor and Yuen's inequality

In order to define $\gamma_{n,k}$ and $\gamma'_{n,k}$, we start from the definition of type one functions. Let V_n be the real vector space of $n \times n$ real symmetric matrices

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and P_n the open cone of positive definite symmetric matrices in V_n. We define the inner product $\langle , \rangle : V_n \times V_n \longrightarrow \mathbf{R}$ by

$$\langle s, t \rangle = \mathrm{Tr}(st)$$

for $s, t \in V_n$. A function $\phi : P_n \longrightarrow \mathbf{R}_{>0}$ is called a type one function if ϕ satisfies

(i) $\phi(\lambda s) = \lambda \phi(s)$ for all $\lambda \in \mathbf{R}_{>0}$, $s \in P_n$, and
(ii) $\phi(s+t) \geqq \phi(s) + \phi(t)$ for all $s, t \in P_n$.

In addition, if ϕ is class invariant, i.e., $\phi({}^t g s g) = \phi(s)$ holds for all $s \in P_n$ and $g \in GL_n(\mathbf{Z})$, then ϕ is called a type one class function. A type one function is continuous on P_n ([PY, Proposition 2.2]).

Let ϕ be a type one function. The dual function $\widehat{\phi} : P_n \longrightarrow \mathbf{R}_{>0}$ of ϕ is defined to be

$$\widehat{\phi}(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle}{\phi(t)} .$$

This $\widehat{\phi}$ is also a type one function, and the dual of $\widehat{\phi}$ equals ϕ. If ϕ is class invariant, then so is $\widehat{\phi}$. A typical example of type one class functions is given by $s \mapsto \sqrt{n}(\det s)^{1/n}$, which is self-dual.

In order to give another example of type one class functions, we fix a positive integer k with $1 \leqq k \leqq n-1$. Let $\overline{P}_n$ be the closure of P_n in V_n, i.e., $\overline{P}_n$ is the closed cone of positive semi-definite matrices. Define the function $m_k : \overline{P}_n \longrightarrow \mathbf{R}_{\geqq 0}$ by

$$m_k(s) = \inf_{\substack{x_1, \cdots, x_k \in \mathbf{Z}^n \\ x_1 \wedge \cdots \wedge x_k \neq 0}} \det(({}^t x_i s x_j)_{i,j})$$

for $s \in \overline{P}_n$. It is obvious that $m_k(s) > 0$ if $s \in P_n$, and $m_k(s) = 0$ otherwise.

Lemma 2.1. *The function $m'_k(s) = m_k(s)^{1/k}$ in $s \in P_n$ is a type one class function.*

Proof. The class invariance and the condition (i) for m'_k are trivial. We show the condition (ii). Since the function $x \mapsto (\det x)^{1/k}$ in $x \in P_k$ is a type one function, we have

$$\begin{aligned} m'_k(s+t) &= \inf_{\substack{x_1, \cdots, x_k \in \mathbf{Z}^n \\ x_1 \wedge \cdots \wedge x_k \neq 0}} \det(({}^t x_i (s+t) x_j)_{i,j})^{1/k} \\ &\geqq \inf_{\substack{x_1, \cdots, x_k \in \mathbf{Z}^n \\ x_1 \wedge \cdots \wedge x_k \neq 0}} \left\{ \det(({}^t x_i s x_j)_{i,j})^{1/k} + \det(({}^t x_i t x_j)_{i,j})^{1/k} \right\} \\ &\geqq \inf_{\substack{x_1, \cdots, x_k \in \mathbf{Z}^n \\ x_1 \wedge \cdots \wedge x_k \neq 0}} \det(({}^t x_i s x_j)_{i,j})^{1/k} + \inf_{\substack{x_1, \cdots, x_k \in \mathbf{Z}^n \\ x_1 \wedge \cdots \wedge x_k \neq 0}} \det(({}^t x_i t x_j)_{i,j})^{1/k} \\ &= m'_k(s) + m'_k(t) . \end{aligned}$$

This shows that m'_k is a type one function. □

If $k = 1$, then $m(s) = m'_1(s)$ is none other than the minimum of the quadratic form $x \mapsto {}^t xsx$ on the set $\mathbf{Z}^n \backslash \{0\}$ of non-zero lattice points. The dual of m is the dyadic trace, i.e.,

$$w(s) = \widehat{m}(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle}{m(t)} ,$$

which plays an important role in Poor – Yuen's theory.

By using m_k, the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$ are defined as follows:

$$\gamma_{n,k} = \sup_{s \in P_n} \frac{m_k(s)}{(\det s)^{k/n}} \quad \text{and} \quad \gamma'_{n,k} = \sup_{s \in P_n} \sqrt{m_k(s) m_k(s^{-1})} \, .$$

A generalization of $w(s)/m(s)$ is given by

$$c_{n,k}(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle^k}{m_k(s) m_k(t)} .$$

We set

$$c_{n,k} = \inf_{s \in P_n} c_{n,k}(s) = \inf_{(s,t) \in P_n \times P_n} \frac{\langle s, t \rangle^k}{m_k(s) m_k(t)} \, .$$

Lemma 2.2. *The infimum*

$$\inf_{(s,t) \in P_n \times P_n} \frac{\langle s, t \rangle^k}{m_k(s) m_k(t)}$$

is attained at some $(s_0, t_0) \in P_n \times P_n$.

Proof. Since $m_k(s) \leqq \gamma_{n,k} (\det s)^{k/n}$ holds for all $s \in \overline{P}_n$, the function $s \mapsto m_k(s)$ is continuous on $\overline{P}_n$. Then, by [PY, Lemma 3.6], the infimum

$$w_k(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle}{m_k(t)^{1/k}}$$

is attained at some $t_s \in P_n$. Since w_k is the dual of m'_k, w_k is also a type one class function, and hence $w_k(s)/m_k(s)^{1/k}$ depends only on the similar $GL_n(\mathbf{Z})$-equivalent class $\mathbf{R}_{>0} \cdot [s]$ of s. Therefore, we have

$$c_{n,k}^{1/k} = \inf_{s \in P_n} \frac{w_k(s)}{m_k(s)^{1/k}} = \inf_{\substack{[s] \in P_n/GL_n(\mathbf{Z}) \\ m_k(s) = 1}} w_k(s) \, .$$

For a constant $c > c_{n,k}$, we consider the non-empty set

$$\Omega_c = \{ [s] \in P_n/GL_n(\mathbf{Z}) \; : \; m_k(s) = 1 \;\text{ and }\; w_k(s) \leqq c^{1/k} \} \, .$$

From the inequality of the arithmetic and the geometric means, it follow that

$$n (\det s)^{1/n} (\det t)^{1/n} \leqq \langle s, t \rangle \tag{2.1}$$

for all $(s,t) \in P_n \times P_n$ (cf. [BC, Theorem 1]). By using (2.1) and the definition of $\gamma_{n,k}$, we have

$$n \cdot \frac{(\det s)^{1/n}}{\gamma_{n,k}^{1/k}} \leqq \frac{\langle s, t_s \rangle}{m_k(t_s)^{1/k}} = w_k(s) \,.$$

This implies that Ω_c is a subset of

$$\Omega'_c = \{[s] \in P_n/GL_n(\mathbf{Z}) \ : \ m_k(s) = 1 \ \text{ and } \ \det s \leqq n^{-n} c^{n/k} (\gamma_{n,k})^{n/k}\} \,.$$

As a consequence, $c_{n,k}^{1/k}$ is represented by

$$c_{n,k}^{1/k} = \inf_{[s] \in \Omega'_c} w_k(s) \,.$$

Since Ω'_c is compact in $P_n/GL_n(\mathbf{Z})$ by [C, Proposition 2.2] and w_k is continuous, this infimum is attained at some $[s_0] \in \Omega'_c$. □

In the case of $k = 1$, Poor and Yuen proved the inequality:

$$\frac{1}{(\gamma_{n,1})^2} \leqq \frac{c_{n,1}}{n} \leqq \frac{1}{(\gamma'_{n,1})^2} \,. \tag{2.2}$$

We note that Barnes and Cohn ([BC]) also proved the first inequality of (2.2).

Theorem 2.1. *For $1 \leqq k \leqq n-1$, we have*

$$\frac{1}{(\gamma_{n,k})^2} \leqq \frac{c_{n,k}}{n^k} \leqq \frac{1}{(\gamma'_{n,k})^2} \,.$$

The equality $(\gamma_{n,k})^{-2} = c_{n,k}/n^k$ holds if and only if $\gamma_{n,k} = \gamma'_{n,k}$.

Proof. By the inequality (2.1), we have

$$n^k \cdot \inf_{s \in P_n} \frac{(\det s)^{k/n}}{m_k(s)} \cdot \inf_{t \in P_n} \frac{(\det t)^{k/n}}{m_k(t)} \leqq c_{n,k} \,,$$

and hence

$$n^k \left(\frac{1}{\gamma_{n,k}} \right)^2 \leqq c_{n,k} \,.$$

By the definition of $c_{n,k}$, the inequality

$$c_{n,k} \leqq \frac{\langle s, s^{-1} \rangle^k}{m_k(s) m_k(s^{-1})} = \frac{n^k}{m_k(s) m_k(s^{-1})}$$

holds for all $s \in P_n$. Therefore,

$$m_k(s) m_k(s^{-1}) \leqq \frac{n^k}{c_{n,k}} \,.$$

This gives

$$(\gamma'_{n,k})^2 \leqq \frac{n^k}{c_{n,k}} \,.$$

It is known that the equality

$$n(\det s)^{1/n}(\det t)^{1/n} = \langle s, t\rangle$$

holds if and only if t is similar to s^{-1}. By Lemma 2.2, there exists $(s_0, t_0) \in P_n \times P_n$ such that

$$c_{n,k} = \frac{\langle s_0, t_0\rangle^k}{m_k(s_0)m_k(t_0)} \,.$$

In general, we have

$$\frac{n^k}{(\gamma_{n,k})^2} \leqq \frac{n^k(\det s_0)^{k/n}(\det t_0)^{k/n}}{m_k(s_0)m_k(t_0)} \leqq \frac{\langle s_0, t_0\rangle^k}{m_k(s_0)m_k(t_0)} = c_{n,k} \,.$$

Thus, if $(\gamma_{n,k})^{-2} = c_{n,k}/n^k$ holds, then t_0 must be similar to s_0^{-1} and $\gamma_{n,k}$ is attained at both s_0 and s_0^{-1}. This implies $\gamma_{n,k} = \gamma'_{n,k}$. □

3. Explicit values of some Hermite–Rankin constants

In [BM], Bergé and Martinet proved several inequalities involving $\gamma_{n,k}$ and $\gamma'_{n,k}$, and determined the values $\gamma'_{2,1} = 2/\sqrt{3}$, $\gamma'_{3,1} = \sqrt{3/2}$ and $\gamma'_{4,1} = \sqrt{2}$. In the inequality (2.2), Poor and Yuen proved the equality $(\gamma'_{n,1})^2 = n/c_{n,1}$ holds for $n \leqq 8$ and $n = 24$ ([PY3, Theorem 2.4]). By using this, they determined the following explicit values of Bergé–Martinet constants:

$$\gamma'_{5,1} = \sqrt{2}, \qquad \gamma'_{6,1} = \sqrt{8/3}, \qquad \gamma'_{7,1} = \sqrt{3}\,.$$

These explicit values give the following:

Theorem 3.1. $\gamma_{8,3} = \gamma_{8,4} = \gamma'_{8,4} = 4$.

Proof. If n is even, then the equality $\gamma_{n,n/2} = \gamma'_{n,n/2}$ holds in general ([M, Corollary 2.8.8]). We recall Rankin's inequality:

$$\gamma_{n,k} \leqq \gamma_{h,k}(\gamma_{n,h})^{k/h} \tag{3.1}$$

holds for $1 \leqq k < h < n$. In the case of $n = 8, h = 4, k = 1$, we have

$$\left(\frac{\gamma_{8,1}}{\gamma_{4,1}}\right)^4 \leqq \gamma_{8,4} \,.$$

Since $\gamma_{8,1} = 2$ and $\gamma_{4,1} = \sqrt{2}$, this gives $4 \leqq \gamma_{8,4}$. On the other hand, the following inequality is known for $1 \leqq k \leqq n/2$ ([M, Theorem 2.8.7]):

$$\gamma'_{n,2k} \leqq (\gamma'_{n-k,k})^2 \,. \tag{3.2}$$

We use (3.2) twice. Namely, if we put $n = 8, k = 2$ and $n = 6, k = 1$, then

$$4 \leqq \gamma_{8,4} = \gamma'_{8,4} \leqq (\gamma'_{6,2})^2 \leqq (\gamma'_{5,1})^4 = 4\,. \tag{3.3}$$

By applying (3.1) to the cases $n = 8, h = 4, k = 3$ and $n = 8, h = 3, k = 1$, we obtain

$$\gamma_{8,3} \leqq \gamma_{4,3}(\gamma_{8,4})^{3/4} = 4^{1/4} \cdot 4^{3/4} = 4$$

and

$$4 = \left(\frac{\gamma_{8,1}}{\gamma_{3,1}}\right)^3 \leqq \gamma_{8,3}\,.$$

□

Corollary 3.1. $\gamma'_{6,2} = 2$, $\gamma_{6,2} = 3^{2/3}$.

Proof. By (3.3), $\gamma'_{6,2} = 2$ is trivial. Rankin's inequality (3.1) gives for $n = 6, h = 2, k = 1$

$$\left(\frac{\gamma_{6,1}}{\gamma_{2,1}}\right)^2 \leqq \gamma_{6,2}\,.$$

Since, $\gamma_{6,1} = (64/3)^{1/6}$ and $\gamma_{2,1} = \sqrt{4/3}$, we have $3^{2/3} \leqq \gamma_{6,2}$. On the other hand, the inequality

$$(\gamma_{n,k})^n \leqq (\gamma_{n-k,k})^{n-k}(\gamma'_{n,k})^{2k} \tag{3.4}$$

is known for $1 \leqq k \leqq n/2$ ([M, Theorem 2.8.7]). By putting $n = 6, k = 2$, we obtain

$$(\gamma_{6,2})^6 \leqq (\gamma_{4,2} \cdot \gamma'_{6,2})^4 = 3^4$$

because of $\gamma_{4,2} = 3/2$. □

Corollary 3.2. $\gamma_{8,2} = \gamma'_{8,2} = 3$.

Proof. Rankin's inequality gives for $n = 8, h = 2, k = 1$

$$3 = \left(\frac{\gamma_{8,1}}{\gamma_{2,1}}\right)^2 \leqq \gamma_{8,2}\,.$$

By (3.2), we have

$$\gamma'_{8,2} \leqq (\gamma'_{7,1})^2 = 3\,.$$

Then, by (3.4),

$$3 \leqq \gamma_{8,2} \leqq (\gamma_{6,2})^{3/4}(\gamma'_{8,2})^{1/2} \leqq 3\,.$$

□

We show that all $\gamma_{8,2}, \gamma_{8,3}$ and $\gamma_{8,4}$ are attained on the E_8 lattice and $\gamma_{6,2}$ is attained on the E_6 lattice. For a full lattice Λ in the Euclidean space $\mathbf{R}^n$, $\det \Lambda$ denotes the determinant of Λ, i.e., $\det \Lambda = (\det g)^2$ if $\Lambda = g\mathbf{Z}^n$

with $g \in GL_n(\mathbf{R})$. The Hermite–Rankin invariant $\gamma_{n,k}(\Lambda)$ and the Bergé–Martinet invarinat $\gamma'_{n,k}(\Lambda)$ of Λ are defined as

$$\gamma_{n,k}(\Lambda) = \inf_{\Lambda'} \gamma(\Lambda, \Lambda')\,, \qquad \text{where } \gamma(\Lambda, \Lambda') = \frac{\det \Lambda'}{(\det \Lambda)^{k/n}}$$

and Λ' runs over all sublattices in Λ of rank k, and then

$$\gamma'_{n,k}(\Lambda) = \sqrt{\gamma_{n,k}(\Lambda)\gamma_{n,k}(\Lambda^*)}\,,$$

where Λ^* denotes the dual lattice of Λ. In terms of lattices, $\gamma_{n,k}$ and $\gamma'_{n,k}$ are given by

$$\gamma_{n,k} = \sup_{\Lambda} \gamma_{n,k}(\Lambda)\,, \qquad \gamma'_{n,k} = \sup_{\Lambda} \gamma'_{n,k}(\Lambda)\,,$$

where Λ runs over all full lattices in $\mathbf{R}^n$.

Proposition 3.1. *One has* $\gamma_{8,2} = \gamma_{8,2}(E_8) = \gamma(E_8, A_2)$, $\gamma_{8,3} = \gamma_{8,3}(E_8) = \gamma(E_8, A_3)$, $\gamma_{8,4} = \gamma_{8,4}(E_8) = \gamma(E_8, D_4)$ *and* $\gamma_{6,2} = \gamma_{6,2}(E_6) = \gamma(E_6, A_2)$.

Proof. From the proof of Rankin's inequality, it follows that

$$\gamma_{n,k}(\Lambda) \leqq \gamma_{h,k} \cdot (\gamma_{n,h}(\Lambda))^{k/h} \tag{3.5}$$

holds for any lattice Λ of rank n and $1 \leqq k < h < n$ (cf. [M, Proof of Theorem 2.8.6]). For $\Lambda = E_8$ and $n = 8, h = 2, k = 1$, (3.5) gives

$$2 = \gamma_{8,1}(E_8) \leqq \frac{2}{\sqrt{3}} \cdot (\gamma_{8,2}(E_8))^{1/2}.$$

This implies

$$3 \leqq \gamma_{8,2}(E_8) \leqq \gamma_{8,2} = 3\,.$$

From the table [M, Table 4.10.13], it follows $\gamma(E_8, A_2) = 3$. By a similar fashion, we have $\gamma_{8,3} = \gamma_{8,3}(E_8)$, $\gamma_{8,4} = \gamma_{8,4}(E_8)$ and $\gamma_{6,2} = \gamma_{6,2}(E_6)$. □

We note that Coulangeon computed $\gamma_{8,2}(E_8) = 3, \gamma_{7,2}(E_7) = 3/2^{2/7}$ and $\gamma_{6,2}(E_6) = 3^{2/3}$ by another method ([C, Théorème 5.1.1]).

Corollary 3.3. $\gamma'_{8,3} = 4$.

Proof. Since E_8 is self-dual, the Bergé–Martinet invariant $\gamma'_{8,3}(E_8)$ coincides with $\gamma_{8,3}(E_8)$, i.e., $\gamma'_{8,3}(E_8) = 4$. Then, by [M, Proposition 2.8.4],

$$4 = \gamma'_{8,3}(E_8) \leqq \gamma'_{8,3} \leqq \gamma_{8,3} = 4\,.$$

□

By Theorem 3.1, Corollaries 3.2, 3.3 and the duality relation ([M, Proposition 2.8.5]), Hermite–Rankin and Bergé–Martinet constants of dimension 8 are completely determined.

Proposition 3.2. *One has*

$$\frac{4}{\sqrt{3}} \leqq \gamma_{6,3} \leqq \sqrt{6} \quad \text{and} \quad 2^{11/7} \leqq \gamma_{7,3} \leqq 2^{4/7} \cdot 3^{2/3}\,. \tag{3.6}$$

These lower bounds are attained on E_6 and E_7, i.e., we have

$$\gamma_{6,3}(E_6) = \gamma(E_6, A_3) = \frac{4}{\sqrt{3}}, \qquad \gamma_{7,3}(E_7) = \gamma(E_7, A_3) = 2^{11/7}$$

and moreover

$$\gamma'_{7,2}(E_7) = \sqrt{6}\,, \qquad \gamma'_{7,3}(E_7) = 2\sqrt{2}\,.$$

Proof. From Rankin's inequality (3.1) for $n = 8, h = 6, k = 3$, it follows $4/\sqrt{3} \leqq \gamma_{6,3}$. All other inequalities of (3.6) are also obtained by Rankin's inequality. Let Λ' be a sublattice of E_6 of rank 3 which attains $\gamma_{6,3}(E_6)$. Since $\gamma(E_8, E_6) = 3$, we have

$$\gamma(E_8, \Lambda') = \det \Lambda' = \gamma_{6,3}(E_6)\gamma(E_8, E_6)^{1/2}\,.$$

Therefore,

$$4 = \gamma_{8,3}(E_8) \leqq \gamma(E_8, \Lambda') = \gamma_{6,3}(E_6)\sqrt{3}\,.$$

We know $\gamma(E_6, A_3) = 4/\sqrt{3}$ by [M, Table 4.10.13], and hence

$$\frac{4}{\sqrt{3}} \leqq \gamma_{6,3}(E_6) \leqq \gamma(E_6, A_3) = \frac{4}{\sqrt{3}}\,.$$

Smilarly, we obtain $\gamma_{7,3}(E_7) = 2^{11/7}$, $\gamma_{7,4}(E_7) = \gamma(E_7, D_4) = 2^{10/7}$, and $\gamma_{7,5}(E_7) = \gamma(E_7, D_5) = 2^{9/7}$. Then $\gamma'_{7,2}(E_7) = \sqrt{\gamma_{7,2}(E_7)\gamma_{7,5}(E_7)} = \sqrt{6}$ and $\gamma'_{7,3}(E_7) = \sqrt{\gamma_{7,3}(E_7)\gamma_{7,4}(E_7)} = 2\sqrt{2}$ follows from the duality relation $\gamma_{n,k}(\Lambda^*) = \gamma_{n,n-k}(\Lambda)$ ([M, Proposition 2.8.5]). □

The same argument as in the proof of Propositon 3.2 yields $\gamma_{6,4}(E_6) = \gamma_{6,2}(E_6^*) = 4/3^{2/3}$. This implies $\gamma'_{6,2}(E_6) = 2 = \gamma'_{6,2}$, namely $\gamma'_{6,2}$ is attained on E_6. Since E_8 is self-dual, all $\gamma'_{8,2}, \gamma'_{8,3}, \gamma'_{8,4}$ are attained on E_8.

Mayer [Ma, Théorème 3.59] extended Theorem 2.1 to generalized Hermite constants over an algebraic number field and applied it to determine the values $\gamma'_{3,1}(\mathbf{Q}(\sqrt{-1})) = \gamma'_{3,1}(\mathbf{Q}(\sqrt{-3})) = 2$, $\gamma'_{4,1}(\mathbf{Q}(\sqrt{-1})) = 4$, $\gamma'_{4,1}(\mathbf{Q}(\sqrt{-3})) = 3$ and $\gamma_{4,2}(\mathbf{Q}(\sqrt{-1})) = \gamma_{4,2}(\mathbf{Q}(\sqrt{-3})) = 4$ ([Ma, Propositions 3.66, 3.67]).

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