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David J. GRYNKIEWICZ, Luz E. MARCHAN et Oscar ORDAZ

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Representation of finite abelian group elements by subsequence sums

par David J. GRYNKIEWICZ, Luz E. MARCHAN et Oscar ORDAZ

Résumé. Soit $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ un groupe abélien fini non trivial avec $n_1|n_2|\ldots|n_r$. Une conjecture d'Hamidoune dit que si $W = w_1 \cdot \ldots \cdot w_n$ est une suite d'entiers, tous, sauf au plus un, premiers à $|G|$, et S une suite d'éléments de G avec $|S| \geq |W| + |G| - 1 \geq |G| + 1$, la multiplicité maximale de S au plus $|W|$, et $\sigma(W) \equiv 0 \mod |G|$, alors il existe un sous-groupe non trivial H tel que tout élément $g \in H$ peut être représenté par une somme pondérée de la forme $g = \sum\limits_{i=1}^{n} w_i s_i$, avec $s_1 \cdot \ldots \cdot s_n$ une sous-suite de S. Nous donnons deux exemples qui montrent que cela n'est pas vrai en général, et nous caractérisons les contre-exemples pour les grands $|W| \geq \frac{1}{2}|G|$.

Un théorème de Gao, généralisant un résultat plus ancien d'Olson, dit que si G est un groupe abélien fini, et S une suite d'éléments de G avec $|S| \geq |G| + \mathsf{D}(G) - 1$, alors, soit tout élément de G peut être représenté par une sous-somme de S à $|G|$ termes, soit il existe une classe $g + H$ telle que tous sauf au plus $|G/H| - 2$ termes de S sont dans $g + H$. Nous établissons quelques cas très spéciaux d'un analogue pondéré de ce théorème, conjecturé par Ordaz et Quiroz, et quelques conclusions partielles dans les autres cas, qui impliquent un résultat récent d'Ordaz et Quiroz. Cela est fait, en partie, en étendant un théorème de Grynkiewicz sur les partitions pondérées, que nous utilisons également pour améliorer le résultat de Gao cité précédemment en montrant que l'hypothèse $|S| \geq |G| + \mathsf{D}(G) - 1$ peut être affaiblie en $|S| \geq |G| + \mathsf{d}^*(G)$, où $\mathsf{d}^*(G) = \sum\limits_{i=1}^{r} (n_i - 1)$. Nous utilisons aussi cette méthode pour déduire une variante de la conjecture d'Hamidoune valide si au moins $\mathsf{d}^*(G)$ des w_i sont premiers à $|G|$.

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Abstract. Let $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite and nontrivial abelian group with $n_1|n_2|\ldots|n_r$. A conjecture of Hamidoune says that if $W = w_1 \cdot \ldots \cdot w_n$ is a sequence of integers, all but at most one relatively prime to $|G|$, and S is a sequence over G with $|S| \geq |W| + |G| - 1 \geq |G| + 1$, the maximum multiplicity of S at most $|W|$, and $\sigma(W) \equiv 0 \mod |G|$, then there exists a nontrivial subgroup H such that every element $g \in H$ can be represented as a weighted subsequence sum of the form $g = \sum_{i=1}^{n} w_i s_i$, with $s_1 \cdot \ldots \cdot s_n$ a subsequence of S. We give two examples showing this does not hold in general, and characterize the counterexamples for large $|W| \geq \frac{1}{2}|G|$.

A theorem of Gao, generalizing an older result of Olson, says that if G is a finite abelian group, and S is a sequence over G with $|S| \geq |G| + \mathsf{D}(G) - 1$, then either every element of G can be represented as a $|G|$-term subsequence sum from S, or there exists a coset $g + H$ such that all but at most $|G/H| - 2$ terms of S are from $g + H$. We establish some very special cases in a weighted analog of this theorem conjectured by Ordaz and Quiroz, and some partial conclusions in the remaining cases, which imply a recent result of Ordaz and Quiroz. This is done, in part, by extending a weighted setpartition theorem of Grynkiewicz, which we then use to also improve the previously mentioned result of Gao by showing that the hypothesis $|S| \geq |G| + \mathsf{D}(G) - 1$ can be relaxed to $|S| \geq |G| + \mathsf{d}^*(G)$, where $\mathsf{d}^*(G) = \sum_{i=1}^{r}(n_i - 1)$. We also use this method to derive a variation on Hamidoune's conjecture valid when at least $\mathsf{d}^*(G)$ of the w_i are relatively prime to $|G|$.

1. Notation

We follow the conventions of [9] and [11] for notation concerning sequences over an abelian group. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Throughout, all abelian groups will be written additively. Let G be an abelian group, and let $A, B \subseteq G$ be nonempty subsets. Then

$$A + B = \{a + b \mid a \in A, b \in B\}$$

denotes their *sumset*. The *stabilizer* of A is defined as $H(A) = \{g \in G \mid g + A = A\}$, and A is called *periodic* if $H(A) \neq \{0\}$, and *aperiodic* otherwise. If A is a union of H-cosets (i.e., $H \leq H(A)$), then we say A is *H-periodic*. The order of an element $g \in G$ is denoted $\mathrm{ord}(g)$, and we use $\phi_H : G \to G/H$ to denote the natural homomorphism. We use $\gcd(a, b)$ to denote the greatest common divisor of $a, b \in \mathbb{Z}$.

For a set P (often with $P = G$ an abelian group), let $\mathcal{F}(P)$ be the free abelian monoid with basis P. The elements of $\mathcal{F}(P)$ are called *sequences* over P. We write sequences $S \in \mathcal{F}(P)$ in the form

$$S = s_1 \cdot \ldots \cdot s_r = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \quad \text{where} \quad \mathsf{v}_g(S) \geq 0 \text{ and } s_i \in G.$$

We call $|S| := r = \sum_{g \in P} \mathsf{v}_g(S)$ the *length* of S, and $\mathsf{v}_g(S)$ the *multiplicity* of g in S. The *support* of S is

$$\operatorname{supp}(S) := \{g \in P \mid \mathsf{v}_g(S) > 0\}.$$

A sequence S_1 is called a *subsequence* of S if $S_1|S$ in $\mathcal{F}(P)$ (equivalently, $\mathsf{v}_g(S_1) \leq \mathsf{v}_g(S)$ for all $g \in P$), and in such case, $S{S_1}^{-1}$ denotes the subsequence of S obtained by removing all terms from S_1. The *sum* of S is

$$\sigma(S) := \sum_{i=1}^{r} s_i = \sum_{g \in G} \mathsf{v}_g(S) g,$$

and we use

$$\mathsf{h}(S) := \max\{\mathsf{v}_g(S) \mid g \in P\}$$

to denote the maximum multiplicity of a term of S. A sequence S is *zero-sum* if $\sigma(S) = 0$. Given any map $\varphi : G \to G'$, we extend φ to a map of sequences, $\varphi : \mathcal{F}(G) \to \mathcal{F}(G')$, by letting $\varphi(S) := \varphi(s_1) \cdot \ldots \cdot \varphi(s_r)$. We say that two sequences $S_1, S_2 \in \mathcal{F}(\mathbb{Z})$ are *congruent modulo* n, and we write $S_1 \equiv S_2 \mod n$, if $\varphi(S_1) = \varphi(S_2)$ for the canonical homomorphism $\varphi \colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. We say that at most n terms of the sequence $S = g_1 \cdot \ldots \cdot g_l$ are from a given subset $A \subseteq G$ if

$$|\{i \in [1, l] \mid g_i \in A\}| \leq n\,.$$

Next we introduce notation for weighted subsequence sums, which we will do in the more general context of R-modules (though the focus of this paper is $R = \mathbb{Z}$). Let R be a ring and G a (left) R-module (thus G is also an abelian group with the two notions coinciding when $R = \mathbb{Z}$). If $w \in R$ and $A \subseteq G$, then $w \cdot A = \{wa \mid a \in A\}$ denotes the dilation of A. Let $S \in \mathcal{F}(G)$, $W \in \mathcal{F}(R)$ and $s = \min\{|S|, |W|\}$. Define

$$W \cdot S = \left\{ \sum_{i=1}^{s} w_i g_i \;\middle|\; \begin{array}{l} w_1 \cdot \ldots \cdot w_s \text{ is a subsequence of } W \text{ and} \\ g_1 \cdot \ldots \cdot g_s \text{ is a subsequence of } S \end{array} \right\},$$

and for $1 \leq n \leq s$, let

$$\Sigma_n(W, S) = \{W' \cdot S' : S'|S,\ W'|W \text{ and } |W'| = |S'| = n\}$$

$$\Sigma_{\leq n}(W, S) = \bigcup_{i=1}^{n} \Sigma_i(W, S) \quad \text{and} \quad \Sigma_{\geq n}(W, S) = \bigcup_{i=n}^{s} \Sigma_i(W, S),$$

$$\Sigma(W, S) = \Sigma_{\leq s}(W, S).$$

If $W = 1^{|S|}$ (with 1 the identity in R), then $\Sigma(W,S)$ (and other such notation) is abbreviated by $\Sigma(S)$, which is the usual notation for the set of *subsequence sums*. Note that $\Sigma_{|W|}(W,S) = W \cdot S$ when $|W| \le |S|$.

Let P denote the set of nonempty subsets of G. The elements of $\mathcal{F}(P)$ will be called *setpartitions* (over G), and an *n-setpartition* B (over G) is an element in $\mathcal{F}(P)$ of length n (in other words, B is a formal product of n nonempty subsets of G). If $B = B_1 \cdot \ldots \cdot B_n \in \mathcal{F}(P)$, with $\emptyset \ne B_i \subseteq G$ for all $i \in [1,n]$, then we say that B is an n-setpartition of the sequence

$$T := \prod_{i=1}^{n} \prod_{b \in B_i} b \ \in \mathcal{F}(G)\,,$$

and we call T the sequence associated to B. Note T is finite if and only if each B_i is finite. Conversely, we say that S has an n-setpartition if S is the associated sequence of some n-setpartition. It is easily shown (see [4] [18] [19]) that S has an n-setpartition if and only if $\mathsf{h}(S) \le n \le |S|$, and if such is the case, then S has an n-setpartition with sets of as near equal a size as possible (i.e., $||B_i| - |B_j|| \le 1$ for all $i,j \in [1,n]$).

2. Introduction

Let

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$$

be a finite abelian group with $n_1 | n_2 | \ldots | n_r$, where C_{n_j} denotes a cyclic group of order $n_j \ge 2$. Thus r is the rank $\mathsf{r}(G)$, $n_1 \cdots n_r$ is the order $|G|$, and n_r is the exponent $\exp(G)$. In 1961, Erdős, Ginzburg and Ziv proved that every sequence $S \in \mathcal{F}(G)$ with $|S| \ge 2|G| - 1$ has $0 \in \Sigma_{|G|}(S)$ [6] [30]. This sparked the field of zero-sum combinatorics, which has now seen much development and become an essential component in Factorization Theory (see [9] [11] for a recent survey and text on the subject).

One of the oldest and most important invariants in this area is the *Davenport constant* of G, denoted $\mathsf{D}(G)$, which is the least integer so that $S \in \mathcal{F}(G)$ with $|S| \ge \mathsf{D}(G)$ implies $0 \in \Sigma(S)$. A very basic argument shows

$$\mathsf{d}^*(G) + 1 \le \mathsf{D}(G) \le |G| \tag{1}$$

(see [11]), where

$$\mathsf{d}^*(G) := \sum_{i=1}^{r} (n_i - 1).$$

Originally, the lower bound was favored as the likely truth, but later examples with $\mathsf{D}(G) > \mathsf{d}^*(G) + 1$ were found (see [8] [12]), and it is not now well understood when $\mathsf{d}^*(G) + 1 = \mathsf{D}(G)$ fails, though it is still thought that equality should hold for many instances (and known to be the case for a few) [11].

Gao later linked the study of zero-sums with the study of $|G|$-term zero-sums (and hence results like the Erdős-Ginzburg-Ziv Theorem), by showing that $\ell(G) = |G| + \mathsf{D}(G) - 1$, where $\ell(G)$ is the least integer so that $S \in \mathcal{F}(G)$ with $|S| \geq \ell(G)$ implies $0 \in \Sigma_{|G|}(S)$ [7]. In the same paper, he also proved the following generalization of an older result of Olson [31].

Theorem A. *Let G be a finite abelian group, and let $S \in \mathcal{F}(G)$ with $|S| \geq |G| + \mathsf{D}(G) - 1$. Then either $\Sigma_{|G|}(S) = G$ or there exist a proper subgroup $H < G$ and some $g \in G$ such that all but at most $|G/H| - 2$ terms of S are from the coset $g + H$.*

Thus the number $\ell(G) = |G| + \mathsf{D}(G) - 1$ also guarantees that *every* element (not just zero) can be represented as an $|G|$-term subsequence sum, provided no coset contains too many of the terms of S.

In this paper, we concern ourselves with weighted zero-sum problems related to the above results, though some of our results are new in the non-weighted case as well. Such variations were initiated by Caro in [5] where he conjectured the following weighted version of the Erdős-Ginzburg-Ziv Theorem, which, after much partial work [3] [10] [22] [23], was recently proven in [14]. (Note the condition $\sigma(W) \equiv 0 \mod \exp(G)$ is necessary, else S with $\mathrm{supp}(S) = \{1\}$ would give a counterexample.)

Theorem B. *Let G be a finite abelian group, and let $S \in \mathcal{F}(G)$ and $W \in \mathcal{F}(\mathbb{Z})$ with $\sigma(W) \equiv 0 \mod \exp(G)$. If $|S| \geq |W| + |G| - 1$, then $0 \in \Sigma_{|W|}(W, S)$.*

Since then, there have been several other results along these lines (see [1] [2] [13] [32] for some examples). However, the following conjecture of Hamidoune remained open [22].

Conjecture 2.1. *Let G be a nontrivial, finite abelian group, and let $S \in \mathcal{F}(G)$ and $W \in \mathcal{F}(\mathbb{Z})$ with $|S| \geq |W| + |G| - 1 \geq |G| + 1$ and $\sigma(W) \equiv 0 \mod |G|$. If $\mathsf{h}(S) \leq |W|$ and there is some $w \in \mathrm{supp}(W)$ such that $\gcd(w', \exp(G)) = 1$ for all $w' \in \mathrm{supp}(w^{-1}W)$, then $\Sigma_{|W|}(W, S)$ contains a nontrivial subgroup.*

Hamidoune verified his conjecture in the case $|W| = |G|$ [22], and under the additional hypothesis of either $\mathsf{h}(S) < |W|$ or $|W| \geq |G|$ or $\gcd(w_i, \exp(G)) = 1$ for all $w_i | W$, Conjecture 2.1 follows from the result in [14]. In Section 3, we give two examples which show that Conjecture 2.1 is false in general, and prove the following theorem, which characterizes the (rather limited) counter-examples for large $|W| \geq \frac{1}{2}|G|$.

Theorem 2.2. *Let G be a finite, nontrivial abelian group, and let $S \in \mathcal{F}(G)$ and $W \in \mathcal{F}(\mathbb{Z})$ with $|S| \geq |W| + |G| - 1 \geq |G| + 1$ and $\sigma(W) \equiv 0 \mod |G|$. Suppose $\mathsf{h}(S) \leq |W|$ and that there is some $w \in \mathrm{supp}(W)$ such that $\gcd(w', \exp(G)) = 1$ for all $w' \in \mathrm{supp}(w^{-1}W)$. If also $|W| \geq \frac{1}{2}|G|$, then either:*

(i) $\Sigma_{|W|}(W, S)$ contains a nontrivial subgroup, or

(ii) $|\mathrm{supp}(S)| = 2$, $|W| = |G| - 1$, $G \cong \mathbb{Z}/2^r\mathbb{Z}$ and

$$W \equiv x^{(n-1)/2}(-x)^{(n-1)/2}0 \mod |G|,$$

for some r, n, $x \in \mathbb{Z}^+$.

Another open conjecture is the following weighted generalization of Theorem A [32]. We remark that, in the same paper, they showed Conjecture 2.3 to be true when $|S| = 2|G| - 1$, and thus for cyclic groups.

Conjecture 2.3. *Let G be a finite abelian group, and let $W \in \mathcal{F}(\mathbb{Z})$ with $|W| = |G|$, $\sigma(W) \equiv 0 \mod \exp(G)$ and $\gcd(w, \exp(G)) = 1$ for all $w \in \mathrm{supp}(W)$. If $S \in \mathcal{F}(G)$ with $|S| = |G| + \mathsf{D}(G) - 1$, then either:*

(i) *$\Sigma_{|G|}(W, S) = G$, or*

(ii) *there exist a proper subgroup $H < G$ and some $g \in G$ that all but at most $|G/H| - 2$ terms of S are from the coset $g + H$.*

In section 5, we prove some limited results related to Conjecture 2.3. In particular, we verify it in the extremal case $\mathsf{h}(S) \geq \mathsf{D}(G) - 1$ (allowing also $|S| \geq |G| + \mathsf{D}(G) - 1$ provided $\mathsf{h}(S) \leq |G|$), and give a corollary that extends the result of [32] and shows, when $\mathsf{h}(S) \leq \mathsf{D}(G) - 1$, that the hypotheses of Conjecture 2.3 (assuming (ii) fails) instead imply $\Sigma_{|S|-|G|}(W, S) = G$. This latter result will follow from the following pair of theorems, which improve (for non-cyclic groups) a corollary from the end of [14] (see also [16] for the non-weighted version, of which this is also an improvement).

Theorem 2.4. *Let G be a finite abelian group, let $S, S' \in \mathcal{F}(G)$ with $S' \mid S$ and let $W = w_1 \cdot \ldots \cdot w_n \in \mathcal{F}(\mathbb{Z})$ be a sequence of integers relatively prime to $\exp(G)$ such that $\mathsf{h}(S') \leq |W| = n \leq |S'|$ and $\mathsf{d}^*(G) \leq |W|$. Then S has a subsequence S'' with $|S''| = |S'|$ such that either:*

(i) *there exists an n-setpartition $A = A_1 \cdot \ldots \cdot A_n$ of S'' such that*

$$|\sum_{i=1}^{n} w_i \cdot A_i| \geq \min\{|G|,\, |S'| - n + 1\},$$

or

(ii) *there exist an n-setpartition $A = A_1 \cdot \ldots \cdot A_n$ of S'', a proper, nontrivial subgroup $H < G$ and some element $g \in G$, such that the following properties are satisfied:*

(a) $(g+H)\cap A_i \neq \emptyset$ *for all* $i \in [1,n]$, *and* $\mathrm{supp}(SS''^{-1}) \subseteq g+H$,

(b) $A_i \subseteq g+H$ *for all* $i \leq \mathsf{d}^*(H)$ *and all* $i > \mathsf{d}^*(H)+\mathsf{d}^*(G/H)$,

(c) $|\sum\limits_{i=1}^{n} w_i \cdot A_i| \geq (e+1)|H|$ *and all but* $e \leq |G/H|-2$ *terms of* S *are from* $g+H$, *and*

(d)
$$\sum_{i=1}^{\mathsf{d}^*(H)} w_i \cdot A_i = \left(\sum_{i=1}^{\mathsf{d}^*(H)} w_i\right) g + H\,.$$

Theorem 2.5. *Let* G *be a finite abelian group, let* $S, S' \in \mathcal{F}(G)$ *with* $S' \mid S$ *and let* $W = w_1 \cdot \ldots \cdot w_n \in \mathcal{F}(\mathbb{Z})$ *be a sequence of integers relatively prime to* $\exp(G)$ *such that* $\mathsf{h}(S') \leq |W| = n \leq |S'|$ *and* $\mathsf{d}^*(G) \leq |W|$. *Suppose there exists a nontrivial subgroup* $K \leq G$ *with the following properties*:

there exist $g' \in G$, $T \in \mathcal{F}(g'+K)$ *with* $T \mid S$, *and a* $\mathsf{d}^*(K)$*-setpartition* $B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(K)}$ *of* T, *such that*
$$\sum_{i=1}^{\mathsf{d}^*(K)} w_i \cdot B_i = \left(\sum_{i=1}^{\mathsf{d}^*(K)} w_i\right) g' + K$$
and $T^{-1}S$ *contains at least* $n-\mathsf{d}^*(K)+|S|-|S'|$ *terms from* $g'+K$.

Let $K^* \leq G$ *be the maximal subgroup having the above properties. Then the following hold.*

(i) *If* $K^* = G$, *then there is an* n*-setpartition* $A = A_1 \cdot \ldots \cdot A_n$ *of a subsequence* S'' *of* S *such that* $|S'| = |S''|$ *and*
$$\sum_{i=1}^{n} w_i \cdot A_i = G.$$

(ii) *If* $K^* \neq G$, *then the conclusion of Theorem 2.4(ii) holds with* $H = K^*$.

Notice that in both Theorems 2.4 and 2.5 one is allowed to chose the ordering on the sequence $W = w_1 \cdot \ldots \cdot w_n$ (given by the choice of indices) in any way, which will affect the implication given by Theorem 2.4(ii)(d). Theorem 2.4 allows the result to applied when $n \geq \mathsf{d}^*(G)$, rather than $n \geq \frac{|G|}{p} - 1$ (as in the original corollary), where p is the smallest prime divisor of $|G|$ (note, for non-cyclic groups, the number $\mathsf{d}^*(G)$ is generally much smaller than $\frac{|G|}{p} - 1$), and contains similar improvements of bounds present in (ii)(b). However, the bound present in (ii)(c) remains unaltered, and improvements here would likely be more difficult. Theorem 2.5 will be used to prove Theorem 2.4, and also gives a way to force Theorem 2.4(ii) to hold.

As a second consequence of Theorems 2.4 and 2.5, we prove the following variation on Theorem 2.2, which extends Hamidoune's result from [23] by showing that it is only necessary to have at least $\mathsf{d}^*(G)$ of the weights relatively prime to $\exp(G)$.

Corollary 2.6. *Let G be a nontrivial, finite abelian group, and let $S \in \mathcal{F}(G)$ and $W \in \mathcal{F}(\mathbb{Z})$ with $|S| \geq |W| + |G| - 1$, $\mathsf{h}(S) \leq |W|$ and $\sigma(W) \equiv 0 \mod \exp(G)$. If W has a subsequence W' such that $|W'| + \mathsf{d}^*(G) \leq |W|$ and $\gcd(w, \exp(G)) = 1$ for all $w \in \operatorname{supp}(W'^{-1}W)$, then $\Sigma_{|W|}(W,S)$ contains a nontrivial subgroup.*

As a third consequence, we improve Theorem A by relaxing the required hypothesis from $|S| \geq |G| + \mathsf{D}(G) - 1$ to $|S| \geq |G| + \mathsf{d}^*(G)$ (recall from (1) that $\mathsf{D}(G) - 1 \geq \mathsf{d}^*(G)$). This should be put in contrast to the fact that $\ell(G) = |G| + \mathsf{D}(G) - 1 > |G| + \mathsf{d}^*(G)$ is in general possible (since $\mathsf{D}(G) - 1 > \mathsf{d}^*(G)$ is possible). The methods of employing Theorems 2.4 and 2.5 from these three applications should also be applicable for other zero-sum problems.

3. On Conjecture 2.1

We begin by giving the two counter examples to Conjecture 2.1.

Example 1. Let $p \equiv -1 \mod 4$ be a prime, let $G = C_p$ be cyclic of prime order, let $n = \frac{p-1}{2}$, let $W = 1^{(n-1)/2}(-1)^{(n-1)/2}0 \in \mathcal{F}(\mathbb{Z})$, and let $S = 0^n g^n (2g)^n$, where $g \in G \setminus \{0\}$. Note that $\mathsf{h}(S) = n = |W|$, that $|S| = 3n = |W| + |G| - 1$, that $\sigma(W) = 0$, and that

$$\Sigma_{|W|}(W,S) = \sum_{i=1}^{(n-1)/2} \{0, g, 2g\} - \sum_{i=1}^{(n-1)/2} \{0, g, 2g\} = G \setminus \{\frac{p+1}{2}g, \frac{p-1}{2}g\}.$$

Thus $G \not\subseteq \Sigma_{|W|}(W,S)$, which, since $|G|$ is prime, implies $\Sigma_{|W|}(W,S)$ does not contain a nontrivial subgroup.

Example 2. Let $m = 2^r$, let $G = C_m$, let $n = m - 1$, let $W = 1^{(n-1)/2}(-1)^{(n-1)/2}0$, and let $S = 0^n g^n$, where $g \in G$ with $\operatorname{ord}(g) = m$. Note that $\mathsf{h}(S) = n = |W|$, that $|S| = 2n = |W| + |G| - 1$, that $\sigma(W) = 0$, and that

$$\Sigma_{|W|}(W,S) = \sum_{i=1}^{(n-1)/2} \{0, g\} - \sum_{i=1}^{(n-1)/2} \{0, g\} = G \setminus \{\frac{m}{2}g\}.$$

Hence, since every nontrivial subgroup of $G \cong \mathbb{Z}/2^r\mathbb{Z}$ contains the unique element of order 2, namely $\frac{m}{2}g$, it follows that $\Sigma_{|W|}(W,S)$ does not contain a nontrivial subgroup.

For the proof of Theorem 2.2, we will need to make use of the Kemperman critical pair theory (though an isoperimetric approach would also be viable, see e.g. [21]). We begin by stating Kneser's Theorem [26] [27] [30] [33].

Theorem C (Kneser's Theorem)**.** *Let G be an abelian group, and let $A_1, \ldots, A_n \subseteq G$ be finite, nonempty subsets. Then*

$$|\sum_{i=1}^{n}\phi_H(A_i)| \geq \sum_{i=1}^{n}|\phi_H(A_i)| - n + 1,$$

where $H = H(\sum_{i=1}^{n} A_i)$.

Note that $|H| \cdot \phi_H(A_i) = |A_i + H|$. Also, if $H = H(A+B)$ and $\rho = |A+H| - |A| + |B+H| - |B|$ is the number of holes in A in B (by a hole in A, with respect to H, we mean an element from $(A+H) \setminus A)$, then Kneser's Theorem implies $|A+B| \geq |A|+|B|-|H|+\rho$. Consequently, if either A or B contains a unique element from some H-coset, then $|A+B| \geq |A|+|B|-1$. More generally, if $\rho = \sum_{i=1}^{n}(|A_i + H| - |A_i|)$ is the total number holes in the A_i, then $|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n}|A_i| - (n-1)|H| + \rho$.

Next we continue with the following two simple cases of Kemperman's Structure Theorem [25, Theorem 5.1]. The reader is directed to [15] [19] [20] [29] for more detailed exposition regarding Kemperman's critical pair theory, including the (somewhat lengthy and involved) statement of the Kemperman Structure Theorem. In what follows, a set $A \subseteq G$ is *quasi-periodic* if there is a nontrivial subgroup H (the *quasi-period*) such that $A = A_0 \cup A_1$ with A_0 nonempty and H-periodic and A_1 a subset of an H-coset.

Lemma 3.1. *Let $A_1, \ldots, A_n$, be a collection of $n \geq 3$ finite subsets in an abelian group G of order m with $0 \in A_i$ and $|A_i| \geq 2$ for all i. Moreover, suppose each A_i is not quasi-periodic and $\langle A_i \rangle = G$. If $\sum_{i=1}^{n} A_i$ is aperiodic and*

$$|\sum_{i=1}^{n}A_i| = \sum_{i=1}^{n}|A_i| - n + 1, \tag{2}$$

then the A_i are arithmetic progressions with common difference.

Proof. We provide a short proof using the formulation (including relevant notation and definitions) of Kemperman's Structure Theorem as given in [25, Theorem 5.1].

Since $\sum\limits_{i=1}^{n} A_i$ is aperiodic, it follows that A_j+A_k is aperiodic for any $j \neq k$. Thus Kneser's Theorem implies $|A_j + A_k| \geq |A_j| + |A_k| - 1$, and we must have

$$|A_j + A_k| = |A_j| + |A_k| - 1,$$

else Kneser's Theorem would imply

$$|\sum_{i=1}^{n} A_i| \geq \sum_{\substack{i=1\\ i\neq j,\, k}}^{n} |A_i| + |A_j + A_k| - (n-1) + 1 \geq \sum_{i=1}^{n} |A_i| - n + 2,$$

contradicting (2). Thus we can apply Kemperman's Structure Theorem to an arbitrary pair A_j and A_k with $j \neq k$.

Since A_i is not quasi-periodic, for $i = j$, k, we conclude from the Kemperman Structure Theorem that (A_j, A_k) is an elementary pair of type (I), (II), (III) or (IV). Since $|A_j|$, $|A_k| \geq 2$ and $A_j + A_k$ is aperiodic, we cannot have type (I) or (III). Since $n \geq 3$, since $|A_i| \geq 2$ for all i, and since $\sum\limits_{i=1}^{n} A_i$ is aperiodic (and in particular, $|\sum\limits_{i=1}^{n} A_i| < |G|$), it follows in view of Kneser's Theorem that $|A_j + A_k| < |\sum\limits_{i=1}^{n} A_i| < |G|$. Thus, in view of $0 \in A_j$ and $\langle A_j \rangle = G$, it follows that we cannot have type (IV) and that $|A_j|$, $|A_k| \leq |G| - 2$. Hence (A_j, A_k) is an elementary pair of type (II), i.e., A_j and A_k are arithmetic progressions of common difference (say) $d \in G$. Note $\mathrm{ord}(d) = |G|$, since $\langle A_j \rangle = G$. Since the difference d of an arithmetic progression A is unique up to sign when $2 \leq |A| \leq \mathrm{ord}(d) - 2$, since $2 \leq |A_j|$, $|A_k| \leq |G| - 2$, and since A_j and A_k with $j \neq k$ were arbitrary, it now follows that all the A_i are arithmetic progressions of common difference d, as desired. ☐

Lemma 3.2. *Let G be an abelian group and let A, $B \subseteq G$ be finite with $|A| \geq 2$ and $|B| = 2$. If neither A nor B is quasi-periodic and $|A + B| = |A| + |B| - 1$, then A and B are arithmetic progressions of common difference.*

Proof. This follows immediately from the Kemperman Structure Theorem or may be taken as an easily verified observation. ☐

The following result from [14] will also be used.

Theorem D. *Let G be a nontrivial, finite abelian group, $S \in \mathcal{F}(G)$ and $W = w_1 \cdot \ldots \cdot w_n \in \mathcal{F}(\mathbb{Z})$ such that $\sigma(W) \equiv 0 \mod \exp(G)$ and $|S| \geq |W| + |G| - 1$. If S has an n-setpartition $A = A_1 \cdot \ldots \cdot A_n$ such that*

$|w_i \cdot A_i| = |A_i|$ for all $i \in [1, n]$, then there exists a nontrivial subgroup H of G and an n-setpartition $A' = A'_1 \cdot \ldots \cdot A'_n$ of S such that

$$H \subseteq \sum_{i=1}^{n} w_i \cdot A'_i \subseteq \Sigma_{|W|}(W, S) \quad \textit{and} \quad |w_i \cdot A'_i| = |A'_i| \quad \textit{for all} \quad i \in [1, n] \,.$$

We now proceed with the proof of Theorem 2.2.

Proof. Let $m = \exp(G)$ and $n = |W|$. By considering G as a $\mathbb{Z}/m\mathbb{Z}$-module (for notational convenience), we may w.l.o.g. consider W as a sequence from $\mathbb{Z}/m\mathbb{Z}$, say w.l.o.g. $W = w_1 \cdot \ldots \cdot w_n$, where $\mathrm{ord}(w_i) = m$ for $i \leq n-1$ (in view of the hypothesis $\gcd(w_i, \exp(G)) = 1$ for $i \leq n-1$). Observe that we may assume $|S| = n + |G| - 1$ (since $n \geq 2$, so that, if $|\mathrm{supp}(S)| \geq 3$, then we can remove terms from S until there are only $n + |G| - 1 \geq 3$ left while preserving that $|\mathrm{supp}(S)| \geq 3$), and that there are distinct $x, y \in G$ with $x^n y^n | S$ such that $w_n(x - y) = 0$, else Theorem D implies the theorem (as if such is not the case, then there would exist, in view of $\mathsf{h}(S) \leq |W| = n$, an n-setpartition of S satisfying the hypothesis of Theorem D). Since $\sigma(W) = 0$, we may w.l.o.g. by translation assume $x = 0$. If $\mathrm{ord}(y) < |G|$, then, since $w_i y \in \langle y \rangle$ and

$$n - 1 \geq \frac{|G|}{2} - 1 \geq \mathrm{ord}(y) - 1 = \mathrm{ord}(w_i y) - 1,$$

for $i \leq n-1$ (in view of $\mathrm{ord}(w_i) = m$ for $i \leq n-1$), it would follow in view of Kneser's Theorem that

$$\langle y \rangle = \sum_{i=1}^{n-1} \{0, w_i y\} = \sum_{i=1}^{n-1} w_i \cdot \{0, y\} + w_n \cdot 0 \subseteq \Sigma_{|W|}(W, S),$$

as desired. Therefore we may assume $\mathrm{ord}(y) = |G|$, whence w.l.o.g. G is cyclic, $m = |G|$ and $y = 1$. Consequently, since $\sigma(W) = 0$, $w_n(x - y) = 0$ and $x = 0$, it follows that $w_n = 0$ and $\sigma(W') = 0$, where $W' := W w_n^{-1}$.

Since $n \geq \frac{m}{2}$, $0^n 1^n | S$ and $|S| = n + m - 1$, it follows that

$$2n \leq |S| \leq 3n - 1. \tag{3}$$

Hence let $A = A_1 \cdot \ldots \cdot A_{n-1}$ be an arbitrary $(n-1)$-setpartition of $S' := S(01)^{-1}$. Note $\{0, 1\} \subseteq A_i$ for all i, so that

$$0 \in \sum_{i=1}^{n-1} w_i \cdot A_i + w_n \cdot 0 \subseteq \Sigma_{|W|}(W, S). \tag{4}$$

Thus we may assume $\sum\limits_{i=1}^{n-1} w_i \cdot A_i$ is aperiodic, else the proof is complete. Consequently, Kneser's Theorem and $\mathrm{ord}(w_i) = m$ for $i \leq n-1$ imply

$|\sum\limits_{i=1}^{n-1} w_i \cdot A_i| \geq \sum\limits_{i=1}^{n-1} |A_i| - (n-1) + 1 = m - 1$, whence

$$|\sum_{i=1}^{n-1} w_i \cdot A_i| = \sum_{i=1}^{n-1} |A_i| - (n-1) + 1 = m - 1, \tag{5}$$

else $G \subseteq \sum\limits_{i=1}^{n-1} w_i \cdot A_i + w_n \cdot 0 \subseteq \Sigma_{|W|}(W, S)$, as desired.

Suppose m is not a prime power. Then we can choose H, $K \leq G$ with $|H|$ and $|K|$ distinct primes, so that $H \cap K = \{0\}$. In view of (5), it follows that $\Sigma_{|W|}(W, S)$ is missing exactly one element, which in view of (4) cannot be zero. Consequently, either $H \subseteq \Sigma_{|W|}(W, S)$ or $K \subseteq \Sigma_{|W|}(W, S)$, as desired. So we may assume $m = p^r$ for some prime p and $r \geq 1$.

Claim A: If $x(-x)|W'$, for some $x \in \mathbb{Z}/m\mathbb{Z}$, then $|S| = 2n$ or $|S| = 3n-1$, else the proof is complete.

Proof. Suppose the claim is false. Thus (3) implies $2n + 1 \leq |S| \leq 3n - 2$, so that $n \geq 3$, and it follows by the pigeonhole principle that $|A_i| \leq 2$ for some i, say $i = n - 1$, and that $|A_j| \geq 3$ for some j, say $j = n - 2$, whence we may w.l.o.g. assume $x = w_{n-2}$ and $-x = w_{n-1}$. Let $g \in A_{n-2} \setminus \{0, 1\}$ (in view of $|A_{n-2}| = |A_j| \geq 3$). Observe that

$$\sum_{i=1}^{n-2} w_i \cdot A_i + w_{n-1} \cdot (A_{n-1} \cup \{g\}) = \left(\sum_{i=1}^{n-1} w_i \cdot A_i\right) \bigcup \left(\sum_{i=1}^{n-3} w_i \cdot A_i + w_{n-2} \cdot (A_{n-2} \setminus \{g\}) + w_{n-1} \cdot (A_{n-1} \cup \{g\})\right) \bigcup \left(\sum_{i=1}^{n-3} w_i \cdot A_i + w_{n-2}g + w_{n-1}g\right). \tag{6}$$

Note that the first two terms of the right hand side of (6) are contained in $\Sigma_{|W|}(W, S)$. Moreover,

$$w_{n-2}g + w_{n-1}g = xg + (-x)g = 0 = w_{n-2} \cdot 0 + w_{n-1} \cdot 0 \in w_{n-2} \cdot A_{n-2} + w_{n-1} \cdot A_{n-1},$$

so that the third term of the right hand side of (6) is contained in $\sum\limits_{i=1}^{n-1} w_i \cdot A_i + w_n \cdot 0 \subseteq \Sigma_{|W|}(W, S)$ as well. Consequently, it follows from (6) that

$$\sum_{i=1}^{n-2} w_i \cdot A_i + w_{n-1} \cdot (A_{n-1} \cup \{g\}) \subseteq \Sigma_{|W|}(W, S). \tag{7}$$

However, since $\sum_{i=1}^{n-1} w_i \cdot A_i$ is aperiodic and $w_{n-1}g \notin w_{n-1} \cdot A_{n-1}$ (in view of $\mathrm{ord}(w_{n-1}) = m$, $|A_{n-1}| = |A_i| = 2$, and $\{0, 1\} \subseteq A_i$), it follows from Kneser's theorem that

$$|\sum_{i=1}^{n-2} w_i \cdot A_i + w_{n-1} \cdot (A_{n-1} \cup \{g\})| > \sum_{i=1}^{n-1} |A_i| - (n-1) + 1 = m - 1.$$

Thus (7) implies that $G \subseteq \Sigma_{|W|}(W, S)$, as desired, completing the proof of Claim A. $\square$

If $n = 2$, then $\sigma(W') = 0$ implies $w_1 = 0$, contradicting $\mathrm{ord}(w_i) = m$ for $i \leq n-1$. Therefore we may assume $n \geq 3$.

Suppose $|S| = 2n$ (so that $S = 0^n 1^n$). Since $|S| = n + m - 1 = 2n$ and $n \geq 3$, it follows that $m = n+1 \geq 4$. In view of (5) and Lemmas 3.1 and 3.2, it follows that each $w_i \cdot A_i = w_i \cdot \{0, 1\} = \{0, w_i\}$ is an arithmetic progression with common difference. Consequently, it follows that $w_i = \pm w_j$ for all $i, j \leq n-1$. Since $n - 1 < m$, since $\sigma(W') = 0$, and since $\mathrm{ord}(w_i) = m$ for all $i \leq n-1$, it follows that the w_i cannot all be equal. As a result, $w_i = \pm w_j$ for all $i, j \leq n-1$ implies that $w_i = \pm x$ for all $i \leq n-1$, with $(x)(-x)|W'$ (for some $x \in \mathbb{Z}/m\mathbb{Z}$), whence $\sigma(W') = 0$ further implies that $W' = x^{(n-1)/2}(-x)^{(n-1)/2}$ with $n-1$ even. Hence, since $m = n+1$ is a prime power, it follows that $m = 2^r$, and we see that (ii) holds. So we may assume $|S| > 2n$.

Suppose $n = 3$. Then $\sigma(W') = 0$ implies that $w_1 = -w_2$. Thus, since $2n < |S|$ and $x(-x)|W'$, where $x = w_1$, it follows in view of Claim A that $|S| = 3n - 1 = 8$. Since $|S| = n + m - 1 = m + 2$, this implies that $m = 6$, contradicting that m is a prime power. So we may assume $n \geq 4$.

In view of (3), choose A such that $|A_i| \in \{2, 3\}$ for all i (possible by the remarks from Section 1). If, for some j, there is $g \in A_j \setminus \{0, 1\}$ such that $\{x, g\}$ is a coset of a cardinality two subgroup H, where $x \in \{0, 1\}$, then

$$\sum_{\substack{i=1 \\ i \neq j}}^{n-1} w_i x + w_j \cdot \{x, g\} + w_n \cdot 0$$

is an H-periodic subset of $\Sigma_{|W|}(W, S)$ that contains $\sum_{i=1}^{n-1} w_i x = \sigma(W') \cdot x = 0$; thus $H \subseteq \Sigma_{|W|}(W, S)$, as desired. Therefore we may assume otherwise, and consequently that no A_j is quasi-periodic (in view of $|A_j| \leq 3$ and $\sum_{i=1}^{n-1} A_i$ aperiodic).

As a result, it follows, in view of $n \geq 4$, (5) and Lemma 3.1, that the $w_i \cdot A_i$ are all arithmetic progressions of common difference. Thus each A_i is an arithmetic progression of length two or three that contains $\{0, 1\}$. Hence, since $n \geq 4$ and $n + m - 1 = |S| > 2n$, so that

$$m \geq 5,$$

it follows that each A_i is an arithmetic progression with difference 1 or $\frac{m+1}{2}$ (which both are of order m), and thus each $w_i \cdot A_i$ is an arithmetic progression with difference w_i or $w_i \cdot \frac{m+1}{2}$.

Thus, since $n-1 \geq 3$, it follows by the pigeonhole principle that there is a pair A_j and A_k, with $j \neq k$, that are arithmetic progressions with common difference d, where $\mathrm{ord}(d) = m$. Thus $w_j \cdot A_k$ and $w_k \cdot A_k$ are arithmetic progression with common difference $w_j d = \pm w_k d$, implying $w_j = \pm w_k$ (since $\mathrm{ord}(d) = m$). Since the indexing for the w_i was arbitrary, then, by applying this argument to all possible permutations of the indices of the w_i (leaving w_n fixed), we conclude that $w_i = \pm w_j$ for all $i, j \leq n-1$. As in the case $|S| = 2n$, we cannot have all the w_i, with $i \leq n-1$, equal to each other (in view of $\sigma(W') = 0$ and $n - 1 < m = \mathrm{ord}(w_i)$), whence $W = x^{(n-1)/2}(-x)^{(n-1)/2}0$ and n is odd, for some $x \in \mathbb{Z}/m\mathbb{Z}$.

Thus, from claim A and $|S| > 2n$, we infer that $n+m-1 = |S| = 3n-1$, implying $2n = m$. Hence m is even. Thus, since m is a prime power, it follows that $m = 2^r$, whence $2n = m = 2^r \geq 5$ implies that n is even, a contradiction, completing the proof. $\square$

4. On $\mathsf{d}^*(G)$

The main goal of this section is to prove the following pair of seemingly innocuous lemmas, which will be needed for the proof of Theorem 2.4. Lemma 4.1 should be compared with the similar [11, Proposition 5.1.11], whose proof is much easier.

Lemma 4.1. *If G is a finite abelian group and $H \leq G$, then*

$$\mathsf{d}^*(H) + \mathsf{d}^*(G/H) \leq \mathsf{d}^*(G).$$

Lemma 4.2. *Let G be a finite abelian group, let $A \subseteq G$ with $|A| \geq 2$, let $H = \langle -a_0 + A \rangle$, where $a_0 \in A$, and let $W = w_1 \cdot \ldots \cdot w_{\mathsf{d}^*(H)}$ be a sequence of integers relatively prime to* $\exp(H)$*. Then*

$$\sum_{i=1}^{\mathsf{d}^*(H)} w_i \cdot A = \left(\sum_{i=1}^{\mathsf{d}^*(H)} w_i \right) a_0 + H.$$

We first gather some basic results from algebra. Proposition 4.3 is easily proved from the machinery of dual groups, and Proposition 4.4 from the notion and basic properties of independent elements.

Proposition 4.3. *Let G be a finite abelian group and $H \le G$. Then there exists $K \le G$ such that $K \cong G/H$ and $G/K \cong H$.*

Proof. Since finite abelian groups are self-dual [28, Theorem I.9.1], this follows from [28, Corollary I.9.3]. $\square$

Proposition 4.4. *Let G be a finite abelian group, say $G \cong \bigoplus_{i=1}^{r} C_{m_i} \cong \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{r} C_{p_i^{k_{i,j}}} \right)$, with $1 < m_1 | \ldots | m_r$, each p_i a distinct prime, and $1 \le k_{i,1} \le \ldots \le k_{i,r}$. If $H \le G$, then*

$$H \cong \bigoplus_{i=1}^{r} C_{m'_i} \cong \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{r} C_{p_i^{k'_{i,j}}} \right),$$

with $1 \le m'_1 | \ldots | m'_r$ and $m'_i | m_i$ and $0 \le k'_{i,1} \le \ldots \le k'_{i,r}$ and $k'_{i,j} \le k_{i,j}$, for all i and j. Moreover, if $m'_s = m_s$ for some s, then $k'_{i,s} = k_{i,s}$ for all i.

Proof. Since $m_j = p_1^{k_{1,j}} p_2^{k_{2,j}} \cdots p_l^{k_{l,j}}$ (see [24, Section II.2]), it suffices to show $k'_{i,j} \le k_{i,j}$ for all i and j. For this, it suffices to consider p-groups (the case $l = 1$). We may assume $k'_{1,1} \le \ldots \le k'_{1,r}$, and now, if the proposition is false, then $k'_{1,j} > k_{1,j}$ for some j, whence H, and hence also G, contains $r - j + 1$ independent elements of order at least $p_1^{k_{1,j}+1}$, say $e_1, \ldots, e_{r-j+1}$. But now $p_1^{k_{1,j}} e_1, \ldots p_1^{k_{1,j}} e_{r-j+1}$ are $r - j + 1$ independent elements in $p_1^{k_{1,j}} \cdot G$ (the image of G under the multiplication by $p_1^{k_{1,j}}$ map), which has total rank $\mathsf{r}^*(p_1^{k_{1,j}} \cdot G)$ at most $r - j$ (in view of $k_{i,1} \le \ldots \le k_{i,r}$), contradicting that the total rank of a group is the maximal number of independent elements (see [11, Apendix A]). $\square$

The next lemma will provide the key inductive mechanism for the proof of Lemma 4.1.

Lemma 4.5. *Let G be a finite abelian group, say $G \cong \bigoplus_{i=1}^{r} C_{m_i}$, with $1 < m_1 | \ldots | m_r$, and let $H \le G$, say $H \cong \bigoplus_{i=1}^{r} C_{m'_i}$, with $1 \le m'_1 | \ldots | m'_r$. If $m'_t = m_t$ for some t, then there exists a subgroup $K \le H$ such that $K \cong C_{m_t}$ and K is a direct summand in both H and G.*

Proof. Let $G \cong \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{r} C_{p_i^{k_{i,j}}} \right)$ and $H \cong \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{r} C_{p_i^{k'_{i,j}}} \right)$, with each p_i a distinct prime, $1 \le k_{i,1} \le \ldots \le k_{i,r}$ and $0 \le k'_{i,1} \le \ldots \le k'_{i,r}$ for all i. In view of Proposition 4.4 and our hypotheses, we have $m'_i | m_i$ and $k'_{i,j} \le k_{i,j}$, for all i and j, and $k'_{i,t} = k_{i,t}$ for all i. Thus it suffices to prove the lemma for p-groups, and so we assume $m_i = p^{s_i}$ and $m'_i = p^{s'_i}$ for some prime p.

By hypothesis, H contains $r-t+1$ independent elements $f_1, \ldots, f_{r-t+1}$ of order at least $m_t = p^{s_t}$ (by an appropriate subselection of elements from a basis of H). Let $e_1, \ldots, e_r$ be a basis for G with $\mathrm{ord}(e_i) = p^{s_i}$, and let $f_j = \sum_{i=1}^{r} \alpha_{j,i} e_i$, where $\alpha_{j,i} \in \mathbb{Z}$. If

$$\mathrm{ord}(f_j) = \mathrm{ord}(\alpha_{j,i} e_i) = \mathrm{ord}(e_i) = p^{s_t},$$

for some i and j, then $e_1, \ldots, e_{i-1}, f_j, e_{i+1}, \ldots, e_r$ is also a basis for G, and the result follows with $K = \langle f_j \rangle$. So we may assume otherwise.

Now $f_1' := p^{s_t-1} f_1, f_2' := p^{s_t-1} f_2, \ldots, f_{r-t+1}' := p^{s_t-1} f_{r-t+1}$ are $r-t+1$ independent elements in $p^{s_t-1} \cdot G$. However, in view of the conclusion of the previous paragraph, each $p^{s_t-1} f_j$ with $\mathrm{ord}(f_j) = p^{s_t}$ must lie in the span of $p^{s_t-1} e_{t+1}, \ldots, p^{s_t-1} e_r$ (as $\mathrm{ord}(e_i) \le p^{s_t}$ for $i \le t$).

Let $\phi_L : p^{s_t-1} \cdot G \to (p^{s_t-1} \cdot G)/L$, where $L = \langle p^{s_t-1} e_1, \ldots, p^{s_t-1} e_t \rangle$, be the natural homomorphism. Then $\phi_L(f_1'), \ldots, \phi_L(f_{r-t+1}')$ are $r-t+1$ independent elements in $\phi_L(p^{s_t-1} \cdot G)$, as the following argument shows. Take any relation

$$0 = \sum_{i=1}^{r-t+1} \alpha_i \phi_L(f_i') = \sum_{i \in I} \alpha_i \phi_L(f_i') + \sum_{i \notin I} \alpha_i \phi_L(f_i'),$$

where $i \in I$ are those indices such that $\mathrm{ord}(f_i') > p$ (and thus $\mathrm{ord}(f_i) > p^{s_t}$) and $\alpha_i \in \mathbb{Z}$. Then, in view of the conclusion of the previous paragraph, we see that

$$0 = \sum_{i=1}^{r-t+1} p^{s'} \alpha_i f_i'$$

is a relation in $p^{s_t-1} \cdot G$, where $s' := \max\{0,\, 1 - \min\{\mathsf{v}_p(\alpha_i) \mid i \in I\}\}$ (here $\mathsf{v}_p(\alpha_i)$ is the p-valuation of $\alpha_i \in \mathbb{Z}$). If $s' = 0$, then the independence of the f_i' implies that $\alpha_i f_i' = 0$, and thus $\phi_L(\alpha_i f_i') = \alpha_i \phi_L(f_i') = 0$, for all i. If $s' = 1$, then the definition of s' implies that $\mathsf{v}_p(\alpha_j) = 0$ for some $j \in I$, whence $\mathrm{ord}(\alpha_j f_j') > p$ follows from the definition of I. As a result, $p\alpha_j f_j' \neq 0$, contradicting that the f_i' are independent. Thus $\phi_L(f_1'), \ldots, \phi_L(f_{r-t+1}')$ are $r-t+1$ independent elements in $\phi_L(p^{s_t-1} \cdot G)$, which is a group of total rank at most $r-t$, contradicting that the total rank is the maximal number of independent elements (see [11, Apendix A]). This completes the proof. □

We can now prove Lemma 4.1.

Proof. If G is cyclic, then $\mathsf{d}^*(G) = |G| - 1$, $\mathsf{d}^*(H) = |H| - 1$ and $\mathsf{d}^*(G/H) = \frac{|G|}{|H|} - 1$. Hence $\mathsf{d}^*(G) \ge \mathsf{d}^*(H) + \mathsf{d}^*(G/H)$ follows from the general inequality $xy \ge x + y - 1$ for $x, y \in \mathbb{Z}_{\ge 1}$. Therefore we may assume $\mathsf{r}(G) \ge 2$ and proceed by induction on the rank $\mathsf{r}(G) = r$.

Let $G \cong \bigoplus_{i=1}^{r} C_{m_i}$, $H \cong \bigoplus_{i=1}^{r} C_{m'_i}$ and $G/H \cong \bigoplus_{i=1}^{r} C_{m''_i}$, with $1 < m_1|\ldots|m_r$ and $1 \leq m'_1|\ldots|m'_r$ and $1 \leq m''_1|\ldots|m''_r$. In view of Propositions 4.4 and 4.3, we see that $m'_i|m_i$ and $m''_i|m_i$ for all i. Hence, if $m'_i < m_i$ and $m''_i < m_i$ for all i, then $m'_i \leq \frac{1}{2}m_i$ and $m''_i \leq \frac{1}{2}m_i$, whence $m'_i - 1 + m''_i - 1 < m_i - 1$; consequently, summing over all i yields the desired bound $\mathsf{d}^*(G) \geq \mathsf{d}^*(H) + \mathsf{d}^*(G/H)$. Therefore we may assume $m'_s = m_s$ or $m''_s = m_s$ for some s, and in view of Proposition 4.3, we may w.l.o.g. assume $m'_s = m_s$.

Now applying Lemma 4.5, we conclude that there are subgroups K, $H_0 \leq H$ and $G_0 \leq G$ such that $H = K \oplus H_0$ and $G = K \oplus G_0$ with $K \cong C_{m_s}$. Moreover, we can choose the complimentary summand H_0 such that $H_0 \leq G_0$. Note $\mathsf{d}^*(H) = \mathsf{d}^*(K) + \mathsf{d}^*(H_0)$ and $\mathsf{d}^*(G) = \mathsf{d}^*(K) + \mathsf{d}^*(G_0)$, while $G/H = (K \oplus G_0)/(K \oplus H_0) \cong G_0/H_0$, so that $\mathsf{d}^*(G_0/H_0) = \mathsf{d}^*(G/H)$. Thus $\mathsf{d}^*(G) \geq \mathsf{d}^*(H) + \mathsf{d}^*(G/H)$ follows by applying the induction hypothesis to G_0 with subgroup H_0. □

Having established Lemma 4.1, we conclude the section with the proof of Lemma 4.2.

Proof. By translation, we may w.l.o.g. assume $a_0 = 0 \in A$ and $H = G$. Since $|A| \geq 2$, we have $\langle A \rangle = H = G$ nontrivial. Let $K \leq H = G$ be the maximal subgroup such that there exists a subset $B \subseteq A$ with $0 \in B$, $K = \langle B \rangle$ and

$$|\sum_{i=1}^{\mathsf{d}^*(K)} w_i \cdot B| = |K|, \tag{8}$$

if such K exists, and otherwise let $K = B = \{0\}$. We may assume $K < H = G$, else the proof is complete.

Since $\langle B \rangle = K \neq G$ and $\langle A \rangle = G$, choose $g \in A \setminus B$ such that $\langle B' \rangle := K' > K$, where $B' = B \cup \{g\}$. Let $L = \langle g \rangle$. Note $K'/K = (K+L)/K \cong L/(K \cap L)$ is cyclic. Hence, in view of Lemma 4.1, we have

$$|K'/K| - 1 = \mathsf{d}^*(K'/K) \leq \mathsf{d}^*(K') - \mathsf{d}^*(K) \leq \mathsf{d}^*(G) - \mathsf{d}^*(K).$$

Thus Kneser's Theorem implies, in view of $w_i g \in L$ and $\gcd(w_i, \exp(H)) = 1$ (so that $\mathrm{ord}(w_i g) = \mathrm{ord}(g)$) and $\langle g \rangle = L$ (so that $\langle \phi_K(g) \rangle = K'/K = (K+L)/K$), that

$$|\sum_{i=\mathsf{d}^*(K)+1}^{\mathsf{d}^*(K')} \phi_K(w_i \cdot B')| = |\sum_{i=\mathsf{d}^*(K)+1}^{\mathsf{d}^*(K')} \phi_K(w_i \cdot \{0, g\})| = |K'/K|,$$

and thus from (8) it follows that

$$|\sum_{i=1}^{\mathsf{d}^*(K')} w_i \cdot B'| = |K'|,$$

contradicting the maximality of K, and completing the proof. $\square$

5. Theorems 2.4 and 2.5

Theorems 2.4 and 2.5 will be derived by an inductive argument from the following result. (Theorem E is easily derived from the proof of [16] using the both the modifications mentioned in [14] and those in [17]; see [19] for a unified presentation of the arguments, though the condition $\gcd(w, \mathrm{ord}(g)) = 1$, for all $w \in \mathrm{supp}(W)$ and all torsion elements $g \in \mathrm{supp}(S)$, is misstated in the statement of Theorem 3.1 in [19], and is corrected below.)

Theorem E. *Let G be an abelian group, let S, $S' \in \mathcal{F}(G)$ with $S'|S$, and let $W = w_1 \cdot \ldots \cdot w_n \in \mathcal{F}(\mathbb{Z})$ be a sequence of integers such that $\mathsf{h}(S') \le n \le |S'|$ and $\gcd(w, \mathrm{ord}(g)) = 1$ for all $w \in \mathrm{supp}(W)$ and all torsion elements $g \in \mathrm{supp}(S)$. Then there exists $H \le G$ and an n-setpartition $A = A_1 \cdot \ldots \cdot A_n$ of a subsequence S'' of S such that $\sum\limits_{i=1}^{n} w_i \cdot A_i$ is H-periodic, $|S'| = |S''|$, and*

$$|\sum_{i=1}^{n} w_i \cdot A_i| \ge ((N-1)n + e + 1)|H|, \tag{9}$$

where

$$N = \frac{1}{|H|} |\bigcap_{i=1}^{n} (A_i + H)| \quad \text{and} \quad e = \sum_{j=1}^{n} (|A_j| - |A_j \cap \bigcap_{i=1}^{n} (A_i + H)|).$$

Furthermore, if H is nontrivial, then $N \ge 1$ and $\mathrm{supp}(S''^{-1}S) \subseteq \bigcap_{i=1}^{n} (A_i + H)$.

The following basic result, which is a simple consequence of the pigeonhole principle, will be used in the proof [11, Lemma 5.2.9].

Proposition F. *Let G be a finite abelian group and let A, $B \subseteq G$ be nonempty subsets. If $|A| + |B| \ge |G| + 1$, then $A + B = G$.*

We proceed with the proof of Theorems 2.4 and 2.5 simultaneously.

Proof. Observe that the hypotheses of Theorem 2.4 allow us to apply Theorem E with G, $S'|S$, $W = w_1 \cdot \ldots \cdot w_n$ and n the same in both theorems. Let H, S'', $A = A_1 \cdot \ldots \cdot A_n$, N and e be as given by Theorem E. If H is trivial, then (9) implies $|\sum\limits_{i=1}^{n} w_i \cdot A_i| \ge |S'| - n + 1$, and if $H = G$, then $\sum\limits_{i=1}^{n} w_i \cdot A_i$ being H-periodic implies $|\sum\limits_{i=1}^{n} w_i \cdot A_i| = |G|$; in either case, (i) follows. Therefore we may assume H is a proper, nontrivial subgroup. This completes the case when $|G|$ is prime in Theorem 2.4.

Concerning Theorem 2.5(i), in view of $\mathsf{h}(S') \le n$ and $|T^{-1}S| \ge n - \mathsf{d}^*(G) + |S| - |S'|$, it is easily seen that the setpartition $B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(G)}$ of T can be extended to a setpartition $A_1 \cdot \ldots \cdot A_n$ of a sequence $S'''|S$, with $B_i \subseteq A_i$ for $i \le \mathsf{d}^*(G)$, $T|S'''$ and $|S'''| = |S'|$, by the following argument. Begin with $A_i = B_i$ for $i \le \mathsf{d}^*(G)$ and $A_i = \emptyset$ for $\mathsf{d}^*(G) < i \le n$. If $W|S$ are all terms with multiplicity at least n and $W' = \prod_{g \in \text{supp}(W)} g^n$, then augment the sets A_i so that $\text{supp}(W) \subseteq A_i$ for all i (that is, simply include each $g \in \text{supp}(W)$ in each set A_i if it was not already there). We must have $|W'^{-1}W| \le |S| - |S'|$, else it would have been impossible that a subsequence of S with length $|S'|$ had an n-setpartition, which we know is the case since $\mathsf{h}(S') \le n \le |S'|$. All remaining terms in $T^{-1}W^{-1}S$ have multiplicity at most $n-1$, and so we can distribute all but $|S| - |S'| - |W'^{-1}W|$ of them among the A_i so that no A_i contains two equal terms, always choosing to place an element in an empty set if available. Since $|T^{-1}S| \ge n - \mathsf{d}^*(G) + |S| - |S'|$, we are either assured that there are enough terms to fill all empty sets in this manner, or that we can move some of the terms from W' (but not from T) placed in the A_i with $i \le \mathsf{d}^*(G)$ so that this is the case, and then the resulting A_i give the n-setpartition with the desired properties.

Consequently, (i) in Theorem 2.5 is trivial, and since the only nontrivial subgroup of G, when $|G|$ is prime, is G, we see that the case $|G|$ prime is complete for Theorem 2.5 as well.

We now proceed by induction on the number of prime factors of $|G|$. We first show that (i) failing in Theorem 2.4 implies the hypotheses of Theorem 2.5 hold (this is Claim B below), from which we infer that Theorem 2.5 implies Theorem 2.4. The remainder of the proof will then be devoted to proving Theorem 2.5 assuming by induction hypothesis that Theorem 2.4 holds in any abelian group G' with $|G'|$ having a smaller number of prime factors than $|G|$.

To this end, we assume (i) fails. Since (i) holds trivially when $n = 1$ (in view of $n \ge \mathsf{h}(S')$), we may assume $n \ge 2$. Let $x := |S| - |S'| \ge 0$. Since (i) fails, it follows from (9) that

$$((N-1)n + e + 1)|H| \le |S'| - n. \tag{10}$$

Much of the proof is contained in the following claim.

Claim B: There exists a nontrivial subgroup K, $g' \in G$, and an $\mathsf{d}^*(K)$-setpartition $B = B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(K)}$ of a subsequence $T|S$ with $T \in \mathcal{F}(g' + K)$, such that

$$\sum_{i=1}^{\mathsf{d}^*(K)} w_i \cdot B_i = \left(\sum_{i=1}^{\mathsf{d}^*(K)} w_i \right) g' + K \tag{11}$$

and $T^{-1}S$ contains at least $n - \mathsf{d}^*(K) + x$ terms from $g' + K$.

Proof. There are two cases.

Case 1: $N \geq 2$. If there does not exist $g' \in \bigcap_{i=1}^{n}(A_i + H)$ and A_j and A_k such that $j \neq k$ and

$$|A_k \cap (g' + H)| + |A_j \cap (g' + H)| \geq |H| + 1, \tag{12}$$

then it would follow from the pigeonhole principle (since $n \geq 2$) that

$$|S'| = |S''| \leq \frac{1}{2}|H|Nn + e,$$

which combined with (10) implies $((N-1)n + e)|H| \leq \frac{1}{2}|H|Nn + e - n$, whence

$$|H|nN \leq 2(|H| - 1)(n - e) \leq 2n(|H| - 1),$$

implying $N < 2$, a contradiction. Therefore we may assume such g' and A_j and A_k exist, and w.l.o.g. $j = 1$ and $k = 2$ (by re-indexing the A_i but *not* the w_i; note we lose the sumset bound given by (9) in doing so, but we will only need the information it implied about the structure of S and not use the bound itself in the remainder of Case 1). By translation we may also assume $g' = 0$.

From Proposition F, (12) and $\gcd(w_i, \exp(G)) = 1$, it follows that

$$|w_1 \cdot (A_1 \cap H) + w_2 \cdot (A_2 \cap H)| = |H|. \tag{13}$$

Let $B_j = A_j \cap \bigcap_{i=1}^{n}(A_i + H)$ for $j = 1, \ldots, n$, and note that $\phi_H(B_i) = \phi_H(B_j)$ for all i and j. Let $K = H + \langle B_i \rangle$ and $T = \prod_{i=1}^{\mathsf{d}^*(K)} B_i \in \mathcal{F}(K)$. From the conclusion of Theorem E, we know $T^{-1}S$ contains at least $n - \mathsf{d}^*(K) + x$ terms from K (since each A_i intersects $\bigcap_{i=1}^{n}(A_i + H)$ in at least $N \geq 1$ points and $\mathrm{supp}(S''^{-1}S) \subseteq \bigcap_{i=1}^{n}(A_i + H)$, both of which were preserved when re-indexing the A_i).

If $\mathsf{d}^*(H) \geq 2$, then from (13) and $g' = 0$ we find that

$$H \subseteq \sum_{i=1}^{\mathsf{d}^*(H)} w_i \cdot B_i. \tag{14}$$

On the otherhand, if $\mathsf{d}^*(H) = 1$, then $|H| = 2$, whence (12) and the pigeonhole principle imply that w.l.o.g. $|A_1 \cap H| = |H|$, and thus (14) holds in this case as well. Since $n \geq \mathsf{d}^*(G) \geq \mathsf{d}^*(K)$, it follows by Lemma 4.1 that

$$n - \mathsf{d}^*(H) \geq \mathsf{d}^*(K) - \mathsf{d}^*(H) \geq \mathsf{d}^*(K/H).$$

Thus, applying Lemma 4.2, taking $\phi_H(B_i)$ for A and G/H for G (recall that $g' = 0$ and $|\phi_H(B_i)| = N \geq 2$), it follows that

$$\sum_{i=\mathsf{d}^*(H)+1}^{\mathsf{d}^*(K)} \phi_H(w_i \cdot B_i) = K/H,$$

which in view of (14) implies that (11) holds. In view of the conclusion of the previous paragraph, this completes the claim.

Case 2: $N = 1$. Let T be the subsequence of S consisting of all terms from $g+H$, let $T'|T$ be the subsequence consisting of all terms with multiplicity at least $\mathsf{d}^*(H)$, and let $B = \text{supp}(T')$. From (10) and Theorem E, it follows that

$$|T| \geq x + |S'| - e \geq (e+1)|H| + n + x - e \geq n + |H| + x. \tag{15}$$

By translation, we may w.l.o.g. assume $0 \in \text{supp}(T)$, and that $0 \in \text{supp}(T')$ if $\text{supp}(T') \neq \emptyset$. We handle two subcases.

Subcase 2.1: Suppose there exists a subsequence $T_0|T$ with $\mathsf{h}(T_0) \leq \mathsf{d}^*(H)$ and $|T_0| = \mathsf{d}^*(H) + |H| - 1$. Then we can apply the induction hypothesis to $T_0|T$ with G taken to be H and n taken to be $\mathsf{d}^*(H)$. Let $B = B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(H)}$ be the resulting setpartition and T_0' the resulting subsequence of T. From (15), we see that

$$|{T_0'}^{-1}T| = |T| - |T_0'| = |T| - |T_0| = |T| - \mathsf{d}^*(H) - |H| + 1 \geq n + x - \mathsf{d}^*(H). \tag{16}$$

If (i) holds, then $|T_0| = \mathsf{d}^*(H) + |H| - 1$ implies that

$$|\sum_{i=1}^{\mathsf{d}^*(H)} w_i \cdot B_i| = |H|,$$

and the claim is complete (in view of (16)) using T_0' for T and H for K. On the otherhand, if (ii) holds with (say) subgroup $K \leq H$, $g' \in H$ and setpartition $B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(H)}$, then (11) follows from (ii)(d) (taking T to be the associated sequence to $B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(K)}$), while (ii)(a) and (15) imply ${T_0''}^{-1}T$ contains at least

$$\mathsf{d}^*(H) - \mathsf{d}^*(K) + |T| - |T_0| = -\mathsf{d}^*(K) + |T| - |H| + 1 \geq n - \mathsf{d}^*(K) + x$$

terms from $g' + K$, whence the claim follows.

Subcase 2.2: There does not exist a subsequence $T_0|T$ with $\mathsf{h}(T_0) \leq \mathsf{d}^*(H)$ and $|T_0| = \mathsf{d}^*(H) + |H| - 1$. Consequently,

$$|\text{supp}(T')|\mathsf{d}^*(H) + |{T'}^{-1}T| \leq \mathsf{d}^*(H) + |H| - 2,$$

which, in view of (15), yields

$$|T'| \geq n + x + 2 + (|\text{supp}(T')| - 1)\mathsf{d}^*(H). \tag{17}$$

Since $\mathsf{v}_g(T') \leq \mathsf{v}_g(T) \leq n + x$ for all $g \in G$ (in view of $\mathsf{h}(S') \leq n$), it follows that $|T'| \leq (n+x)|\text{supp}(T')|$. Thus, in view of $n \geq \mathsf{d}^*(G) \geq \mathsf{d}^*(H)$ and $x \geq 0$, we conclude from (17) that $|\text{supp}(T')| \geq 2$.

Let $K = \langle \text{supp}(T') \rangle \leq H$ and let $T_0 := \prod_{g \in \text{supp}(T')} g^{\mathsf{d}^*(K)}$ be the subsequence of T' (recall the definition of T') obtained by taking each term with multiplicity exactly $\mathsf{d}^*(K) \le \mathsf{d}^*(H)$. Observe, in view of (17) and $\mathsf{d}^*(K) \le \mathsf{d}^*(H)$, that

$$\begin{aligned}(18)\quad |T_0^{-1}T'| &= |T'| - |T_0| = |T'| - |\mathrm{supp}(T')|\mathsf{d}^*(K)\\ &\ge n + x + 2 + (|\mathrm{supp}(T')| - 1)(\mathsf{d}^*(H) - \mathsf{d}^*(K)) - \mathsf{d}^*(K)\\ &\ge n + x - \mathsf{d}^*(K).\end{aligned}$$

Applying Lemma 4.2 with A taken to be $\mathrm{supp}(T')$, we conclude (recall $0 \in \mathrm{supp}(T')$) that

$$\sum_{i=1}^{\mathsf{d}^*(K)} w_i \cdot B_i = K,$$

where $B_i = \mathrm{supp}(T')$ for $i = 1, \ldots, \mathsf{d}^*(K)$. Hence, in view of (18), we see that the claim follows (taking T to be T_0). $\square$

Having now established Claim B, we see that it suffices to prove Theorem 2.5 to complete the inductive proofs of Theorems 2.4 and 2.5. Let K be a maximal subgroup satisfying Claim B, and let g', T and $B_1 \cdot \ldots \cdot B_{\mathsf{d}^*(K)}$ be as given by Claim B. By translation we may w.l.o.g. assume $g' = 0$. Let $S_0|S$ be the subsequence consisting of all terms x with $\phi_K(x) \ne 0$, and let $e := |S_0|$. As remarked earlier, if $K = G$, then Theorem 2.5(i) follows trivially. Therefore assume $K < G$. Observe that Claim B implies

$$(19)\qquad |T^{-1}S_0^{-1}S| \ge n - \mathsf{d}^*(K) + x.$$

Suppose $\mathsf{h}(\phi_K(S_0)) \ge \mathsf{d}^*(G/K)$. Then let $g \in \mathrm{supp}(S_0)$ with $\mathsf{v}_{\phi_K(g)}(\phi_K(S_0)) \ge \mathsf{d}^*(G/K)$ and let $L = K + \langle g \rangle$. By Lemma 4.1, we have

$$(20)\qquad \mathsf{d}^*(L) \ge \mathsf{d}^*(K) + \mathsf{d}^*(L/K).$$

In view of (19), $\mathsf{h}(\phi_K(S_0)) \ge \mathsf{d}^*(G/K) \ge \mathsf{d}^*(L/K)$ and $n \ge \mathsf{d}^*(G) \ge \mathsf{d}^*(L)$, we can find a subsequence $T'|T^{-1}S$ such that $\phi_K(T') = \phi_K(g)^{\mathsf{d}^*(L/K)}0^{\mathsf{d}^*(L)-\mathsf{d}^*(K)}$, and thus such that $(TT')^{-1}S$ contains at least

$$(21)\qquad n - \mathsf{d}^*(K) + x - (\mathsf{d}^*(L) - \mathsf{d}^*(K)) = n - \mathsf{d}^*(L) + x$$

terms from L. In view of (20), let $B_{\mathsf{d}^*(K)+1} \cdot \ldots \cdot B_{\mathsf{d}^*(L)}$ be a setpartition of T' such that $|B_i| = 2$ and $\phi_K(B_i) = \{0, \phi_K(g)\}$, for $i = \mathsf{d}^*(K)+1, \ldots, \mathsf{d}^*(K) + \mathsf{d}^*(L/K)$, and $|B_i| = 1$ and $\phi_K(B_i) = \{0\}$, for $i = \mathsf{d}^*(K) + \mathsf{d}^*(L/K) + 1, \ldots, \mathsf{d}^*(L)$.

Applying Lemma 4.2 to $\{0, \phi_K(g)\}$, we conclude that

$$|\sum_{i=\mathsf{d}^*(K)+1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} \phi_K(w_i \cdot B_i)| = |L/K|,$$

and consequently (in view of (11) and (20)) that

$$|\sum_{i=1}^{\mathsf{d}^*(L)} w_i \cdot B_i| = |L|.$$

But now, in view also of (21), we see that the maximality of K is contradicted by L. So we may instead assume $\mathsf{h}(\phi_K(S_0)) < \mathsf{d}^*(G/K)$.

Let R be a subsequence of $T^{-1}S$ such that $S_0|R$ and $|R| = |S_0| + \mathsf{d}^*(G/K)$ (possible in view of (19), $x \geq 0$, $n \geq \mathsf{d}^*(G)$ and Lemma 4.1). Moreover, from (19),

$$|(TR)^{-1}S| \geq n + x - \mathsf{d}^*(K) - \mathsf{d}^*(G/K), \tag{22}$$

with all term of $(TR)^{-1}S$ contained in K (since $S_0|R$).

Since $\mathsf{h}(\phi_K(S_0)) < \mathsf{d}^*(G/K)$, since $\phi_K(y) = 0$ for $y|S_0^{-1}S$, and since $\phi_K(y) \neq 0$ for $y|S_0$, it follows that $\mathsf{h}(\phi_K(R)) \leq \mathsf{d}^*(G/K)$. Thus we can apply the induction hypothesis to the subsequence $\phi_K(R)|\phi_K(R)0^{|G/K|-1}$ with $n = \mathsf{d}^*(G/K)$ and G taken to be G/K. Let $\phi_K(B_{\mathsf{d}^*(K)+1}) \cdot \ldots \cdot \phi_K(B_{\mathsf{d}^*(K)+\mathsf{d}^*(G/K)})$ be the resulting setpartition and $\phi_K(R')$ the resulting sequence, where $R'|R0^{|G/K|-1}$ and $B_{\mathsf{d}^*(K)+1} \cdot \ldots \cdot B_{\mathsf{d}^*(K)+\mathsf{d}^*(G/K)}$ is a setpartition of R'. Observe, since $\mathsf{v}_0(\phi_K(R)) = \mathsf{d}^*(G/K)$, that $\text{supp}(\phi_K(R')^{-1}\phi_K(R)0^{|G/K|-1}) = \{0\}$, and thus that we can w.l.o.g. assume $R' = R$ and likewise that $B_{\mathsf{d}^*(K)+1} \cdot \ldots \cdot B_{\mathsf{d}^*(K)+\mathsf{d}^*(G/K)}$ is a setpartition of R.

Suppose (ii) holds and let L/K be the corresponding subgroup. Since $\mathsf{v}_0(\phi_K(R)0^{|G/K|-1}) \geq |G/K| - 1$, it follows in view of (ii)(c) that w.l.o.g. $g = 0$ (where g is as given by (ii)). But then (ii)(d) implies

$$\sum_{i=\mathsf{d}^*(K)+1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} w_i \cdot \phi_K(B_i) = L/K,$$

whence (11) implies

$$\sum_{i=1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} w_i \cdot B_i = L.$$

In view of (ii)(a) and (22), it follows that there are still at least

$$n + x - \mathsf{d}^*(K) - \mathsf{d}^*(G/K) + (\mathsf{d}^*(G/K) - \mathsf{d}^*(L/K)) = n + x - \mathsf{d}^*(K) - \mathsf{d}^*(L/K) \tag{23}$$

terms remaining in $\left(\prod_{i=1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} B_i\right)^{-1} S$ that are contained in L. Thus (in view of Lemma 4.1) by appending on an additional $\mathsf{d}^*(L) - \mathsf{d}^*(L/K) - \mathsf{d}^*(K) \geq 0$ terms B_i, for $i = \mathsf{d}^*(K) + \mathsf{d}^*(L/K) + 1, \ldots, \mathsf{d}^*(L)$, with each such new B_i consisting of a single element from L contained in

$\left(\prod_{i=1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} B_i\right)^{-1} S$ (that is, $\mathrm{supp}(\prod_{i=\mathsf{d}^*(K)+\mathsf{d}^*(L/K)+1}^{\mathsf{d}^*(L)} B_i) \subseteq L$ where we have $\prod_{i=\mathsf{d}^*(K)+\mathsf{d}^*(L/K)+1}^{\mathsf{d}^*(L)} B_i | \left(\prod_{i=1}^{\mathsf{d}^*(K)+\mathsf{d}^*(L/K)} B_i\right)^{-1} S$), we see that

$$\sum_{i=1}^{\mathsf{d}^*(L)} w_i \cdot B_i = L$$

and with $(\prod_{i=1}^{\mathsf{d}^*(L)} B_i)^{-1} S$ containing at least (in view of (23))

$$n + x - \mathsf{d}^*(K) - \mathsf{d}^*(L/K) - (\mathsf{d}^*(L) - \mathsf{d}^*(L/K) - \mathsf{d}^*(K)) = n + x - \mathsf{d}^*(L)$$

terms from L. Hence L contradicts the maximality of K. So we may assume instead that (i) holds.

As above, let B_i, for $i = \mathsf{d}^*(K) + \mathsf{d}^*(G/K) + 1, \ldots, n$ (in view of (22)), be defined by partitioning, as singleton terms (i.e., $|B_i| = 1$), $n - \mathsf{d}^*(K) - \mathsf{d}^*(G/K)$ of the terms of the sequence $\left(\prod_{i=1}^{\mathsf{d}^*(K)+\mathsf{d}^*(G/K)} B_i\right)^{-1} S = (TR)^{-1}S$ (which are all from K in view of the comment after (22)).

If

$$\sum_{i=\mathsf{d}^*(K)+1}^{\mathsf{d}^*(K)+\mathsf{d}^*(G/K)} w_i \cdot \phi_K(B_i) = G/K, \tag{24}$$

then (11), $n \geq \mathsf{d}^*(G)$ and Lemma 4.1 imply that

$$\sum_{i=1}^{\mathsf{d}^*(G)} w_i \cdot B_i = G.$$

Thus, in view of (22), we see that Claim B holds with $K = G$, contrary to assumption. Therefore we can assume (24) fails, which, in view of $|R| = |S_0| + \mathsf{d}^*(G/K)$ and (i) holding for $\phi_K(R)$ with $n = \mathsf{d}^*(G/K)$, implies that $e := |S_0| \leq |G/K| - 2$ and, in view of (11), that

$$|\sum_{i=1}^{n} w_i \cdot B_i| \geq (e+1)|K|.$$

The remaining conclusions for (ii) now follow easily from Claim B holding with K (by the same arguments used for establishing Theorem 2.5(i)), so that (ii) holds for S' with subgroup K, as desired. This completes the proof. $\square$

With the proof of Theorems 2.4 and 2.5 complete, the improvement to Theorem A follows as a simple corollary.

Corollary 5.1. *Let G be a finite abelian group, and let $S \in \mathcal{F}(G)$ with $|S| \geq |G| + \mathsf{d}^*(G)$. Then either*

(i) $\Sigma_{|G|}(S) = G$, *or*

(ii) *there exist a proper subgroup $H < G$ and some $g \in G$ such that all but at most $|G/H| - 2$ terms of S are from the coset $g + H$.*

Proof. Let $|S| = |G| + \mathsf{d}^*(G) + x$, so $x \geq 0$. We assume (ii) fails for every H and prove (i) holds. Note (ii) failing with H trivial implies $\mathsf{h}(S) \leq \mathsf{d}^*(G) + x + 1$.

Suppose $\mathsf{h}(S) \leq \mathsf{d}^*(G) + x$. Then we can apply Theorem 2.4 with $n = \mathsf{d}^*(G) + x$, $S = S'$ and $w_i = 1$ for all i. If Theorem 2.4(ii) holds, then Theorem 2.4(ii)(c) implies Corollary 5.1(ii), contrary to assumption. If instead Theorem 2.4(i) holds, then from $|S| = |G| + \mathsf{d}^*(G) + x$ we conclude that $\Sigma_{\mathsf{d}^*(G)+x}(S) = G$. Since $\Sigma_n(S) = \sigma(S) - \Sigma_{|S|-n}(S)$ holds trivially for any n (there is a natural correspondence between $S_0|S$ and ${S_0}^{-1}S|S$), it now follows that (i) holds for S, as desired. So we may assume $\mathsf{h}(S) = \mathsf{d}^*(G) + x + 1$.

By translation, we may w.l.o.g. assume 0 is a term with multiplicity $\mathsf{h}(S)$ in S. We may also assume there is a nonzero $g \in G$ with $\mathsf{v}_g(S) = \mathsf{v}_0(S) = \mathsf{h}(S)$, else applying Theorem 2.4 to $0^{-1}S|S$ completes the proof as in the previous paragraph. Let $S'|S$ be a maximal length subsequence with $\mathsf{h}(S') = \mathsf{d}^*(G) + x$, let $A = \mathrm{supp}(S'^{-1}S)$, and let $K = \langle A \rangle$. Notice $\{0, g\} \subseteq A$. Hence, since $\mathsf{h}(S) = \mathsf{d}^*(G) + x + 1$, it follows from Lemma 4.2 that the hypotheses of Theorem 2.5 hold with $n = \mathsf{d}^*(G) + x$, $S'|S$, K, and $w_i = 1$ and $B_i = A$ for all i. If Theorem 2.5(i) holds, then $|G| = |\Sigma_{\mathsf{d}^*(G)+x}(S)| = |\Sigma_{|G|}(S)|$ (as in the case $\mathsf{h}(S) \leq \mathsf{d}^*(G) + x$), yielding (i). On the otherhand, Theorem 2.5(ii) implies (ii) holds (in view of Theorem 2.4(ii)(c)). Thus the proof is complete. □

Next, the related corollary concerning Conjecture 2.3. Note the coset condition assumed below for H trivial implies $\mathsf{h}(S) \leq |S| - |G| + 1$, so the hypothesis $\mathsf{h}(S) \leq h \leq |S| - |G| + 1$ is not vacuous. The case $h = |G|$ and $|S| = 2|G| - 1$ in Corollary 5.2 is the result from [32].

Corollary 5.2. *Let G be a finite abelian group, let $S \in \mathcal{F}(G)$, let $h \in \mathbb{Z}$ with $\max\{\mathsf{h}(S), \mathsf{d}^*(G)\} \leq h \leq |S| - |G| + 1$, and let $W \in \mathcal{F}(\mathbb{Z})$ be a sequence of integers relatively prime to $\exp(G)$ with $|W| \geq h$. Suppose there does not exist a proper subgroup $H < G$ and $g \in G$ such that all but at most $|G/H| - 2$ terms of S are from the coset $g + H$. Then $\Sigma_h(W, S) = G$. In particular, $\Sigma(W, S) = G$*

Proof. The proof is identical to the case $\mathsf{h}(S) \leq \mathsf{d}^*(G) + x$ in Corollary 5.1 using $n = h$, the only other exception being that the identity $|\Sigma_n(W, S)| = |\Sigma_{|S|-n}(W, S)|$ is not necessarily valid for arbitrary W, S and n, thus preventing the proof of Conjecture 2.3 itself. □

Now we derive Corollary 2.6 from Theorem 2.5.

Proof. Let $m=\exp(G)$, $n=|W|$ and $t=|W'|$. By considering G as a $\mathbb{Z}/m\mathbb{Z}$-module (for notational convenience), we may w.l.o.g. consider W as a sequence from $\mathbb{Z}/m\mathbb{Z}$, say w.l.o.g. $W=w_1\cdot\ldots\cdot w_n$, where $\mathrm{ord}(w_i)=m$ for $i\leq n-t$ (in view of the hypothesis $\gcd(w,\exp(G))=1$ for all $w\in\mathrm{supp}({W'}^{-1}W)$). Observe that we may assume $|S|=n+|G|-1$ and that there are distinct x, $y\in G$ with $x^{n-t+1}y^{n-t+1}|S$, else Theorem D implies the theorem (as if such is not the case, then in view of $\mathsf{h}(S)\leq n=|W|$ there would exist a n-setpartition of S with t sets of cardinality one). Since $\sigma(W)=0$, we may w.l.o.g. by translation assume $x=0$.

Let $A\subseteq\mathrm{supp}(S)$ be all those elements with multiplicity at least $n-t$, let $K=\langle A\rangle$, let $R|S$ be the maximal subsequence with $\mathrm{supp}(R)=A$, let $T:=\prod_{g\in A}g^{\mathsf{d}^*(K)}$, and let $T_0=\prod_{g\in A}g^{n-t}$. Notice $\{0,y\}\subseteq A$. Hence, since $\mathsf{h}(S)\leq n$ and $|W|-t=n-t\geq\mathsf{d}^*(G)$ by hypothesis, it follows from Lemma 4.2 applied to A that the hypotheses of Theorem 2.5 hold with n taken to be $n-t$, $B_i=A$ for $i=1,\ldots,\mathsf{d}^*(K)$, and $S'=T_0(R^{-1}S)|S$.

If $|R|\leq|A|(n-t)+t$, then Theorem D once more completes the proof (as then there exists an n-setpartition of S with at least t sets of cardinality one, in view of $\mathsf{h}(S)\leq n$). Therefore $|R|\geq|A|(n-t)+t+1$, and so

$$|S|-|S''|=|S|-|S'|=|R|-|T_0|\geq t+1, \tag{25}$$

where S'' is as given by Theorem 2.5. Consequently, if Theorem 2.5(i) holds, then it follows that $\Sigma_{n-t}({W'}^{-1}W,S'')=G$ with $|{S''}^{-1}S|\geq t$, whence $\Sigma_n(W,S)=\Sigma_{|W|}(W,S)=G$ follows, as desired. On the otherhand, if Theorem 2.5(ii) holds, then Theorem 2.4(ii)(a)(d) implies

$$\left(\sum_{i=1}^{n-t}w_i\right)g+H\subseteq\sum_{i=1}^{n-t}w_i\cdot A_i\subseteq\Sigma_{n-t}({W'}^{-1}W,S''),$$

where g, H and the A_i are as given by Theorem 2.4(ii), whence (25), $\mathrm{supp}({S''}^{-1}S)\subseteq g+H$ (in view of (ii)(a)), and $\sigma(W)=0$ imply

$$H=\left(\sum_{i=1}^{n}w_i\right)g+H\subseteq\Sigma_n(W,S)=\Sigma_{|W|}(W,S),$$

as desired. □

Finally, we show Conjecture 2.3 holds when $\mathsf{h}(S)\geq\mathsf{D}(G)-1$. For this, we need the following modification of a result from [7].

Lemma 5.3. *Let R be a ring, G an R-module, $W\in\mathcal{F}(R)$ and $S\in\mathcal{F}(G)$ with $|S|\geq|W|+\mathsf{D}(G)-1$. If $\mathsf{v}_0(S)=\mathsf{h}(S)\geq\mathsf{D}(G)-1$, then*

$$\Sigma(W,S)=\Sigma_{|W|}(W,S)\,.$$

Proof. Let $S'|S$ be the subsequence consisting of all nonzero terms. Let $g \in \Sigma(W,S)$ be arbitrary. Since $\Sigma_{|W|}(W,S) \subseteq \Sigma(W,S)$, we need to show that $g \in \Sigma_{|W|}(W,S)$.

If $g = 0$ and $\mathsf{h}(S) \geq |W|$, then $0 \in \Sigma_{|W|}(W, 0^{\mathsf{h}(S)}) \subseteq \Sigma_{|W|}(W,S)$ (in view of $\mathsf{v}_0(S) = \mathsf{h}(S)$), as desired. If $g = 0$ and $\mathsf{h}(S) \leq |W| - 1$, then $\mathsf{h}(S) \geq \mathsf{D}(G) - 1$ implies $|W| \geq \mathsf{D}(G)$, while $|S'| \geq |W| + \mathsf{D}(G) - 1 - \mathsf{h}(S) \geq \mathsf{D}(G)$. Thus

$$g \in \Sigma(W,S') \tag{26}$$

follows from the definition of $\mathsf{D}(G)$ applied to the sequence $(w_1s_1)(w_2s_2) \cdot \ldots \cdot (w_{\mathsf{D}(G)}s_{\mathsf{D}(G)}) \in \mathcal{F}(G)$, where $w_1 \cdot \ldots \cdot w_{\mathsf{D}(G)}|W$ and $s_1 \cdot \ldots \cdot s_{\mathsf{D}(G)}|S'$. On the otherhand, if $g \neq 0$, then (26) holds trivially. Thus we can assume (26) regardless, and we choose $W_1|W$ and $S_1|S'$ such that $W_1 = w_1 \cdot \ldots \cdot w_t$, $S_1 = s_1 \cdot \ldots \cdot s_t$ and $g = \sum\limits_{i=1}^{t} w_i s_i$, with t maximal.

Note $t \leq |W|$. If $t \geq |W| - \mathsf{h}(S)$, then $g \in \Sigma_{|W|}(W, S_1 0^{\mathsf{h}(S)}) \subseteq \Sigma_{|W|}(W,S)$, as desired. So we may assume

$$t \leq |W| - \mathsf{h}(S) - 1. \tag{27}$$

Hence

$$|S_1^{-1}S'| \geq |W| + \mathsf{D}(G) - 1 - \mathsf{h}(S) - t \geq \mathsf{D}(G). \tag{28}$$

Observe, in view of (27) and the hypotheses, that

$$|W_1^{-1}W| = |W| - t \geq \mathsf{h}(S) + 1 \geq \mathsf{D}(G). \tag{29}$$

Let $S' = s_1 \cdot \ldots \cdot s_t s_{t+1} \cdot \ldots \cdot s_{|S|-\mathsf{h}(S)}$ and $W = w_1 \cdot \ldots \cdot w_t w_{t+1} \cdot \ldots \cdot w_n$. In view of (28) and (29), let

$$T := (w_{t+1}s_{t+1})(w_{t+2}s_{t+2}) \cdot \ldots \cdot (w_{t+\mathsf{D}(G)}s_{t+\mathsf{D}(G)}) \in \mathcal{F}(G).$$

Observe $|T| = \mathsf{D}(G)$, whence the definition of $\mathsf{D}(G)$ implies T has a zero-sum subsequence, say (by re-indexing if necessary) $(w_{t+1}s_{t+1})(w_{t+2}s_{t+2}) \cdot \ldots \cdot (w_{t+r}s_{t+r})$, where $r \geq 1$. But now the sequences $w_1 \cdot \ldots \cdot w_{t+r}$ and $s_1 \cdot \ldots \cdot s_{t+r}$ contradict the maximality of t, completing the proof. $\square$

Note that Corollary 5.4(ii) failing with H trivially implies $\mathsf{h}(S) \leq |G|$ for $|S| \leq 2|G| - 1$, and that $2|G| - 1 \geq |G| + \mathsf{D}(G) - 1$ in view of the trivial bound $\mathsf{D}(G) \leq |G|$ (see [11]). Thus the restriction $\mathsf{h}(S) \leq |G|$ in Corollary 5.4 can be dropped when $|S| \leq 2|G| - 1$, and thus, in particular, when $|S| = |G| + \mathsf{D}(G) - 1$.

Corollary 5.4. *Let G be a finite abelian group, $S \in \mathcal{F}(G)$ with $|S| \geq |G| + \mathsf{D}(G) - 1$ and $|G| \geq \mathsf{h}(S) \geq \mathsf{D}(G) - 1$, and let $W \in \mathcal{F}(\mathbb{Z})$ with $|W| = |G|$ and $\gcd(w, \exp(G)) = 1$ for all $w \in \mathrm{supp}(W)$. Then either*

(i) $\Sigma_{|G|}(W,S) = G$, *or*

(ii) *there exist a proper subgroup $H < G$ and some $g \in G$ such that all but at most $|G/H| - 2$ terms of S are from the coset $g + H$.*

Proof. We may w.l.o.g. assume $\mathsf{v}_0(S) = \mathsf{h}(S)$. Thus our hypotheses allow us to apply Lemma 5.3, whence

$$\Sigma(W, S) = \Sigma_{|G|}(W, S). \tag{30}$$

Since we may assume (ii) fails with H trivial, it follows that $\mathsf{h}(S) \leq |S| - |G| + 1$. Consequently, since $|W| = |G| \geq \mathsf{h}(S)$, then the result follows from (30) and Corollary 5.2 applied with $h = \mathsf{h}(S)$ (in view of $\mathsf{D}(G) \geq \mathsf{d}^*(G) + 1$). □

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David J. Grynkiewicz
Institut für Mathematik und Wissenschaftliches Rechnen
Karl-Franzens-Universität Graz
Heinrichstraße 36
8010 Graz, Austria.

Luz E. Marchan
Departamento de Matemáticas
Decanato de Ciencias y Tecnologías
Universidad Centroccidental Lisandro Alvarado
Barquisimeto, Venezuela.

Oscar Ordaz
Departamento de Matemáticas y Centro ISYS
Facultad de Ciencias
Universidad Central de Venezuela
Ap. 47567
Caracas 1041-A, Venezuela.