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Victor ABRASHKIN

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## Modified proof of a local analogue of the Grothendieck conjecture

par VICTOR ABRASHKIN

RÉSUMÉ. L'analogie locale de la conjecture de Grothendieck peut être formulé comme une équivalence entre la catégorie des corps  $K$  complets pour une valuation discrète à corps résiduel fini de caractéristique  $p \neq 0$  et la catégorie des groupes de Galois absolus des corps  $K$  munis de la filtration de ramification. Le cas des corps de caractéristique 0 a été étudié par Mochizuki il y a quelques années. Ensuite, l'auteur de cet article a établi, par une méthode différente l'analogie de la conjecture de Grothendieck dans le cas  $p > 2$  (mais  $K$  de caractéristique quelconque). Nous proposons ici une modification de cette approche qui inclut le cas  $p = 2$  dans la preuve, contient des simplifications considérables et remplace le groupe de Galois par son pro- $p$ -quotient maximal. Une attention particulière est accordée au procédé de la reconstruction de l'isomorphisme de corps à partir d'un isomorphisme de groupe de Galois compatible avec les filtrations de ramification correspondantes.

ABSTRACT. A local analogue of the Grothendieck Conjecture is an equivalence between the category of complete discrete valuation fields  $K$  with finite residue fields of characteristic  $p \neq 0$  and the category of absolute Galois groups of fields  $K$  together with their ramification filtrations. The case of characteristic 0 fields  $K$  was studied by Mochizuki several years ago. Then the author of this paper proved it by a different method in the case  $p > 2$  (but with no restrictions on the characteristic of  $K$ ). In this paper we suggest a modified approach: it covers the case  $p = 2$ , contains considerable technical simplifications and replaces the Galois group of  $K$  by its maximal pro- $p$ -quotient. Special attention is paid to the procedure of recovering field isomorphisms coming from isomorphisms of Galois groups, which are compatible with the corresponding ramification filtrations.

### Introduction

Throughout this paper  $p$  is a prime number. If  $E$  is a complete discrete valuation field then we shall assume that its residue field has characteristic

$p$ . We will consider  $E$  to be a subfield of a fixed separable closure  $E_{\text{sep}}$ . Define  $\Gamma_E = \text{Gal}(E_{\text{sep}}/E)$ . We denote by  $E(p)$  the maximal  $p$ -extension of  $E$  in  $E_{\text{sep}}$  and we let  $\Gamma_E(p) = \text{Gal}(E(p)/E)$ .

Assume that  $E$  and  $E'$  are complete discrete valuation fields with finite residue fields and there is a continuous field isomorphism  $\mu : E \rightarrow E'$ . Then  $\mu$  can be extended to a field isomorphism  $\bar{\mu} : E(p) \rightarrow E'(p)$ . With the conventions about compositions of morphisms which are described at the end of this introduction, the correspondence  $\tau \mapsto \bar{\mu}^{-1}\tau\bar{\mu}$  defines a continuous group isomorphism  $\bar{\mu}^* : \Gamma_E(p) \rightarrow \Gamma_{E'}(p)$  such that for any  $v \geq 0$ ,  $\bar{\mu}^*(\Gamma_E(p)^{(v)}) = \Gamma_{E'}(p)^{(v)}$ . Here  $\Gamma_E(p)^{(v)}$  is the ramification subgroup of  $\Gamma_E(p)$  in the upper numbering.

The principal result of this paper is the following theorem.

**Theorem A.** *Suppose  $E$  and  $E'$  are complete discrete valuation fields with finite residue fields and there is a continuous group isomorphism  $g : \Gamma_E(p) \rightarrow \Gamma_{E'}(p)$  such that for all  $v \geq 0$ ,  $g(\Gamma_E(p)^{(v)}) = \Gamma_{E'}(p)^{(v)}$ . Then there is a continuous field isomorphism  $\bar{\mu} : E(p) \rightarrow E'(p)$  such that  $\bar{\mu}(E) = E'$  and  $g = \bar{\mu}^*$ .*

This theorem implies easily a corresponding statement, where the maximal  $p$ -extensions  $E(p)$  and  $E'(p)$  and their Galois groups  $\Gamma_E(p)$  and  $\Gamma_{E'}(p)$  are replaced, respectively, by the separable closures  $E_{\text{sep}}$  and  $E'_{\text{sep}}$  and the Galois groups  $\Gamma_E$  and  $\Gamma_{E'}$ . Such a statement is known as a local analogue of the Grothendieck Conjecture. Mochizuki [7] proved this local analogue for local fields of characteristic 0. His method is based on an elegant application of Hodge-Tate theory. Under the restriction  $p > 2$  the case of local fields of arbitrary characteristic was proved by another method by the author [3]. This proof is based on an explicit description of the ramification subgroups  $\Gamma_K(p)^{(v)}$  modulo the subgroup  $C_3(\Gamma_K(p))$  of commutators of order  $\geq 3$  in  $\Gamma_K(p)$ , where  $K = k((t))$ , and  $k$  is a finite field of characteristic  $p > 2$ . The restriction  $p \neq 2$  appears because the proof uses the equivalence of the category of  $p$ -groups and of Lie  $\mathbb{Z}_p$ -algebras of nilpotent class 2, which holds only under the assumption  $p > 2$ .

The statement of Theorem A is free from the restriction  $p \neq 2$ . Its proof follows mainly the strategy from [3] but there are several essential changes.

Firstly, instead of working with the ramification subgroups  $\Gamma_K(p)^{(v)}$ ,  $v \geq 0$ , we fix the simplest possible embedding of  $\Gamma_K(p)$  into its Magnus's algebra  $\mathcal{A}$  and study the induced filtration by the ideals  $\mathcal{A}^{(v)}$ ,  $v \geq 0$ , of  $\mathcal{A}$ . As a result, we obtain an explicit description of the ideals  $\mathcal{A}^{(v)} \bmod \mathcal{J}^3$ , where  $\mathcal{J}$  is the augmentation ideal in  $\mathcal{A}$ . This corresponds to the description of the groups  $\Gamma_K(p)^{(v)} \bmod C_3(\Gamma_K(p))$  in [1] but it is easier to obtain and it works for all prime numbers  $p$  including  $p = 2$ .

Secondly, any continuous group automorphism of  $\Gamma_K(p)$  which is compatible with the ramification filtration induces a continuous algebra automorphism  $f$  of  $\mathcal{A}$  such that for any  $v \geq 0$ ,  $f(\mathcal{A}^{(v)}) = \mathcal{A}^{(v)}$ . Similarly to [3], the conditions  $f(\mathcal{A}^{(v)}) \bmod \mathcal{J}^3 = \mathcal{A}^{(v)} \bmod \mathcal{J}^3$  imply non-trivial properties of the restriction of the original automorphism of  $\Gamma_K(p)$  to the inertia subgroup  $I_K(p)^{\text{ab}}$  of the Galois group of the maximal abelian extension of  $K$ . These properties are studied in detail in this paper. This allows us to give a more detailed and effective version of the final stage of the proof of the local analogue of the Grothendieck Conjecture even in the case  $p \neq 2$ . In particular, this clarifies why it holds with the absolute Galois groups replaced by the Galois groups of maximal  $p$ -extensions.

The methods of this paper can be helpful for understanding the relations between fields and their Galois groups in the context of the global Grothendieck Conjecture. For example, suppose  $F$  is an algebraic number field,  $\bar{F}$  is its algebraic closure,  $\Gamma_F = \text{Gal}(\bar{F}/F)$ ,  $\wp$  is a prime divisor in  $F$ ,  $\bar{\wp}$  is its extension to  $\bar{F}$  and  $F_\wp, \bar{F}_{\bar{\wp}}$  are the corresponding completions of  $F$  and  $\bar{F}$ , respectively. Then  $\Gamma_{F,\bar{\wp}} = \text{Gal}(\bar{F}_{\bar{\wp}}/F_\wp) \subset \Gamma_F$  is the decomposition group of  $\bar{\wp}$ . Suppose  $F$  is Galois over  $\mathbb{Q}$  and  $g_\wp : \Gamma_{F,\bar{\wp}} \rightarrow \Gamma_{F,\bar{\wp}}$  is a continuous group automorphism which is compatible with the ramification filtration on  $\Gamma_{F,\bar{\wp}}$ . By the local analogue of the Grothendieck Conjecture,  $g_\wp$  is induced by a field automorphism  $\bar{\mu}_\wp : \bar{F}_{\bar{\wp}} \rightarrow \bar{F}_{\bar{\wp}}$  such that  $\bar{\mu} := \bar{\mu}_\wp|_{\bar{F}}$  maps  $\bar{F}$  to  $\bar{F}$  (because  $\bar{\mu}(\mathbb{Q}) = \mathbb{Q}$ ), and, therefore,  $F$  to  $F$  (because  $F$  is Galois over  $\mathbb{Q}$ ). So,  $\bar{\mu}$  induces a group automorphism  $g$  of  $\Gamma_F$ , which extends the automorphism  $g_\wp$  of  $\Gamma_{F,\bar{\wp}}$ , and we obtain the following criterion:

**Criterion.** *A group automorphism  $g_\wp \in \text{Aut}\Gamma_{F,\bar{\wp}}$  can be extended to a group automorphism  $g \in \text{Aut}\Gamma_F$  if and only if  $g_\wp$  is compatible with the ramification filtration on  $\Gamma_{F,\bar{\wp}}$ .*

It would be interesting to understand how “global” information about the embedding of  $\Gamma_{F,\bar{\wp}}$  into  $\Gamma_F$  is reflected in “local” properties of the ramification filtration of  $\Gamma_{F,\bar{\wp}}$ .

Everywhere in the paper we use the following agreement about compositions of morphisms: if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are morphisms then their composition will be denoted by  $fg$ , in other words, if  $a \in A$  then  $(fg)(a) = g(f(a))$ . One of the reasons is that when operating on morphisms (rather than on their values in  $a \in A$ ) the notation  $fg$  reflects much better the reality that it is the composition of the first morphism  $f$  and the second one  $g$ .

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## 1. An analogue of the Magnus algebra for $\Gamma(p)$

In this section  $K = k((t_K))$  is the local field of formal Laurent series with residue field  $k = \mathbb{F}_{q_0}$ , where  $q_0 = p^{N_0}$ ,  $N_0 \in \mathbb{N}$ , and  $t_K$  is a fixed uniformiser of  $K$  (in most cases  $t_K$  will be denoted just by  $t$ ). We fix a choice of a separable closure  $K_{\text{sep}}$  of  $K$ , denote by  $K(p)$  the maximal  $p$ -extension of  $K$  in  $K_{\text{sep}}$  and set  $\Gamma = \text{Gal}(K_{\text{sep}}/K)$ ,  $\Gamma(p) = \text{Gal}(K(p)/K)$ .

**1.1. Liftings.** Notice first, that the uniformiser  $t_K$  of  $K$  can be taken as a  $p$ -basis for any finite extension  $L$  of  $K$  in  $K_{\text{sep}}$ . For  $M \in \mathbb{N}$ , set

$$O_M(L) = W_M(\sigma^{M-1}L)[t_{K,M}] \subset W_M(L),$$

where  $W_M$  is the functor of Witt vectors of length  $M$ ,  $\sigma$  is the  $p$ -th power map and  $t_{K,M} = [t_K] = (t_K, 0, \dots, 0) \in W_M(L)$  is the Teichmüller representative of  $t_K$ . Very often we shall use the simpler notation  $t$  for  $t_{K,M}$  (as well as for  $t_K$ ).  $O_M(L)$  is a lifting of  $L$  modulo  $p^M$  or, in other words, it is a flat  $W_M(\mathbb{F}_p)$ -module such that  $O_M(L) \bmod p = L$ . This is a special case of the construction of liftings in [4].

Let  $O_M(K_{\text{sep}})$  be the inductive limit of all  $O_M(L)$ , where  $L \subset K_{\text{sep}}$ ,  $[L : K] < \infty$ . Then we have a natural action of  $\Gamma$  on  $O_M(K_{\text{sep}})$  and  $O_M(K_{\text{sep}})^\Gamma = O_M(K) = W_M(k)((t))$ . We shall use again the notation  $\sigma$  for the natural extension of  $\sigma$  to  $O_M(K_{\text{sep}})$ . Clearly,  $O_M(K_{\text{sep}})|_{\sigma=\text{id}} = W_M(\mathbb{F}_p)$ . Introduce the absolute liftings  $O(K) = \varprojlim_M O_M(K)$  and  $O(K_{\text{sep}}) = \varprojlim_M O_M(K_{\text{sep}})$ . Again we have  $O(K_{\text{sep}})^\Gamma = O(K)$  and  $O(K_{\text{sep}})|_{\sigma=\text{id}} = W(\mathbb{F}_p)$ .

We can also consider the liftings  $O_M(K(p))$  and  $O(K(p))$  with the natural action of  $\Gamma(p)$  and similar properties.

Notice that for any  $j \in O(K(p))$  there is an  $i \in O(K(p))$  such that  $\sigma(i) - i = j$ .

**1.2. The algebra  $\mathcal{A}$ .** Set  $\mathbb{Z}(p) = \{a \in \mathbb{N} \mid (a, p) = 1\}$  and  $\mathbb{Z}^0(p) = \mathbb{Z}(p) \cup \{0\}$ . Let  $\mathcal{A}_k$  be the profinite associative  $W(k)$ -algebra with the set of pro-free generators  $\{D_{an} \mid a \in \mathbb{Z}(p), n \in \mathbb{Z} \bmod N_0\} \cup \{D_0\}$ .

This means that  $\mathcal{A}_k = \varprojlim_{C, M} \mathcal{A}_{CMk}$ , where  $C, M \in \mathbb{N}$ ,

$$\mathcal{A}_{CMk} = W_M(k)[[\{D_{an} \mid a \leq C, n \in \mathbb{Z} \bmod N_0\}]]$$

and the connecting morphisms  $\mathcal{A}_{C_1 M_1 k} \longrightarrow \mathcal{A}_{C_2 M_2 k}$  are defined for  $C_1 \geq C_2$ ,  $M_1 \geq M_2$  and induced by the correspondences  $D_{an} \mapsto 0$  if  $C_2 < a \leq C_1$  and  $D_{an} \mapsto D_{an}$  if  $a \leq C_2$ , and by the morphism  $W_{M_1}(k) \longrightarrow W_{M_2}(k)$  of reduction modulo  $p^{M_2}$ .

Denote again by  $\sigma$  the extension of the automorphism  $\sigma$  of  $W(k)$  to  $\mathcal{A}_k$  via the correspondences  $\sigma : D_{an} \mapsto D_{a, n+1}$ , where  $a \in \mathbb{Z}(p)$ ,  $n \in \mathbb{Z} \bmod N_0$ , and the correspondence  $D_0 \mapsto D_0$ . Then  $\mathcal{A} := \mathcal{A}_k|_{\sigma=\text{id}}$  is a pro-

free  $\mathbb{Z}_p$ -algebra: if  $\beta_1, \dots, \beta_{N_0}$  is a  $\mathbb{Z}_p$ -basis of  $W(k)$  and, for  $a \in \mathbb{Z}(p)$  and  $1 \leq r \leq N_0$ ,

$$D_a^{(r)} := \sum_{n \in \mathbb{Z} \bmod N_0} \sigma^n(\beta_r) D_{an},$$

then  $\{D_a^{(r)} \mid a \in \mathbb{Z}(p), 1 \leq r \leq N_0\} \cup \{D_0\}$  is a set of pro-free generators of  $\mathcal{A}$ . Notice also that if  $\alpha_1, \dots, \alpha_{N_0} \in W(k)$  is a dual basis for  $\beta_1, \dots, \beta_{N_0}$  (i.e.  $\text{Tr}(\alpha_i \beta_j) = \delta_{ij}$ , where  $1 \leq i, j \leq N_0$  and  $\text{Tr}$  is the trace of the field extension  $W(k) \otimes \mathbb{Q}_p$  over  $\mathbb{Q}_p$ ) then for any  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$ , it holds

$$D_{an} = \sum_{1 \leq r \leq N_0} \sigma^n(\alpha_r) D_a^{(r)}.$$

Denote by  $\mathcal{J}$ , resp.  $\mathcal{J}_{CM}$ , the augmentation ideal in  $\mathcal{A}$ , resp.  $\mathcal{A}_{CM}$ . Set  $\mathcal{A}_K := \mathcal{A} \hat{\otimes} O(K)$ ,  $\mathcal{A}_{CMK} = \mathcal{A}_{CM} \hat{\otimes} O(K)$ ,  $\mathcal{A}_{K(p)} = \mathcal{A} \hat{\otimes} O(K(p))$ . We shall also use similar notation in other cases of extensions of scalars, e.g.  $\mathcal{J}_k = \mathcal{J} \hat{\otimes} W(k)$ ,  $\mathcal{J}_K = \mathcal{J} \hat{\otimes} O(K)$ ,  $\mathcal{J}_{K(p)} = \mathcal{J} \hat{\otimes} O(K(p))$ .

**1.3. The embeddings  $\psi_f$ .** Take  $\alpha_0 \in W(k)$  such that  $\text{Tr}(\alpha_0) = 1$ , where again  $\text{Tr}$  is the trace of the field extension  $W(k) \otimes \mathbb{Q}_p \supset \mathbb{Q}_p$ . For all  $n \in \mathbb{Z} \bmod N_0$ , set  $D_{0n} = \sigma^n(\alpha_0) D_0$  and introduce the element

$$e = 1 + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0} \in 1 + \mathcal{J}_K.$$

We shall use the same notation  $e$  for the projections of  $e$  to any of  $\mathcal{A}_{CMK} \bmod \mathcal{J}_{CMK}^n$ , where  $C, M, n \in \mathbb{N}$ .

**Proposition 1.1.** *There is an  $f \in 1 + \mathcal{J}_{K(p)}$  such that  $\sigma(f) = fe$ .*

*Proof.* For  $C, M, n \in \mathbb{N}$ , set

$$S_{CMn} = \left\{ f \in 1 + \mathcal{J}_{CMK(p)} \bmod \mathcal{J}_{CMK(p)}^n \mid \sigma f = fe \bmod \mathcal{J}_{CMK(p)}^n \right\}.$$

We use induction on  $n \in \mathbb{N}$  to prove that for all  $C, M, n \in \mathbb{N}$ ,  $S_{CMn} \neq \emptyset$ .

Clearly,  $S_{CM1} = \{1\} \neq \emptyset$ .

Suppose that  $S_{CMn} \neq \emptyset$  and  $f \in S_{CMn}$ . Then  $\sigma(f) = fe \bmod \mathcal{J}_{CMK(p)}^n$ .

Let

$$\pi : 1 + \mathcal{J}_{CMK(p)} \bmod \mathcal{J}_{CMK(p)}^{n+1} \longrightarrow 1 + \mathcal{J}_{CMK(p)} \bmod \mathcal{J}_{CMK(p)}^n$$

be the natural projection. If  $f_1 \in 1 + \mathcal{J}_{CMK(p)} \bmod \mathcal{J}_{CMK(p)}^{n+1}$  is such that  $\pi(f_1) = f$  then  $\sigma(f_1) = f_1 e + j \bmod \mathcal{J}_{CMK(p)}^{n+1}$ , where  $j \in \mathcal{J}_{CMK(p)}^n$ . There is an  $i \in \mathcal{J}_{CMK(p)}^n$  such that  $\sigma(i) - i = j$ , cf. n.1.1. Therefore,

$$\sigma(f_1 - i) = f_1 e + j - (i + j) = (f_1 - i) e \bmod \mathcal{J}_{CMK(p)}^{n+1},$$

using that  $ie = i \bmod \mathcal{J}_{CMK(p)}^{n+1}$ , and  $S_{CM, n+1} \neq \emptyset$  because it contains  $f_1 - i$ .

Notice that each  $S_{CMn}$  is a finite set and each  $f \in S_{CMn}$  has a finite field of definition. This follows from the fact that for any  $C, M, n \in \mathbb{N}$ , the  $\mathbb{Z}_p$ -module  $\mathcal{A}_{CM} \bmod \mathcal{J}_{CM}^n$  has finitely many free generators and, therefore, the equation  $\sigma f = fe$  is equivalent to finitely many usual polynomial equations. Also notice that  $\{S_{CMn} \mid C, M, n \in \mathbb{N}\}$  has a natural structure of projective system. Therefore,  $\varprojlim_{C, M, n} CMn \neq \emptyset$ , and any element  $f$  of this projective limit satisfies  $f \in 1 + \mathcal{J}_{K(p)}$  and  $\sigma(f) = fe$ .

The proposition is proved.  $\square$

For any  $f \in 1 + \mathcal{J}_{K(p)}$  such that  $\sigma(f) = fe$  and  $\tau \in \Gamma(p)$ , set  $\psi_f(\tau) = (\tau f)f^{-1}$ . Clearly,  $\sigma(\psi_f(\tau)) = \tau(\sigma f)(\sigma f)^{-1} = (\tau f)ee^{-1}f = \psi_f(\tau)$ . Therefore,  $\psi_f(\tau) \in (1 + \mathcal{J}_{K(p)})|_{\sigma=\text{id}} = 1 + \mathcal{J}$ .

**Proposition 1.2.** a)  $\psi_f$  is a closed group embedding of  $\Gamma(p)$  into  $(1 + \mathcal{J})^\times$ .  
 b)  $\psi_f$  induces an isomorphism  $\psi_f^{\text{ab}}$  of the topological groups  $\Gamma(p)^{\text{ab}}$  and  $(1 + \mathcal{J})^\times \bmod \mathcal{J}^2$ .  
 c) If  $f_1 \in 1 + \mathcal{J}_{K(p)}$  is such that  $\sigma(f_1) = f_1e$  then there is an element  $c \in 1 + \mathcal{J}$  such that for any  $\tau \in \Gamma(p)$ ,  $\psi_{f_1}(\tau) = c\psi_f(\tau)c^{-1}$ .  
 d)  $\psi_f$  induces an embedding of the group of all continuous automorphisms  $\text{Aut}\Gamma(p)$  into the group  $\text{Aut}\mathcal{A}$  of continuous automorphisms of the  $\mathbb{Z}_p$ -algebra  $\mathcal{A}$ .

*Proof.* a) Clearly,  $\psi_f$  can be treated as a pro- $p$ -version of the embedding of the group  $\Gamma(p)$  into its Magnus algebra. Therefore, by [8], Ch 1, Sec 6,  $\psi_f$  induces, for all  $n \in \mathbb{N}$ , the closed embeddings of the quotients  $C_n(\Gamma(p))/C_{n+1}(\Gamma(p))$  of commutator subgroups in  $\Gamma(p)$  into  $1 + \mathcal{J}^n \bmod \mathcal{J}^{n+1}$ . This implies that  $\psi_f$  induces, for all  $n \geq 1$ , the closed group embeddings of  $\Gamma(p)/C_n(\Gamma(p))$  into  $1 + \mathcal{J} \bmod \mathcal{J}^n$ , and therefore,  $\psi_f$  is a closed group monomorphism.

b) Consider the profinite  $\mathbb{Z}_p$ -basis  $\{D_a^{(r)} \mid a \in \mathbb{Z}(p), 1 \leq r \leq N_0\} \cup \{D_0\}$  for  $\mathcal{J} \bmod \mathcal{J}^2$  from n.1.2. For  $1 \leq r \leq N_0$ , as earlier, consider  $\alpha_r \in W(k)$ , which form the dual basis of the basis  $\{\beta_r \mid 1 \leq r \leq N_0\}$  chosen in n.1.2 to define the generators  $D_a^{(r)}$ . Then

$$e = 1 + \sum_{1 \leq r \leq N_0, a \in \mathbb{Z}(p)} \alpha_r t^{-a} D_a^{(r)} + \alpha_0 D_0$$

and

$$f = 1 + \sum_{1 \leq r \leq N_0, a \in \mathbb{Z}(p)} f_a^{(r)} D_a^{(r)} + f_0 D_0 \bmod \mathcal{J}_{K(p)}^2,$$

where for  $1 \leq r \leq N_0$  and  $a \in \mathbb{Z}(p)$ ,  $f_a^{(r)}$  and  $f_0$  belong to  $O(K(p)) \subset W(K(p))$  and satisfy the equations  $\sigma f_a^{(r)} - f_a^{(r)} = \alpha_r t^{-a}$  and  $\sigma f_0 - f_0 = \alpha_0$ .

Then for any  $\tau \in \Gamma(p)$ ,

$$\psi_f(\tau) = 1 + \sum_{a,r} (\tau f_a^{(r)} - f_a^{(r)}) D_a^{(r)} + (\tau f_0 - f_0) D_0 \bmod \mathcal{J}_{K(p)}^2$$

and the identification  $\psi_f : \Gamma(p)^{\text{ab}} \simeq (1 + \mathcal{J})^\times \bmod \mathcal{J}^2$  is equivalent to the identifications of Witt-Artin-Schreier theory

$$\bigoplus_{a \in \mathbb{Z}(p)} W(k)t^{-a} \oplus W(\mathbb{F}_p)\alpha_0 = O(K)/(\sigma - \text{id})O(K) = \text{Hom}_{\text{cts}}(\Gamma(p), W(\mathbb{F}_p)).$$

c) Clearly,  $\sigma(f_1 f^{-1}) = \sigma(f_1)\sigma(f)^{-1} = f_1 e e^{-1} f^{-1} = f_1 f^{-1}$ . Therefore,

$$f_1 f^{-1} = c \in (1 + \mathcal{J}_{K(p)}) \cap \mathcal{A} = 1 + \mathcal{J}$$

and for any  $\tau \in \Gamma(p)$ ,

$$\psi_{f_1}(\tau) = \tau(f_1) f_1^{-1} = \tau(cf)(cf)^{-1} = c(\tau f) f^{-1} c^{-1} = c \psi_f(\tau) c^{-1}.$$

d) This also follows from the above mentioned interpretation of  $\mathcal{A}$  as a profinite analogue of the Magnus algebra for  $\Gamma(p)$ .  $\square$

**1.4. The identification  $\psi_f^{\text{ab}}$ .** As it was already mentioned in the proof of proposition 1.2 the identification  $\psi_f^{\text{ab}}$  comes from the isomorphism of Witt-Artin-Schreier theory

$$\Gamma(p)^{\text{ab}} = \text{Hom}(O(K)/(\sigma - \text{id})O(K), W(\mathbb{F}_p))$$

and does not depend on the choice of  $t = t_K$  and  $f \in 1 + \mathcal{J}_{K(p)}$ . Suppose  $\tau_0 \in \Gamma(p)^{\text{ab}}$  is such that  $\psi_f^{\text{ab}}(\tau_0) = 1 + D_0$  and for  $a \in \mathbb{Z}(p)$  and  $1 \leq r \leq N_0$ , the elements  $\tau_a^{(r)} \in \Gamma(p)^{\text{ab}}$  are such that  $\psi_f^{\text{ab}}(\tau_a^{(r)}) = 1 + D_a^{(r)} \bmod \mathcal{J}^2$ . Then the element

$$e = 1 + \alpha_0 D_0 + \sum_{a,r} \alpha_r t^{-a} D_a^{(r)}$$

corresponds to the diagonal element  $\alpha_0 \otimes \tau_0 + \sum_{a,r} \alpha_r t^{-a} \otimes \tau_a^{(r)}$  from  $O(K) \otimes \Gamma(p)^{\text{ab}} =$

$$O(K) \otimes \text{Hom}(O(K)/(\sigma - \text{id})O(K), \mathbb{Z}_p) = \text{Hom}(O(K)/(\sigma - \text{id})O(K), O(K)),$$

which comes from the following natural embedding

$$O(K)/(\sigma - \text{id})O(K) = \bigoplus_{a \in \mathbb{Z}(p)} W(k)t^{-a} \oplus W(\mathbb{F}_p)\alpha_0 \subset O(K).$$

The above elements  $\tau_0$ , resp.  $\tau_a^{(r)}$ , correspond to  $t$ , resp.  $E(\beta_r, t^a)^{1/a}$ , by the reciprocity map of local class field theory. (Here  $\beta_1, \dots, \beta_{N_0} \in W(k)$  were chosen in n.1.2 and for  $\beta \in W(k)$ ,

$$E(\beta, X) = \exp(\beta X + (\sigma\beta)X^p/p + \dots + (\sigma^n\beta)X^{p^n}/p^n + \dots) \in W(k)[[X]]$$



is the generalisation of the Artin-Hasse exponential introduced by Shafarevich [9].) This fact follows from the Witt explicit reciprocity law, cf. [5]. Then the elements  $D_{an}$ , where  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$ , correspond to

$$\sum_{1 \leq r \leq N_0} \sigma^n(\alpha_r) \otimes E(\beta_r, t^a)^{1/a} \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{G}_a,$$

where the (multiplicative) group  $\mathcal{G}_a := \{E(\gamma, t^a) \mid \gamma \in W(k)\}$  is identified with the  $\mathbb{Z}_p$ -module of Witt vectors  $W(k)$  via the map  $E(\gamma, t^a)^{1/a} \mapsto \gamma$ . Consider the identification

$$W(k) \otimes_{\mathbb{Z}_p} W(k) = \bigoplus_{m \in \mathbb{Z} \bmod N_0} W(k)_m$$

given by the correspondence  $\alpha \otimes \beta \mapsto \{\sigma^{-m}(\alpha)\beta\}_{m \in \mathbb{Z} \bmod N_0}$ . Under this identification the element  $D_{an}$  corresponds to the vector  $\delta_n \in \bigoplus_m W(k)_m$ , which has  $n$ -th coordinate 1 and all remaining coordinates 0. This interpretation of the generators  $D_{an}$  will be applied below in the following situation. Suppose  $[k' : k] < \infty$ ,  $k' \simeq \mathbb{F}_{q'_0}$  with  $q'_0 = p^{N'_0}$ . Clearly,  $N'_0 \equiv 0 \bmod N_0$ . For  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N'_0$  denote by  $D'_{an}$  an analogue of  $D_{an}$  constructed for  $K' = k'((t_{K'}))$  with  $t_{K'} = t$ . Let  $\Gamma' = \text{Gal}(K_{\text{sep}}/K')$  and let  $\Gamma'(p)$  be the Galois group of the maximal  $p$ -extension  $K'(p)$  of  $K'$  in  $K_{\text{sep}}$ . With the above notation we have the following property:

**Proposition 1.3.** *For any  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N'_0$ ,  $D'_{an}$  is mapped to  $D_{a, n \bmod N_0}$  under the map  $\Gamma'(p)^{\text{ab}} \rightarrow \Gamma(p)^{\text{ab}}$ , which is induced by the natural embedding  $\Gamma' \subset \Gamma$ .*

## 2. Action of analytic automorphisms on $I^{\text{ab}}(p)$

As earlier,  $K = k((t))$ ,  $k \simeq \mathbb{F}_{q_0}$  with  $q_0 = p^{N_0}$  and  $\Gamma(p) = \text{Gal}(K(p)/K)$ . Let  $I(p)$  be the inertia subgroup of  $\Gamma(p)$  and let  $I(p)^{\text{ab}}$  be its image in the maximal abelian quotient  $\Gamma(p)^{\text{ab}}$  of  $\Gamma(p)$ .

**2.1.** Consider the group  $\text{Aut}K$  of continuous field automorphisms of  $K$ . Let  $\text{Fr}(t) \in \text{Aut}K$  be such that  $\text{Fr}(t)|_k = \sigma$  and  $\text{Fr}(t) : t \mapsto t$ . Then any element of  $\text{Aut}K$  is the composition of a power  $\text{Fr}(t)^n$ , where  $n \in \mathbb{Z} \bmod N_0$ , and a field automorphism from  $\text{Aut}^0(K) := \{\eta \in \text{Aut}K \mid \eta|_k = \text{id}\}$ . Notice that any  $\eta \in \text{Aut}^0 K$  is uniquely determined by the image  $\eta(t)$  of  $t$ , which is again a uniformizer in  $K$ .

Let  $\text{Aut}_K K(p)$  be the group of continuous automorphisms  $\bar{\eta}$  of  $K(p)$  such that  $\bar{\eta}|_K \in \text{Aut}K$ . Then  $\text{Aut}_K K(p)$  acts on  $\Gamma(p)$ : if  $\bar{\eta} \in \text{Aut}_K K(p)$  and  $\tau \in \Gamma(p)$  then the action of  $\bar{\eta}$  is given by the correspondence  $\tau \mapsto \bar{\eta}^*(\tau) = \bar{\eta}^{-1}\tau\bar{\eta}$ , i.e.  $\bar{\eta}^*(\tau) : K(p) \xrightarrow{\bar{\eta}^{-1}} K(p) \xrightarrow{\tau} K(p) \xrightarrow{\bar{\eta}} K(p)$ , cf. the introduction for the agreement about compositions of maps. The action

induced by  $\bar{\eta}^* \in \text{Aut}_K K(p)$  on  $\Gamma(p)^{\text{ab}}$  depends only on  $\eta := \bar{\eta}|_K$  and will be denoted simply by  $\eta^*$ .

**2.2.** Let  $\mathcal{M} = I(p)^{\text{ab}} \otimes \mathbb{F}_p$ . If  $U_K$  is the group of principal units in  $K$  then we shall use the identification  $\mathcal{M} = U_K/U_K^p$ , which is given by the reciprocity map of local class field theory. Notice that, with respect of this identification, for any  $\eta \in \text{Aut}K$ , the action  $\eta^*$  comes from the natural action of  $\eta$  on  $K$ . We shall denote the  $k$ -linear extension of the action of  $\eta$  to  $\mathcal{M}_k := \mathcal{M} \otimes_{\mathbb{F}_p} k$  by the same symbol  $\eta^*$ .

Use the map  $m \mapsto (\psi_f^{\text{ab}}(m) - 1) \bmod p$  to identify  $\mathcal{M}_k$  with a submodule of  $\mathcal{J}_k \bmod(p, \mathcal{J}_k^2)$ . For  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$ , consider the images of the elements  $D_{an}$ , where  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$  (cf. n.1), in  $\mathcal{J}_k \bmod(p, \mathcal{J}_k^2)$ . Denote these images by same symbols. Then they give a set of free topological generators of the  $k$ -module  $\mathcal{M}_k$ . The action of  $\eta \in \text{Aut}K$  on  $\mathcal{M}_k$  in terms of these generators is as follows.

**Proposition 2.1.** 1)  $\text{Fr}(t)^*(D_{an}) = D_{a,n-1}$ ;  
 2) if  $\eta \in \text{Aut}^0 K$ , then

$$\sum_{a \in \mathbb{Z}(p)} t^{-a} \eta^*(D_{a0}) \equiv \sum_{a \in \mathbb{Z}(p)} \eta^{-1}(t)^{-a} D_{a0} \bmod(k + (\sigma - \text{id})K) \otimes \mathcal{M}.$$

*Proof.* 1) Consider the generators  $\alpha_r D_a^{(r)}$  of  $\mathcal{A}$  from n.1.2, where  $a \in \mathbb{Z}(p)$ ,  $1 \leq r \leq N_0$ . Note that the residue of the corresponding element  $e - 1$  modulo  $(\sigma - \text{id})K \otimes (\mathcal{J} \bmod \mathcal{J}^2)$  does not depend on the choice of  $t$  or of the elements  $\alpha_1, \alpha_2, \dots, \alpha_{N_0}$ , because this is the diagonal element of Artin-Schreier duality. Therefore, if  $\text{Fr}(t)^*(D_a^{(r)}) = D_a'^{(r)}$  and  $\text{Fr}(t)^*(D_0) = D_0'$  then

$$\begin{aligned} (2.1) \quad e - 1 &\equiv \sigma(\alpha_0) \otimes D_0' + \sum_{a,r} \sigma(\alpha_r) t^{-a} \otimes D_a'^{(r)} \\ &\equiv \alpha_0 \otimes D_0 + \sum_{a,r} \alpha_r t^{-a} \otimes D_a^{(r)} \bmod(\sigma - \text{id})K \otimes (\mathcal{J} \bmod \mathcal{J}^2). \end{aligned}$$

So, for any  $a \in \mathbb{Z}(p)$ , we see that in  $k \otimes_{\mathbb{F}_p} \mathcal{M} = \mathcal{M}_k$

$$D_{a0} = \sum_r \alpha_r \otimes D_a^{(r)} = \sum_r \sigma(\alpha_r) \otimes D_a'^{(r)}.$$

Denoting the  $k$ -linear extension of  $\text{Fr}(t)^*$  by the same symbol, as usual, we have

$$\text{Fr}(t)^*(D_{a0}) = \sum_r \alpha_r \otimes \text{Fr}(t)^*(D_a^{(r)}) = \sum_r \alpha_r \otimes D_a'^{(r)} = \sigma^{-1} D_{a0} = D_{a,-1}.$$

Therefore, for any  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$ ,  $\text{Fr}(t)^*(D_{an}) = D_{a,n-1}$ . Notice also that congruence (2.1) implies that  $\text{Fr}(t)^* D_0 = D_0$ .

2) Using that  $\eta$  is a  $k$ -linear automorphism of  $K$  and proceeding similarly to the above part 1) we obtain that

$$\sum_{a \in \mathbb{Z}(p)^0} \eta(t)^{-a} \eta^*(D_{a0}) \equiv \sum_{a \in \mathbb{Z}(p)^0} t^{-a} D_{a0} \pmod{(\sigma - \text{id})K \otimes \mathcal{M}}.$$

Now apply  $(\eta^{-1} \otimes \text{id})$  to both sides of this congruence and notice that we can omit the terms with index  $a = 0$  when working modulo  $(k + (\sigma - \text{id})K) \otimes \mathcal{M}$ , because they belong to  $\mathcal{M}_k$ . The lemma is proved.  $\square$

**2.3.** If  $f$  is a continuous automorphism of the  $\mathbb{F}_p$ -module  $\mathcal{M}$ , we agree to use the same notation  $f$  for its  $k$ -linear extension to an automorphism of  $\mathcal{M}_k$ . For any  $a \in \mathbb{Z}(p)$ , set

$$f(D_{a0}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ n \in \mathbb{Z} \bmod N_0}} \alpha_{abn}(f) D_{bn}.$$

Then all coefficients  $\alpha_{abn}(f)$  are in  $k$ . Sometimes we shall use the notation  $\alpha_{abn}(f)$  if  $a$  or  $b$  are divisible by  $p$ , then it is assumed that  $\alpha_{abn}(f) = 0$ . Notice that for any  $m \in \mathbb{Z} \bmod N_0$ ,

$$f(D_{am}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ n \in \mathbb{Z} \bmod N_0}} \sigma^m(\alpha_{abn}(f)) D_{b,n+m}.$$

**Definition.** For any  $v \in \mathbb{N}$ , let  $\mathcal{M}^{(v)}$  be the minimal closed  $\mathbb{F}_p$ -submodule in  $\mathcal{M}$  such that  $\mathcal{M}_k^{(v)} := \mathcal{M}^{(v)} \otimes k$  is topologically generated over  $k$  by all  $D_{an}$ , where  $a \in \mathbb{Z}(p)$ ,  $a \geq v$  and  $n \in \mathbb{Z} \bmod N_0$ . (Notice that  $\mathcal{M} = \mathcal{M}^{(1)}$ .)

**Definition.**  $\text{Aut}_{\text{adm}} \mathcal{M}$  is the subset in the group  $\text{Aut} \mathcal{M}$ , consisting of all continuous  $\mathbb{F}_p$ -linear automorphisms  $f$  satisfying  $\alpha_{a,b,m \bmod N_0}(f) = 0$  if  $bp^m < a$ , for any  $a, b \in \mathbb{Z}(p)$  and  $-N_0 < m \leq 0$ .

It is easy to see that:

- (1)  $\text{Aut}_{\text{adm}} \mathcal{M}$  is a subgroup of  $\text{Aut} \mathcal{M}$ ;
- (2) if  $f \in \text{Aut}_{\text{adm}} \mathcal{M}$  then for any  $a \in \mathbb{N}$ ,  $f(\mathcal{M}^{(a)}) \subset \mathcal{M}^{(a)}$ , i.e.  $f$  is compatible with the image of the ramification filtration in  $\mathcal{M}$ ;
- (3) if  $f \in \text{Aut}_{\text{adm}} \mathcal{M}$  then for any  $a \in \mathbb{Z}(p)$ ,  $\alpha_{aa0} \in k^*$  and  $\alpha_{aan}(f) = 0$  if  $n \neq 0$ .

**Definition.** For  $f \in \text{Aut} \mathcal{M}$ , let  $f_{\text{an}} \in \text{End} \mathcal{M}$  be such that for all  $a \in \mathbb{Z}(p)$ ,

$$f_{\text{an}}(D_{a0}) = \sum_{b \in \mathbb{Z}(p)} \alpha_{ab0}(f) D_{b0}.$$

**Proposition 2.2.** *If  $f, g \in \text{Aut}_{\text{adm}}\mathcal{M}$  then for any  $a, b \in \mathbb{Z}(p)$  such that  $a \leq b < ap^{N_0}$ ,*

$$\alpha_{ab0}(fg) = \sum_c \alpha_{ac0}(f)\alpha_{cb0}(g).$$

**Corollary 2.3.** *If  $v < p^{N_0}$  then the correspondence  $f \mapsto f_{\text{an}}$  is a group homomorphism from  $\text{Aut}_{\text{adm}}\mathcal{M}$  to  $\text{Aut}_{\text{adm}}\mathcal{M} \bmod \mathcal{M}^{(v)}$ .*

*Proof.* We have  $\alpha_{ab0}(fg) =$

$$\sum_{\substack{m+n \equiv 0 \bmod N_0 \\ 0 \geq n, m > -N_0}} \alpha_{a,c,n \bmod N_0}(f) \sigma^n(\alpha_{c,b,m \bmod N_0}(g)) D_{b,(m+n) \bmod N_0}.$$

Then  $\alpha_{a,c,n \bmod N_0}(f) \neq 0$  implies that  $cp^n \geq a$  and  $\alpha_{c,b,m \bmod N_0}(g) \neq 0$  implies that  $bp^m \geq c$ . So, if the corresponding coefficient for  $D_{b,(m+n) \bmod N_0}$  is not zero then  $bp^{m+n} \geq a$ , i.e.  $m+n > -N_0$  and, therefore,  $m = n = 0$ .  $\square$

The following proves that  $\text{Aut}^0 K \subset \text{Aut}_{\text{adm}}\mathcal{M}$ .

**Proposition 2.4.** *If  $\eta \in \text{Aut}^0 K$  then  $\eta^* \in \text{Aut}_{\text{adm}}\mathcal{M}$ .*

*Proof.* For  $a \in \mathbb{Z}(p)$ , set

$$\eta^{-1}(t)^{-a} \equiv \sum_{\substack{b \in \mathbb{Z}(p) \\ s \geq 0}} \gamma_{abs} t^{-bp^s} \bmod k[[t]].$$

Clearly,  $\gamma_{abs} = 0$  if  $bp^s > a$ . It follows from part 2) of proposition 2.1 that

$$\eta^*(D_{b0}) = \sum_{\substack{a \in \mathbb{Z}(p) \\ s \geq 0}} \sigma^{-s}(\gamma_{abs}) D_{a,-s \bmod N_0}.$$

Therefore, for  $0 \leq m < N_0$ ,

$$\alpha_{b,a,-m \bmod N_0}(\eta^*) = \sum_{\substack{s \equiv m \bmod N_0 \\ s \geq 0}} \sigma^{-s}(\gamma_{abs})$$

and  $a/p^m < b$  implies for  $s \equiv m \bmod N_0$ ,  $s \geq 0$ , that  $a/p^s < b$ . So,  $bp^s > a$ ,  $\gamma_{abs} = 0$  and  $\alpha_{b,a,-m \bmod N_0}(\eta^*) = 0$ .

The proposition is proved.  $\square$

**2.4.** In this subsection we prove three technical propositions. Notice that in proposition 2.5 we treat the case of fields of characteristic  $p \neq 2$  and in proposition 2.6 the characteristic of  $K$  is 2. Propositions 2.5-2.7 will be used later in section 5. If  $a, b \in \mathbb{N}$  then  $\delta_{ab}$  is the Kronecker symbol.

**Proposition 2.5.** *Suppose  $p \neq 2$ ,  $w_0 \in \mathbb{N}$ ,  $w_0 + 1 \leq p^{N_0}$  and  $f \in \text{Aut}_{\text{adm}}\mathcal{M}$  is such that  $\alpha_{1a0}(f) = \delta_{1a}$  if  $1 \leq a < w_0$  and  $\alpha_{2a0}(f) = 0$  if  $a \equiv 1 \bmod p$  and  $a \leq w_0$ . Then there is an  $\eta \in \text{Aut}^0 K$  such that  $\eta(t) \equiv t \bmod t^{w_0}$ ,  $\alpha_{1a0}(f\eta^*) = \delta_{1a}$  if  $1 \leq a < w_0 + 1$ , and  $\alpha_{2a0}(f\eta^*) = 0$  if  $a \equiv 1 \bmod p$  and  $a \leq w_0 + 1$ .*

*Proof.* Take  $\eta \in \text{Aut}^0 K$  such that  $\eta^{-1}(t) = t(1 + \gamma t^{w_0-1})$  with  $\gamma \in k$ . Then for any  $a \in \mathbb{Z}(p)$ ,  $\eta^{-1}(t^{-a}) = t^{-a}(1 - a\gamma t^{w_0-1}) \pmod{t^{-a+w_0}}$ , and part 2) of proposition 2.1 implies that  $\alpha_{aa0}(\eta^*) = 1$ ,  $\alpha_{ab0}(\eta^*) = 0$  if  $a < b < a + w_0 - 1$ ,  $\alpha_{a,a+w_0-1,0}(\eta^*) = -(a + w_0 - 1)\gamma$ .

Therefore, by proposition 2.2  $\alpha_{1a0}(f\eta^*) = \delta_{1a}$  if  $1 \leq a < w_0$  and  $\alpha_{2a0}(f\eta^*) = 0$  if  $a \equiv 1 \pmod{p}$ ,  $a \leq w_0$ .

Suppose  $w_0 \not\equiv 0 \pmod{p}$ . Then by proposition 2.2

$$\alpha_{1w_00}(f\eta^*) = -w_0\gamma + \alpha_{1w_00}(f) = 0$$

if  $\gamma = w_0^{-1}\alpha_{1w_00}(f)$ . This proves the proposition in the case  $w_0 \not\equiv 0 \pmod{p}$ , because  $w_0 + 1 \not\equiv 1 \pmod{p}$  and no conditions are required for  $\alpha_{2,w_0+1,0}(f\eta^*)$ .

Suppose  $w_0 \equiv 0 \pmod{p}$ . Then there are no conditions for  $\alpha_{1w_00}(f\eta^*)$  and by proposition 2.2

$$\begin{aligned} \alpha_{2,w_0+1,0}(f\eta^*) &= \alpha_{220}(f)\alpha_{2,w_0+1,0}(\eta^*) + \alpha_{2,w_0+1,0}(f)\alpha_{w_0+1,w_0+1,0}(\eta^*) \\ &= -\alpha_{220}(f)\gamma + \alpha_{2,w_0+1,0}(f) = 0 \end{aligned}$$

if  $\gamma = \alpha_{2,w_0+1,0}(f)\alpha_{220}(f)^{-1}$ . (Using that  $f \in \text{Aut}_{\text{adm}}\mathcal{M}$  hence  $\alpha_{220}(f) \in k^*$ .)

The proposition is proved.  $\square$

**Proposition 2.6.** *Let  $M \in \mathbb{N}$ ,  $p = 2$ ,  $w_0 = 4M$  and  $w_0 + 1 < 2^{N_0}$ . Suppose  $f \in \text{Aut}_{\text{adm}}\mathcal{M}$  is such that  $\alpha_{1a0}(f) = \delta_{1a}$  if  $1 \leq a \leq w_0 - 3$  and  $\alpha_{3a0}(f) = \delta_{3a}$  if  $3 \leq a \leq w_0 - 1$ . Then there is an  $\eta \in \text{Aut}^0 K$  such that  $\alpha_{1a0}(f\eta^*) = \delta_{1a}$  and  $\alpha_{3a0}(f\eta^*) = \delta_{3a}$  if  $a \leq w_0 + 1$ .*

*Proof. 1st step.*

Take  $\eta_1 \in \text{Aut}^0 K$  such that  $\eta_1^{-1}(t) = t(1 + \gamma_1 t^{4M-2})$  with  $\gamma_1 \in k$ . Then for  $a \in \mathbb{Z}(2)$ ,  $\eta_1^{-1}(t^{-a}) \equiv t^{-a}(1 + \gamma_1 t^{4M-2}) \pmod{t^{-a+4M}}$  and by part 2) of proposition 2.1,  $\alpha_{aa0}(\eta_1^*) = 1$ ,  $\alpha_{ab0}(\eta_1^*) = 0$  if  $a < b < a + 4M - 2$ , and  $\alpha_{a,a+4M-2,0}(\eta_1^*) = \gamma_1$ .

So by proposition 2.2,  $\alpha_{1a0}(f\eta_1^*) = \alpha_{1a0}(f)$  if  $a \leq 4M - 3 = w_0 - 3$ ,  $\alpha_{3a0}(f\eta_1^*) = \alpha_{3a0}(f)$  if  $a \leq 4M - 1 = w_0 - 1$ ,  $\alpha_{1,w_0-1,0}(f\eta_1^*) = \alpha_{1,w_0-1,0}(f) + \alpha_{1,w_0-1,0}(\eta_1^*) = 0$  if  $\gamma_1 = \alpha_{1,w_0-1,0}(f)$ .

*2nd step.*

By the above first step we can now assume that  $\alpha_{1,w_0-1,0}(f) = 0$ .

Take  $\eta_2 \in \text{Aut}^0 K$  such that  $\eta_2^{-1}(t) = t(1 + \gamma_2 t^{2M-1})$ . Then for  $a \in \mathbb{Z}(2)$ ,  $\eta_2^{-1}(t^{-a}) \equiv t^{-a}(1 + \gamma_2 t^{2M-1} + \delta(a)\gamma_2^2 t^{4M-2}) \pmod{t^{-a+4M}}$ , where  $\delta(a) = a(a+1)/2$ .

So by part 2) of proposition 2.1,  $\alpha_{aa0}(\eta_2^*) = 1$ ,  $\alpha_{ab0}(\eta_2^*) = 0$  if  $a < b < a + 4M - 2$  (notice that  $-a + 2M - 1 \equiv 0 \pmod{2}$ ), and  $\alpha_{a,a+4M-2,0}(\eta_2^*) = \delta(a + 4M - 2)\gamma_2^2$  (notice that  $\delta(a + 4M - 2) = 0$  if  $a \equiv 1 \pmod{4}$  and  $\delta(a + 4M - 2) = 1$  if  $a \equiv 3 \pmod{4}$ ).

Again by proposition 2.2,  $\alpha_{1a0}(f\eta_2^*) = \alpha_{1a0}(f)$  if  $a \leq 4M - 1 = w_0 - 1$  (use that  $\alpha_{1,w_0-1,0}(f) = \alpha_{1,w_0-1,0}(\eta_2^*) = 0$ ),  $\alpha_{3a0}(f\eta_2^*) = \alpha_{3a0}(f)$  if  $a \leq 4M - 1 = w_0 - 1$ ,  $\alpha_{3,w_0+1,0}(f\eta_2^*) = \alpha_{3,w_0+1,0}(f) + \alpha_{3,w_0+1,0}(\eta_2^*) = 0$  if  $\gamma_2 \in k$  is such that  $\gamma_2^2 = \alpha_{3,w_0+1,0}(f)$ .

*3rd step.*

Now we can assume that  $\alpha_{1,w_0-1,0}(f) = \alpha_{3,w_0+1,0}(f) = 0$ .

Take  $\eta_3 \in \text{Aut}^0 K$  such that  $\eta_3^{-1}(t) = t(1 + \gamma_3 t^{4M})$ . Then for  $a \in \mathbb{Z}(2)$ ,  $\eta_3^{-1}(t^{-a}) \equiv t^{-a}(1 + \gamma_3 t^{4M}) \pmod{t^{-a+4M+2}}$ ,  $\alpha_{aa0}(\eta_3^*) = 1$ ,  $\alpha_{ab0}(\eta_3^*) = 0$  if  $a < b < a + 4M$ , and  $\alpha_{a,a+4M,0}(\eta_3^*) = \gamma_3$ .

This implies that  $\alpha_{1a0}(f\eta_3^*) = \alpha_{1a0}(f)$  if  $a \leq 4M - 1 = w_0 - 1$ ,  $\alpha_{1,w_0+1,0}(f\eta_3^*) = \alpha_{1,w_0+1,0}(f) + \alpha_{1,w_0+1,0}(\eta_3^*) = 0$  if  $\gamma_3 = \alpha_{1,w_0+1,0}(f)$  and  $\alpha_{3a0}(f\eta_3^*) = \alpha_{3a0}(f)$  if  $a \leq w_0 + 1$ .

The proposition is proved.  $\square$

**Proposition 2.7.** *Suppose  $a \in \mathbb{Z}(p)$ ,  $w_0 \leq ap^{N_0}$ , where  $w_0 \in p\mathbb{N}$ ,  $w_0 > a + 1$  if  $p \neq 2$  and  $w_0 \in 4\mathbb{N}$ ,  $w_0 > a + 2$  if  $p = 2$ . Suppose  $\eta, \eta_1 \in \text{Aut}^0 K$  are such that for any  $b, c \in \mathbb{Z}(p)$  satisfying the restrictions  $a \leq c \leq b < w_0 \leq ap^{N_0}$ , we have the equality*

$$\alpha_{cb0}(\eta^*) = \alpha_{cb0}(\eta_1^*).$$

Then  $\eta(t) \equiv \eta_1(t) \pmod{t^{v_0}}$ , where  $v_0 = w_0 - a + 1$  if  $p \neq 2$  and  $v_0 = (w_0 - a + 1)/2$  if  $p = 2$ .

**Remark.** With notation from Subsection 2.3 this proposition implies that if  $\eta_{1\text{an}}^* \equiv \eta_{\text{an}}^* \pmod{\mathcal{M}^{(w_0)}}$  then  $\eta(t) \equiv \eta_1(t) \pmod{t^{v_0}}$ .

*Proof.* Use proposition 2.2 to reduce the proof to the case  $\eta_1(t) = t$ .

Suppose, first, that  $\eta^{-1}(t) = \alpha t \pmod{t^2}$ . Then

$$(2.2) \quad \alpha_{cc0}(\eta^*) = \alpha^{-c} = 1.$$

If  $a + 1 \in \mathbb{Z}(p)$  then  $p \neq 2$  and we can use formula (2.2) for  $c = a, a + 1$  to prove that  $\alpha = 1$ . Suppose  $a + 1 \notin \mathbb{Z}(p)$ . If  $p = 2$  use (2.2) for  $c = a, a + 2 < w_0$ , and if  $p \neq 2$  use (2.2) for  $c = a + 2, a + 3 < w_0$  to prove again that  $\alpha = 1$ .

Assume now that  $p \neq 2$ .

Suppose  $\eta^{-1}(t) \equiv t + \alpha t^{v-1} \pmod{t^v}$  with  $v \geq 3$  and  $\alpha \in k^*$ . If  $a + v - 2 \in \mathbb{Z}(p)$  then by part 2) of proposition 2.1  $\alpha_{a,a+v-2,0}(\eta^*) \neq 0$ . This implies that  $a + v - 2 \geq w_0 + 1$ , i.e.  $v \geq w_0 - a + 1$ , as required. If  $a + v - 2 \equiv 0 \pmod{p}$  then by part 2) of proposition 2.1  $\alpha_{a+1,a+v-1,0}(\eta^*) \neq 0$ . This implies that  $a + v - 1 \geq w_0 + 1$  and  $v \geq w_0 - a + 2 > w_0 - a + 1$ . The case  $p \neq 2$  is considered.

Assume now that  $p = 2$ .

Suppose that  $M \in \mathbb{N}$  is such that

$$\eta^{-1}(t) = t \left( 1 + \sum_{r \geq 2M-1} \gamma_r t^r \right) \equiv t \pmod{t^{2M}}$$

with either  $\gamma_{2M-1} \neq 0$  or  $\gamma_{2M} \neq 0$ .

Therefore, if  $r \equiv 0 \pmod{2}$ ,  $r \geq 2M-1$  and  $a+r < ap^{N_0}$  then by part 2) of proposition 2.1  $\alpha_{a,a+r,0}(\eta^*) = \gamma_r$ . This implies that either  $2M \geq w_0$  (and the proposition is proved) or  $2M \leq w_0 - 2$ ,  $\gamma_{2M} = 0$  and  $\gamma_{2M-1} \neq 0$ .

Suppose  $a + 4M < w_0$ . Then with the notation from the second step in the proof of proposition 2.6, we have

$$\begin{aligned} \alpha_{a,a+4M-2,0}(\eta^*) &= \gamma_{4M-2} + \gamma_{2M-1}^2 \delta(a+4M-2) \\ \alpha_{a+2,a+4M,0}(\eta^*) &= \gamma_{4M-2} + \gamma_{2M-1}^2 \delta(a+4M). \end{aligned}$$

The sum of the right hand sides of the above two equalities is  $\gamma_{2M-1}^2 \neq 0$ , because  $\delta(a+4M-2) + \delta(a+4M) = 1$ . Therefore, at least one of their left hand sides is not zero. This means that the assumption about  $a+4M < w_0$  was wrong. Therefore,  $4M > w_0 - a$  and  $2M \geq (w_0 - a + 1)/2$ .

The proposition is proved.  $\square$

### 3. Compatible systems of group morphisms

For any  $s \in \mathbb{Z}_{\geq 0}$ , let  $K_s$  be the unramified extension of  $K$  in  $K(p)$  of degree  $p^s$ . Then  $K_s = k_s((t))$ , where  $t = t_K$  is a fixed uniformiser,  $k \subset k_s$ ,  $[k_s : k] = p^s$ ,  $k_s \simeq \mathbb{F}_{q_s}$ ,  $q_s = p^{N_s}$  with  $N_s = N_0 p^s$ .

Let  $K_{\text{ur}}$  be the union of all  $K_s$ ,  $s \geq 0$ . This is the maximal unramified extension of  $K$  in  $K(p)$  and its residue field coincides with the residue field  $k(p)$  of  $K(p)$ . Let  $I_{K_{\text{ur}}}(p)^{\text{ab}}$ , resp.  $I_{K_s}(p)^{\text{ab}}$ , for  $s \in \mathbb{Z}_{\geq 0}$ , be the images of the inertia subgroups of  $\text{Gal}(K(p)/K_{\text{ur}})$ , resp.  $\text{Gal}(K(p)/K_s)$ , in the corresponding maximal abelian quotients. Then  $I_{K_{\text{ur}}}(p)^{\text{ab}} = \varprojlim_s I_{K_s}(p)^{\text{ab}}$ .

**3.1.** For  $s \geq 0$ , introduce the  $\mathbb{F}_p$ -modules  $\mathcal{M}_{K_s} = I_{K_s}(p)^{\text{ab}} \otimes \mathbb{F}_p$  and  $\mathcal{M}_{K_{\text{ur}}} = I_{K_{\text{ur}}}(p)^{\text{ab}} \otimes \mathbb{F}_p$  with the corresponding  $k(p)$ -modules  $\bar{\mathcal{M}}_{K_s} = \mathcal{M}_{K_s} \hat{\otimes}_{\mathbb{F}_p} k(p)$  and  $\bar{\mathcal{M}}_{K_{\text{ur}}} = \mathcal{M}_{K_{\text{ur}}} \hat{\otimes}_{\mathbb{F}_p} k(p)$ . Then for all  $s \geq 0$ , we have natural connecting morphisms  $j_s : \mathcal{M}_{K_{s+1}} \rightarrow \mathcal{M}_{K_s}$  and  $\bar{j}_s : \bar{\mathcal{M}}_{K_{s+1}} \rightarrow \bar{\mathcal{M}}_{K_s}$  (both are induced by the natural group embeddings  $\Gamma_{K_{s+1}} \rightarrow \Gamma_{K_s}$ ). Therefore, we have projective systems  $\{\mathcal{M}_{K_s}, j_s\}$  and  $\{\bar{\mathcal{M}}_{K_s}, \bar{j}_s\}$  and natural identifications  $\mathcal{M}_{K_{\text{ur}}} = \varprojlim_s \mathcal{M}_{K_s}$  and  $\bar{\mathcal{M}}_{K_{\text{ur}}} = \varprojlim_s \bar{\mathcal{M}}_{K_s}$ .

Let  $\mathcal{M}_{K_\infty}$  be the  $k(p)$ -submodule in  $\bar{\mathcal{M}}_{K_{\text{ur}}}$  which is topologically generated by all  $D_{an}^\infty := \varprojlim_s D_{a,n \pmod{N_s}}^{(s)}$ , where  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z}$ . Here for

$a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_s$ ,  $D_{an}^{(s)}$  are generators for  $\bar{\mathcal{M}}_{K_s}$ , which are analogues of the generators  $D_{an}$  introduced in Section 2 for the  $k$ -module  $\mathcal{M}_k$ . Notice that the generators  $D_{an}^{(s)}$  depend on the choice of the uniformising element  $t$  in  $K$ .

**Proposition 3.1.** *The  $k(p)$ -submodule  $\mathcal{M}_{K_\infty}$  of  $\bar{\mathcal{M}}_{K_{\text{ur}}}$  does not depend on the choice of  $t$ .*

*Proof.* Let  $t_1$  be another uniformiser in  $K$ . Introduce  $\eta \in \text{Aut}^0(K_{\text{ur}})$  such that  $\eta(t) = t_1$ . The proposition will be proved if we show that  $\eta^*(\mathcal{M}_{K_\infty}) = \mathcal{M}_{K_\infty}$ .

For  $s \geq 0$ , let  $\eta_s = \eta|_{K_s} \in \text{Aut}^0 K_s$ . Then for  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_s$ ,

$$\eta_s^*(D_{an}^{(s)}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_s}} \sigma^n \alpha_{abm}(\eta_s^*) D_{b,m+n}^{(s)},$$

where the coefficients  $\alpha_{abm}(\eta_s^*) \in k_s$  satisfy the following compatibility conditions (using that  $j_s(D_{an}^{(s)}) = D_{a,n \bmod N_{s-1}}^{(s-1)}$ ):

if  $a, b \in \mathbb{Z}(p)$  and  $m \in \mathbb{Z} \bmod N_{s-1}$  then

$$\sum_{n \bmod N_{s-1} = m} \alpha_{abn}(\eta_s^*) = \alpha_{abm}(\eta_{s-1}^*).$$

By proposition 2.4, if  $0 \leq m < N_s$  and  $b/p^m < a$  then  $\alpha_{a,b,-m \bmod N_s}(\eta_s^*) = 0$ . Therefore, if  $s$  is such that  $b/p^{N_s} < a$  then  $\alpha_{a,b,-m}^\infty(\eta^*) := \alpha_{a,b,-m \bmod N_s}(\eta_s^*)$  does not depend on  $s$  and for any  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z}_{\geq 0}$ ,

$$\eta^*(D_{an}^\infty) = \sum_{b \in \mathbb{Z}(p), m \geq 0} \sigma^n \alpha_{a,b,-m}^\infty(\eta^*) D_{b,n-m}^\infty \in \mathcal{M}_{K_\infty}.$$

The proposition is proved.  $\square$

**3.2.** Consider the identification of class field theory  $I_{K_s}(p)^{\text{ab}} = U_{K_s}$ , where  $U_{K_s}$  is the group of principal units of  $K_s$ . Define the continuous morphism of topological  $k(p)$ -modules

$$\pi_{K_s} : \bar{\mathcal{M}}_{K_s} = I_{K_s}(p)^{\text{ab}} \hat{\otimes} k(p) \longrightarrow \hat{\Omega}_{O_{K_{\text{ur}}}}^1,$$

by  $\pi_{K_s}(u \otimes \alpha) = \alpha \text{d}(u)/u$  for  $u \in U_{K_s}$  and  $\alpha \in k(p)$ . Here  $\hat{\Omega}_{O_{K_{\text{ur}}}}^1$  is the completion of the module of differentials of the valuation ring  $O_{K_{\text{ur}}}$  with respect to the  $t$ -adic topology. Notice that for any  $a \in \mathbb{Z}(p)$  and  $0 \leq n < N_s$ ,

$$D_{a,n \bmod N_s}^{(s)} = \sum_{0 \leq i < N_s} u_i \otimes (\sigma^n \alpha_i \bmod p).$$



Here  $\{\alpha_i \mid 1 \leq i \leq N_s\}$  is a  $\mathbb{Z}_p$ -basis of  $W(k_s)$ . If  $\{\beta_i \mid 1 \leq i \leq N_s\}$  is its dual basis then for  $1 \leq i \leq N_s$ ,  $u_i = E(\beta_i, t^a)^{1/a}$ , cf. Subsection 1.4. Therefore,

$$\pi_{K_s}(D_{a, n \bmod N_s}^{(s)}) = \left( \sum_{i \geq 0} t^{ap^{n+iN_s}} \right) \frac{d(t)}{t}.$$

It is easy to see that  $\pi_{K_{\text{ur}}} := \varprojlim \pi_{K_s}$  is a continuous map from  $\bar{\mathcal{M}}_{K_{\text{ur}}}$  to  $\hat{\Omega}_{O_{K_{\text{ur}}}}^1$ .

Notice that if  $\bar{n} = \varprojlim (n_s \bmod N_s) \in \varprojlim \mathbb{Z}/N_s\mathbb{Z}$ , where all  $n_s \in [0, N_s)$  and if  $D_{a\bar{n}}^\infty = \varprojlim_s D_{a, n_s \bmod N_s}^{(s)}$ , for  $a \in \mathbb{Z}(p)$ , then  $\pi_{K_{\text{ur}}}(D_{a\bar{n}}^\infty) = 0$  if  $\bar{n} \notin \mathbb{Z}_{\geq 0} \subset \varprojlim \mathbb{Z}/N_s\mathbb{Z}$ , and  $\pi_{K_{\text{ur}}}(D_{a\bar{n}}^\infty) = t^{ap^{\bar{n}-1}} d(t)$  if  $\bar{n} = n \in \mathbb{Z}_{\geq 0}$ .

Let  $\pi_{K_\infty} := \pi_{K_{\text{ur}}}|_{\mathcal{M}_{K_\infty}}$ . Then one can easily prove the following proposition.

**Proposition 3.2.** 1)  $\pi_{K_\infty} : \mathcal{M}_{K_\infty} \longrightarrow \hat{\Omega}_{O_{K_{\text{ur}}}}^1$  is a continuous epimorphism of  $k(p)$ -modules;  
2)  $\ker \pi_{K_\infty}$  is the  $k(p)$ -submodule in  $\mathcal{M}_{K_\infty}$  topologically generated by all  $D_{an}^\infty$  with  $n < 0$ .

**3.3. Admissible systems of group morphisms.** Suppose  $K' = k((t')) \subset K(p)$  has the same residue field as  $K$ . Using  $K'$  instead of  $K$  we can introduce analogues  $\mathcal{M}_{K'_s}$ ,  $\bar{\mathcal{M}}_{K'_s}$ ,  $\mathcal{M}_{K'_\infty}$ , etc. of  $\mathcal{M}_{K_s}$ ,  $\bar{\mathcal{M}}_{K_s}$ ,  $\mathcal{M}_{K_\infty}$ , etc.

**Definition.**  $f_{KK'} = \{f_{KK'_s}\}_{s \geq 0}$  is a family of continuous morphisms of  $\mathbb{F}_p$ -modules  $f_{KK'_s} : \mathcal{M}_{K_s} \longrightarrow \mathcal{M}_{K'_s}$  which are always assumed to be compatible, i.e. for all  $s \geq 0$ ,  $f_{KK', s+1} j'_s = j_s f_{KK'_s}$ . Here  $j_s : \mathcal{M}_{K, s+1} \longrightarrow \mathcal{M}_{K_s}$  and  $j'_s : \mathcal{M}_{K', s+1} \longrightarrow \mathcal{M}_{K'_s}$  are connecting morphisms.

We shall denote the  $k(p)$ -linear extension of  $f_{KK'_s}$  by the same symbol  $f_{KK'_s}$ . Set

$$f_{KK'_{\text{ur}}} := \varprojlim_s f_{KK'_s} : \bar{\mathcal{M}}_{K_{\text{ur}}} \longrightarrow \bar{\mathcal{M}}_{K'_{\text{ur}}}.$$

**Definition.** With the above notation  $f_{KK'}$  is called admissible if:

**A1.** There is a continuous  $k(p)$ -linear isomorphism  $f_{KK'_\infty} : \hat{\Omega}_{O_{K_{\text{ur}}}}^1 \longrightarrow \hat{\Omega}_{O_{K'_{\text{ur}}}}^1$  such that  $f_{KK'_{\text{ur}}} \pi_{K'_{\text{ur}}} = \pi_{K_{\text{ur}}} f_{KK'_\infty}$ ;

**A2.**  $f_{KK'_\infty}$  commutes with the Cartier operators  $C$  and  $C'$  on  $\hat{\Omega}_{O_{K_{\text{ur}}}}^1$  and, resp.,  $\hat{\Omega}_{O_{K'_{\text{ur}}}}^1$ ;

**A3.** For all  $m \in \mathbb{N}$ ,  $f_{KK'_\infty} (t^m \hat{\Omega}_{O_{K_{\text{ur}}}}^1) \subset t^m \hat{\Omega}_{O_{K'_{\text{ur}}}}^1$ .

**Remark.** Recall that the Cartier operator  $C : \hat{\Omega}_{O_{K_{\text{ur}}}}^1 \longrightarrow \hat{\Omega}_{O_{K_{\text{ur}}}}^1$  is uniquely determined by the following properties:

- a)  $C(d(\hat{O}_{K_{\text{ur}}})) = 0$ ;
- b) if  $f \in t\hat{O}_{K_{\text{ur}}}$  then  $C(f^p d(t)/t) = f d(t)/t$ .

It can be shown that the definition of  $C$  does not depend on the choice of the uniformiser  $t$ ,  $C$  is  $\sigma^{-1}$ -linear and  $\text{Ker}C = d(\hat{O}_{K_{\text{ur}}})$ .

The following properties of admissible systems  $f_{KK'} = \{f_{KK's}\}_{s \geq 0}$  follow directly from the above definition:

- (1) the map  $f_{KK'\infty}$  is uniquely determined by  $f_{KK'\text{ur}}$ ;
- (2) if  $K'' = k((t'')) \subset K(p)$  and  $g_{K'K''} = \{g_{K'K''s}\}_{s \geq 0}$  is admissible then so is the composition  $(fg)_{KK''} := \{f_{KK's}g_{K'K''s}\}_{s \geq 0}$  and it holds  $(fg)_{KK''\infty} = f_{KK'\infty}g_{K'K''\infty}$ ;
- (3)  $f_{KK'\infty}(d\hat{O}_{K_{\text{ur}}}) \subset d\hat{O}_{K'_{\text{ur}}}$ ;
- (4) for all  $a, b \in \mathbb{Z}(p)$  and  $m \in \mathbb{Z}_{\geq 0}$ , there are unique  $\alpha_{a,b,-m}^\infty(f_{KK'}) \in k(p)$  such that if  $n \geq 0$  then

$$(3.1) \quad f_{KK'\infty} \left( t^{ap^n} \frac{d(t)}{t} \right) = \sum_{\substack{b \in \mathbb{Z}(p) \\ 0 \leq m \leq n}} \sigma^n \alpha_{a,b,-m}^\infty(f_{KK'}) t^{bp^n - m} \frac{d(t')}{t'};$$

- (5) the above coefficients  $\alpha_{a,b,-m}^\infty(f_{KK'})$  satisfy the following property: if  $b/p^m < a$  then  $\alpha_{a,b,-m}^\infty(f_{KK'}) = 0$ .

**Definition.** With the above notation an admissible compatible system  $f_{KK'}$  will be called special admissible if  $f_{KK'\text{ur}}(\mathcal{M}_{K\infty}) \subset \mathcal{M}_{K'\infty}$ .

Notice that the composition of special admissible systems is again special admissible.

**3.4. Characterisation of special admissible systems.** Let  $f_{KK'} = \{f_{KK's}\}_{s \geq 0}$  be a compatible system. Then for any  $s \geq 0$ , the  $k(p)$ -linear morphism  $f_{KK's} : \bar{\mathcal{M}}_{Ks} \longrightarrow \bar{\mathcal{M}}_{K's}$  is defined over  $\mathbb{F}_p$ , i.e. it comes from a  $\mathbb{F}_p$ -linear morphism  $f_{KK's} : \mathcal{M}_{Ks} \longrightarrow \mathcal{M}_{K's}$ . Therefore, in terms of the standard generators  $D_{an}^{(s)}$  and  $D_{an}'^{(s)}$  (which correspond to the uniformisers  $t = t_K$  and, resp.,  $t' = t_{K'}$ ), we have for any  $s \geq 0$  and  $a \in \mathbb{Z}(p)$  that

$$f_{KK's}(D_{a0}^{(s)}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_s}} \alpha_{abm}(f_{KK's}) D_{bm}'^{(s)},$$

where all  $\alpha_{abm}(f_{KK's}) \in k_s \subset k(p)$ . Notice that for all  $n \in \mathbb{Z} \bmod N_s$ , it holds

$$f_{KK's}(D_{an}^{(s)}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_s}} \sigma^n \alpha_{abm}(f_{KK's}) D'_{b,m+n}{}^{(s)}.$$

**Proposition 3.3.** *Suppose  $f_{KK'} = \{f_{KK's}\}_{s \geq 0}$  is a compatible system. Then it is special admissible if and only if for any  $s \geq 0$ , there are  $v_s \in \mathbb{N}$  such that  $v_s \rightarrow \infty$  if  $s \rightarrow \infty$ , and if  $a, b < v_s$ ,  $m \geq 0$  and  $b/p^m < a$  then  $\alpha_{a,b,-m \bmod N_s}(f_{KK's}) = 0$ .*

*Proof.* Suppose  $f_{KK'}$  is special admissible. Then  $f_{KK' \text{ur}}(\mathcal{M}_{K\infty}) \subset \mathcal{M}_{K'\infty}$  and for all  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z}$ ,

$$f_{KK' \text{ur}}(D_{an}^\infty) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z}}} \beta_{abm} D'_{b,n+m}{}^\infty.$$

Here all coefficients  $\beta_{abm} \in k(p)$  and because  $f_{KK' \text{ur}}$  commutes with  $\sigma$ , there are  $\gamma_{abm} \in k(p)$  such that  $\beta_{abm} = \sigma^n(\gamma_{abm})$ . Therefore, if  $a, b \in \mathbb{Z}(p)$ ,  $m \in \mathbb{Z}$  and  $\gamma_{abm} \neq 0$  then  $m \leq 0$  and  $\alpha_{abm}^\infty(f_{KK'}) = \gamma_{abm}$ .

If  $s \geq 0$ ,  $a \in \mathbb{Z}(p)$ ,

$$f_{KK's}(D_{a0}^{(s)}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_s}} \alpha_{a,b,-m}(f_{KK's}) D'_{b,-m}{}^{(s)}$$

and  $b/p^{N_s} < a$  then for any  $m \geq 0$ ,  $\alpha_{a,b,-m \bmod N_s}(f_{KK's}) = \alpha_{a,b,-m}^\infty(f_{KK'})$ . This implies that  $\alpha_{a,b,-m \bmod N_s}(f_{KK's}) = 0$  if  $a, b < p^{N_s}$  and  $b/p^m < a$ . Therefore, we can take  $v_s = p^{N_s}$ . This proves the “only if” part of the proposition.

Suppose now that  $v_s \rightarrow \infty$  if  $s \rightarrow \infty$  and for  $a, b \in \mathbb{Z}(p)$ ,  $m \geq 0$ ,

$$\alpha_{a,b,-m \bmod N_s}(f_{KK's}) = 0$$

if  $a, b < v_s$  and  $b/p^m < a$ . If in addition  $p^{N_s} > b$  then  $\alpha_{a,b,-m \bmod N_s}(f_{KK's})$  does not depend on  $s$  and can be denoted by  $\alpha_{a,b,-m}^\infty$ . Clearly,  $\alpha_{a,b,-m}^\infty = 0$  if  $b/p^m < a$ . Let  $a \in \mathbb{Z}(p)$  and

$$d = f_{KK' \text{ur}}(D_{a0}^\infty) - \sum_{\substack{b \in \mathbb{Z}(p) \\ m \geq 0}} \alpha_{a,b,-m}^\infty D'_{b,-m}{}^\infty.$$

Let  $s \geq 0$  and let  $d_s \in \bar{\mathcal{M}}_{K_s}$  be the image of  $d$  under the natural projection  $\bar{\mathcal{M}}_{K \text{ur}} \rightarrow \bar{\mathcal{M}}_{K_s}$ . If  $s_1 \geq s$  then the corresponding projection  $d_{s_1} \in \bar{\mathcal{M}}_{K_{s_1}}$  is a linear combination of  $D_{bm}^{(s_1)}$  with  $b > p^{N_{s_1}}$ . Therefore,  $d_s$  also does not contain the terms  $D_{bm}^{(s)}$  for which  $b > p^{N_{s_1}}$ . Because  $\lim_{s_1 \rightarrow \infty} N_{s_1} = \infty$ , this implies that  $d_s = 0$  for all  $s \geq 0$  and, therefore,  $d = 0$ . So,  $f_{KK' \text{ur}}(\mathcal{M}_{K\infty}) \subset \mathcal{M}_{K'\infty}$ .

Set  $\alpha_{a,b,-m}^\infty(f_{KK'}) := \alpha_{a,b,-m}^\infty$  and define  $f_{KK'\infty} : \hat{\Omega}_{O_{K_{\text{ur}}}}^1 \longrightarrow \hat{\Omega}_{O_{K'_{\text{ur}}}}^1$  by formula (3.1). It is easy to see that  $f_{KK'\infty}$  satisfies the requirements **A1-A3** from the definition of admissible system in Subsection 3.3. This proves the “if” part of our proposition.  $\square$

**Remark.** Any special admissible  $f_{KK'}$  can be defined as a  $k(p)$ -linear isomorphism  $f_{KK'\text{ur}} : \mathcal{M}_{K_\infty} \longrightarrow \mathcal{M}_{K'\infty}$  such that

- (1)  $f_{KK'\text{ur}}$  commutes with  $\sigma$ ;
- (2) if  $a \in \mathbb{Z}(p)$  then

$$f_{KK'\text{ur}}(D_{a0}^\infty) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \geq 0}} \alpha_{a,b,-m} D_{b,-m}'^\infty$$

where  $\alpha_{a,b,-m} = 0$  if  $b/p^m < a$ .

**3.5. Analytic compatible systems.** Suppose  $K, K' \subset K(p)$ . Then the corresponding residue fields  $k$  and  $k'$  are subfields of the residue field  $k(p) \subset \bar{\mathbb{F}}_{q_0}$ . Therefore, if  $K \simeq K'$  then  $k = k'$  and we can introduce the set  $\text{Iso}^0(K, K')$  of field isomorphisms  $\eta : K \longrightarrow K'$  such that  $\eta|_k = \text{id}$ . Notice that any  $\eta \in \text{Iso}^0(K, K')$  induces a  $k(p)$ -linear map  $\Omega^1(\eta) : \hat{\Omega}_{O_{K_{\text{ur}}}}^1 \longrightarrow \hat{\Omega}_{O_{K'_{\text{ur}}}}^1$ .

For all  $s \geq 0$ , any  $\eta \in \text{Iso}^0(K, K')$  can be naturally extended to  $\eta_s \in \text{Iso}^0(K_s, K'_s)$ . Then  $\eta_{KK'}^* = \{\eta_s^*\}_{s \geq 0}$  is a compatible system and  $\eta_{KK'\infty} = \Omega^1(\eta)$ . Propositions 2.4 and 3.3 imply that  $\eta_{KK'}^*$  is a special admissible system.

Consider the opposite situation. Choose a uniformiser  $t_K$  in  $K$  and introduce  $\text{Fr}(t_K) \in \text{Aut}(K_{\text{ur}})$  such that  $\text{Fr}(t_K) : t_K \mapsto t_K$  and  $\text{Fr}(t_K)|_{k(p)} = \sigma$ . Then for all  $s \geq 0$ ,  $\text{Fr}(t_K)$  induces an automorphism of  $K_s$  which will be denoted by  $\text{Fr}(t_K)_s$ . Then  $\text{Fr}(t_K)^* = \{\text{Fr}(t_K)_s\}_{s \geq 0}$  is a compatible system, but this system is not admissible: the corresponding map  $\text{Fr}(t_K)_\infty$  coincides with the Cartier operator and, therefore, is not  $k(p)$ -linear.

More generally, consider a compatible system  $\theta_{KK'} = \{\theta_{KK'_s}\}_{s \geq 0}$  where for all  $s \geq 0$ ,  $\theta_{KK'_s} = \theta_s^*$  and  $\theta_s \in \text{Iso}(K_s, K'_s)$ . Then after choosing a uniformising element  $t_{K'}$  in  $K'$  we have  $\theta_s = \eta_s \text{Fr}(t_{K'})^{n_s}$ , for all  $s \geq 0$ , where  $\eta_s \in \text{Iso}^0(K_s, K'_s)$  and  $n_{s+1} \equiv n_s \pmod{N_s}$ . If  $\bar{n} = \varprojlim_s n_s \in \varprojlim_s \mathbb{Z}/N_s \mathbb{Z}$

then  $\theta_{KK'}$  is the composite of the special admissible system  $\{\eta_s^*\}_{s \geq 0}$  and the system  $\text{Fr}(t_{K'})^{\bar{n}*}$  which is special admissible if and only if  $\bar{n} = 0$ . Therefore,  $\theta_{KK'}$  is special admissible if and only if it comes from a compatible system of field isomorphisms  $\eta_s \in \text{Iso}^0(K_s, K'_s)$ .

### 3.6. Locally analytic systems.

**Definition.** If  $f_{KK'}$  is an admissible system, then  $f_{KK'\text{an}} := f_{KK'\infty}|_{\text{d}(\hat{O}_{K_{\text{ur}}})}$ .

**Remark.** Notice the following similarity to the definition of  $f_{\text{an}}$  for  $f \in \text{Aut}\mathcal{M}$  from n.2.3. If  $f_{KK} = \{f_{KKs}\}_{s \geq 0}$  is any admissible system then  $g_{KK} := \{f_{KK\text{san}}\}_{s \geq 0}$  is also admissible and  $f_{KK\text{an}} = g_{KK\text{an}}$ .

**Definition.** An admissible system  $f_{KK'} = \{f_{KK's}\}_{s \geq 0}$  will be called locally analytic if for any  $s \geq 0$ , there are  $v_s \in \mathbb{N}$  and  $\eta_s \in \text{Iso}^0(K, K')$  such that  $v_s \rightarrow +\infty$  as  $s \rightarrow \infty$  and  $f_{KK'\text{an}} \equiv \text{d}(\eta_s) \hat{\otimes}_k k(p) \pmod{t^{v_s}}$ .

**Proposition 3.4.** *Suppose that  $f_{KK'} = \{f_{KK's}\}_{s \geq 0}$  is special admissible and locally analytic. Then there is an  $\eta \in \text{Iso}^0(K, K')$  such that  $f_{KK'\text{an}} = \text{d}(\eta) \hat{\otimes}_k k(p)$ .*

*Proof.* If  $s \geq 0$  and  $a, b \in \mathbb{Z}(p)$  are such that  $v_s/p^{N_0} < a, b < v_s$ , then

$$\alpha_{ab0}^\infty(f_{KK'}) = \alpha_{ab0}(\eta_s^*) = \alpha_{ab0}(f_{KK's}) = \alpha_{ab0}(f_{KK'0}) \in k.$$

Therefore, by Proposition 2.7, all conjugates of  $\eta_s$  over  $K$  are congruent modulo  $t^{v_s(1-p^{-N_0})/\delta_p}$ , and  $\eta_s(t) \in k[[t]] \pmod{t^{v_s(1-p^{-N_s})/\delta_p}}$ , where  $\delta_p$  is 1 if  $p \neq 2$  and  $\delta_p = 2$  if  $p = 2$ . This implies that  $\alpha_{ab0}(f_{KK's}) \in k$  if  $a, b < v_s(1-p^{-N_s})/\delta_p$ .

If  $b < p^{N_s}$  then  $\alpha_{ab0}(f_{KK's}) = \alpha_{ab0}^\infty(f_{KK'})$ . So,  $\alpha_{ab0}^\infty(f_{KK'}) \in k$  if  $b < c_s := \min\{p^{N_s}, v_s(1-p^{-N_s})/\delta_p\}$ . But  $c_s \rightarrow \infty$  if  $s \rightarrow \infty$  and, therefore,  $\alpha_{ab0}^\infty(f_{KK'}) \in k$  for all  $a, b \in \mathbb{Z}(p)$ .

As we have already noticed, if  $b < \min\{p^{N_s}, v_s\}$  then

$$\alpha_{ab0}(f_{KK's}) = \alpha_{ab0}(\eta_s^*) = \alpha_{ab0}^\infty(f_{KK'}).$$

Therefore, by Proposition 2.7 there exists  $\varprojlim_s \eta_s := \eta \in \text{Iso}^0(K, K')$  and

$$f_{KK'\text{an}} = \text{d}(\eta) \hat{\otimes}_k k(p).$$

The proposition is proved.  $\square$

**3.7. Comparability of admissible systems.** With the above notation suppose  $L, L'$  are finite field extensions of  $K$ , resp.  $K'$ , in  $K(p)$ . Let  $g_{LL'} = \{g_{LL's}\}_{s \geq 0}$  be a compatible family of continuous field isomorphisms  $g_{LL's} : L_s \rightarrow L'_s$ . Then the natural embeddings  $\Gamma_L(p) \subset \Gamma_K(p)$  and  $\Gamma_{L'}(p) \subset \Gamma_{K'}(p)$  induce embeddings  $\Gamma_{L_s}(p) \subset \Gamma_{K_s}(p)$  and  $\Gamma_{L'_s}(p) \subset \Gamma_{K'_s}(p)$ , for any  $s \geq 0$ .

**Definition.** With the above assumptions the systems  $g_{LL'}$  and  $f_{KK'}$  will be called comparable if, for all  $s \geq 0$ , there is the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{M}_{L_s} & \xrightarrow{g_{LL'_s}} & \mathcal{M}_{L'_s} \\
 \downarrow j_s & & \downarrow j'_s \\
 \mathcal{M}_{K_s} & \xrightarrow{f_{KK'_s}} & \mathcal{M}_{K'_s}
 \end{array}$$

where the vertical arrows  $j_s$  and  $j'_s$  are induced by the embeddings  $\Gamma_{L_s}(p) \subset \Gamma_{K_s}(p)$  and, resp.,  $\Gamma_{L'_s}(p) \subset \Gamma_{K'_s}(p)$ .

If  $g_{LL'}$  and  $f_{KK'}$  are comparable then we have the following commutative diagram

$$(3.2) \quad \begin{array}{ccc}
 \bar{\mathcal{M}}_{L_{\text{ur}}} & \xrightarrow{g_{LL'_{\text{ur}}}} & \bar{\mathcal{M}}_{L'_{\text{ur}}} \\
 \downarrow j_{\text{ur}} & & \downarrow j'_{\text{ur}} \\
 \bar{\mathcal{M}}_{K_{\text{ur}}} & \xrightarrow{f_{KK'_{\text{ur}}}} & \bar{\mathcal{M}}_{K'_{\text{ur}}}
 \end{array}$$

where  $j_{\text{ur}} := \varprojlim_s j_s \hat{\otimes}_{k_s} k(p)$  and  $j'_{\text{ur}} := \varprojlim_s j'_s \hat{\otimes}_{k_s} k(p)$ . Notice that  $j_{\text{ur}}$  and  $j'_{\text{ur}}$  are epimorphic. Indeed, let  $U_{L_s}, U_{K_s}$  be principal units in  $L_s$ , resp.  $K_s$ . Then  $\mathcal{M}_{L_{\text{ur}}} = \varprojlim_s U_{L_s}/U_{L_s}^p$  and  $\mathcal{M}_{K_{\text{ur}}} = \varprojlim_s U_{K_s}/U_{K_s}^p$  contain as dense subsets the images of the groups of principal units  $U_{L_{\text{ur}}}$ , resp.  $U_{K_{\text{ur}}}$ , of the fields  $L_{\text{ur}}$ , resp.  $K_{\text{ur}}$ . By class field theory,  $j_{\text{ur}}$  is induced by the norm map  $N = N_{L_{\text{ur}}/K_{\text{ur}}}$  from  $L_{\text{ur}}^*$  to  $K_{\text{ur}}^*$ . By [6], Ch 2,  $N(U_{L_{\text{ur}}})$  is dense in  $U_{K_{\text{ur}}}$  and, therefore,  $j_{\text{ur}}$  (together with  $j'_{\text{ur}}$ ) is surjective.

Suppose  $L/K$  and  $L'/K'$  are Galois extensions. Denote their inertia subgroups by  $I_{L/K}$  and  $I_{L'/K'}$ . Then we have identifications  $I_{L/K} = \text{Gal}(L_{\text{ur}}/K_{\text{ur}})$  and  $I_{L'/K'} = \text{Gal}(L'_{\text{ur}}/K'_{\text{ur}})$ .

Consider the following condition:

**C.** *There is a group isomorphism  $\kappa : I_{L/K} \longrightarrow I_{L'/K'}$  such that for any  $\tau \in I_{L/K}$ ,  $\tau_{LL_{\text{ur}}}^* g_{LL'_{\text{ur}}} = g_{LL'_{\text{ur}}} \kappa(\tau)_{L'L'_{\text{ur}}}^*$ .*

**Proposition 3.5.** *Suppose  $g_{LL'}$  and  $f_{KK'}$  are comparable and  $g_{LL'}$  satisfies the above condition **C**. If  $g_{LL'}$  is admissible then  $f_{KK'}$  is also admissible.*

*Proof.* Because  $g_{LL'}$  is admissible we have the following commutative diagram

$$(3.3) \quad \begin{array}{ccc} \bar{\mathcal{M}}_{L_{\text{ur}}} & \xrightarrow{g_{LL'_{\text{ur}}}} & \bar{\mathcal{M}}_{L'_{\text{ur}}} \\ \downarrow \pi_{L_{\text{ur}}} & & \downarrow \pi_{L'_{\text{ur}}} \\ \hat{\Omega}_{O_{L_{\text{ur}}}}^1 & \xrightarrow{g_{LL'_{\infty}}} & \hat{\Omega}_{O_{L'_{\text{ur}}}}^1 \end{array}$$

If  $\tau \in I_{L/K} \subset \text{Aut}^0(L_{\text{ur}})$  then it follows from the definition of  $\pi_{L_{\text{ur}}}$  that

$$(3.4) \quad \tau^* \pi_{L_{\text{ur}}} = \pi_{L_{\text{ur}}} \Omega(\tau).$$

This means that  $\pi_{L_{\text{ur}}}$  transforms the natural action of  $I_{L/K}$  on  $\bar{\mathcal{M}}_{L_{\text{ur}}}$  into the natural action of  $I_{L/K}$  on  $\hat{\Omega}_{O_{L_{\text{ur}}}}^1$ . Because  $j_{\text{ur}}$  is induced by the norm map of the field extension  $L_{\text{ur}}/K_{\text{ur}}$ , this gives us the following commutative diagram

$$(3.5) \quad \begin{array}{ccc} \bar{\mathcal{M}}_{L_{\text{ur}}} & \xrightarrow{\pi_{L_{\text{ur}}}} & \hat{\Omega}_{O_{L_{\text{ur}}}}^1 \\ \downarrow j_{\text{ur}} & & \downarrow \text{Tr} \\ \bar{\mathcal{M}}_{K_{\text{ur}}} & \xrightarrow{\pi_{K_{\text{ur}}}} & \hat{\Omega}_{O_{K_{\text{ur}}}}^1 \end{array}$$

where  $\text{Tr}$  is induced by the trace of the extension  $L_{\text{ur}}/K_{\text{ur}}$ . Similarly, we have the commutative diagram

$$(3.6) \quad \begin{array}{ccc} \bar{\mathcal{M}}_{L'_{\text{ur}}} & \xrightarrow{\pi_{L'_{\text{ur}}}} & \hat{\Omega}_{O_{L'_{\text{ur}}}}^1 \\ \downarrow j'_{\text{ur}} & & \downarrow \text{Tr}' \\ \bar{\mathcal{M}}_{K'_{\text{ur}}} & \xrightarrow{\pi_{K'_{\text{ur}}}} & \hat{\Omega}_{O_{K'_{\text{ur}}}}^1 \end{array}$$

We have already seen that  $\pi_{L_{\text{ur}}}$ ,  $\pi_{L'_{\text{ur}}}$ ,  $j_{\text{ur}}$  and  $j'_{\text{ur}}$  are surjective. The traces  $\text{Tr}$  and  $\text{Tr}'$  are also surjective. Indeed, suppose  $t_L$ , resp.  $t_K$ , are uniformising elements for  $L$ , resp.  $K$ . Then

$$\hat{\Omega}_{O_{L_{\text{ur}}}}^1 = \{f \, \text{d}(t_L) \mid f \in \hat{O}_{L_{\text{ur}}}\} = \{g \, \text{d}(t_K) \mid g \in \mathcal{D}(L/K)^{-1} \hat{O}_{L_{\text{ur}}}\},$$

where  $\mathcal{D}(L/K)$  is the different of the extension  $L/K$ . It remains to notice that  $\text{Tr}(\mathcal{D}(L/K)^{-1} \hat{O}_{L_{\text{ur}}}) = \hat{O}_{K_{\text{ur}}}$ .

Because  $g_{LL'}$  and  $f_{KK'}$  are comparable, we have the following commutative diagram

$$(3.7) \quad \begin{array}{ccc} \bar{\mathcal{M}}_{L_{\text{ur}}} & \xrightarrow{g_{LL'_{\text{ur}}}} & \bar{\mathcal{M}}_{L'_{\text{ur}}} \\ \downarrow j_{\text{ur}} & & \downarrow j'_{\text{ur}} \\ \bar{\mathcal{M}}_{K_{\text{ur}}} & \xrightarrow{f_{KK'_{\text{ur}}}} & \bar{\mathcal{M}}_{K'_{\text{ur}}} \end{array}$$

Suppose  $\omega_K \in \hat{\Omega}_{O_{K_{\text{ur}}}}^1$ . As it has been proved there is an  $\omega_L \in \hat{\Omega}_{O_{L_{\text{ur}}}}^1$  such that

$$\text{Tr}(\omega_L) = \sum_{\tau \in I_{L/K}} \Omega(\tau)(\omega_L) = \omega_K.$$

Then

$$(3.8) \quad \begin{aligned} g_{LL'_{\infty}}(\omega_K) &= \sum_{\tau \in I_{L/K}} g_{LL'_{\infty}}(\Omega(\tau)(\omega_L)) \\ &= \sum_{\tau' \in I_{L'/K'}} \Omega(\tau')(g_{LL'_{\infty}}(\omega_L)) = \text{Tr}'(g_{LL'_{\infty}}(\omega_L)) \in \hat{\Omega}_{O_{K'_{\text{ur}}}}^1 \end{aligned}$$

because  $\Omega(\tau)g_{LL'_{\infty}} = g_{LL'_{\infty}}\Omega(\kappa(\tau))$ , for any  $\tau \in I_{L/K}$ . This equality is implied by the following computations (we use the commutative diagrams (3.3), (3.4) and condition **C**)

$$\begin{aligned} \pi_{L_{\text{ur}}}\Omega(\tau)g_{LL'_{\infty}} &= \tau^*\pi_{L_{\text{ur}}}g_{LL'_{\infty}} = \tau^*g_{LL'_{\text{ur}}}\pi_{L'_{\text{ur}}} \\ &= g_{LL'_{\text{ur}}}\kappa(\tau)^*\pi_{L'_{\text{ur}}} = g_{LL'_{\text{ur}}}\pi_{L'_{\text{ur}}}\Omega(\kappa(\tau)) \\ &= \pi_{L_{\text{ur}}}g_{LL'_{\infty}}\Omega(\kappa(\tau)), \end{aligned}$$

because  $\pi_{L_{\text{ur}}}$  is surjective.

Let  $f_{KK'_{\infty}}$  be the restriction of  $g_{LL'_{\infty}}$  on  $\hat{\Omega}_{O_{K_{\text{ur}}}}^1$ . Then formula (3.8) implies that  $f_{KK'_{\infty}}(\hat{\Omega}_{O_{K_{\text{ur}}}}^1) \subset \hat{\Omega}_{O_{K'_{\text{ur}}}}^1$  and we have the following commutative diagram

$$(3.9) \quad \begin{array}{ccc} \hat{\Omega}_{O_{L_{\text{ur}}}}^1 & \xrightarrow{g_{LL'_{\infty}}} & \hat{\Omega}_{O_{L'_{\text{ur}}}}^1 \\ \downarrow \text{Tr} & & \downarrow \text{Tr}' \\ \hat{\Omega}_{O_{K_{\text{ur}}}}^1 & \xrightarrow{f_{KK'_{\infty}}} & \hat{\Omega}_{O_{K'_{\text{ur}}}}^1 \end{array}$$

We now verify that  $f_{KK'_{\infty}}$  satisfies the requirements **A1-A3** from n.3.3. Property **A1** means that we have the following commutative diagram



$$\begin{array}{ccc}
\bar{\mathcal{M}}_{K_{\text{ur}}} & \xrightarrow{f_{KK'_{\text{ur}}}} & \bar{\mathcal{M}}_{K'_{\text{ur}}} \\
\downarrow \pi_{K_{\text{ur}}} & & \downarrow \pi_{K'_{\text{ur}}} \\
\hat{\Omega}_{O_{K_{\text{ur}}}}^1 & \xrightarrow{f_{KK'_{\infty}}} & \hat{\Omega}_{O_{K'_{\text{ur}}}}^1
\end{array}$$

Its commutativity is implied by the following computations (we use commutative diagrams (3.2), (3.5), (3.3) and (3.9))

$$\begin{aligned}
j_{\text{ur}} f_{KK'_{\text{ur}}} \pi_{K'_{\text{ur}}} &= g_{LL'_{\text{ur}}} j'_{\text{ur}} \pi_{K'_{\text{ur}}} = g_{LL'_{\text{ur}}} \pi_{L'_{\text{ur}}} \text{Tr}' \\
&= \pi_{L_{\text{ur}}} g_{LL'_{\infty}} \text{Tr}' = \pi_{L_{\text{ur}}} \text{Tr} f_{KK'_{\infty}} = j_{\text{ur}} \pi_{K_{\text{ur}}} f_{KK'_{\infty}}
\end{aligned}$$

because  $j_{\text{ur}}$  is surjective.

Let  $C_K$ ,  $C_{K'}$ ,  $C_L$  and  $C_{L'}$  be the Cartier operators on, resp.,  $\hat{\Omega}_{O_{K_{\text{ur}}}}^1$ ,  $\hat{\Omega}_{O_{K'_{\text{ur}}}}^1$ ,  $\hat{\Omega}_{O_{L_{\text{ur}}}}^1$  and  $\hat{\Omega}_{O_{L'_{\text{ur}}}}^1$ . Clearly,  $C_L \text{Tr} = \text{Tr} C_K$  and  $C_{L'} \text{Tr}' = \text{Tr}' C_{K'}$ . Then it follows from the commutative diagram (3.9) and property **A2** for  $g_{LL'_{\infty}}$  that

$$\begin{aligned}
\text{Tr} C_K f_{KK'_{\infty}} &= C_L \text{Tr} f_{KK'_{\infty}} = C_L g_{LL'_{\infty}} \text{Tr} \\
&= g_{LL'_{\infty}} C_{L'} \text{Tr}' = g_{LL'_{\infty}} \text{Tr} C_{K'} = \text{Tr} f_{KK'_{\infty}} C_{K'}.
\end{aligned}$$

Property **A2** for  $f_{KK'_{\infty}}$  follows because  $\text{Tr}$  is surjective.

By condition **C**, the ramification indices  $e$  and  $e'$  of the extensions  $L_{\text{ur}}/K_{\text{ur}}$  and  $L'_{\text{ur}}/K'_{\text{ur}}$  are equal. Then we use the condition **A3** for  $g_{LL'_{\infty}}$  to deduce that for any  $n \geq 0$ ,

$$g_{LL'_{\infty}}(t_K^n \hat{\Omega}_{O_{L_{\text{ur}}}}^1) = g_{LL'_{\infty}}(t_L^{en} \hat{\Omega}_{O_{L_{\text{ur}}}}^1) = t_L^{e'n} \hat{\Omega}_{O_{L'_{\text{ur}}}}^1 = t_{K'}^n \hat{\Omega}_{O_{L'_{\text{ur}}}}^1.$$

Therefore, it follows from the commutativity of diagram (3.9) that

$$\begin{aligned}
t_{K'}^n \hat{\Omega}_{O_{K'_{\text{ur}}}}^1 &= t_{K'}^n \text{Tr}'(\hat{\Omega}_{O_{L'_{\text{ur}}}}^1) = \text{Tr}'(g_{LL'_{\infty}}(t_K^n \hat{\Omega}_{O_{L_{\text{ur}}}}^1)) \\
&= f_{KK'_{\infty}}(\text{Tr}(t_K^n \hat{\Omega}_{O_{L_{\text{ur}}}}^1)) = f_{KK'_{\infty}}(t_{K'}^n \hat{\Omega}_{O_{K_{\text{ur}}}}^1).
\end{aligned}$$

The proposition is proved.  $\square$

**Remark.** Using the embeddings of the Galois groups  $\Gamma_{L_s}(p)$  and  $\Gamma_{K_s}(p)$  into their Magnus's algebras from n.1.3, one can prove in addition that if  $g_{LL'}$  is special then  $f_{KK'}$  is also special. In other words, under condition **C**,  $j_{\text{ur}}(\mathcal{M}_{L_{\infty}}) \subset \mathcal{M}_{K_{\infty}}$ .

Suppose  $g_{LL'}$  and  $f_{KK'}$  are comparable systems. Suppose also that  $g_{LL'}$  and  $f_{KK'}$  are special admissible, locally analytic and satisfy condition **C**. Then there are  $\eta_{LL'} \in \text{Iso}^0(L, L')$  and  $\eta_{KK'} \in \text{Iso}^0(K, K')$  such that  $f_{KK'_{\infty}}|_{\text{d}\hat{O}_{K_{\text{ur}}}} = \text{d}(\eta_{KK'}) \hat{\otimes}_k k(p)$  and  $g_{LL'_{\infty}}|_{\text{d}\hat{O}_{L_{\text{ur}}}} = \text{d}(\eta_{LL'}) \hat{\otimes}_{k_L} k_L(p)$ .

**Proposition 3.6.** *With the above notation and assumptions,  $\eta_{LL'}|_K = \eta_{KK'}$ .*

*Proof.* Clearly, for any  $\tau \in I_{L/K}$ , condition **C** implies that  $\tau_{LL\infty}^* g_{LL'\infty} = g_{LL'\infty} \kappa(\tau)_{L'L'\infty}^*$ . Restricting this equality to  $d\hat{O}_{L_{\text{ur}}}$ , we obtain

$$d(\tau) d(\eta_{LL'}) = d(\eta_{LL'}) d(\kappa(\tau)).$$

Then it follows from proposition 2.7 that  $\tau\eta_{LL'} = \eta_{LL'}\kappa(\tau)$ . Therefore,  $\eta_{LL'}|_K$  induces a ring isomorphism from  $\hat{O}_{K_{\text{ur}}}$  onto  $\hat{O}_{K'_{\text{ur}}}$ .

Suppose  $a \in \text{Tr}(\hat{O}_{L_{\text{ur}}}) \subset \hat{O}_{K_{\text{ur}}}$ . If  $a = \text{Tr}(b)$  with  $b \in \hat{O}_{L_{\text{ur}}}$  then it follows from diagram (3.9) and condition **C** that

$$\begin{aligned} d(\eta_{KK'}(a)) &= \text{Tr}'(d(\eta_{LL'}(b))) = \sum_{\tau' \in I_{L'/K'}} d(\tau') (d(\eta_{LL'}(b))) \\ &= \sum_{\tau \in I_{L/K}} d(\eta_{LL'})(d(\tau(b))) = d_{\eta_{LL'}}(da) = d(\eta_{LL'}(a)). \end{aligned}$$

Therefore, for a sufficiently large  $M \in \mathbb{N}$ ,  $d(\eta_{LL'}|_K)$  and  $d\eta_{KK'}$  coincide on  $t_K^M \hat{O}_{K_{\text{ur}}}$ . Then proposition 2.7 implies that  $\eta_{LL'}|_K = \eta_{KK'}$ .

The proposition is proved.  $\square$

#### 4. Explicit description of the ramification ideals $\mathcal{A}^{(v)}$ mod $\mathcal{J}^3$

We return to the notation from Section 1. In particular,  $\mathcal{A}$  is the  $\mathbb{Z}_p$ -algebra from Subsection 1.2,  $\mathcal{J}$  is its augmentation ideal,  $\mathcal{A}_k = \mathcal{A} \otimes W(k)$ ,  $\mathcal{J}_k = \mathcal{J} \otimes W(k)$ ,  $\mathcal{A}_K = \mathcal{A} \otimes O(K)$ , etc. are the corresponding extensions of scalars,  $e \in \mathcal{A}_K$  is the element introduced in Subsection 1.3. We fix an  $f \in \mathcal{A}_{K(p)}$  such that  $\sigma f = fe$  and denote the embedding  $\psi_f : \Gamma(p) \rightarrow (1 + \mathcal{J})^\times$  by  $\psi$ .

##### 4.1. Ramification filtration on $\mathcal{A}$

For any  $v \geq 0$ , consider the ramification subgroup  $\Gamma(p)^{(v)}$  of  $\Gamma(p)$  in the upper numbering. Denote by  $\mathcal{A}^{(v)}$  the minimal 2-sided closed ideal in  $\mathcal{A}$  containing the elements  $\psi(\tau) - 1$ , for all  $\tau \in \Gamma(p)^{(v)}$ . Then  $\{\mathcal{A}^{(v)} \mid v \geq 0\}$  is a decreasing filtration by closed ideals of  $\mathcal{A}$ . In particular, if  $\mathcal{A}_{CM}^{(v)}$  mod  $\mathcal{J}_{CM}^n$  are the projections of  $\mathcal{A}^{(v)}$  to  $\mathcal{A}_{CM}$  mod  $\mathcal{J}_{CM}^n$ , for  $C, M, n \in \mathbb{N}$ , then  $\mathcal{A}^{(v)} = \varprojlim_{C, M, n} \mathcal{A}_{CM}^{(v)}$  mod  $\mathcal{J}_{CM}^n$ . Notice also that the ramification filtration

$\{\Gamma(p)^{(v)}\}_{v \geq 0}$  is left-continuous, i.e.  $\Gamma(p)^{(v_0)} = \bigcap_{v < v_0} \Gamma(p)^{(v)}$ , for any  $v_0 > 0$ .

This implies a corresponding analogous property for the filtration  $\{\mathcal{A}^{(v)} \mid v \geq 0\}$  on each finite level, i.e. for any  $C, M, n \in \mathbb{N}$ , we have the following property.

**Proposition 4.1.** *For any  $C, M, n \in \mathbb{N}$  and  $v_0 > 0$ , there is a  $0 < \delta < v_0$  such that  $\mathcal{A}_{CM}^{(v)} \bmod \mathcal{J}_{CM}^n = \mathcal{A}_{CM}^{(v_0)} \bmod \mathcal{J}_{CM}^n$ , for any  $v \in (v_0 - \delta, v_0)$ .*

*Proof.* This follows directly from the definition of the ramification filtration and the fact that the field of definition of each projection  $f_{CM} \bmod \mathcal{J}_{CM}^n$  of  $f$  to  $\mathcal{A}_{CMK(p)} \bmod \mathcal{J}_{CMK(p)}^n$  is a finite extension of  $K$ , cf. Subsection 1.3.  $\square$

Notice also that the class field theory implies the following property.

**Proposition 4.2.** *If  $v \geq 0$  and  $\mathcal{A}_k^{(v)} := \mathcal{A}^{(v)} \otimes W(k)$  then  $\mathcal{A}_k^{(v)} \bmod \mathcal{J}_k^2$  is topologically generated by all elements  $p^s D_{an}$ , for  $n \in \mathbb{Z} \bmod N_0$ ,  $a \in \mathbb{Z}(p)$ ,  $s \geq 0$  and  $p^s a \geq v$ .*

**4.2. The filtration  $\mathcal{A}(v)$ ,  $v \geq 0$ .** For any  $\gamma \geq 0$ , introduce  $\mathcal{F}_\gamma \in \mathcal{A}_k$  as follows.

If  $\gamma = 0$  let  $\mathcal{F}_\gamma = D_0$ .

If  $\gamma > 0$  let  $\mathcal{F}_\gamma =$

$$p^{v_\gamma} a_\gamma D_{a_\gamma v_\gamma} - \sum_{\substack{a_1, a_2 \in \mathbb{Z}(p) \\ n \geq 0 \\ p^n(a_1 + a_2) = \gamma}} p^n a_1 D_{a_1 n} D_{a_2 n} - \sum_{\substack{a_1, a_2 \in \mathbb{Z}(p) \\ n_1 \geq 0, n_2 < n_1 \\ p^{n_1} a_1 + p^{n_2} a_2 = \gamma}} p^{n_1} a_1 [D_{a_1 n_1}, D_{a_2 n_2}].$$

Here the first two terms appear only if  $\gamma \in \mathbb{N}$ , and the corresponding  $v_\gamma \in \mathbb{Z}_{\geq 0}$  and  $a_\gamma \in \mathbb{Z}(p)$  are uniquely determined from the equality  $\gamma = p^{v_\gamma} a_\gamma$ . If  $\gamma \notin \mathbb{Z}$  then the above formula for  $\mathcal{F}_\gamma$  contains only the last sum.

For any  $v \geq 0$ , let  $\mathcal{A}(v)$  be the minimal closed ideal in  $\mathcal{A}$  such that  $\mathcal{F}_\gamma \in \mathcal{A}(v) := \mathcal{A}^{(v)} \otimes W(k)$ , for all  $\gamma \geq v$ . Equivalently,  $\mathcal{A}(v)$  is the minimal  $\sigma$ -invariant closed ideal of  $\mathcal{A}_k$ , which contains all  $\mathcal{F}_\gamma$  with  $\gamma \geq v$ .

**Remark.** a) For any  $v \geq 0$ ,  $\mathcal{A}^{(v)} \bmod \mathcal{J}^2 = \mathcal{A}(v) \bmod \mathcal{J}^2$ .

b) The filtration  $\{\mathcal{A}(v) \mid v \geq 0\}$  is left-continuous.

c) If  $C, M \in \mathbb{N}$  and  $\mathcal{A}_{CM}(v) \bmod \mathcal{J}_{CM}^n$  is the image of  $\mathcal{A}(v)$  in  $\mathcal{A}_{CM} \bmod \mathcal{J}_{CM}^n$ , then  $\mathcal{A}(v) \bmod \mathcal{J}^n = \varinjlim_{\overline{C, M}} \mathcal{A}_{CM}(v) \bmod \mathcal{J}_{CM}^n$ .

If  $\gamma \geq v_0 \geq 0$ , denote by  $\tilde{\mathcal{F}}_\gamma(v_0)$  the elements in  $\mathcal{A}_k$  given by the same expressions as  $\mathcal{F}_\gamma$  but with the additional restriction  $p^{n_1} a_1, p^{n_1} a_2 < v_0$  for all degree 2 terms  $p^{n_1} a_1 D_{a_1 n_1} D_{a_2 n_2}$  or  $p^{n_1} a_1 [D_{a_1 n_1}, D_{a_2 n_2}]$ . Clearly, we have the following property.

**Proposition 4.3.** a)  $\mathcal{A}(v_0) \bmod \mathcal{J}^3$  is the minimal ideal of  $\mathcal{A}$  such that  $\mathcal{A}_k(v_0)$  is generated by all elements  $\tilde{\mathcal{F}}_\gamma(v_0)$  with  $\gamma \geq v_0$ .

b) If  $\gamma \geq 2v_0$ , then  $\tilde{\mathcal{F}}_\gamma(v_0) = \gamma D_{a_\gamma v_\gamma}$ .

The following theorem is the main technical result about the structure of the ramification filtration that we need in this paper.

**Theorem B.** For any  $v \geq 0$ ,  $\mathcal{A}^{(v)} \bmod \mathcal{J}^3 = \mathcal{A}(v) \bmod \mathcal{J}^3$ .

This theorem gives an explicit description of the ramification filtration  $\{\mathcal{A}^{(v)}\}_{v \geq 0}$  on the level of  $p$ -extensions of nilpotent class 2. (On the level of abelian  $p$ -extensions such a description is given by the above Remark a.) Theorem B can also be stated in the following equivalent form, where we use the index  $M + 1$  instead of  $M$  to simplify the notation in its proof below.

**Theorem B'.** Suppose  $C \in \mathbb{N}$ ,  $M \in \mathbb{Z}_{\geq 0}$  and  $v_0 > 0$ . If, for all  $v > v_0$ ,

$$\mathcal{A}_{C,M+1}^{(v)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v) \bmod \mathcal{J}_{C,M+1}^3,$$

then

$$\mathcal{A}_{C,M+1}^{(v_0)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v_0) \bmod \mathcal{J}_{C,M+1}^3.$$

Clearly, Theorem B' follows from theorem B.

Conversely, notice first that, for a given  $C \in \mathbb{N}$ ,  $M \geq 0$  and  $v \gg 0$ ,

$$\mathcal{A}_{C,M+1}^{(v)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v) \bmod \mathcal{J}_{C,M+1}^3 = 0.$$

Indeed, this is obvious for the ideals  $\mathcal{A}_{C,M}(v)$ , because they are generated by the elements obtained from the above elements  $\tilde{\mathcal{F}}_\gamma(v)$  by adding the restrictions  $a_1, a_2, a_\gamma < C$  and  $n_1, n_2, v_\gamma \leq M$ . But then, for  $\gamma \geq 2p^M C$ , the conditions  $p^{n_1} a_1 + p^{n_2} a_2 = \gamma$  (where  $n_2 \leq n_1$ ) and  $p^{v_\gamma} a_\gamma = \gamma$  are never satisfied. For the filtration  $\{\mathcal{A}^{(v)}\}_{v \geq 0}$ , we notice, as earlier, that the field of definition  $K_{C,M+1,3}(f)$  of the image of  $f$  in  $\mathcal{A}_{C,M+1,K(p)} \bmod \mathcal{J}_{C,M+1,K(p)}^3$  is of finite degree over the basic field  $K$ . Therefore, for  $v \gg 0$ , the ramification subgroup  $\Gamma(p)^{(v)}$  acts trivially on  $K_{C,M+1,3}(f)$  and  $\mathcal{A}_{C,M+1}^{(v)} \bmod \mathcal{J}_{C,M+1}^3 = 0$ .

Now we can apply descending transfinite induction on  $v \geq 0$ . Let

$$S_{C,M+1} = \{v \geq 0 \mid \mathcal{A}_{C,M+1}^{(v)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v) \bmod \mathcal{J}_{C,M+1}^3\}.$$

Then  $S_{C,M+1} \neq \emptyset$ . Let  $v_0 = \inf S_{C,M+1}$ .

If  $v_0 > 0$  then  $\mathcal{A}_{C,M+1}^{(v_0)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v_0) \bmod \mathcal{J}_{C,M+1}^3$  by Theorem B'. By the left-continuity property of both filtrations, there is a  $\delta \in (0, v_0)$  such that  $\mathcal{A}_{C,M+1}^{(v)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(v) \bmod \mathcal{J}_{C,M+1}^3$  whenever  $v \in (v_0 - \delta, v_0)$ . So,  $v_0 = \inf S_{C,M+1} \leq v_0 - \delta$ . This is a contradiction, hence  $v_0 = 0$ . In this case we have  $\mathcal{A}_{C,M+1}^{(0)} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1} \bmod \mathcal{J}_{C,M+1}^3 = \mathcal{A}_{C,M+1}(0) \bmod \mathcal{J}_{C,M+1}^3$ . This implies that  $S_{C,M+1} = \mathbb{R}_{\geq 0}$ , and Theorem B is deduced from Theorem B'.

The rest of this section is concerned with a proof of Theorem B'.

### 4.3. Auxiliary results.

**4.3.1. The field  $K(N^*, r^*)$ .** Suppose  $N^* \in \mathbb{N}$ ,  $q = p^{N^*}$  and  $r^* = m^*/(q-1)$ , where  $m^* \in \mathbb{Z}(p)$ . Then there is a field  $K_1 := K(N^*, r^*) \subset K_{\text{sep}}$  such that

- a)  $[K_1 : K] = q$ ;
- b) the Herbrand function  $\varphi_{K_1/K}(x)$  has only one corner point  $(r^*, r^*)$ ;
- c)  $K_1 = k((t_{K_1}))$ , where  $t_{K_1}^q E(-1, t_{K_1}^{m^*}) = t_K$  and  $E$  is the generalised Artin-Hasse exponential introduced in n.1.4.

The field  $K(N^*, r^*)$  appears as a subfield of  $K(U)$ , where  $U^q - U = u^{-m^*}$  and  $u^{q-1} = t_K$ . It is of degree  $q$  over  $K$ . Its construction is explained in all detail in [2].

**4.3.2. Relation between liftings of  $K$  and  $K_1$  modulo  $p^{M+1}$ ,  $M \geq 0$ .** Recall that we use the uniformiser  $t_K$  in  $K$  to construct the liftings modulo  $p^{M+1}$  of  $K$ ,  $O_{M+1}(K) = W_{M+1}(k)((t))$  and of  $K(p)$ ,  $O_{M+1}(K(p))$ , where  $t = t_{K, M+1}$ . We use the uniformiser  $t_{K_1}$  from above n.4.3.1 c) to construct analogous liftings for  $K_1$ ,  $O'_{M+1}(K_1) = W_{M+1}(k)((t_1))$  and for  $K_1(p) \supset K(p)$ ,  $O'_{M+1}(K_1(p))$ . (Here  $t_1 = t_{K_1, M+1}$  is the Teichmüller representative of  $t_{K_1}$  in  $W_{M+1}(K_1(p))$ .)

Note that, with the above notation the field embedding  $K \subset K_1$  does not induce an embedding  $O_{M+1}(K) \subset O'_{M+1}(K_1)$  for  $M \geq 1$ , because the Teichmüller representative  $t_1 = t_{K_1, M+1} = [t_{K_1}]$  cannot be expressed in terms of the Teichmüller representative  $t = t_{K, M+1} = [t_K]$ . This difficulty can be overcome as follows. Take  $t_K^{p^M}$  as a uniformising element for  $\sigma^M K$  and consider the corresponding liftings modulo  $p^{M+1}$ ,  $O_{M+1}(\sigma^M K) = W_{M+1}(k)((t^{p^M}))$  and  $O_{M+1}(\sigma^M K(p)) \subset O_{M+1}(K(p))$ . From the definition of liftings it follows that

$$\begin{aligned} O_{M+1}(\sigma^M K) \subset W_{M+1}(\sigma^M K) \subset W_{M+1}(\sigma^M K_1) \\ \subset O'_{M+1}(K_1) \subset W_{M+1}(K_1), \end{aligned}$$

$$\begin{aligned} O_{M+1}(\sigma^M K(p)) \subset W_{M+1}(\sigma^M K(p)) \subset W_{M+1}(\sigma^M K_1(p)) \\ \subset O'_{M+1}(K_1(p)) \subset W_{M+1}(K_1(p)). \end{aligned}$$

**Lemma 4.4.** *With respect to the above embedding  $O_{M+1}(\sigma^M K) \subset O'_{M+1}(K_1)$  we have*

$$t^{p^M} = t_1^{qp^M} E(-1, t_1^{m^*})^{p^M}.$$

*Proof.* If  $V$  is the Verschiebung morphism on  $W_{M+1}(K_1)$  then property c) from n.4.3.1 is equivalent to the relation

$$t \equiv t_1^{qp^M} E(-1, t_1^{m^*}) \pmod{VW_{M+1}(K_1)}.$$

Then, for any  $s \geq 0$ , we have

$$t^{p^s} \equiv t_1^{qp^s} E(-1, t_1^{m^*})^{p^s} \pmod{V^{s+1}W_{M+1}(K_1)}.$$

(Using that for any  $w_1, w_2 \in W_M(K_1)$ ,  $(Vw_1)(Vw_2) = V^2(F(w_1w_2))$  and  $pV(w_1) = V^2(Fw_1)$ .) For  $s = M$  we obtain the statement of the lemma.  $\square$

**4.3.3. A criterion.** Consider  $\sigma^M e = 1 + \sum_{a \in \mathbb{Z}^0(p)} t^{-ap^M} D_{a,M} \in \mathcal{A} \otimes O(\sigma^M K)$ , where  $O(\sigma^M K) = \varprojlim_n O_n(\sigma^M K)$ . Then  $\sigma^M f \in \mathcal{A} \otimes O(\sigma^M K(p))$  satisfies the relation  $\sigma(\sigma^M f) = (\sigma^M f)(\sigma^M e)$  and induces the same morphism  $\psi : \Gamma(p) \rightarrow \mathcal{A}$  as  $f$ . Indeed, for any  $\tau \in \Gamma(p)$ ,

$$\tau(\sigma^M f)(\sigma^M f)^{-1} = \sigma^M(\tau(f)f^{-1}) = \sigma^M(\psi(\tau)) = \psi(\tau)$$

because  $\sigma$  acts trivially on  $\mathcal{A}$ .

This means that we can still study the structure of the ramification filtration  $\{\mathcal{A}^{(v)} \pmod{p^{M+1}}\}_{v \geq 0}$  by working inside the lifting  $O'_{M+1}(K_1(p)) \supset O_{M+1}(\sigma^M K(p))$  associated with our auxiliary field  $K_1$  and its uniformiser  $t_{K_1}$ .

Set  $\mathcal{B} = \mathcal{A}_{C,M+1} \pmod{\mathcal{J}_{C,M+1}^3}$  and for any rational number  $v \geq 0$ ,  $\mathcal{B}^{(v)} = \mathcal{A}_{C,M+1}^{(v)} \pmod{\mathcal{J}_{C,M+1}}$ . We shall also use the notation  $\mathcal{B}_k = \mathcal{B} \otimes W_{M+1}(k)$ ,  $\mathcal{B}_{K_1} = \mathcal{B} \otimes O'_{M+1}(K_1)$ , and  $\mathcal{B}_{K_1(p)} = \mathcal{B} \otimes O'_{M+1}(K_1(p))$ . Denote again by  $\mathcal{J}$  the augmentation ideal in  $\mathcal{B}$ . Its extensions of scalars will be denoted similarly by  $\mathcal{J}_k, \mathcal{J}_{K_1}$  and  $\mathcal{J}_{K_1(p)}$ .

Consider an abstract continuous field isomorphism  $\alpha : K \rightarrow K_1$ , which is the identity on the residue fields and sends  $t_K$  to  $t_{K_1}$ . Consider its extension to the field isomorphism  $\hat{\alpha} : K(p) \rightarrow K_1(p)$ . Then we have an induced isomorphism of liftings  $\hat{\alpha} : O_{M+1}(K(p)) \rightarrow O'_{M+1}(K_1(p))$ . Use it to define the morphism

$$\text{id} \otimes \hat{\alpha} : \mathcal{A}_{C,M+1,K(p)} \rightarrow \mathcal{B}_{K_1(p)}$$

and set  $f_1 := (\text{id} \otimes \hat{\alpha})(f) \in \mathcal{B}_{K_1(p)}$ . Then  $\sigma(f_1) = f_1 e_1$ , where  $e_1 = (\text{id} \otimes \hat{\alpha})(e) = 1 + \sum_{a \in \mathbb{Z}^0(p)} t_1^{-a} D_{a0}$ .

If  $N^* \equiv 0 \pmod{N_0}$ , then  $\sigma^{M+N^*}(D_{a0}) = \sigma^M(D_{a0}) = D_{aM}$  and we can relate the elements  $\sigma^M e = 1 + \sum_{a \in \mathbb{Z}^0(p)} t^{-ap^M} D_{a,M}$  and  $\sigma^{M+N^*} e_1 = 1 + \sum_{a \in \mathbb{Z}^0(p)} t_1^{-ap^{M+q}} D_{a,M}$  by the use of the relation between  $t$  and  $t_1$  from lemma 4.4. So, it will be natural to compare the elements  $\sigma^M f$  and  $\sigma^{M+N^*} f_1$  in  $\mathcal{B}_{K_1(p)}$  by introducing  $X \in \mathcal{B}_{K_1(p)}$  such that  $(\sigma^M f)(1 + X) = \sigma^{M+N^*} f_1$ . This element will be used for the characterisation of the ideal  $\mathcal{B}^{(v_0)}$  in proposition 4.5 below.

Notice first, that  $\mathcal{B}^{(v_0)}$  is the minimal 2-sided ideal in  $\mathcal{B}$  such that the field of definition of  $f \pmod{\mathcal{B}_{K_1(p)}^{(v_0)}}$  is invariant under the action of the group

$\Gamma(p)^{(v_0)}$ . In other words, if  $I$  is a 2-sided ideal in  $\mathcal{B}$  and  $K(f, I)$  is the field of definition of  $f \bmod I_{K_1(p)}$ , then  $I$  contains  $\mathcal{B}^{(v_0)}$  if and only if the largest upper ramification number  $v(K(f, I)/K)$  (= the 2nd coordinate of the last vertex of the graph of the Herbrand function  $\varphi_{K(f, I)/K}$ ) is less than  $v_0$ .

With the above notation we have the following criterion.

**Proposition 4.5.** *Suppose  $r^* = v(K_1/K) < v_0$ . Then  $\mathcal{B}^{(v_0)}$  is the minimal element in the set of all 2-sided ideals  $I$  such that if  $K_1(X, I)$  is the field of definition of  $X \bmod I_{K_1(p)}$  over  $K_1$  then its largest upper ramification number satisfies  $v(K_1(X, I)/K_1) < qv_0 - r^*(q - 1)$ .*

*Proof.* We must prove that for any 2-sided ideal  $I$  in  $\mathcal{B}$ ,

$$v := v(K(f, I)/K) < v_0 \iff v_1(X) := v(K_1(X, I)/K_1) < qv_0 - r^*(q - 1).$$

The following proof is similar to the proof of the corresponding statement from [1, 2].

Suppose  $v < v_0$ . The existence of the field isomorphism  $\hat{\alpha}$  implies that  $v(K_1(f_1, I)/K_1) = v$ . Then

$$(4.1) \quad v_1 := v(K_1(f_1, I)/K) = \max\{r^*, \varphi_{K_1/K}(v)\}$$

Indeed, it is sufficient to look at the maximal vertex of the Herbrand function for the extension  $K_1(f_1, I)/K$  and to use the composition property for the corresponding Herbrand functions  $\varphi_{K_1(f_1, I)/K}(x) = \varphi_{K_1/K}(\varphi_{K_1(f_1, I)/K_1}(x))$ . This implies that  $v_1 = r^*$  if  $r^* \geq v$  and  $v_1 < v$  if  $v > r^*$ , where we have used that  $\varphi_{K_1/K}(v) = r^* + (v - r^*)/q < v$  if  $v > r^*$ . Therefore, the largest upper ramification number of the composite  $K(f, I)$  and  $K_1(f_1, I)$  over  $K$  is  $\max\{r^*, v\} < v_0$ . Clearly,  $K_1(X, I)$  is contained in this composite and, therefore,  $v(X) := v(K_1(X, I)/K) < v_0$ . Similarly to formula (4.1) we obtain that  $v(X) = \max\{r^*, \varphi_{K_1/K}(v_1(X))\}$ . Therefore,  $\varphi_{K_1/K}(v_1(X)) < v_0$  and  $v_1(X) < qv_0 - r^*(q - 1)$ .

Conversely, assume that  $v_1(X) < qv_0 - r^*(q - 1)$ . Then

$$v(X) = \max\{r^*, \varphi_{K_1/K}(v_1(X))\} < v_0.$$

Suppose  $v = v(K(f, I)/K) \geq v_0$ . As earlier, the existence of  $\hat{\alpha}$  implies that  $v(K_1(f_1, I)/K_1) = v$  and similarly to (4.1) we have

$$v(K_1(f_1, I)/K) = \max\{r^*, \varphi_{K_1/K}(v)\} = \varphi_{K_1/K}(v) < v.$$

Therefore, the largest upper ramification number of the composite of  $K_1(X, I)$  and  $K_1(f_1, I)$  over  $K$  equals

$$\max\{v(K_1(X, I)/K), v(K_1(f_1, I)/K)\} = \max\{v(X), \varphi_{K_1/K}(v)\}.$$

Because  $K(f, I)$  is contained in this composite, we have

$$v \leq \max\{v(X), \varphi_{K_1/K}(v)\}.$$

But  $v \geq v_0 > v(X)$  and  $v > \varphi_{K_1/K}(v)$ . This contradiction proves the proposition.  $\square$

**4.3.4. Choosing  $N^*$  and  $r^*$ .** In order to apply the criterion from Proposition 4.5 we shall use the special choice of  $K_1 = K(N^*, r^*)$ , where  $N^* \in \mathbb{N}$  and  $r^* < v_0$  are specified as follows.

Introduce  $\delta_1 := \min\{v_0 - p^s a \mid p^s a < v_0, a \leq C, a \in \mathbb{Z}^0(p)\}$ , and  $\delta_2 :=$

$$\min\{v_0 - (p^{s_1} a_1 + p^{s_2} a_2) \mid p^{s_1} a_1 + p^{s_2} a_2 < v_0, a_1, a_2 \leq C, a_1, a_2 \in \mathbb{Z}^0(p), s_1, s_2 \in \mathbb{Z}\}.$$

One can see that for a sufficiently large natural number  $N^* \equiv 0 \pmod{N_0}$ , there exists  $r^* = m^*/(q-1) < v_0$  with  $q = p^{N^*}$  and  $m^* \in \mathbb{Z}(p)$  such that

- a)  $-(v_0 - \delta_1)q + r^*(q-1) > Cp^M$ ;
- b)  $-(v_0 - \delta_2)q + r^*(q-1) > 0$ ;
- c)  $v_0 q < 2r^*(q-1)$ .

So, we may assume that  $K_1 = K(N^*, r^*)$  where  $N^* \equiv 0 \pmod{N_0}$  and the above inequalities a)-c) hold.

**4.4. A recurrence formula for  $X$ .** Set  $\Theta^* = t_1^{r^*(q-1)}$ . Then

$$\omega = \sigma^M e - \sigma^{M+N^*} e_1 = \sum_{a \in \mathbb{Z}^0(p)} t_1^{-ap^M q} (E(a, \Theta^*)^{p^M} - 1) D_{aM} \in \mathcal{J}_{K_1}.$$

The relation  $1 + X = (\sigma^M f)^{-1}(\sigma^{M+N^*} f_1)$  implies that

$$1 + \sigma X = (\sigma^M e)^{-1}(1 + X)(\sigma^{M+N^*} e_1)$$

and

$$(4.2) \quad X - \sigma X = \omega + (\sigma^M e - 1)\sigma X - X(\sigma^{M+N^*} e_1 - 1).$$

If  $\bar{X} := X \pmod{\mathcal{J}_{K_1(p)}^2}$ , then the above relation (4.2) gives  $\bar{X} - \sigma \bar{X} = \omega \pmod{\mathcal{J}_{K_1(p)}^2}$ . We shall use this relation in Subsection 4.5 below to study  $\bar{X}$ . Now (4.2) can be rewritten as

$$(4.3) \quad X - \sigma X = \omega - \omega(\sigma^{M+N^*} e_1 - 1) - [\sigma \bar{X}, \sigma^{M+N^*} e_1 - 1] + \omega \sigma(\bar{X}),$$

using that  $X \equiv \omega + \sigma X \pmod{\mathcal{J}_{K_1(p)}^2}$ . We shall use this relation in nn.4.6-4.7 below to study the field of definition of  $X$ .



**4.5. The study of  $\bar{X}$ .** For  $0 \leq r \leq M$  and  $b \in \mathbb{Z}_p$ , introduce  $\mathcal{E}_r(b, T) \in \mathbb{Z}_p[[T]]$  as follows:

$\mathcal{E}_0(b, T) = E(b, T) - 1$ , where  $E(b, T)$  is the generalisation of the Artin-Hasse exponential from n.1.4;

$$\mathcal{E}_1(b, T) = E(b, T)^p - E(b, T^p) = (\exp(pbT) - 1)E(b, T^p),$$

.....

$$\mathcal{E}_M(b, T) = E(b, T)^{p^M} - E(b, T^p)^{p^{M-1}} = (\exp(p^M bT) - 1)E(b, T^p)^{p^{M-1}}.$$

Notice the following simple properties:

- (1)  $E(b, T)^{p^M} - 1 = \mathcal{E}_0(b, T^{p^M}) + \mathcal{E}_1(b, T^{p^{M-1}}) + \cdots + \mathcal{E}_M(b, T)$ ;
- (2)  $\mathcal{E}_r(b, T) = p^r T + p^r T^2 g_r(T)$ , where  $0 \leq r \leq M$  and  $g_r \in \mathbb{Z}_p[[T]]$ .

Consider the decomposition  $\omega = \sum_{r+s=M} \sigma^r \omega_s$  (cf. Subsection 4.4 for the definition of  $\omega$ ), where

$$\omega_s := \sum_{a \in \mathbb{Z}^0(p)} t_1^{-ap^s q} \mathcal{E}_s(a, \Theta^*) D_{as},$$

for  $0 \leq s \leq M$ . Note that  $p^s D_{as} \in \mathcal{B}_k^{(v_0)} \bmod \mathcal{J}_k^2$ , whenever  $p^s a \geq v_0$ , cf. proposition 4.2. Also, if  $p^s a < v_0$  then  $-ap^s q + r^*(q-1) > Cp^M$ , cf. Subsubsection 4.3.4, and we have  $t_1^{-ap^s q} \mathcal{E}_s(a, \Theta^*) \in t_1^{Cp^M} \mathfrak{m}_1$ , where  $\mathfrak{m}_1 := t_1 W_M(k)[[t_1]]$ .

So, for  $0 \leq s \leq M$ ,

$$(4.4) \quad \omega_s \in \mathcal{B}_{K_1}^{(v_0)} + t_1^{Cp^M} \mathcal{J}_{\mathfrak{m}_1} + \mathcal{J}_{K_1}^2,$$

where  $\mathcal{J}_{\mathfrak{m}_1} = \mathcal{J} \otimes \mathfrak{m}_1$ .

For  $0 \leq s \leq M$ , consider  $X_s \in \mathcal{B}_{K_1(p)}$  such that  $X_s - \sigma X_s = \omega_s$ . Because of (4.4), we may assume that  $X_s \equiv \sum_{u \geq 0} \sigma^u \omega_s \bmod (\mathcal{B}_{K_1(p)}^{(v_0)} + \mathcal{J}_{K_1(p)}^2)$ . Notice that

$$\bar{X} \equiv \sum_{r+s=M} \sigma^r (X_s) \bmod \mathcal{J}_{K_1(p)}^2,$$

and after replacing the infinite sum  $\sum_{u \geq 0}$  by its first  $(N^* - s)$  terms in the above congruence for  $X_s$ , we obtain

$$(4.5) \quad \bar{X} = \sum_{\substack{u+s \geq M \\ u < N^*}} \sigma^u \omega_s \bmod (\mathcal{B}_{K_1(p)}^{(v_0)} + \mathcal{J}_{K_1(p)}^2 + t_1^{Cp^M q} \mathcal{J}_{\mathfrak{m}_1}).$$

**4.6. The study of  $X$ .** From the above formulas (4.4) it follows that  $\bar{X}$  and  $\sigma(\bar{X})$  belong to  $\mathcal{B}_{K_1(p)}^{(v_0)} + t_1^{Cp^M} \mathcal{J}_{m_1} + \mathcal{J}_{K_1(p)}^2$ . This implies that

$$\omega\sigma(\bar{X}) \in \mathcal{B}_{K_1(p)}^{(v_0)} \mathcal{J}_{K_1(p)} + \mathcal{J}_{m_1}.$$

Therefore, when solving equation (4.3) for  $X$ , this term will not have any influence on the field of definition of  $X \bmod \mathcal{B}_{K_1(p)}^{(v_0)} \mathcal{J}_{K_1(p)}$ .

For a similar reason, we may replace  $\bar{X}$  in (4.3) by the right hand side from (4.5) without affecting the field of definition of  $X \bmod \mathcal{B}_{K_1(p)}^{(v_0)} \mathcal{J}_{K_1(p)}$ . The new right hand side will be then equal to

$$\begin{aligned} & \sum_{\substack{a \in \mathbb{Z}^0(p) \\ 0 \leq s \leq M}} t_1^{-ap^M q} \mathcal{E}_s(a, \Theta^{*p^{M-s}}) \\ & \quad - \sum_{\substack{a_1, a_2 \in \mathbb{Z}^0(p) \\ 0 \leq s \leq M}} t_1^{-(a_1+a_2)p^M q} \mathcal{E}_s(a_1, \Theta^{*p^{M-s}}) D_{a_1 M} D_{a_2 M} \\ & \quad - \sum_{\substack{0 \leq s_1 \leq M, a_1, a_2 \in \mathbb{Z}^0(p) \\ N^* > u > M - s_1}} t_1^{-a_1 p^{s_1+u} - a_2 p^M q} \mathcal{E}_{s_1}(a_1, \Theta^{*p^u}) [D_{a_1, s_1+u}, D_{a_2, M}]. \end{aligned}$$

Finally we can apply the Witt-Artin-Schreier equivalence to the last formula to deduce that modulo any ideal containing the ideal  $\mathcal{B}_{K_1(p)}^{(v_0)} \mathcal{J}_{K_1(p)}$ , the elements  $X$  and  $X'$ , where  $X' - \sigma X' =$

$$\begin{aligned} & \sum_{0 \leq s \leq M} t_1^{-ap^s q} \mathcal{E}_s(a_1, \Theta^*) D_{as} - \sum_{0 \leq s \leq M} t_1^{-(a_1+a_2)p^s q} \mathcal{E}_s(a_1, \Theta^*) D_{a_1 s} D_{a_2 s} \\ & \quad - \sum_{\substack{0 \leq s_1 \leq M \\ M - N^* < s_2 < s_1}} t_1^{-(a_1 p^{s_1} + a_2 p^{s_2}) q} \mathcal{E}_{s_1}(a, \Theta^*) [D_{a_1 s_1}, D_{a_2 s_2}] \end{aligned}$$

have the same field of definition.

We can use this relation to find the minimal ideal  $I$  in  $\mathcal{B}$  such that  $X \bmod I_{K_1(p)}$  is defined over an extension of  $K_1$  with upper ramification number less than  $qv_0 - r^*(q-1)$ . Indeed, we know that  $I \bmod \mathcal{J}^2 = \mathcal{B}^{(v_0)} \bmod \mathcal{J}^2$  and therefore, we may always assume that  $I \supset \mathcal{B}^{(v_0)} \mathcal{J}$ . As before, we are also allowed to change the right hand side of (4.6) by any element of  $\mathcal{B} \otimes \mathcal{J}_{m_1}$ . We may always assume that  $I \supset \mathcal{B}^{(v)}$  for any  $v > v_0$ , because  $I$  must contain all  $\mathcal{B}^{(v)}$  with  $v > v_0$  and, by the inductive assumption,  $\mathcal{B}^{(v)}$  coincides with  $\mathcal{B}^{(v)}$ . So, we can assume that  $I$  contains the ideal  $\mathcal{B}^{(v_0+)}$  generated by  $\mathcal{B}^{(v_0)} \mathcal{J}$  and all  $\mathcal{B}^{(v)}$  with  $v > v_0$ .

**4.7. Final simplification of (4.6).** For  $0 \leq s \leq M$ , consider the identity  $\mathcal{E}_s(a, \Theta^*) = p^s a t_1^{r^*(q-1)} + p^s t_1^{2r^*(q-1)} g_r(t_1)$  from Subsection 4.5.

**Lemma 4.6.**  $p^s t_1^{-(a_1+a_2)p^s q + 2r^*(q-1)} D_{a_1 s} D_{a_2 s} \in \mathcal{B}_{K_1}^{(v_0)} \mathcal{J}_{K_1} + \mathcal{J}_{m_1}$ .

*Proof.* Indeed, if  $p^s a_1 \geq v_0$  (resp. if  $p^s a_2 \geq v_0$ ) then  $p^s D_{a_1 s}$  (resp.  $p^s D_{a_2 s}$ ) belongs to  $\mathcal{B}_k^{(v_0)} \bmod \mathcal{J}_k^2$ .

If both  $p^s a_1, p^s a_2$  are less than  $v_0$  then we use the fact that

$$-(a_1 + a_2)p^s q + 2r^*(q-1) > Cp^M + Cp^M > 0,$$

cf. Subsubsection 4.3.4, to conclude that the corresponding term belongs to  $\mathcal{J}_{m_1}$ .

The lemma is proved □

The following lemma deals with the terms coming from the third sum and can be proved similarly.

**Lemma 4.7.**  $p^{s_1} t_1^{-(p^{s_1} a_1 + p^{s_2} a_2)q + 2r^*(q-1)} [D_{a_1 s_1}, D_{a_2 s_2}] \in \mathcal{B}_{K_1}^{(v_0)} \mathcal{J}_{K_1} + \mathcal{J}_{m_1}$ .

The next lemma deals with the terms coming from the first sum.

**Lemma 4.8.**  $p^s t_1^{-ap^s q + 2r^*(q-1)} D_{as} \in \mathcal{B}_{K_1}^{(v_0+)} + \mathcal{J}_{m_1}$ .

*Proof.* There is nothing to prove if  $-ap^s q + 2r^*(q-1) > 0$ .

Assume now that  $ap^s q \geq 2r^*(q-1)$ . Consider the expression for  $\mathcal{F}_{ap^s}$ , cf. Subsection 4.2. Notice that  $ap^s > v_0$  (use estimate c) from n.4.3.4) and, therefore,  $\mathcal{F}_{ap^s} \in \mathcal{B}_k(ap^s) = \mathcal{B}_k^{(ap^s)}$ .

It will be sufficient to show that any term of degree 2 in the expression of  $\mathcal{F}_{ap^s}$  belongs to  $\mathcal{B}_k^{(v_0)} \mathcal{J}_k$ . Indeed, it then follows that the linear term  $p^s a D_{as}$  of  $\mathcal{F}_{ap^s}$  belongs to  $\mathcal{B}_k^{(ap^s)} + \mathcal{B}_k^{(v_0)} \mathcal{J}_k \subset \mathcal{B}_k^{(v_0+)}$  and the statement of our lemma is proved.

In order to prove this property of degree 2 terms notice that all of them contain as a factor either a product  $p^{s_1} D_{a_1 s_1} D_{a_2 s_2}$  or a product  $p^{s_1} D_{a_2 s_2} D_{a_1 s_1}$ , where  $s_1 \geq s_2$  and  $p^{s_1} a_1 + p^{s_2} a_2 = p^s a$ . Then we have the following two cases:

- (1) if either  $p^{s_1} a_1 \geq v_0$  or  $p^{s_1} a_2 \geq v_0$  then this product belongs to  $\mathcal{B}_k^{(v_0)} \mathcal{J}_k$ ;
- (2) if both  $p^{s_1} a_1$  and  $p^{s_1} a_2$  are less than  $v_0$ , then  $p^{s_1} a_1 < v_0 - \delta_1$  and  $p^{s_2} a_2 \leq p^{s_1} a_2 < v_0 - \delta_1$ . Therefore,

$$2r^*(q-1) \leq p^s a q = (p^{s_1} a_1 + p^{s_2} a_2)q < 2q(v_0 - \delta_1).$$

This contradicts the assumption a) from Subsubsection 4.3.4.

The lemma is completely proved. □

By the above three lemmas, we can everywhere replace the factors  $\mathcal{E}_s(a, \Theta^*)$  by  $p^s a t_1^{r^*(q-1)}$  and, therefore, the right hand side of (4.6) is congruent modulo  $\mathcal{B}_{K_1}^{(v_0+)} + \mathcal{J}_{m_1}$  to the sum  $\sum_{\gamma \geq 0} t_1^{-q\gamma+r^*(q-1)} \mathcal{F}'_\gamma$ , where  $\mathcal{F}'_\gamma$  is given by the same formula as  $\mathcal{F}_\gamma$ , cf. Subsection 4.2, but with the additional restriction  $n_2 > M - N^*$  in the last sum.

**Lemma 4.9.** *If  $\gamma \geq v_0$  then  $\mathcal{F}'_\gamma \equiv \mathcal{F}_\gamma \pmod{\mathcal{B}_k^{(v_0)} \mathcal{J}_k}$ .*

*Proof.* Suppose the term  $p^{n_1} a_1 [D_{a_1 n_1}, D_{a_2 n_2}]$  enters into the formula for  $\mathcal{F}_\gamma$  but does not enter into the formula for  $\mathcal{F}'_\gamma$ .

Then  $a_1, a_2 \leq C$ ,  $p^{n_1} a_1 + p^{n_2} a_2 = \gamma \geq v_0$  and  $n_2 \leq M - N^*$ . Then

$$p^{n_1} a_1 = \gamma - p^{n_2} a_2 \geq v_0 - p^M q^{-1} C > r^*(1 - q^{-1}) - p^M q^{-1} C > v_0 - \delta_1$$

(use 4.3.2 a)). Therefore,  $p^{n_1} a_1 \geq v_0$ ,  $p^{n_1} D_{a_1 n_1} \in \mathcal{B}_k^{(v_0)} \mathcal{J}_k^2$  and  $p^{n_1} a_1 [D_{a_1 n_1}, D_{a_2 n_2}] \in \mathcal{B}_k^{(v_0)} \mathcal{J}_k$ .

The lemma is proved.  $\square$

Now notice that:

- if  $\gamma > v_0$ , then the term  $t_1^{-q\gamma+r^*(q-1)} \mathcal{F}_\gamma$  belongs to  $\mathcal{B}_{K_1}(\gamma) = \mathcal{B}_{K_1}^{(\gamma)}$ ;
- if  $\gamma < v_0$ , then the term  $t_1^{-q\gamma+r^*(q-1)} \mathcal{F}'_\gamma$  belongs to  $\mathcal{J}_{m_1}$ .

So, the ideal  $\mathcal{B}^{(v_0)}$  appears as the minimal ideal  $I$  of  $\mathcal{B}$  such that  $I$  contains the ideal  $\mathcal{B}^{(v_0+)}$  and such that the largest upper ramification number of the field of definition over  $K_1$  of the solution  $X'' \in \mathcal{B}_{K_1(p)} \pmod{I_{K_1(p)}}$  of the equation

$$X'' - \sigma X'' = \mathcal{F}_{v_0} t_1^{-qv_0+r^*(q-1)} \pmod{I_{K_1(p)}}$$

is less than  $qv_0 - r^*(q - 1)$ .

It only remains to notice that  $p\mathcal{F}_{v_0} \in \mathcal{B}_k^{(v_0+)}$ , and if  $\mathcal{F}_{v_0} \notin I_k$  then the upper ramification number of the field of definition  $K_1(X'', I)$  over  $K_1$  is equal to  $qv_0 - r^*(q - 1)$ .

The theorem is proved.

## 5. Compatibility with ramification filtration

In this section with the notation from Section 1,  $A = \mathcal{A} \pmod{\mathcal{J}^3}$ ,  $A_k = A \otimes W(k)$ . For any  $v \geq 0$ ,  $A^{(v)} = \mathcal{A}^{(v)} \pmod{\mathcal{J}^3}$ ,  $A_k^{(v)} := A^{(v)} \otimes W(k)$ . We also set  $J = \mathcal{J} \pmod{\mathcal{J}^3}$  with the corresponding extension of scalars  $J_k = J \otimes W(k)$ . Suppose  $f$  is a continuous automorphism of the  $\mathbb{Z}_p$ -algebra  $A$  such that, for any  $v \geq 0$ ,  $f(A^{(v)}) = A^{(v)}$ . Consider the identification  $\mathcal{J} \pmod{\mathcal{J}^2} = \Gamma(p)^{\text{ab}}$  from part b) of proposition 1.2 and denote again by  $f$  the continuous automorphism of  $\mathcal{M} = I(p)^{\text{ab}} \pmod{p}$  induced by  $f$ . Consider

the standard topological generators  $D_{an}$ ,  $a \in \mathbb{Z}(p)$ ,  $n \in \mathbb{Z} \bmod N_0$ , for  $\mathcal{M}$  and set, for any  $a \in \mathbb{Z}(p)$ ,

$$f(D_{a0}) = \sum_{b,m} \alpha_{abm}(f) D_{bm},$$

where the coefficients  $\alpha_{abm}(f) \in k$ . With the above notation, the principal results of this section are:

if  $\alpha_{110}(f) \neq 0$  and  $N_0 \geq 3$  then

- there is an  $\eta \in \text{Aut}^0 K$  such that for any  $a, b \in \mathbb{Z}(p)$  and  $a \leq b < p^{N_0-3}$ , it holds  $\alpha_{ab0}(f) = \alpha_{ab0}(\eta^*)$ ;
- if  $a \leq b < p^{N_0-3}$  and  $m \in \mathbb{N}$  is such that  $b/p^m < a$  then  $\alpha_{a,b,-m \bmod N_0}(f) = 0$ .

**5.1. The elements  $\mathcal{F}_\gamma(v)$ .** By Theorem B, cf. Subsection 4.2, for any  $v \geq 0$ , the ideal  $A_k^{(v)}$  is the minimal closed  $\sigma$ -invariant ideal in  $A_k$  containing the explicitly given elements  $\mathcal{F}_\gamma$ , for all  $\gamma \geq v$ . For any  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_0$ , set  $\Delta_{a0} = (1/a)\mathcal{F}_a$  and  $\Delta_{an} = \sigma^n \Delta_{a0}$ . Then  $\Delta_{an} \equiv D_{an} \bmod \mathcal{J}_k^2$  and  $\{\Delta_{an} \mid a \in \mathbb{Z}(p), n \in \mathbb{Z} \bmod N_0\} \cup \{D_0\}$  is a new system of topological generators for  $A_k$ . The elements of this new set of generators together with their pairwise products form a topological basis of the  $W(k)$ -module  $A_k$ .

For any  $\gamma \geq v \geq 0$ , consider the following elements  $\mathcal{F}_\gamma(v)$  (these elements have already been mentioned in Subsection 4.2):

If  $\gamma = ap^m$  with  $a \in \mathbb{Z}(p)$  and  $m \in \mathbb{Z}_{\geq 0}$  set

$$\mathcal{F}_\gamma(v) = p^m a \Delta_{am} - \sum_{\substack{n \geq 0, a_1, a_2 \in \mathbb{Z}(p) \\ p^n(a_1+a_2) = \gamma \\ p^n a_1, p^n a_2 < v}} p^n a_1 \Delta_{a_1 n} \Delta_{a_2 n};$$

If  $\gamma \notin \mathbb{Z}$  set

$$\mathcal{F}_\gamma(v) = - \sum_{\substack{n_1 \geq 0, a_1, a_2 \in \mathbb{Z}(p) \\ p^{n_1} a_1 + p^{n_2} a_2 = \gamma \\ p^{n_1} a_1, p^{n_1} a_2 < v}} p^{n_1} a_1 [\Delta_{a_1 n_1}, \Delta_{a_2 n_2}].$$

Similarly to Subsection 4.2, we have the following property.

**Proposition 5.1.** *For any  $v \geq 0$ ,  $A_k^{(v)}$  is the minimal  $\sigma$ -invariant closed ideal of  $A_k$  containing the elements  $\mathcal{F}_\gamma(v)$  for all  $\gamma \geq v$ .*

## 5.2. The submodules $A_{\text{tr}}^{(v)}$ and $A_{\text{adm}}^{(v)}$ . Suppose $v \geq 0$ .

Let  $A_{\text{tr}}^{(v)}$  be the  $W(k)$ -submodule in  $A_k$  generated by the following elements:

$$\text{tr}_1) p^s \Delta_{an} \text{ with } s \geq 0 \text{ and } p^s a \geq 2v;$$

$\text{tr}_2) p^s \Delta_{a_1 n_1} \Delta_{a_2 n_2}$  with  $a_1, a_2 \in \mathbb{Z}(p)$ ,  $s \geq 0$  and  $n_1, n_2 \in \mathbb{Z} \bmod N_0$  such that  $\max\{p^s a_1, p^s a_2\} \geq v$ .

Let  $A_{\text{adm}}^{(v)}$  be the minimal closed  $W(k)$ -submodule in  $A_k$  containing  $A_{\text{tr}}^{(v)}$  and the following elements:

$$\text{adm}_1) p^s \Delta_{an}, \text{ with } s \geq 0, a \in \mathbb{Z}(p) \text{ and } p^s a \geq v;$$

$\text{adm}_2) p^s \Delta_{a_1 n_1} \Delta_{a_2 n_2}$ , where  $a_1, a_2 \in \mathbb{Z}(p)$ ,  $n_1, n_2 \in \mathbb{Z} \bmod N_0$  and  $s = s(a_1, a_2) \in \mathbb{Z}_{\geq 0}$  are such that:

- (1)  $v/p \leq \max\{p^s a_1, p^s a_2\} < v$ ;
- (2)  $\max\left\{p^s \left(a_1 + \frac{a_2}{p^{n_{12}}}\right), p^s \left(\frac{a_1}{p^{n_{21}}} + a_2\right)\right\} \geq v$ , where  $0 \leq n_{12}, n_{21} < N_0$ ,  $n_{12} \equiv n_1 - n_2 \bmod N_0$  and  $n_{21} \equiv n_2 - n_1 \bmod N_0$ ;
- (3) if  $n_1 = n_2$  then  $a_1 + a_2 \equiv 0 \bmod p$ .

**Proposition 5.2.** *For any  $v \geq 0$ ,*

- 1)  $f(A_{\text{tr}}^{(v)}) = A_{\text{tr}}^{(v)}$ ;
- 2)  $A_{\text{adm}}^{(v)} \supset A_k^{(v)} \supset A_{\text{tr}}^{(v)} \supset pA_{\text{adm}}^{(v)}$ ;
- 3) *the elements from  $\text{adm}_1)$  and  $\text{adm}_2)$  form a  $k$ -basis of  $A_{\text{adm}}^{(v)} \bmod A_{\text{tr}}^{(v)}$ .*

*Proof.* 1) It is sufficient to notice that  $A_{\text{tr}}^{(v)}$  is the minimal  $\sigma$ -invariant  $W(k)$ -submodule in  $A$  containing  $\sum_{\gamma \geq 2v} \mathcal{F}_\gamma(v)W(k) + \sum_{\gamma \geq v} \mathcal{F}_\gamma(v)J_k$ .

2) From the above n.1) it follows that  $A_k^{(v)} \supset A_{\text{tr}}^{(v)}$ . The embedding  $A_k^{(v)} \subset A_{\text{adm}}^{(v)}$  follows from the definition of  $A_{\text{adm}}^{(v)}$ : as a matter of fact,  $A_{\text{tr}}^{(v)}$  is spanned by all summands of elements  $\sigma^s \mathcal{F}_\gamma$  with  $s \in \mathbb{Z} \bmod N_0$  and  $\gamma \geq v$ .

The embedding  $pA_{\text{adm}}^{(v)} \subset A_{\text{tr}}^{(v)}$  follows from the fact that each element listed in  $\text{adm}_1)$  and  $\text{adm}_2)$  belongs to  $A_{\text{tr}}^{(v)}$  after multiplication by  $p$ .

3) It is easy to see that any  $k$ -linear combination of the elements from  $\text{adm}_1)$  and  $\text{adm}_2)$  does not belong to  $A_{\text{tr}}^{(v)} \bmod pA_{\text{adm}}^{(v)}$ .  $\square$

**Proposition 5.3.** *Suppose  $v \geq 0$  and  $p^s \Delta_{a_1 n_1} \Delta_{a_2 n_2}$  is one of elements listed in  $\text{adm}_2)$ . Let  $n = \min\{n_{12}, n_{21}\}$ . If*

$$v/p^{N_0-n} \leq d(v) := \min\{v - a \mid a \in \mathbb{Z}, a < v\}$$

*then there are unique  $m \in \mathbb{Z} \bmod N_0$  and  $\gamma \geq v$  such that  $p^s a_1 \Delta_{a_1 n_1} \Delta_{a_2 n_2}$  appears (with non-zero coefficient) in the expression of  $\sigma^m \mathcal{F}_\gamma(v)$ .*

**Remark.** We are going to apply this proposition in the following situations:

- (1)  $v \in \mathbb{N}$ ,  $v < p^{N_0}$ ,  $n_1 = n_2 = 0$ ;
- (2)  $v = c + 1/p$ ,  $n_1 = 0$ ,  $n_2 = -1$ , where  $c \in \mathbb{N}$  and  $c < p^{N_0-2}$ .

*Proof.* By symmetry we may assume that  $n = n_{12}$ .

If  $n_{12} \neq 0$  we have  $p^s \left( a_1 + \frac{a_2}{p^n} \right) = \gamma \geq v$ , because of property  $\text{adm}_2$ (2), and

$$p^s \left( \frac{a_1}{p^{N_0-n}} + a_2 \right) < \frac{v}{p^{N_0-n}} + p^s a_2 \leq d(v) + (v - d(v)) = v \leq \gamma.$$

Therefore, the term  $p^s \Delta_{a_1 n_1} \Delta_{a_2 n_2}$  appears in the expression of  $\sigma^{n_1-s} \mathcal{F}_\gamma(v)$ . This term will appear in the expression of another  $\sigma^{n'} \mathcal{F}_{\gamma'}(v)$ , where  $\gamma' \geq v$ , if and only if  $p^s \left( a_1 + \frac{a_2}{p^{n+mN_0}} \right) \geq v$  or  $p^s \left( \frac{a_1}{p^m N_0 - n} + a_2 \right) \geq v$ , where  $m \in \mathbb{N}$ . But the condition  $v/p^{N_0-n} < d(v)$  implies that all such numbers are less than  $v$ .

If  $n_{12} = 0$  then  $\gamma = p^s(a_1 + a_2) \geq v$  and  $p^s \Delta_{a_1 n_1} \Delta_{a_2 n_2}$  appears in the expression of  $\sigma^{n_1-s} \mathcal{F}_\gamma(v)$ . This element can appear in the expression of another  $\sigma^{n'} \mathcal{F}_{\gamma'}(v)$ , where  $\gamma' \geq v$ , if and only if  $\gamma' = p^s \left( a_1 + \frac{a_2}{p^m N_0} \right) \geq v$  or  $\gamma' = p^s \left( \frac{a_1}{p^m N_0} + a_2 \right) \geq v$ , where  $m \in \mathbb{N}$ . As earlier,  $\gamma' < v$  in both cases.

The proposition is proved.  $\square$

**Remark.** If  $v/p^{N_0/2} < d(v)$ , then elements of the set

$$\{\sigma^s \mathcal{F}_\gamma^{(v)} \bmod A_{\text{adm}}^{(v)} \mid 0 \leq s < N_0, \gamma \geq v\}$$

are linear combinations of disjoint groups of elements listed in  $\text{adm}_1$ ) and  $\text{adm}_2$ ).

**5.3.** Denote by the same symbol  $f$  the morphism of  $W(k)$ -modules

$$A^{(v)} \bmod A_{\text{tr}}^{(v)} \longrightarrow A^{(v)} \bmod A_{\text{tr}}^{(v)},$$

which is induced by  $f : A \longrightarrow A$ . As earlier, denote again by  $f$  the  $k$ -linear extension of the automorphism of  $\mathcal{M}$ , which is induced by  $f$ . Because the images of  $D_{an}$  and  $\Delta_{an}$  coincide in  $\mathcal{M}_k$ , we have, for any  $a \in \mathbb{Z}(p)$ ,

$$f(\Delta_{a0}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_0}} \alpha_{abm}(f) \Delta_{bm}.$$

It will be convenient sometimes to set  $\alpha_{ab0}(f) = 0$  if  $a$  or  $b$  are divisible by  $p$ .

**Proposition 5.4.** *Suppose  $\alpha_{110}(f) = \alpha \in k^*$ . Then  $\alpha_{aa0}(f) = \alpha^a$ , for any  $a \in \mathbb{Z}(p)$  such that  $a < p^{N_0-1}$  if  $p \neq 2$  and  $N_0 \geq 2$ , and such that  $a < 2^{N_0}$  if  $p = 2$  and  $N_0 \geq 3$ .*

*Proof.* By proposition 5.3, for any  $v \leq p^{N_0}$  such that  $v \equiv 0 \pmod{p}$ , the formula for  $f(\mathcal{F}_v(v)) \bmod A_{\text{tr}}^{(v)}$  must contain all terms  $a_1 \Delta_{a_1 0} \Delta_{a_2 0}$ , for which  $a_1 + a_2 = v$ , and the term  $p^s a \Delta_{as}$ , where  $p^s a = v$  and  $a \in \mathbb{Z}(p)$ , with the same coefficient. In other words, for such indices  $a_1, a_2, a \in \mathbb{Z}(p)$ ,

$$(5.1) \quad \alpha_{a_1 a_1 0}(f) \alpha_{a_2 a_2 0}(f) = \sigma^s \alpha_{aa0}(f).$$

For  $a \in \mathbb{Z}(p)$ ,  $a < p^{N_0}$ , set  $\gamma(a) = \alpha_{aa0}(f) \alpha_{110}(f)^{-1}$ . Then  $\gamma(1) = 1$  and  $\gamma(a_1) \gamma(a_2) = \gamma(a)^{p^s}$  if  $a_1 + a_2 = p^s a$ .

Suppose  $p \neq 2$ .

First, we prove that for  $n \in \mathbb{Z}(p)$  satisfying  $1 \leq n < p^{N_0-1}$ , we have

$$(5.2) \quad \gamma(n) = \gamma(2)^{n-1}.$$

This is obviously true for  $n = 1$  and  $n = 2$ .

Assume that  $n \geq 2$  and that  $\gamma(m) = \gamma(2)^{m-1}$  holds for all  $m \in \mathbb{Z}(p)$  such that  $m \leq n$ . Consider a special case of relation (5.1) with  $n \in \mathbb{Z}(p)$

$$(5.3) \quad \gamma(1) \gamma(np - 1) = \gamma(n)^p.$$

If  $n \not\equiv -1 \pmod{p}$  then use the relation  $\gamma(p-1) \gamma(p+1) = \gamma(2)^p$ , which is again a special case of (5.1), to deduce from (5.3) that

$$\gamma(n+1) = \gamma(1) \gamma(n+1) = \gamma(n) \gamma(2) = \gamma(2)^n.$$

If  $n \equiv -1 \pmod{p}$  and  $p \neq 3$  then  $n \geq 4$  and by the inductive assumption  $\gamma(3) = \gamma(2)^2$ . Apply the relation  $\gamma(p-1) \gamma(2p+1) = \gamma(3)^p = \gamma(2)^{2p}$  to deduce from (5.3) that

$$\gamma(n+1) = \gamma(1) \gamma(n+2) = \gamma(n) \gamma(2)^2 = \gamma(2)^{n+1}.$$

If  $p = 3$  then  $\gamma(p-1) \gamma(2p+1) = \gamma(1)^{p^2}$  and we obtain from (5.3) that

$$\gamma(n+1) = \gamma(1) \gamma(n+2) = \gamma(n) = \gamma(2)^{n-1} = \gamma(2)^{n+1},$$

because  $\gamma(2) = 1$  (using that  $\gamma(1) \gamma(2) = \gamma(1)^3$ ).

So, relation (5.2) is proved.

Still assuming that  $p \neq 2$  prove that  $\gamma(2) = 1$ . The relation  $\gamma(1) \gamma(p-1) = \gamma(1)^p$  implies that  $\gamma(2)^{p-2} = \gamma(p-1) = 1$ . The equality  $\gamma(1) \gamma(p^2-1) = \gamma(1)^{p^2}$  implies that  $\gamma(2)^{p^2-2} = \gamma(p^2-1) = 1$ . Then  $\gamma(2) = 1$  because  $p^2-2$  and  $p-2$  are coprime. This completes the case  $p \neq 2$ .

Consider now the case  $p = 2$ .

Notice that for any  $n \in \mathbb{Z}(2)$  such that  $1 < n < 2^{N_0}$ , we have  $n+1 = 2^s a$ , where  $a \in \mathbb{Z}(2)$ ,  $s \in \mathbb{N}$  and  $a < n$ . Therefore,  $\gamma(1) \gamma(n) = \gamma(a)^{2^s}$  and the equality  $\gamma(n) = 1$  follows by induction on  $n \geq 1$  for all  $n < 2^{N_0}$ .  $\square$

**Corollary 5.5.** *If  $\alpha_{110}(f) = 1$  then  $\alpha_{aa0}(f) = 1$  whenever  $a < p^{N_0-1}$ ,  $p \neq 2$  or  $a < 2^{N_0}$ ,  $p = 2$ .*

**Proposition 5.6.** *Suppose  $N_0 \geq 3$ ,  $\alpha_{110}(f) \in k^*$ ,  $a, b \in \mathbb{Z}(p)$ ,  $a, b < p^{N_0-2}$ . If  $0 \leq m < N_0$  and  $b/p^m < a$  then  $\alpha_{a,b,-m \bmod N_0}(f) = 0$ .*



*Proof.* For a given  $b \in \mathbb{Z}(p)$ ,  $b < p^{N_0-2}$  and  $1 \leq m < N_0$ , let  $a \in \mathbb{Z}(p)$  be the minimal integer such that  $\alpha_{a',b,-m}(f) = 0$  if  $a' > a$ . If such an  $a$  does not exist then  $\alpha_{a,b,-m}(f) = 0$  for all  $a$  and there is nothing to prove.

If  $p \neq 2$  put  $v = p^{N_0-1}$  and consider  $f(\mathcal{F}_v(v)) \bmod (A_{\text{tr}}^{(v)} + pA_{\text{adm}}^{(v)})$ .

We prove that the term  $\Delta_{v-a,0}\Delta_{b,-m}$  enters in  $f(\mathcal{F}_v(v))$  with the coefficient

$$(5.4) \quad (v-a)\alpha_{v-a,v-a,0}(f)\alpha_{a,b,-m}(f) = -a\alpha_{v-a,v-a,0}(f)\alpha_{a,b,-m}(f).$$

Indeed,  $\mathcal{F}_v(v) \bmod (A_{\text{tr}}^{(v)} + pA_{\text{adm}}^{(v)})$  is a sum of the terms of the form  $a_1\Delta_{a_1,0}\Delta_{a_2,0}$  with  $a_1, a_2 \in \mathbb{Z}(p)$  such that  $a_1 + a_2 = v$ . Therefore,  $f(a_1\Delta_{a_1,0}\Delta_{a_2,0})$  contains  $\Delta_{v-a,0}\Delta_{b,-m}$  with coefficient

$$a_1\alpha_{a_1,v-a,0}(f)\alpha_{a_2,b,-m}(f).$$

Now notice that  $\alpha_{a_2,b,-m}(f) = 0$  if  $a_2 > a$ , and  $\alpha_{a_1,v-a,0}(f) = 0$  if  $a_1 > v-a$  or, equivalently, if  $a_2 < a$ . So,  $a_1 = v-a$  and the coefficient is given by formula (5.4).

By the choice of  $a$ , the coefficient (5.4) is not zero. Therefore,  $\Delta_{v-a,0}\Delta_{b,-m} \in A_{\text{adm}}^{(v)}$ . Notice that

$$\max \left\{ v-a + \frac{b}{p^m}, \frac{v-a}{p^{N_0-m}} + b \right\} = v-a + \frac{b}{p^m}$$

and  $b/p^m \geq a$ . Indeed, we can use that

$$\frac{v-a}{p^{N_0-m}} + b < \frac{p^{N_0-1}}{p} + p^{N_0-2} < 2p^{N_0-2} < p^{N_0-1} - p^{N_0-2} < v-a + \frac{b}{p^m}.$$

Therefore,  $v-a + b/p^m \geq v$ , i.e.  $b/p^m \geq a$  and the proposition is proved in the case  $p \neq 2$ .

If  $p = 2$  we can take  $v = 2^{N_0}$  and repeat the above arguments by using in the last step the inequality

$$\frac{v-a}{2^{N_0-m}} + b < \frac{2^{N_0}}{2} + 2^{N_0-2} < 2^{N_0} - a \left( 1 - \frac{1}{2^m} \right) \leq v-a + \frac{b}{2^m}.$$

The proposition is completely proved.  $\square$

**5.4.** Suppose  $r \in \mathbb{N}$  is such that  $\alpha_{aa'0}(f) = 0$  for any  $a, a' \in \mathbb{Z}(p)$  such that  $a < a' < a+r < p^{N_0-2}$ .

Let  $\delta(p)$  be  $p$  if  $p \neq 2$  and  $\delta(p) = 4$  if  $p = 2$ .

**Proposition 5.7.** *Assume that  $\alpha_{110}(f) = 1$ . If  $b, b_1 \in \mathbb{Z}(p)$ ,  $b_1 = b+r$  and  $b_1 + \delta(p) < p^{N_0-2}$  then  $\alpha_{bb_10}(f) = \alpha_{b-\delta(p), b_1-\delta(p), 0}(f)$ .*

*Proof.* Let  $a_0 = p^{N_0-2} - 1$ ,  $v_0 = a_0 + 1/p$ ,  $v = a_0 + \frac{b}{p}$ . We need the following lemma.

**Lemma 5.8.** *If  $a', b', c \leq a_0$  and  $a' + b'/p = v$  then  $\alpha_{a',c,-1}(f) = 0$ .*

*Proof.* It follows from the inequalities

$$\frac{c}{p} \leq \frac{a_0}{p} \leq a_0 - \frac{a_0}{p} < v - \frac{b'}{p} = a'$$

and proposition 5.6.  $\square$

We continue the proof of proposition 5.7. Consider

$$\mathcal{F}_v(v_0) = - \sum_{\substack{a'+b'/p=v \\ a',b' \leq a_0}} a' [\Delta_{a',0}, \Delta_{b',-1}] \bmod pA_{\text{adm}}^{(v)}.$$

Using that  $v_0/p^{N_0-1} < d(v_0) = 1/p$ , cf. proposition 5.3, we can find now the coefficient for  $[\Delta_{a_0,0}, \Delta_{b_1,-1}]$  in  $f(\mathcal{F}_v(v_0))$ . By the above lemma  $\alpha_{a',b,-1}(f) = 0$ , therefore the image of the term  $a'[\Delta_{a',0}, \Delta_{b',-1}]$  gives a coefficient

$$a' \alpha_{a'a_0,0}(f) \sigma^{-1}(\alpha_{b'b_1,0}(f)).$$

If  $a' < a_0$  and  $\alpha_{a'a_0,0}(f) \neq 0$  then  $a' \leq a_0 - r$ ,  $b' \geq b + rp > b_1$  and  $\alpha_{b'b_1,0}(f) = 0$ . So, the coefficient is non-zero only for  $a' = a_0$ . Then by Corollary 5.5  $\alpha_{a'a_0,0}(f) = 1$  and the coefficient will be equal to  $a_0 \sigma^{-1}(\alpha_{bb_1,0}(f))$ .

If  $p \neq 2$  we can proceed similarly to find the coefficient for  $[\Delta_{a_0-1,0}, \Delta_{b_1+p,-1}]$  in  $f(\mathcal{F}_v(v_0))$ . It equals  $(a_0 - 1) \sigma^{-1}(\alpha_{b+p,b_1+p,0}(f))$ . Therefore, by proposition 5.3

$$\alpha_{bb_1,0}(f) = \alpha_{b+p,b_1+p,0}(f)$$

and the case  $p \neq 2$  is completely considered.

If  $p = 2$ , we similarly find similarly the coefficient for  $[\Delta_{a_0-2,0}, \Delta_{b_1+4,-1}]$  in  $f(\mathcal{F}_v(v_0))$ . It equals  $(a_0 - 2) \sigma^{-1}(\alpha_{b+4,b_1+4,0}(f))$  and we obtain

$$\alpha_{bb_1,0}(f) = \alpha_{b+4,b_1+4,0}(f).$$

The proposition is proved.  $\square$

**5.5.** Now we come to the central point of this section.

**Proposition 5.9.** *Suppose  $\alpha_{110}(f) \neq 0$  and  $N_0 \geq 3$ . Then there is an  $\eta \in \text{Aut}^0 K$  such that  $\alpha_{ab0}(f\eta^*) = \delta_{ab}$ , for any  $a, b \in \mathbb{Z}(p)$  with  $a \leq b < p^{N_0-3}$ , where  $\delta_{ab}$  is the Kronecker symbol.*

*Proof.* Proposition 5.4 together with part 2) of proposition 2.1 imply that after replacing  $f$  by  $f\eta^*$  for some  $\eta \in \text{Aut}^0 K$  such that  $\eta(t) = \alpha_{110}(f)t$ , we can assume that  $\alpha_{aa0}(f) = 1$  if  $a < p^{N_0-1}$ .

Let  $r = r(f) \in \mathbb{N}$  be the maximal subject to the condition that  $\alpha_{ab0}(f) = 0$ , for any  $a, b \in \mathbb{Z}(p)$  with  $a, b < p^{N_0-2}$  and  $a < b < a + r$ .

If  $r \geq p^{N_0-3} - 1$  then there is nothing to prove. Therefore, we can assume that  $r \leq p^{N_0-3} - 2$ . For  $1 \leq a < p^{N_0-2}$ , set  $\alpha_a(r) = \alpha_{a,a+r,0}(f)$  if  $a \in \mathbb{Z}(p)$  and  $\alpha_a(r) = 0$ , otherwise.

By proposition 5.7  $\alpha_a(r)$  depends only on the residue  $a \bmod \delta(p)$  and by the choice of  $r$  the function  $a \mapsto \alpha_a(r)$  is not identically zero. The proposition will be proved if we show the existence of  $\eta \in \text{Aut}^0 K$  such that  $r(f\eta^*) > r(f)$ .

In the case  $p \neq 2$  apply proposition 2.5 with  $w_0 = 1 + r$ . Let  $\eta$  will be the corresponding character. If  $r(f\eta^*) > r(f)$ , then the proposition is proved. So, assume that  $r(f\eta^*) = r(f)$ . Therefore, by replacing  $f$  by  $f\eta^*$  we can assume the following normalisation conditions:

- a)  $\alpha_1(r) = 0$  if  $r \not\equiv -1 \pmod p$ ;
- b)  $\alpha_2(r) = 0$  if  $r \equiv -1 \pmod p$ .

In the case  $p = 2$ , apply proposition 2.6 with either  $w_0 = r + 2$  if  $r \equiv 2 \pmod 4$  or  $w_0 = r$  if  $r \equiv 0 \pmod 4$ . In the first case we have the normalisation condition

- c)  $\alpha_1(r) = \alpha_3(r) = 0$ ;

in the second case we obtain only that

- d)  $\alpha_1(r) = 0$ .

*The case  $p \neq 2$ .*

If  $r = p^{N_0-3} - 2$  then  $\alpha_1(r) = \alpha_{ab0}(f) = 0$  if  $a = 1, b = p^{N_0-3} - 1$ . For all other couples  $a, b \in \mathbb{Z}(p)$  such that  $a < b < p^{N_0-3}$ , we have  $\alpha_{ab0}(f) = 0$  because  $b - a < r$ . Therefore, we can assume that  $r \leq p^{N_0-3} - 3$ .

Let  $c_j = p(r+1) + j$  for  $j = 1, 2, \dots, p-1$ . Then  $c_j \leq p(p^{N_0-3} - 2) + p - 1 < p^{N_0-2}$ , for all  $j$ . Set  $v_j = c_j + 1/p$  and consider the coefficient for  $\mathcal{F}_{v_j+r}(v_j)$  in the image  $f(\mathcal{F}_{v_j}(v_j)) \in A_{\text{adm}}^{(v_j)} \bmod A_{\text{tr}}^{(v_j)} + pA_{\text{adm}}^{(v_j)}$ .

Similarly to the proof of proposition 5.7, we see that the term  $[\Delta_{c_j,0}, \Delta_{1+rp,-1}]$  from the expression of  $\mathcal{F}_{v_j+r}(v_j)$  can appear with non-zero coefficient only as image of one of the following two terms from  $\mathcal{F}_{v_j}(v_j)$ :  $(c_j - r)[\Delta_{c_j-r,0}, \Delta_{1+rp,-1}]$  and  $c_j[\Delta_{c_j,0}, \Delta_{1,-1}]$ . This coefficient is equal to

$$(c_j - r)\alpha_{c_j-r}(r) + c_j\alpha_{1,1+rp,0}(f).$$

Similarly, the term  $[\Delta_{c_j-1,0}, \Delta_{1+(r+1)p,-1}]$  from the expression of  $\mathcal{F}_{v_j+r}(v_j)$  can appear with non-zero coefficient only in the image of either  $(c_j - 1 - r)[\Delta_{c_j-1-r,0}, \Delta_{1+(r+1)p,-1}]$  or  $(c_j - 1)[\Delta_{c_j-1,0}, \Delta_{1+p,-1}]$ . Therefore, this coefficient will be equal to

$$(c_j - 1 - r)\alpha_{c_j-1-r}(r) + (c_j - 1)\sigma^{-1}\alpha_{1+p,1+(r+1)p,0}(f)$$

and we obtain the following relation

$$(5.5) \quad \frac{c_j - r}{c_j}\alpha_{c_j-r}(r) = \frac{c_j - 1 - r}{c_j - 1}\alpha_{c_j-1-r}(r) + X,$$

where  $X = \sigma^{-1}(\alpha_{1+p,1+(r+1)p,0}(f)) - \sigma^{-1}(\alpha_{1,1+rp,0}(f))$ .

For  $j = 1, \dots, p-1$ , set  $\beta_j = \frac{c_j - r}{c_j} \alpha_{j-r}(r)$ . Then the above relation (5.5) implies that  $\beta_2 = \beta_1 + X, \beta_3 = \beta_2 + X, \dots, \beta_{p-1} = \beta_{p-2} + X$ .

*The case  $r \not\equiv 0 \pmod{p}$ ,  $p \neq 2$ .*

In this case the normalisation conditions imply that

— if  $r \not\equiv -1 \pmod{p}$  then  $\beta_{r+1} = 0$ ;

— if  $r \equiv -1 \pmod{p}$  then  $\beta_{r+2} = 0$ .

In both cases  $\beta_r = 0$ . This implies that  $\beta_1 = \dots = \beta_{p-1} = 0$ . Therefore,  $\alpha_a(r) = 0$ , for all  $a$ . This is a contradiction.

So, in the case  $r \not\equiv 0 \pmod{p}$ ,  $p \neq 2$  the proposition is proved.

*The case  $r \equiv 0 \pmod{p}$ ,  $p \neq 2$*

In this case we only have the normalisation condition  $\beta_1 = 0$ . Therefore, for  $i = 1, \dots, p-1$ , we have  $\beta_i = (i-1)X$  and  $\alpha_a(r) = (a-1)X$  for any  $a \in \mathbb{Z}(p)$ ,  $a < p^{N_0-3}$ .

Let  $v = (p-1)r + p$  and consider the coefficient for  $\mathcal{F}_{v+r}(v)$  in the image  $f(\mathcal{F}_v(v))$ . Following the images of terms of degree 2 we see that this coefficient equals  $-2X$ . Now notice that the linear terms in  $\mathcal{F}_v(v)$  (resp.  $\mathcal{F}_{v+r}(v)$ ) have coefficients with  $p$ -adic valuation  $v_p((p-1)r+p)$  (resp.  $v_p(pr+p)$ ). Clearly, if  $1 = v_p(pr+p)$  and if  $1 < v_p((p-1)r+p)$  then the linear term of  $\mathcal{F}_{v+r}(v)$  cannot appear in the image  $f(\mathcal{F}_v(v))$ . Therefore,  $1 = v_p(pr+p) = v_p((p-1)r+p)$  and the linear terms in  $\mathcal{F}_v(v)$  (resp.  $\mathcal{F}_{v+r}(v)$ ) are multiples of  $\Delta_{r+1-r/p,1}$  (resp.  $\Delta_{r+1,1}$ ). But then  $(r+1) - (r+1-r/p) = r/p < r$  and by the definition of  $r$ ,  $\Delta_{r+1,1}$  will not appear in the image  $F(\Delta_{r+1-r/p,1})$ . This contradiction proves the proposition in the case  $r \equiv 0 \pmod{p}$ ,  $p \neq 2$ .

*The case  $p = 2$ .*

Here  $r \equiv 0 \pmod{2}$ . If  $r \equiv 2 \pmod{4}$  then the normalisation conditions imply that  $\alpha_a(r) = 0$  for all  $a$  and the proposition is proved.

If  $r \equiv 0 \pmod{4}$  then we only have one normalisation condition  $\alpha_a(r) = 0$  if  $a \equiv 1 \pmod{4}$ . Let  $\alpha_a(r) = \alpha$  where  $a \equiv 3 \pmod{4}$ . Consider

$$\mathcal{F}_{r+4}(r+4) = (r+4)\Delta_{\frac{r+4}{2^s},s} + \sum_{a+b=r+4a,b<r+4} \Delta_{a0}\Delta_{b0} \in A_{\text{adm}}^{(r+4)} \pmod{A_{\text{tr}}^{(r+4)}},$$

where  $s = v_2(r+4) \geq 2$ . Then  $f(\mathcal{F}_{r+4}(r+4))$  contains  $\Delta_{r+1,0}\Delta_{r+3,0}$  with coefficient

$$\alpha_{1,r+1,0}(f) + \alpha_{3,3+r,0}(f) = \alpha,$$

and therefore it contains  $\mathcal{F}_{2r+4}(r+4)$  with coefficient  $\alpha$ . Similarly to the case  $p \neq 2$ , we obtain the equality  $v_2(r+4) = v_2(2r+4) = 2$  and consequently the fact that  $f(\Delta_{r/2+1,2})$  cannot contain  $\Delta_{r/4+1,2}$  with non-zero coefficient because  $(r/2+1) - (r/4+1) = r/4 < r$ . The proposition is completely proved.  $\square$

## 6. Proof of the main theorem — the characteristic $p$ case

Suppose  $E$  is a field of characteristic  $p$ .

Then  $E'$  is also a field of characteristic  $p$ , because the topological groups  $\Gamma_E(p)^{\text{ab}}$  and  $\Gamma_{E'}(p)^{\text{ab}}$  are isomorphic. Looking at the ramification filtrations of these groups we deduce that the residue fields of  $E$  and  $E'$  are isomorphic. Therefore,  $E$  and  $E'$  are isomorphic complete discrete valuation fields and we can identify the maximal  $p$ -extensions  $E(p)$  of  $E$  and  $E'(p)$  of  $E'$ .

Let  $K$  be a finite Galois extension of  $E$  in  $E(p)$ . Then  $E(p)$  is a maximal  $p$ -extension of  $K$  and  $\Gamma_K(p) = \text{Gal}(E(p)/K)$ . Let  $K'$  be the extension of  $E'$  in  $E(p)$  such that  $g(\Gamma_K(p)) = \Gamma_{K'}(p)$  (recall that  $g$  is a group isomorphism). If  $s \geq 0$  and  $K_s$  is the unramified extension of  $K$  in  $E(p)$  such that  $[K_s : K] = p^s$  then  $g(\Gamma_{K_s}(p)) = \Gamma_{K'_s}(p)$ , where  $K'_s$  is the unramified extension of  $K'$  in  $E(p)$  of degree  $p^s$ . Therefore, with the notation from Section 3 we have a compatible system  $g_{KK'} = \{g_{KK'_s}\}_{s \geq 0}$  of  $\mathbb{F}_p$ -linear continuous automorphisms  $g_{KK'_s} : \bar{\mathcal{M}}_{K_s} \rightarrow \bar{\mathcal{M}}_{K'_s}$ .

Now choose uniformising elements  $t_K$  and  $t_{K'}$  in  $K$  and, resp.,  $K'$ . Consider the corresponding standard generators  $D_{an}^{(s)}$  (resp.  $D_{an}'^{(s)}$ ), where  $a \in \mathbb{Z}(p)$  and  $n \in \mathbb{Z} \bmod N_s$ , of  $\bar{\mathcal{M}}_{K_s} = \mathcal{M}_{K_s} \hat{\otimes}_k k(p)$  (resp.,  $\bar{\mathcal{M}}_{K'_s} = \mathcal{M}_{K'_s} \hat{\otimes}_k k(p)$ ). Here, as usual,  $k \simeq \mathbb{F}_{q_0}$  is the residue field of  $K$ ,  $q_0 = p^{N_0}$ ,  $N_s = N_0 p^s$ . Then

$$g_{KK'_s}(D_{a0}^{(s)}) = \sum_{\substack{b \in \mathbb{Z}(p) \\ m \in \mathbb{Z} \bmod N_s}} \alpha_{abm}(g_{KK'_s}) D_{bm}'^{(s)}$$

with  $\alpha_{abm}(g_{KK'_s}) \in k_s \subset k(p)$ .

For each  $s \geq 0$ , choose  $n_s \in \mathbb{Z} \bmod N_s$  such that  $\alpha_{11n_s}(g_{KK'_s}) \neq 0$ :  $n_s$  exists, because  $g_{KK'_s}$  induces a  $k(p)$ -linear isomorphism of  $\bar{\mathcal{M}}_{K_s} \bmod \bar{\mathcal{M}}_{K_s}^{(2)}$  and  $\bar{\mathcal{M}}_{K'_s} \bmod \bar{\mathcal{M}}_{K'_s}^{(2)}$ .

Let  $\text{Fr}(t_{K'}) \in \text{Aut} K'_{\text{ur}}$  be such that  $\text{Fr}(t_{K'}) : t_{K'} \mapsto t_{K'}$  and  $\text{Fr}(t_{K'})|_{k(p)} = \sigma$ . Let  $\xi \in \text{Iso}^0(K'_{\text{ur}}, K_{\text{ur}})$  be such that  $\xi(t_{K'}) = t_K$ .

For any  $s \geq 0$ ,  $\text{Fr}(t_{K'})$  (resp.  $\xi$ ) induces a continuous field isomorphism  $K'_s \rightarrow K'_s$  (resp.  $K'_s \rightarrow K_s$ ). It will be denoted by  $\text{Fr}(t_{K'})_s$  (resp.  $\xi_s$ ). With notation from n.3, we introduce continuous group isomorphisms

$$g_{KK'_s}^0 = g_{KK'_s} \text{Fr}(t_{K'})_s^{n_s*} : \bar{\mathcal{M}}_{K_s} \rightarrow \bar{\mathcal{M}}_{K'_s}.$$

Clearly,  $h_s := g_{KK'_s}^0 \xi_s^*$  is induced by an automorphism of  $\Gamma_{K_s}(p)$  which is compatible with the ramification filtration. Notice also that, by proposition 2.1, if  $a \in \mathbb{Z}(p)$ ,  $n \in \mathbb{Z} \bmod N_s$  and

$$h_s(D_{a0}^{(s)}) = \sum_{b,m} \alpha_{abm}(h_s) D_{bm}^{(s)},$$

then  $\alpha_{a,b,m-n_s}(h_s) = \sigma^{n_s} \alpha_{abm}(g_{KK's})$ . In particular,  $\alpha_{110}(h_s) \neq 0$ . Therefore, applying proposition 5.6, we obtain that for all  $s \geq 0$ ,

$$h_s \in \text{Aut}_{\text{adm}} \mathcal{M}_{K_s} \text{ mod } \mathcal{M}_{K_s}^{(p^{N_s-2})},$$

the residues  $n_s \in \mathbb{Z} \text{ mod } N_s$  are unique, and  $n_{s+1} \text{ mod } N_s = n_s$ . Here we use that  $D_{\text{an}}^{(s+1)} \mapsto D_{\text{an}}^{(s)}$  under the natural morphism from  $\bar{\mathcal{M}}_{K,s+1}$  to  $\bar{\mathcal{M}}_{K,s}$ . Then  $h_{KK} := \{h_s\}_{s \geq 0}$  and  $g_{KK'}^0 := \{g_{KK's}^0\}_{s \geq 0}$  are compatible systems and, by propositions 3.3 and 5.9, they are special admissible locally analytic systems. By proposition 3.4 there is an  $\eta_{KK'} \in \text{Iso}^0(K, K')$  such that  $g_{KK'\text{an}}^0 = d(\eta_{KK'}) \hat{\otimes}_k k(p)$ . Notice also that if  $\bar{n}_{KK'} := \varprojlim_s n_s \in \varprojlim_s \mathbb{Z}/N_s \mathbb{Z}$  then  $g_{KK'} = g_{KK'}^0 \text{Fr}(t_{K'})^{-\bar{n}_{KK'}^*}$ , where  $\text{Fr}(t_{K'})^* = \{\text{Fr}(t_{K'})_s\}_{s \geq 0}$  is the compatible system from Subsection 3.5.

Suppose  $L$  is a finite Galois extension of  $E$  in  $E(p)$  containing  $K$ . Proceed similarly to obtain  $L' \subset E(p)$  such that  $g$  induces an isomorphism of  $\Gamma_L(p)$  and  $\Gamma_{L'}(p)$ , the corresponding compatible system  $g_{LL'} = \{g_{LL's}\}_{s \geq 0}$  and the special admissible locally analytic system  $g_{LL'}^0 = \{g_{LL's}^0\}_{s \geq 0}$ , where  $g_{LL'} = g_{LL'}^0 \text{Fr}(t_{L'})^{-\bar{n}_{LL'}^*}$ , together with the corresponding  $\eta_{LL'} \in \text{Iso}^0(L, L')$  such that  $g_{LL'\text{an}}^0 = d(\eta_{LL'}) \hat{\otimes}_{k_L} k_L(p)$ . Here  $k_L$  is the residue field of  $L$ ,  $k_L \simeq \mathbb{F}_{p^{M_0}}$  and  $\bar{n}_{LL'} \in \varprojlim_s \mathbb{Z}/p^{M_0 p^s} \mathbb{Z}$ . Notice that all these maps depend on some choice of uniformising elements  $t_L$  and  $t_{L'}$  in, respectively,  $L$  and  $L'$ .

The systems  $g_{LL'}$  and  $g_{KK'}$  are comparable because both come from the group isomorphisms  $\Gamma_L(p) \rightarrow \Gamma_{L'}(p)$  and  $\Gamma_K(p) \rightarrow \Gamma_{K'}(p)$  which are induced by  $g$ . If  $I_{L/K}$  is the inertia subgroup of  $\text{Gal}(L/K)$  then there is a natural group embedding  $I_{L/K} \subset \text{Aut}^0(L) \subset \text{Aut}^0(L_{\text{ur}})$ . Similarly, we have a group embedding for the inertia subgroup  $I_{L'/K'}$  of  $\text{Gal}(L'/K')$  into  $\text{Aut}^0(L')$ .

Let  $\kappa : I_{L/K} \rightarrow I_{L'/K'}$  be the group isomorphism induced by  $g$ . Then  $\tau^* g_{LL's} = g_{LL's} \kappa(\tau)^*$ , for any  $\tau \in I_{L/K}$  and any  $s \geq 0$ . This implies that

$$\tau^* g_{LL'\text{ur}} = g_{LL'\text{ur}} \kappa(\tau)^*,$$

i.e. condition C from Subsection 3.7 holds in this case.

Let  $\mu_{KK'} = \eta_{KK'} \text{Fr}(t_{K'})^{-\bar{n}_{KK'}}$   $\in \text{Iso}(K, K')$  and  $\mu_{LL'} = \eta_{LL'} \text{Fr}(t_{L'})^{-\bar{n}_{LL'}}$   $\in \text{Iso}(L, L')$ .

**Proposition 6.1.** *With the above notation:*

- a)  $\mu_{LL'}|_K = \mu_{KK'}$ ;
- b) for any  $\tau \in I_{L/K}$ ,  $\tau \mu_{LL'} = \mu_{LL'} \kappa(\tau)$ .

*Proof.* Let  $\alpha = \text{Fr}(t_{L'})^{\bar{n}_{LL'}}$ . Consider  $K'_{\text{ur}}$  as a subfield in  $L'_{\text{ur}}$  and set  $K''_{\text{ur}} = \alpha(K'_{\text{ur}}) \subset L'_{\text{ur}}$ . Then  $K''_{\text{ur}}$  is the maximal unramified  $p$ -extension of the complete discrete valuation field  $K'' := \alpha(K') \subset E(p)$  in  $E(p)$ .

Let  $\beta = \alpha|_{K'_{\text{ur}}}$ . Consider the following commutative diagram

$$\begin{array}{ccccccc} \bar{\mathcal{M}}_{L_{\text{ur}}} & \xrightarrow{g_{LL'_{\text{ur}}}} & \bar{\mathcal{M}}_{L'_{\text{ur}}} & \xrightarrow{\alpha^*_{L'_{\text{ur}}}} & \bar{\mathcal{M}}_{L'_{\text{ur}}} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \bar{\mathcal{M}}_{K_{\text{ur}}} & \xrightarrow{g_{KK'_{\text{ur}}}} & \bar{\mathcal{M}}_{K'_{\text{ur}}} & \xrightarrow{\beta^*_{K'_{\text{ur}}}} & \bar{\mathcal{M}}_{K''_{\text{ur}}} & & \end{array}$$

where the vertical arrows come from natural embeddings of the corresponding Galois groups.

The systems  $g_{LL'}^0 = g_{LL'}\alpha^*_{L'_{\text{ur}}}$  and  $f_{KK''} := g_{KK'}\beta^*_{K'_{\text{ur}}}$  are comparable, because they come from the compatible group isomorphisms  $\Gamma_L(p) \rightarrow \Gamma_{L'}(p)$  and  $\Gamma_K(p) \xrightarrow{f} \Gamma_{K''}(p)$ . In this situation, condition **C** is automatically satisfied and, by proposition 3.5, the admissibility of  $g_{LL'}^0$  implies the admissibility of  $f_{KK''}$ . Because the group homomorphism  $f$  is compatible with ramification filtrations, we can apply the results of section 5 to deduce that  $f_{KK''}$  is special admissible locally analytic and that there is an  $\eta^1_{KK''} \in \text{Iso}^0(K, K'')$  such that  $f_{KK''_{\text{an}}} = d(\eta^1_{KK''}) \hat{\otimes}_k k(p)$  and  $\eta_{LL'}|_K = \eta^1_{KK''}$ .

Consider  $\psi := \eta_{KK'}^{-1}\eta_{LL'}|_K \in \text{Iso}^0(K', K'')$ . Then

$$\begin{aligned} \psi_{\text{an}} &= \eta_{KK'_{\text{an}}}^{-1}\eta^1_{KK''_{\text{an}}} = (g_{KK'_{\text{an}}}^0)^{-1}(g_{KK'}\beta^*_{K'_{\text{ur}}})_{KK''_{\text{an}}} \\ &= \left( g_{KK'}^0 \quad^{-1} g_{KK'}\beta^*_{K'_{\text{ur}}} \right)_{K'_{\text{ur}}} = \left( \text{Fr}(t_{K'})^{-\bar{n}_{KK'}} \beta \right)_{\text{an}}. \end{aligned}$$

Therefore by proposition 2.7,

$$\eta_{KK'}^{-1}\eta_{LL'}|_K = \text{Fr}(t_{K'})^{-\bar{n}_{KK'}} \text{Fr}(t_{L'})^{\bar{n}_{LL'}}|_K$$

or  $\mu_{LL'}|_K = \mu_{KK'}$ .

Part a) of our proposition is proved.

Consider the inertia subgroups  $I_{L/K} \subset \text{Gal}(L_{\text{ur}}/K_{\text{ur}})$ ,  $I_{L'/K'} \subset \text{Gal}(L'_{\text{ur}}/K'_{\text{ur}})$  and  $I_{L'/K''} \subset \text{Gal}(L'_{\text{ur}}/K''_{\text{ur}})$ . As it was noticed earlier, the correspondence

$$\tau^* \mapsto \tau'^* = g_{LL'_{\text{ur}}}^{-1}\tau^*g_{LL'_{\text{ur}}}$$

induces a group isomorphism  $\kappa : I_{L/K} \rightarrow I_{L'/K'}$  such that  $\kappa(\tau) = \tau'$ .

We use the correspondence

$$\alpha^* : \tau' \mapsto \tau'' = \alpha^{-1}\tau'\alpha$$

to define the group isomorphism  $\kappa_\alpha : I_{L'/K'} \rightarrow I_{L'/K''}$  such that  $\kappa_\alpha(\tau') = \tau''$ . With this notation we have the following equality of compatible systems

$$\tau_{LL'}^* g_{LL'}^0 = g_{LL'}^0 \tau_{L'L'}''^*,$$

where as earlier,  $g_{LL'}^0 = g_{LL'}\alpha^*_{L'_{\text{ur}}}$ .

Therefore, the equality  $(\tau\eta_{LL'})_{\text{an}} = (\tau_{LL'}^*g_{LL'}^0)_{\text{an}} = (g_{LL'}^0\tau''_{L'L'}^*)_{\text{an}} = (\eta_{LL'}\tau'')_{\text{an}}$  together with proposition 2.7 and the definition of  $\tau''$  imply that  $\tau\eta_{LL'} = \eta_{LL'}\tau'' = \eta_{LL'}\alpha^{-1}\tau'\alpha$ , i.e.  $\tau\mu_{LL'} = \mu_{LL'}\tau'$ .

The proposition is proved.  $\square$

Let  $\mu := \lim_{\rightarrow} \mu_{KK'} : E(p) \longrightarrow E(p)$ . Clearly, it is a continuous field isomorphism and  $\mu(E) = E'$ .

**Proposition 6.2.**  $\mu^* = g$ .

*Proof.* As earlier, let  $K$  and  $K'$  be Galois extensions of  $E$  and  $E'$ , respectively, such that  $g(\Gamma_K(p)) = \Gamma_{K'}(p)$ .

By part b) of the above proposition 6.1, the correspondences  $\mu^* : \tau \mapsto \mu^{-1}\tau\mu$  and  $g : \tau \mapsto g(\tau)$  induce the same isomorphism of the inertia subgroups  $I_K(p) \longrightarrow I_{K'}(p)$ . Consider the induced isomorphism  $I_K(p)^{\text{ab}} \longrightarrow I_{K'}(p)^{\text{ab}}$ . With respect to the identifications of class field theory  $I_K(p)^{\text{ab}} = U_K$  and  $I_{K'}(p)^{\text{ab}} = U_{K'}$ , where  $U_K$  and  $U_{K'}$  are groups of principal units in  $K$  and  $K'$ , respectively, this homomorphism is induced by the restriction of the field isomorphism  $\mu_{KK'}$  on  $U_K$ . In addition,  $\mu_{KK'}$  transforms the natural action of any  $\tau \in \Gamma_E(p)$  on  $U_K$  into the natural action of  $g(\tau) \in \Gamma_{E'}(p)$  on  $U_{K'}$ . Therefore, the two field automorphisms  $\mu^{-1}\tau\mu|_{K'}$  and  $g(\tau)|_{K'}$  of  $K'$  become equal after restricting on  $U_{K'}$ . This implies that they coincide on the whole field  $K'$ , i.e.  $\mu^{-1}\tau\mu \equiv g(\tau) \pmod{\Gamma_{K'}(p)}$ , for any  $\tau \in \Gamma_E(p)$ . Because  $K$  is an arbitrary Galois extension of  $E$  in  $E(p)$  this implies that  $g = \mu^*$ .

So, proposition 6.2 together with the characteristic  $p$  case of the Main Theorem are completely proved.  $\square$

## 7. Proof of the main theorem — the mixed characteristic case

In this section we assume that  $E$  is a field of characteristic 0. Clearly, this implies that the field  $E'$  is also of characteristic 0.

**7.1.** Following the paper [10] introduce the categories  $\Psi$ ,  $\tilde{\Psi}$  and the functor  $\Phi : \Psi \longrightarrow \tilde{\Psi}$ .

The objects of  $\Psi$  are the field extensions  $L/K$ , where  $[K : \mathbb{Q}_p] < \infty$ ,  $L$  is an infinite Galois extension of  $K$  in a fixed maximal  $p$ -extension  $K(p)$  of  $K$  and  $\Gamma_{L/K} = \text{Gal}(L/K)$  is a  $p$ -adic Lie group. A morphism from  $L/K$  to an object  $L'/K'$  in  $\Psi$  is a continuous field embedding  $f : L \longrightarrow L'$  such that  $[L' : f(L)] < \infty$  and  $f|_K$  is a field isomorphism of  $K$  and  $K'$ .

The objects of  $\tilde{\Psi}$  are couples  $(\mathcal{K}, G)$  where  $\mathcal{K}$  is a complete discrete valuation field of characteristic  $p$  with finite residue field and  $G$  is a closed subgroup of the group of all continuous automorphisms of  $\mathcal{K}$ . In addition, with respect to the induced topology  $G$ , is a compact finite dimensional



$p$ -adic Lie group. A morphism from  $(\mathcal{K}, G)$  to an object  $(\mathcal{K}', G')$  in  $\tilde{\Psi}$  is a closed field embedding  $f : \mathcal{K} \rightarrow \mathcal{K}'$  such that  $\mathcal{K}'$  is a finite separable extension of  $f(\mathcal{K})$ . In addition,  $f(\mathcal{K})$  is  $G'$ -invariant and the correspondence  $\tau \mapsto \tau|_{f(\mathcal{K})}$  induces a group epimorphism from  $G'$  to  $G$ .

Let  $X$  be the Fontaine-Wintenberger field-of-norm functor, cf. [11]. Then the correspondence  $L/K \mapsto (X(L), G_{L/K})$ , where  $G_{L/K} = \{X(\tau) \mid \tau \in \Gamma_{L/K}\}$ , induces the functor  $\Phi : \Psi \rightarrow \tilde{\Psi}$ .

One of main results in [10] states that the functor  $\Phi$  is fully faithful.

**7.2.** Let  $\{E_\alpha/E, i_{\alpha\beta}\}_{\mathcal{I}}$  be an inductive system of objects in the category  $\Psi$ . From now on  $\mathcal{I}$  is a set of indices  $\alpha$  with a suitable partial ordering. The connecting morphisms  $i_{\alpha\beta} \in \text{Hom}_\Psi(E_\alpha, E_\beta)$  are the natural field embeddings defined for suitable couples  $\alpha, \beta \in \mathcal{I}$ . We can choose this inductive system to be large enough to satisfy the requirement  $\varinjlim E_\alpha = E(p)$ .

By applying the functor  $\Phi$ , we obtain the inductive system  $\{(\mathcal{E}_\alpha, G_\alpha), \tilde{i}_{\alpha\beta}\}_{\mathcal{I}}$  in the category  $\tilde{\Psi}$ , where  $(\mathcal{E}_\alpha, G_\alpha) = \Phi(E_\alpha/E)$  and  $\tilde{i}_{\alpha\beta} = \Phi(i_{\alpha\beta})$ , for all  $\alpha \in \mathcal{I}$ . Then  $\varinjlim \mathcal{E}_\alpha = \mathcal{E}(p)$  is a maximal  $p$ -extension for each field  $\mathcal{E}_\alpha$ ,  $\alpha \in \mathcal{I}$ .

Notice that the field embeddings  $\tilde{i}_{\alpha\beta}$  induce group epimorphisms  $\tilde{j}_{\alpha\beta} : G_\beta \rightarrow G_\alpha$  with corresponding projective system  $\{G_\alpha, \tilde{j}_{\alpha\beta}\}_{\mathcal{I}}$  such that  $\varprojlim G_\alpha$  is identified via the functor  $X$  with  $\Gamma_E(p)$ . For any  $\alpha \in \mathcal{I}$ , we then have the identifications  $\Gamma_{E_\alpha}(p) = \Gamma_{\mathcal{E}_\alpha}(p)$ . These identifications are compatible with the ramification filtrations. This means that one can define the Herbrand function  $\varphi_\alpha$  for the infinite extension  $E_\alpha/E$  as the limit of Herbrand functions of all finite subextensions in  $E_\alpha$  over  $E$  and

$$\Gamma_E(p)^{(v)} \cap \Gamma_{E_\alpha}(p) = \Gamma_{\mathcal{E}_\alpha}(p)^{(\varphi_\alpha(v))},$$

for all  $v \geq 0$ .

**7.3.** Consider the group isomorphism  $g : \Gamma_E(p) \rightarrow \Gamma_{E'}(p)$  from the statement of the Theorem. For  $\alpha \in \mathcal{I}$ , let  $E'_\alpha \subset E'(p)$  be such that  $g(\Gamma_{E_\alpha}(p)) = \Gamma_{E'_\alpha}(p)$ . Then we have the corresponding injective system  $\{E'_\alpha, i'_{\alpha\beta}\}_{\mathcal{I}}$  and  $\varinjlim E'_\alpha = E'(p)$ .

Clearly, for any  $\alpha \in \mathcal{I}$ ,

- $E'_\alpha/E'$  is an object of  $\Psi$ ;
- $\bar{g}_\alpha := g_\alpha \bmod \Gamma_{E_\alpha}(p) : \Gamma_{E_\alpha/E} \rightarrow \Gamma_{E'_\alpha/E'}$  is a group isomorphism which is compatible with the ramification filtrations; in particular, this implies that the Herbrand functions for the infinite extensions  $E_\alpha/E$  and  $E'_\alpha/E'$  are equal;

- for any  $v \geq 0$ ,  $g_\alpha := g|_{\Gamma_{E_\alpha}(p)}$  induces a continuous group isomorphism of  $\Gamma_E(p)^{(v)} \cap \Gamma_{E_\alpha}(p)$  and  $\Gamma_{E'}(p)^{(v)} \cap \Gamma_{E'_\alpha}(p)$ .

For  $\alpha \in \mathcal{I}$ , set  $\Phi(E'_\alpha/E') = (\mathcal{E}'_\alpha, G'_\alpha)$  and  $\Phi(i'_{\alpha\beta}) = \tilde{i}'_{\alpha\beta}$ . Then we have an inductive system  $\{(\mathcal{E}'_\alpha, G'_\alpha), \tilde{i}'_{\alpha\beta}\}_{\mathcal{I}}$  and  $\varinjlim \mathcal{E}'_\alpha := \mathcal{E}'(p)$  is a maximal  $p$ -extension for each  $\mathcal{E}'_\alpha$ . As earlier, we obtain the projective system  $\{G'_\alpha, \tilde{j}'_{\alpha\beta}\}_{\mathcal{I}}$  and the field-of-norms functor allows us to identify the topological groups  $\varprojlim G'_\alpha$  and  $\Gamma_{E'}(p)$ . Therefore, for any  $\alpha \in \mathcal{I}$ , we have an identification of the groups  $\Gamma_{E'_\alpha}(p)$  and  $\Gamma_{\mathcal{E}'_\alpha}(p)$ .

This implies that for all  $\alpha \in \mathcal{I}$ , we have the following isomorphisms of topological groups:

- $\tilde{g}_\alpha := X(g_\alpha) : \Gamma_{\mathcal{E}_\alpha}(p) \longrightarrow \Gamma_{\mathcal{E}'_\alpha}(p)$  such that, for any rational number  $v \geq 0$ ,  $\tilde{g}_\alpha(\Gamma_{\mathcal{E}_\alpha}(p)^{(v)}) = \Gamma_{\mathcal{E}'_\alpha}(p)^{(v)}$ ;
- $X(\tilde{g}_\alpha) : G_\alpha \longrightarrow G'_\alpha$  which maps the projective system  $\{G_\alpha, \tilde{j}_{\alpha\beta}\}_{\mathcal{I}}$  to the projective system  $\{G'_\alpha, \tilde{j}'_{\alpha\beta}\}_{\mathcal{I}}$ .

**7.4.** By the characteristic  $p$  case of the Main Theorem for all  $\alpha \in \mathcal{I}$ , there are continuous field isomorphisms  $\tilde{\mu}_\alpha : \mathcal{E}_\alpha \longrightarrow \mathcal{E}'_\alpha$  such that

- $\{\tilde{\mu}_\alpha\}_{\alpha \in \mathcal{I}}$  maps the inductive system  $\{\mathcal{E}_\alpha, \tilde{i}_{\alpha\beta}\}_{\mathcal{I}}$  to the inductive system  $\{\mathcal{E}'_\alpha, \tilde{i}'_{\alpha\beta}\}_{\mathcal{I}}$ ;
- $X(\tilde{g}_\alpha)$  is induced by  $\tilde{\mu}_\alpha$ , i.e. if  $\tau \in G_\alpha$  and  $\tau' = X(\tilde{g}_\alpha) \in G'_\alpha$  then  $\tau\tilde{\mu}_\alpha = \tilde{\mu}_\alpha\tau'$ .

Because  $\Phi$  is a fully faithful functor, for all indices  $\alpha \in \mathcal{I}$ , there is a  $\mu_\alpha \in \text{Hom}_\Psi(E_\alpha/E, E'_\alpha/E')$  such that

- $\{\mu_\alpha\}_{\alpha \in \mathcal{I}}$  transforms the inductive system  $\{E_\alpha/E, i_{\alpha\beta}\}_{\mathcal{I}}$  into the inductive system  $\{E'_\alpha/E', i'_{\alpha\beta}\}_{\mathcal{I}}$ ;
- if  $\tau \in \Gamma_{E_\alpha/E}$  and  $\tau' = \tilde{g}_\alpha(\tau) \in \Gamma_{E'_\alpha/E'}$  then  $\tau\mu_\alpha = \mu_\alpha\tau'$ .

Therefore,  $\mu := \varinjlim \mu_\alpha$  is a continuous field isomorphism from  $E(p)$  to  $E'(p)$  such that  $\tau\mu = \mu g(\tau)$ , i.e.  $g(\tau) = \mu^{-1}\tau\mu$ , for  $\tau \in \varprojlim \Gamma_{E_\alpha/E} = \Gamma_E(p)$  and  $g(\tau) \in \varprojlim \Gamma_{E'_\alpha/E'} = \Gamma_{E'}(p)$ .

The Main Theorem is completely proved.

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Victor ABRASHKIN

Math Dept of Durham University

Sci Laboratories, South Road

DH7 7QR Durham, UK

*E-mail*: victor.abrashkin@durham.ac.uk

*URL*: <http://www.maths.dur.ac.uk/~dma0va/>