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The integral logarithm in Iwasawa theory: an exercise

par Jürgen RITTER et Alfred WEISS

Résumé. Soient l un nombre premier impair et H un groupe fini abélien. Nous décrivons le groupe d'unités de $\Lambda_\wedge[H]$ (la complétion du localisé de $\mathbb{Z}_l[[T]][H]$ en l) ainsi que le noyau et le conoyau du logarithme intégral $L : \Lambda_\wedge[H]^\times \to \Lambda_\wedge[H]$, qui apparaît dans la théorie d'Iwasawa non-commutative.

Abstract. Let l be an odd prime number and H a finite abelian l-group. We describe the unit group of $\Lambda_\wedge[H]$ (the completion of the localization at l of $\mathbb{Z}_l[[T]][H]$) as well as the kernel and cokernel of the integral logarithm $L : \Lambda_\wedge[H]^\times \to \Lambda_\wedge[H]$, which appears in non-commutative Iwasawa theory.

1. Introduction

Let $\Lambda = \mathbb{Z}_l[[T]]$ denote the ring of power series in one variable over the l-adic integers $\mathbb{Z}_l$, where l is an odd prime number. We localize Λ at the prime ideal $l \cdot \Lambda$ to arrive at $\Lambda_\bullet$ and then form the completion

$$\Lambda_\wedge = \varprojlim_n \Lambda_\bullet / l^n \Lambda_\bullet \ .$$

The integral logarithm $\mathbb{L} : \Lambda_\wedge^\times \to \Lambda_\wedge$ is defined by

$$\mathbb{L}(e) = \frac{1}{l} \log \frac{e^l}{\psi(e)} \ ,$$

where $\psi : \Lambda_\wedge \to \Lambda_\wedge$ is the $\mathbb{Z}_l$-algebra homomorphism induced by $\psi(T) = (1+T)^l - 1$ and with 'log' defined by the usual power series.

In this paper, the unit group $\Lambda_\wedge^\times$ as well as $\ker(\mathbb{L})$ and $\operatorname{coker}(\mathbb{L})$ are studied – more precisely, we study the analogous objects when $\Lambda_\wedge$ is replaced by the group ring $\Lambda_\wedge[H]$ of a finite abelian l-group H.

The interest in doing so comes from recent work in Iwasawa theory in which refined 'main conjectures' are formulated in terms of the K-theory of completed group algebras $\mathbb{Z}_l[[G]]$ with G an l-adic Lie group (see [5],[9]). For l-adic Lie groups of dimension 1, use of the integral logarithm $\mathbb{L}$ has reduced the 'main conjecture' to questions of the existence of special elements ("pseudomeasures") in $K_1(\mathbb{Z}_l[[G]]_\bullet)$, by Theorem A of [10], and,

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more recently ([11],[12]), to still unproved *logarithmic congruences* between Iwasawa L-functions. Moreover, $\mathbb{L}$ has been indispensable for the proof of the 'main conjecture' in the few special cases ([6], [13]) which have been settled so far.

The integral logarithm $\mathbb{L}$ when applied to $K_1(\mathbb{Z}_l[[T]]_\bullet)$ takes its values only in $\mathbb{Z}_l[[T]]_\wedge$, which is why we need to consider completions. As for finite G (see [8] and [4]), $\mathbb{L}$ can be used to obtain structural information about K_1; in particular, $\operatorname{coker}(\mathbb{L})$ can be detected on the abelianization G^{ab} of G (by [10], Theorem 8). Then

$G^{\mathrm{ab}} = H \times \Gamma$, with $\Gamma \simeq \mathbb{Z}_l$ and H as before,
$K_1(\mathbb{Z}_l[[G^{\mathrm{ab}}]]_\wedge) = \mathbb{Z}_l[[G^{\mathrm{ab}}]]_\wedge^\times$ (see [1], 40.31 and 40.32 (ii)),
$\mathbb{Z}_l[[G^{\mathrm{ab}}]]$, $\mathbb{Z}_l[[G^{\mathrm{ab}}]]_\wedge$ are $\Lambda[H]$ and $\Lambda_\wedge[H]$, respectively,
and ψ is induced by the map $g \mapsto g^l$ on G^{ab}.

For these reasons it seems worthwhile to present a rather complete understanding of $\mathbb{L}$ in the abelian situation, which is the purpose of our exercise.

The content of the paper is as follows. In section 2 we consider Λ and define an integral exponential $\mathbb{E}$ on $T^2\Lambda$ which is inverse to $\mathbb{L}$ (on $1+T^2\Lambda$). As a consequence, we obtain the decomposition

$$\Lambda^\times = \mathbb{Z}_l{}^\times \times (1+T)^{\mathbb{Z}_l} \times \mathbb{E}(T^2\Lambda)$$

for the unit group $\Lambda^\times$ of Λ (which reminds us of [2], Theorem 1). Applying $\mathbb{L}$ to the decomposition yields a generalization of the Oliver congruences [8], Theorem 6.6.

The third section centers around $\Lambda_\wedge$ and two important subgroups

$$\Xi = \{\sum_{k=-\infty}^{\infty} x_k T^k \in \Lambda_\wedge : x_k = 0 \text{ for } l|k\} \quad \text{and} \quad \Xi_2 = \{\sum_{k\geq 2} x_k T^k \in \Xi\}\ .$$

In terms of these we exhibit natural decompositions of $\Lambda_\wedge$ and $\Lambda_\wedge^\times$, which leads immediately to $\ker(\mathbb{L})$ and $\operatorname{im}(\mathbb{L})$, $\operatorname{coker}(\mathbb{L})$.

Section 4 is still concerned with $\Lambda_\wedge$: we determine the kernel and cokernel of its endomorphism $1-\psi$.

This will be used in the last section, §5, where we extend most of the results to the group ring $\Lambda_\wedge[H]$ of a finite abelian l-group H over $\Lambda_\wedge$ and determine $\ker(\mathbb{L})$ and $\operatorname{coker}(\mathbb{L})$ here.

2. The integral exponential $\mathbb{E}$ and $\Lambda^\times$

Recall that Λ is the ring $\mathbb{Z}_l[[T]]$ of formal power series $\sum_{k\geq 0} y_k T^k$ with coefficients $y_k \in \mathbb{Z}_l$, and that the integral logarithm is defined on the units $e \in \Lambda^\times$ of Λ by

$$\mathbb{L}(e) = \frac{1}{l}\log\frac{e^l}{\psi(e)} \quad \text{where} \quad \psi(T) = (1+T)^l - 1\ .$$

Moreover, note that $1+T^2\Lambda$ is a subgroup of $\Lambda^\times$ since $T \in \mathrm{rad}(\Lambda) = \langle l, T\rangle$.

We now turn to the integral exponential $\mathbb{E}$ on $T^2\Lambda$: This is the formal power series, with coefficients in $\mathbb{Q}_l$, defined by

$$\mathbb{E}(y) = \exp\Big(\sum_{i\geq 0} \frac{\psi^i(y)}{l^i}\Big) \in \mathbb{Q}_l[[T]] \quad \text{for each} \quad y \in T^2\Lambda\ .$$

Observe that $\mathbb{E}$ and ψ commute.

Lemma 2.1. $\mathbb{E}(y) \in 1 + T^2\Lambda$*, and* $\mathbb{E}$ *and* $\mathbb{L}$ *are inverse to each other*

$$T^2\Lambda \underset{\mathbb{L}}{\overset{\mathbb{E}}{\rightleftarrows}} 1 + T^2\Lambda\ .$$

Proof. The proof is an adaptation of that of the Dwork-Dieudonné lemma (see [7], 14 §2): if $f(T) \in 1 + T^2\mathbb{Q}_l[[T]]$ satisfies $\frac{f(T)^l}{\psi(f(T))} \in 1 + lT^2\mathbb{Z}_l[[T]]$, then $f(T) \in 1 + T^2\mathbb{Z}_l[[T]]$.

First, $\psi^i(T) \equiv l^iT \bmod T^2\mathbb{Z}_l[[T]]$ implies $\psi^i(T^k) \equiv l^{ik}T^k \bmod T^{k+i}\mathbb{Z}_l[[T]]$, and thus, if $y \in y_kT^k + T^{k+1}\mathbb{Z}_l[[T]]$ with $y_k \in \mathbb{Z}_l$, then

$$\psi^i(y)/l^i \in y_k l^{(k-1)i}T^k + T^{k+1}\mathbb{Q}_l[[T]]\,;$$

whence for $k \geq 2$,

$$\begin{aligned}\sum_{i\geq 0}\frac{\psi^i(y)}{l^i} &\in \Big(\sum_{i\geq 0} y_k l^{(k-1)i}T^k\Big) + T^{k+1}\mathbb{Q}_l[[T]] \\ &= y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Q}_l[[T]]\,,\end{aligned}$$

so $\mathbb{E}(y) \in 1 + y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Q}_l[[T]]$.

Second,

$$(*) \qquad \begin{aligned}\mathbb{E}(y)^l/\mathbb{E}(\psi(y)) &= \exp\Big(l\sum_{i\geq 0}\frac{\psi^i(y)}{l^i} - \sum_{i\geq 0}\frac{\psi^{i+1}(y)}{l^i}\Big) \\ &= \exp(ly) \in 1 + lT^2\mathbb{Z}_l[[T]]\,,\end{aligned}$$

which brings us in a position to employ the Dwork-Dieudonné argument to obtain $\mathbb{E}(y) \in 1 + T^2\mathbb{Z}_l[[T]] = 1 + T^2\Lambda$ and, in particular, $\mathbb{E}(y) \in 1 + y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Z}_l[[T]]$.

Moreover, given $b_k \in \mathbb{Z}_l$ and setting $a_k = (1-l^{k-1})b_k$, then $\mathbb{E}(a_kT^k) \in 1 + b_kT^k + T^{k+1}\mathbb{Z}_l[[T]]$ for all k, which implies that $\mathbb{E}(T^2\Lambda) = 1 + T^2\Lambda$.

We finish the proof of the lemma by showing $\mathbb{L}\mathbb{E}(y) = y$, and $\mathbb{E}\mathbb{L}(1+y) = 1+y$ whenever $y \in T^2\Lambda$, so $1 + y = \mathbb{E}(\tilde{y})$:

$$\begin{aligned}\mathbb{L}\mathbb{E}(y) &= \frac{1}{l}\log\frac{\mathbb{E}(y)^l}{\mathbb{E}(\psi(y))} \overset{(*)}{=} \frac{1}{l}\log\exp(ly) = y\,,\\ \mathbb{E}\mathbb{L}(1+y) &= \mathbb{E}\mathbb{L}(\mathbb{E}(\tilde{y})) = \mathbb{E}(\tilde{y}) = y\,.\end{aligned}$$

□

Corollary 2.1.

$$\mathbb{L}(1+l\Lambda) = (l-\psi)\Lambda\,,$$
$$\exp(ly) = (1+T)^{ly_1}\mathbb{E}((l-\psi)y)$$

if $y \equiv y_1 T \mod T^2\Lambda$ *(and* $y_1 \in \mathbb{Z}_l$*).*

Proof. Since $\exp(l\Lambda) = 1 + l\Lambda$, for the first assertion it suffices to compute

$$\mathbb{L}(\exp(ly)) = \frac{1}{l}\,\log(\exp(ly)^{l-\psi}) = \frac{1}{l}\,\log\exp(l(l-\psi)y) = (l-\psi)y\,.$$

The second assertion holds for $y = T$ (so $y_1 = 1$):

$$(1+T)^{-l}\exp(lT) \in 1 + T^2\mathbb{Z}_l[[T]]\,,$$

hence $(1+T)^{-l}\exp(lT) = \mathbb{E}(z)$ for some $z \in T^2\mathbb{Z}_l[[T]]$.

Apply $\mathbb{L}$ and get $(l-\psi)(T) = z$ from the last but one displayed formula and as $\mathbb{L}(1+T) = \frac{1}{l}\,\log\,\frac{(1+T)^l}{1+(1+T)^l-1} = 0$.

Next, take $y \in T^2\mathbb{Z}_l[[T]]$, so $\exp(ly) \in 1 + T^2\mathbb{Z}_l[[T]]$ and again $\exp(ly) = \mathbb{E}((l-\psi)y)$.

The two special cases can be combined on writing $y = y_1T + (y - y_1T)$. $\square$

Denote by μ_{l-1} the group of roots of unity in $\mathbb{Z}_l{}^\times$.

Corollary 2.2.

$$\Lambda^\times = \mathbb{Z}_l{}^\times \times (1+T)^{\mathbb{Z}_l} \times \mathbb{E}(T^2\Lambda)\,,$$
$$\ker(\mathbb{L}) = \mu_{l-1} \times (1+T)^{\mathbb{Z}_l}\,,$$
$$\mathrm{im}\,(\mathbb{L}) = \mathbb{Z}_l \oplus T^2\Lambda$$

Proof. The first coefficient e_0 of $e = \sum_{k\geq 0} e_k T^k \in \Lambda^\times$ is a unit in $\mathbb{Z}_l$. Replacing e by $e_0^{-1}\cdot e = 1 + e_1' T + \cdots$ and then multiplying by $(1+T)^{-e_1'}$ gives the new unit $1 + \tilde{e}_2T^2 + \cdots \in 1 + T^2\Lambda$. Thus, by Lemma 2.1, $\Lambda^\times = \mathbb{Z}_l{}^\times \cdot (1+T)^{\mathbb{Z}_l} \cdot \mathbb{E}(T^2\Lambda)$, and the product is obviously direct. Since, on $\mathbb{Z}_l{}^\times$, $\mathbb{L}(\zeta) = 0$ precisely for $\zeta \in \mu_{l-1}$, and since $\mathbb{L}(1+T) = 0$, we also get the claimed description of the kernel and image of $\mathbb{L}$. $\square$

3. The integral logarithm $\mathbb{L}$ on $\Lambda_\wedge$

We recall that $\Lambda_\bullet$ denotes the localization of Λ at the prime ideal $l\Lambda$ and that $\Lambda_\wedge = \varprojlim_n \Lambda_\bullet/l^n\Lambda_\bullet$. In particular, $\Lambda_\bullet$ and $\Lambda_\wedge$ have the same residue field $\mathbb{F}_l((\overline{T}))$ (which carries the natural $\overline{T}$-valuation $v_{\overline{T}}$). It follows that

$$\Lambda_\wedge = \{x = \sum_{k\in\mathbb{Z}} x_kT^k : x_k \in \mathbb{Z}_l\,,\ \lim_{k\to-\infty} x_k = 0\}\,.$$

Such large rings are basic objects in the theory of higher dimensional local fields [3]; the map ψ on $\Lambda_\wedge$ is extra structure which remembers the group Γ.

In what follows we frequently use the decomposition

$$\Lambda_\wedge = {\Lambda_\wedge}^- \oplus \mathbb{Z}_l \oplus {\Lambda_\wedge}^+ , \quad \text{where } {\Lambda_\wedge}^\pm = \begin{cases} \{x \in \Lambda_\wedge : x_k = 0 \text{ for } k \le 0\} \\ \{x \in \Lambda_\wedge : x_k = 0 \text{ for } k \ge 0\} . \end{cases}$$

Note that the three summands are subrings which are preserved by ψ. As a consequence, we see that $\Lambda \cap (l-\psi)\Lambda_\wedge = (l-\psi)\Lambda$.

Definition.

$$\Xi = \{x = \sum_{k\in\mathbb{Z}} x_k T^k \in \Lambda_\wedge : x_k = 0 \text{ when } l \text{ divides } k\} ,$$

$$\Xi_s = \{x = \sum_{k\ge s} x_k T^k \in \Xi\} , \quad \text{where } s \in \mathbb{Z} .$$

Lemma 3.1.
(1) $l-\psi$ *is injective on* $\Lambda_\wedge$ *and has image* $\mathbb{L}(1+l\Lambda_\wedge)$ *,*
(2) $\Lambda_\wedge = \Xi \oplus (l-\psi)\Lambda_\wedge$

Proof. For the first assertion we make use of the commuting diagram

$$(1) \qquad \begin{array}{ccccc} \Lambda_\wedge & \overset{l}{\rightarrowtail} & \Lambda_\wedge & \twoheadrightarrow & \mathbb{F}_l((\overline{T})) \\ l-\psi \downarrow & & l-\psi \downarrow & & -\overline{\psi} \downarrow \\ \Lambda_\wedge & \overset{l}{\rightarrowtail} & \Lambda_\wedge & \twoheadrightarrow & \mathbb{F}_l((\overline{T})) \end{array}$$

with exact rows and with $\overline{\psi}(\overline{T}) = \overline{T}^l$, so $\overline{\psi}(\overline{x}) = \overline{x}^l$ for $\overline{x} \in \mathbb{F}_l((\overline{T}))$. In particular, $-\overline{\psi}$ is injective and hence the snake lemma implies $\ker(l-\psi) = l \cdot \ker(l-\psi)$ from which $\ker(l-\psi) = 0$ follows by $\bigcap_{n\ge 0} l^n \Lambda_\wedge = 0$.

Regarding the image, we observe that $\exp(l\Lambda_\wedge) = 1 + l\Lambda_\wedge$ and recall $\mathbb{L}(\exp(ly)) = (l-\psi)y$ from the proof of Corollary 2.1 (but now with $y \in \Lambda_\wedge$).

For the second assertion we make use of the commuting diagram

$$(2) \qquad \begin{array}{ccccc} \Xi & \overset{l}{\rightarrowtail} & \Xi & \twoheadrightarrow & \Xi/l\Xi \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda_\wedge/(l-\psi)\Lambda_\wedge & \overset{l}{\rightarrowtail} & \Lambda_\wedge/(l-\psi)\Lambda_\wedge & \twoheadrightarrow & \mathbb{F}_l((\overline{T}))/\overline{\psi}(\mathbb{F}_l((T))) \end{array}$$

with natural vertical maps (which we denote by $\tilde{}$). Its bottom row is the sequence of cokernels of diagram (1) and thus exact. Its right vertical map is an isomorphism, by the definition of Ξ and by $\overline{\psi}(\overline{x}) = \overline{x}^l$. Consequently, the other vertical map, $\Xi \to \Lambda_\wedge/(l-\psi)\Lambda_\wedge$, is injective, by the snake lemma and $\bigcap_{n\ge 0} l^n \Lambda_\wedge = 0$. To finish the proof of the lemma we are left with showing the surjectivity of $\Xi \to \Lambda_\wedge/(l-\psi)\Lambda_\wedge$. Starting, in (2), with $\tilde{x} \in \Lambda_\wedge/(l-\psi)\Lambda_\wedge$ (the middle term in the bottom row) we find elements $y_0 \in \Xi$

and $\tilde{x}_1 \in \Lambda_\wedge/(l-\psi)\Lambda_\wedge$ such that $\tilde{x} - \tilde{y}_0 = l\tilde{x}_1$. Continuing, we get $\tilde{x} = \tilde{y}_0 + l\tilde{y}_1 + l^2\tilde{y}_2 + \cdots$, with $y_0 + ly_1 + l^2y_2 + \cdots \in \Xi$. $\square$

Corollary 3.1. $T^2\Lambda = \Xi_2 \oplus (l-\psi)T\Lambda$

Proof. Since $\Xi \cap {\Lambda_\wedge}^+ = \Xi_1$, Lemma 3.1 gives ${\Lambda_\wedge}^+ = \Xi_1 \oplus (l-\psi){\Lambda_\wedge}^+$, i.e., $T\Lambda = \Xi_1 \oplus (l-\psi)T\Lambda$. We intersect with $T^2\Lambda$ and obtain the corollary from $(l-\psi)T\Lambda \subset T^2\Lambda$ and $\Xi_1 \cap T^2\Lambda = \Xi_2$. $\square$

Proposition 3.1. $\Lambda_\wedge^\times = T^{\mathbb{Z}} \times \mu_{l-1} \times (1+T)^{\mathbb{Z}_l} \times \mathbb{E}(\Xi_2) \times (1 + l\Lambda_\wedge)$

Proof. Given $e = \sum_{k\in\mathbb{Z}} e_k T^k \in \Lambda_\wedge^\times$, we will modify e by factors in $T^{\mathbb{Z}}$, $\mu_{l-1} \times (1 + l\Lambda_\wedge)$ and $(1+T)^{\mathbb{Z}_l}$ to arrive at a new unit $\mathbb{E}(y)$ for some $y \in \Xi_2$. This confirms the claimed product decomposition of $\Lambda_\wedge^\times$ but not yet that it is a direct product.

(1) Going modulo l, let $\overline{e} = \sum_{k\geq k_0} \overline{e}_k \overline{T}^k \in \mathbb{F}_l((\overline{T}))$ have coefficient $\overline{e}_{k_0} \neq 0$. Multiplying e by $T^{-k_0} \in T^{\mathbb{Z}}$ gives a new unit with zero coefficient not divisible by l but all coefficients with negative index divisible by l; we denote it again by e.

(2) Now $e_0 \in {\mathbb{Z}_l}^\times = \mu_{l-1} \times (1 + l\mathbb{Z}_l) \subset \mu_{l-1} \times (1 + l\Lambda_\wedge)$, and multiplying e by e_0^{-1} allows us to assume that $e = le^- + 1 + e^+$, where $e^- \in {\Lambda_\wedge}^-$ and $e^+ \in {\Lambda_\wedge}^+$, so $1 + e^+ \in \Lambda^\times \leq \Lambda_\wedge^\times$ and $e(1+e^+)^{-1} = 1 + l(e^-(1+e^+)^{-1}) \in 1 + l\Lambda_\wedge$, i.e., $e \equiv 1 + e^+ \mod 1 + l\Lambda_\wedge$.

(3) If $1 + e^+ = 1 + e_1T + e_2T^2 + \cdots$, then multiplying $1 + e^+$ by $(1+T)^{-e_1} \in (1+T)^{\mathbb{Z}_l}$ produces $1 + T^2\tilde{y}$ with $\tilde{y} \in \Lambda$ (note $(1+T)^z \equiv 1 + zT \mod T^2\Lambda$). Hence, by Lemma 2.1, modulo $T^{\mathbb{Z}} \cdot \mu_{l-1} \cdot (1+T)^{\mathbb{Z}_l} \cdot (1 + l\Lambda_\wedge)$, the original unit e satisfies $e \equiv \mathbb{E}(y')$ with $y' \in T^2\Lambda$.

(4) As $\mathbb{E}(T^2\Lambda) = \mathbb{E}(\Xi_2) \times \mathbb{E}((l-\psi)T\Lambda)$ by the above corollary, multiplying $\mathbb{E}(y')$ with $\mathbb{E}(y)$ for a suitable $y \in \Xi_2$ yields an element $\mathbb{E}((l-\psi)y'')$ with $y'' \in y_1''T + T^2\Lambda$. It follows from Corollary 2.1 to Lemma 2.1 that $\mathbb{E}((l-\psi)y'') = (1+T)^{-ly_1''}\exp(ly'')$. The first factor is in $(1+T)^{\mathbb{Z}_l}$ and the second in $1 + l\Lambda$.

We now prove that we actually have a direct product.

We have already used $(1+T)^z \equiv 1 + zT \mod T^2\Lambda$. Together with $\mathbb{E}(\Xi_2) \subset 1 + T^2\Lambda_\wedge$ it implies that the product $T^{\mathbb{Z}} \cdot \mu_{l-1} \cdot (1+T)^{\mathbb{Z}_l} \cdot \mathbb{E}(\Xi_2)$ is direct. Moreover, an element in it which also lies in $1 + l\Lambda_\wedge$ must equal $\mathbb{E}(y)$ with $y \in \Xi_2$. Indeed, $(1+T)^z \equiv 1 \mod l$ gives $z \equiv 0 \mod l$, hence $(1+T)^{\frac{z}{l}} \equiv 1 \mod l$, since modulo l we are in characteristic l. Thus $z = 0$.

So assume $\mathbb{E}(y) = 1 + lz$. Applying $\mathbb{L}$ gives $y = \mathbb{L}(1 + lz) = (l-\psi)z' \in (l-\psi)\Lambda_\wedge$, by Corollary 2.1. As $y \in \Xi_2 \subset T^2\Lambda$, the zero coefficient of z' vanishes and Corollary 3.1 implies $y = 0$. This completes the proof of the proposition. $\square$

Definition. $\xi : \Lambda_\wedge = \Xi \oplus (l-\psi)\Lambda_\wedge \to \Xi$ is the identity on Ξ and zero on $(l-\psi)\Lambda_\wedge$

Corollary 3.2. *We have an exact sequence*

$$\mu_{l-1} \times (1+T)^{\mathbb{Z}_l} \rightarrowtail {\Lambda_\wedge}^\times \xrightarrow{\mathbb{L}} \Lambda_\wedge \twoheadrightarrow \Xi \,/\, (\mathbb{Z}\cdot\xi((\mathbb{L}(T)) \oplus \Xi_2)\,.$$

Proof. For the proof note that $\xi(\mathbb{L}(T))$ is in ${\Lambda_\wedge}^-$ and non-zero: writing $\frac{T^l}{\psi(T)} = \frac{1}{1-lv}$ with $v = -\frac{1}{l}\sum_{i=1}^{l-1}\binom{l}{i}T^{-i}$ we have

$$\mathbb{L}(T) = -\log(1-lv) = \sum_{j\geq 1}\frac{l^{j-1}}{j}\,v^j \in {\Lambda_\wedge}^-$$

with

$$\xi(\mathbb{L}(T)) \equiv \xi(v) = v \equiv \sum_{i=1}^{l-1}\frac{(-1)^i}{i}\,T^{-i} \mod l\,.$$

Recall that $\mu_{l-1}\times(1+T)^{\mathbb{Z}_l} \subset \ker(\mathbb{L})$, that $\mathbb{L}\mathbb{E}$ is the identity on Ξ_2, and that $1+l\Lambda_\wedge = \exp(l\Lambda_\wedge)$.

Suppose now that $e = T^b\zeta(1+T)^z\mathbb{E}(x)\exp(ly)$ is in $\ker(\mathbb{L})$ (with $b\in\mathbb{Z}$, $\zeta\in\mu_{l-1}$, $z\in\mathbb{Z}_l$, $x\in\Xi_2$, $y\in\Lambda_\wedge$). Then $-b\mathbb{L}(T) = x + (l-\psi)y$ implies $-b\xi(\mathbb{L}(T)) = x$ is in ${\Lambda_\wedge}^-\cap\Xi_2 = 0$, hence $b = 0 = x$ and then $y = 0$ by 1. of Lemma 3.1, as required.

Concerning $\operatorname{coker}(\mathbb{L})$, it suffices to show that $\operatorname{im}(\mathbb{L}) = \mathbb{Z}\cdot\xi(\mathbb{L}(T))\oplus\Xi_2\oplus(l-\psi)\Lambda_\wedge$. By Proposition 3.1, 1. of Lemma 3.1 and $\mathbb{L}(T)-\xi(\mathbb{L}(T)) \in (l-\psi)\Lambda_\wedge$ this again follows from $\xi(\mathbb{L}(T)) \notin \Xi_2$.

This finishes the proof of the corollary. ☐

Remark. When $l = 2$, more effort is needed, since $-1 \in 1+2\Lambda_\wedge$ and 'log, exp' are no longer inverse to each other.

4. Kernel and cokernel of $1-\psi$ on $\Lambda_\wedge$

Lemma 4.1. *There is an exact sequence*

$$0 \to \mathbb{Z}_l \to \Lambda_\wedge \xrightarrow{1-\psi} \Lambda_\wedge \to (\Xi/\Xi_1)\oplus\mathbb{Z}_l \to 0\,.$$

Proof. We start its proof from the obvious diagram below and show that $\ker(\overline{1}-\overline{\psi}) = \mathbb{F}_l$, the constants in $\mathbb{F}_l((\overline{T})) = \Lambda_\wedge/l\Lambda_\wedge$.

$$\begin{array}{ccccc} \Lambda_\wedge & \overset{l}{\rightarrowtail} & \Lambda_\wedge & \twoheadrightarrow & \Lambda_\wedge/l\Lambda_\wedge \\ {\scriptstyle 1-\psi}\downarrow & & {\scriptstyle 1-\psi}\downarrow & & {\scriptstyle \overline{1}-\overline{\psi}}\downarrow \\ \Lambda_\wedge & \overset{l}{\rightarrowtail} & \Lambda_\wedge & \twoheadrightarrow & \Lambda_\wedge/l\Lambda_\wedge \end{array}$$

Indeed,

$$(\overline{1}-\overline{\psi})(\sum_{k\geq -n}\overline{z}_k\overline{T}^k) = 0 \iff \sum_{k\geq -n}\overline{z}_k\overline{T}^k = \sum_{k\geq -n}\overline{z}_k\overline{T}^{lk} = (\sum_{k\geq -n}\overline{z}_k\overline{T}^k)^l\,,$$

and the only $l-1^{\mathrm{st}}$ roots of unity in the field $\mathbb{F}_l((\overline{T}))$ are the constants $\neq 0$. The above implies $\ker(1-\psi) = \mathbb{Z}_l + l\ker(1-\psi)$. By successive approximation this gives $\ker(1-\psi) = \mathbb{Z}_l$.

Turning back to the diagram, we obtain from the snake lemma the short exact sequence

$$\operatorname{coker}(1-\psi) \overset{l}{\rightarrowtail} \operatorname{coker}(1-\psi) \twoheadrightarrow \operatorname{coker}(\overline{1}-\overline{\psi}).$$

We compute its right end. Because $\mathbb{F}_l((\overline{T}))$ is complete in the $v_{\overline{T}}$-topology, $\sum_{n\geq 0}\overline{z}^{l^n}$ converges for every element $\overline{z} = \sum_{k\geq 1}\overline{z}_k\overline{T}^k$, hence

$$(\overline{1}-\overline{\psi})(\sum_{n\geq 0}\overline{z}^{l^n}) = \overline{z}$$

implies that these $\overline{z}$ all belong to $\operatorname{im}(\overline{1}-\overline{\psi})$. Also, $\overline{T}^i - \overline{T}^{li} = (\overline{1}-\overline{\psi})(\overline{T}^i) \in \operatorname{im}(\overline{1}-\overline{\psi})$. Thus, $\operatorname{coker}(\overline{1}-\overline{\psi})$ is spanned by the images of $\overline{T}^j$ with $j = 0$ or $j < 0$ & $l \nmid j$. These elements are actually linearly independent over $\mathbb{F}_l$. To see this, read an equation

$$\sum_{\substack{-n\leq k<0 \\ l\nmid k}} \overline{z}_k\overline{T}^k + z_0 = (\overline{1}-\overline{\psi})(\overline{x}) = \sum_{-n\leq k<0}\overline{x}_k(\overline{T}^k - \overline{T}^{lk})$$

coefficientwise from $k=-n$ to $k=0$.

Going back to the short exact sequence displayed above, we now realize that $\Xi/\Xi_1 \oplus \mathbb{Z}_l$ maps onto $\operatorname{coker}(1-\psi)$, since $\Lambda_\wedge$ is l-complete. And by the last paragraph, this surjection is, in fact, an isomorphism. □

5. Kernel and cokernel of $\mathbb{L}$ on $\boldsymbol{\Lambda_\wedge[H]}$

As in the introduction, H is a finite abelian l-group and $\Lambda_\wedge[H]$ is its group ring over $\Lambda_\wedge$. Perhaps the description $\Lambda_\wedge[H] = \mathbb{Z}_l[[\Gamma\times H]]_\wedge$, with Γ denoting the cyclic pro-l group generated by $1+T$, gives a better understanding of the ring homomorphism ψ on $\Lambda_\wedge[H]$: ψ is induced by $\psi(g) = g^l$ for $g\in\Gamma\times H$. And the integral logarithm $\mathbb{L}:\Lambda_\wedge[H]^\times \to \Lambda_\wedge[H]$, as before, takes a unit $e\in\Lambda_\wedge[H]^\times$ to $\mathbb{L}(e) = \frac{1}{l}\log\frac{e^l}{\psi(e)}$.

For the discussion of its kernel and cokernel we first invoke the augmentation map $\Lambda_\wedge[H]\to\Lambda_\wedge$, $h\mapsto 1$ for $h\in H$, so that we can employ our earlier results. Let $\mathfrak{g}$ denote its kernel and note that $1+\mathfrak{g}\subset\Lambda_\wedge[H]^\times$, as $\mathfrak{g}\subset\mathfrak{r}\overset{\mathrm{def}}{=}\operatorname{rad}(\Lambda_\wedge[H]) = \mathfrak{g}+l\Lambda_\wedge[H]$; moreover, for the same reason, $\Lambda_\wedge[H]^\times\to\Lambda_\wedge^\times$ is surjective.

Proposition 5.1. $\mathbb{L}:\Lambda_\wedge[H]^\times\to\Lambda_\wedge[H]$ *has*

$$\ker(\mathbb{L}) = \mu_{l-1}\times(1+T)^{\mathbb{Z}_l}\times H \quad (=\mu_{l-1}\times(\Gamma\times H)),$$

and coker $(\mathbb{L})$ *is described by the split exact sequence*

$$(\Xi/\Xi_1 \oplus \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} H \rightarrowtail \operatorname{coker}(\mathbb{L}) \twoheadrightarrow \Xi \,/\, (\mathbb{Z}\xi(\mathbb{L}(T)) \oplus \Xi_2)\ .$$

Proof. The proof begins with the commutative diagram

$$\begin{array}{ccccc} 1+\mathfrak{g} & \rightarrowtail & \Lambda_\wedge[H]^\times & \twoheadrightarrow & \Lambda_\wedge^\times \\ \mathbb{L}\downarrow & & \mathbb{L}\downarrow & & \mathbb{L}\downarrow \\ \mathfrak{g} & \rightarrowtail & \Lambda_\wedge[H] & \twoheadrightarrow & \Lambda_\wedge \end{array}$$

with exact rows which are split by the *same* inclusion $\Lambda_\wedge \rightarrowtail \Lambda_\wedge[H]$ of rings. Here the right square commutes because ψ and 'log' both commute with augmentation, and thus induces the left square since the sequences are exact.

The right vertical $\mathbb{L}$ fits into the exact sequence of Corollary 3.2. Similarly we will need

Lemma 5.1. *There is an exact sequence*

$$H \rightarrowtail 1+\mathfrak{g} \xrightarrow{\mathbb{L}} \mathfrak{g} \twoheadrightarrow (\Xi/\Xi_1 \oplus \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} H\ .$$

Proposition 5.1 follows from Lemma 5.1 and the snake lemma : for $\mu_{l-1} \times (1+T)^{\mathbb{Z}_l} \times H \subset \ker(\mathbb{L})$ maps onto the kernel of the right vertical $\mathbb{L}$, and the cokernel sequence splits because the natural splittings in the commutative diagram are compatible. $\square$

So it remains to prove Lemma 5.1, which we do next.

Proof. a) $\mathfrak{g}/\mathfrak{g}^2 \simeq \Lambda_\wedge \otimes_{\mathbb{Z}_l} H$ *by* $h-1 \mod \mathfrak{g}^2 \mapsto h$

This is a consequence of $\Lambda_\wedge[H] = \Lambda_\wedge \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[H]$ and the natural isomorphism $\Delta H/\Delta^2 H \simeq H$, $h-1 \mapsto h$, where $\Delta H = \langle h-1 : h \in H\rangle_{\mathbb{Z}_l}$ is the augmentation ideal of the group ring $\mathbb{Z}_l[H]$, so $\mathfrak{g} = \Lambda_\wedge \otimes_{\mathbb{Z}_l} \Delta H$.

b) *If* $e = 1 + \sum_{1\neq h\in H} e_h(h-1) \in 1+\mathfrak{g}$ *(with* $e_h \in \Lambda_\wedge$*), then*

$$\mathbb{L}(e) \equiv \textstyle\sum_h (e_h - \psi(e_h))(h-1) \mod \mathfrak{g}^2\,.$$

Indeed, modulo $l\mathfrak{g}^2$ we have

$$\begin{aligned} e^l &\equiv 1 + l\sum_h e_h(h-1) + \sum_h e_h^l (h-1)^l \\ &\equiv 1 + l\sum_h e_h(h-1) + \sum_h \psi(e_h)(h-1)^l \\ &\equiv 1 + l\sum_h e_h(h-1) + \sum_h \psi(e_h)(h^l-1) - l\sum_h \psi(e_h)(h-1) \\ &\equiv \psi(e) + l\sum_h (e_h - \psi(e_h))(h-1)\,, \end{aligned}$$

so

$$\begin{aligned} \frac{e^l}{\psi(e)} &\equiv 1 + \psi(e)^{-1} l \sum_h (e_h - \psi(e_h))(h-1) \\ &\equiv 1 + l \sum_h (e_h - \psi(e_h))(h-1) \mod l\mathfrak{g}^2 \end{aligned}$$

as $\psi(e)^{-1} \in 1 + \mathfrak{g}$. Now apply '$\frac{1}{l}$ log'.

From a), b) we get the right square of the commutative diagram

$$\begin{array}{ccccc} 1+\mathfrak{g}^2 & \rightarrowtail & 1+\mathfrak{g} & \twoheadrightarrow & \Lambda_\wedge \otimes_{\mathbb{Z}_l} H \\ \mathbb{L}\downarrow & & \mathbb{L}\downarrow & & (1-\psi)\otimes 1 \downarrow \\ \mathfrak{g}^2 & \rightarrowtail & \mathfrak{g} & \twoheadrightarrow & \Lambda_\wedge \otimes_{\mathbb{Z}_l} H \end{array}$$

with left square induced by the exactness of the rows. The map $(1-\psi)\otimes 1$ has kernel and cokernel given by tensoring the sequence in Lemma 4.1 with H: it remains exact since it is composed of two short exact sequences of torsionfree $\mathbb{Z}_l$-modules. So the snake lemma reduces Lemma 5.1 to proving that $\mathbb{L} : 1 + \mathfrak{g}^2 \to \mathfrak{g}^2$ is an isomorphism.

We do this by induction on $|H|$ and, to that end, choose an element $h_0 \in H$ of order l and let $H \xrightarrow{\sim} \tilde{H} = H/\langle h_0 \rangle$ be the natural map.

Recalling that $\mathfrak{r} = \mathrm{rad}(\Lambda_\wedge[H]) = \mathfrak{g} + l\Lambda_\wedge[H]$, we start with the right square of the diagram

$$\begin{array}{ccccc} 1+(h_0-1)\mathfrak{r} & \rightarrowtail & 1+\mathfrak{g}^2 & \twoheadrightarrow & 1+\tilde{\mathfrak{g}}^2 \\ \mathbb{L}\downarrow & & \mathbb{L}\downarrow & & \tilde{\mathbb{L}}\downarrow \\ (h_0-1)\mathfrak{r} & \rightarrowtail & \mathfrak{g}^2 & \twoheadrightarrow & \tilde{\mathfrak{g}}^2 \end{array} ,$$

which commutes since ψ, 'log' commute with $\tilde{\ }$. Since $\tilde{\mathbb{L}}$ is an isomorphism by the induction hypothesis, it suffices to show that the kernels in the rows are as shown and that the left $\mathbb{L}$ is an isomorphism:

i. $\mathfrak{g}^2 \to \tilde{\mathfrak{g}}^2$ *has kernel* $(h_0-1)\mathfrak{r}$. Since $(h_0-1)\mathfrak{r}$ is in the kernel of $\tilde{\ }$ and $l(h_0-1)$ is in $\mathfrak{g}^2$, by $l(h_0-1) \equiv h_0^l - 1 \mod \mathfrak{g}^2$, it remains to check

$$(h_0-1)\Lambda_\wedge[H] \cap \mathfrak{g}^2 \subset (h_0-1)\mathfrak{r}.$$

If $(h_0-1)b = (h_0-1)\sum_{h\in H} b_h h \in \mathfrak{g}^2$ (with $b_h \in \Lambda_\wedge$), then the isomorphism $\mathfrak{g}/\mathfrak{g}^2 \simeq \Lambda_\wedge \otimes_{\mathbb{Z}_l} H$ takes $(h_0-1)b$ to $0 = \sum_{h\in H} b_h \otimes h_0 = (\sum_{h\in H} b_h)\otimes h_0$, whence $\sum_{h\in H} b_h \in l\Lambda_\wedge$, since h_0 has order l. Thus, $(h_0-1)b \in (h_0-1)(\sum_{h\in H} b_h(h-1) + l\Lambda_\wedge) \subset (h_0-1)\mathfrak{r}$.

The same argument applies to the kernel in the top row. It follows that $\mathbb{L}(1+(h_0-1)\mathfrak{r}) \subset (h_0-1)\mathfrak{r}$.

ii. $\mathbb{L} : 1 + (h_0-1)\mathfrak{r} \to (h_0-1)\mathfrak{r}$ *is an isomorphism*. If $x \in \mathfrak{r}$, then $\psi(h_0-1) = 0$ implies that $\mathbb{L}(\exp((h_0-1)x)) = (h_0-1)x$, hence $\mathbb{L}$

is onto, and

$$\begin{aligned}\mathbb{L}(1-(h_0-1)x) &= \log(1-(h_0-1)x) \\ &= -(h_0-1)(x-x^l)+(h_0-1)^2x^2\lambda_x \\ &= -(h_0-1)x+(h_0-1)x^2\lambda'_x\end{aligned}$$

with some $\lambda_x, \lambda'_x \in \Lambda_\wedge[H]$ by (†) in [10], p.40 (with z replaced by h_0). If this is zero, then $(h_0-1)x(1-x\lambda'_x)=0$ with $1-x\lambda'_x \in \Lambda_\wedge[H]^\times$. So $\mathbb{L}$ is injective.

□

Remark. Admittedly, Proposition 5.1 is closer to Corollary 3.1 than to Proposition 3.1 itself, as $\Lambda_\wedge[H]^\times$ has not been determined.

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