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par Stanley J. GURAK

Résumé. Soit K un corps de degré n sur $\mathbf{Q}_p$, le corps des nombres p-adiques, de degré résiduel f, indice de ramification e et valuation de la différente d. Soient O l'anneau des entiers de K et P son unique idéal premier. Les applications trace et norme de $K/\mathbf{Q}_p$ sont notées Tr et N, respectivement. Fixons $q = p^r$, une puissance du nombre premier p, et η un caractère défini modulo q et d'ordre $o(\eta)$. Ce caractère η s'étend naturellement à l'anneau des entiers p-adiques $\mathbb{Z}_p$; précisément $\eta(u) = \eta(\tilde{u})$, où $\tilde{u}$ désigne la classe résiduelle de u modulo q, et de même pour la racine de l'unité $\zeta_q^u = exp(2\pi i\tilde{u}/q)$. Fixons un entier positif $\gamma \geq re-d$ pour lequel $N(1+P^\gamma) \subseteq 1+q\mathbb{Z}_p$, de sorte que les sommes (todues) de Kloosterman

$$R(\eta, z) = \sum_{\alpha \in (O/P^\gamma)^*} \eta(N\alpha)\zeta_q^{Tr\ \alpha + z/N\alpha} \qquad (z \in \mathbf{Z}/q\mathbf{Z}^*)$$

sont bien définies.

Saliè a déterminé explicitement $R(\eta, z)$ dans le cas classique $n = 1$ (donc $K = \mathbf{Q}_p$) pour $q = p^r$ avec $r > 1$ et $o(\eta) = 1$ ou 2. Ici, je généralise le résultat de Saliè dans le cas général $n > 1$ pour des caractères η avec $o(\eta)|p-1$ (et aussi $o(\eta) = 2$ quand $p = 2$), et pour tout $\gamma \geq re - d > 1$ sauf un petit nombre de valeurs exceptionnelles de r. Mon évaluation repose sur la détermination récente et explicite par l'auteur des sommes de Gauss pour les puissances de nombres premiers dans les corps p-adiques, et des sommes d'exponentielles de la forme $\sum \chi(x)^{ax}\zeta_q^{bx}$.

Abstract. Let K be a field of degree n over $\mathbf{Q}_p$, the field of rational p-adic numbers, say with residue degree f, ramification index e and differential exponent d. Let O be the ring of integers of K and P its unique prime ideal. The trace and norm maps for $K/\mathbf{Q}_p$ are denoted Tr and N, respectively. Fix $q = p^r$, a power of a prime p, and let η be a numerical character defined modulo q and of order $o(\eta)$. The character η extends to the ring of p-adic integers $\mathbb{Z}_p$ in the natural way; namely $\eta(u) = \eta(\tilde{u})$, where $\tilde{u}$ denotes the residue class of u modulo q, and similarly for the root of unity $\zeta_q^u = exp(2\pi i\tilde{u}/q)$. Fix a positive integer $\gamma \geq re - d$ for which $N(1 + P^\gamma) \subseteq 1 + q\mathbb{Z}_p$ so that the (twisted) Kloosterman sums

$$R(\eta, z) = \sum_{\alpha \in (O/P^\gamma)^*} \eta(N\alpha)\zeta_q^{Tr\ \alpha + z/N\alpha} \qquad (z \in \mathbf{Z}/q\mathbf{Z}^*)$$

are well-defined.

Saliè explicitly determined $R(\eta, z)$ in the classical case $n = 1$ (so $K = \mathbf{Q}_p$) for $q = p^r$ with $r > 1$ and $o(\eta) = 1$ or 2. Here I generalize Saliè's result for the general case $n > 1$ for characters η with $o(\eta)|p-1$ (also $o(\eta) = 2$ when $p = 2$), and for all $\gamma \geq re - d > 1$ but for a few small exceptional values r. My evaluation relies on the author's recent explicit determination of Gauss sums for prime powers in p-adic fields and exponential sums of the form $\sum \chi(x)^{ax}\zeta_q^{bx}$.

1. Introduction

Let K be a finite extension of the p-adic rational field $\mathbf{Q}_p$ of degree n with residue degree f and ramification index e. Let T denote the maximal unramified subfield of K. The trace and norm maps for $K/\mathbf{Q}_p$ will be denoted $Tr = Tr_{K/Q_p}$ and $N = N_{K/Q_p}$, respectively. The rings of integers of K, T and $\mathbf{Q}_p$ are denoted by O, O_T and $\mathbb{Z}_p$ respectively. Fix a uniformizant Π to generate the prime ideal P of O. It is known that Π satisfies an Eisenstein polynomial of degree e over T with $\Pi^e = pu$ for some unit u of K (when $K/\mathbf{Q}_p$ is tamely ramified Π may be chosen so $u \in O_T$). The differential exponent d of $K/\mathbf{Q}_p$ is the largest non-negative integer r such that TrP^{-r} is contained in $\mathbb{Z}_p$. The ideal P^d is known as the different of $K/\mathbf{Q}_p$. It is known that $d \geq e-1$ with $d = e-1$ if and only if $K/\mathbf{Q}_p$ is tamely ramified; otherwise $K/\mathbf{Q}_p$ is wildly ramified and $p|e$. Furthermore, for any integer r,

$$Tr\ P^r = p^{r'}\mathbb{Z}_p$$

where $r' = [(r+d)/e]$. (Here $[x]$ denotes the largest integer less than or equal to x.)

Now fix $q = p^r$, a power of a prime, writing $r = 2s$ or $2s+1$ with $s' = s$ or $s+1$ according as r is even or odd. Consider a numerical character η defined modulo q and of conductor $f(\eta)$ and order $o(\eta)$. Any such character η modulo q extends to $\mathbb{Z}_p$ in the natural way; namely, $\eta(u) = \eta(\tilde{u})$ where $\tilde{u}$ denotes the residue class of u modulo q, and similarly for the root of unity $\zeta_q^u = exp(2\pi i\tilde{u}/q)$. Fix a positive integer γ satisfying

$$\gamma \geq re - d \quad \text{and} \tag{1.1}$$

$$N(1 + P^\gamma) \subseteq 1 + q\mathbb{Z}_p \tag{1.2}$$

Then for any $z \in \mathbf{Z}/q\mathbf{Z}^*$ one may form the (twisted) Kloosterman sum

$$R(\eta, z) = \sum_{\alpha \in (O/P^\gamma)^*} \eta(N\alpha)\zeta_q^{Tr\ \alpha + z/N\alpha}. \tag{1.3}$$

Conditions (1.1) and (1.2) ensure that $R(\eta, z)$ is well-defined.

The twisted Kloosterman sums (1.3) lie in $\mathbf{Q}(\zeta_{q(p-1)})$ and are easily seen to satisfy for $(v, q(p-1)) = 1$

$$\sigma_v(R(\eta, z)) = \bar{\eta}^{vn}(v)R(\eta^v, zv^{n+1}), \tag{1.4}$$

where σ_v is the automorphism induced by sending $\zeta_{q(p-1)}$ to $\zeta_{q(p-1)}^v$. Indeed,

$$\begin{aligned}
\sigma_v(R(\eta, z)) &= \sum_{\alpha \in (O/P^\gamma)^*} \eta^v(N\alpha)\zeta_q^{v(Tr\ \alpha + z/N\alpha)} \\
&= \sum_{\alpha \in (O/P^\gamma)^*} \eta^v(\bar{v}^n)\eta^v(N\alpha)\zeta_q^{Tr\ \alpha + zv^{n+1}/N\alpha} \\
&= \bar{\eta}^{vn}(v)R(\eta^v, zv^{n+1}).
\end{aligned}$$

Replacing η by $\eta^{\bar{v}}$ in (1.4) above where $\bar{v}$ is the multiplicative inverse of v modulo $q(p-1)$, one finds

$$R(\eta, z'v^{n+1}) = \eta^n(v)\sigma_v(R(\eta^{\bar{v}}, z')) \tag{1.5}$$

for any $z' \in \mathbf{Z}/q\mathbf{Z}^*$.

The twisted Kloosterman sums can be elegantly expressed in terms of Gauss sums. For any numerical character χ defined modulo q one forms the Gauss sum

$$G_{P^\gamma}(\chi) = \sum_{\alpha \in (O/P^\gamma)^*} \chi(N\alpha)\zeta_q^{Tr\ \alpha}, \tag{1.6}$$

which is well-defined in view of (1.1) and (1.2). A routine exercise (see [Lemma 3.1, 6] or [14, p. 47] for instance) yields the relation

$$R(\eta, z) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(z)G_q(\chi)G_{P^\gamma}(\chi\eta), \tag{1.7}$$

where $G_q(\chi) = \sum_{x \in Z/qZ^*} \chi(x)\zeta_q^x$ is the ordinary Gauss sum over $\mathbf{Z}/q\mathbf{Z}^*$. The sum here is over all the numerical characters χ defined modulo q.

Now set $\gamma_0 = max(\frac{e}{p-1}, re - d)$. Recently, I evaluated the Gauss sum (1.6) for $s' \geq \delta$ and $\gamma \geq \gamma_0$, where $\delta = 1 + \epsilon$ if $K/\mathbf{Q}_p$ is tamely ramified but otherwise

$$\delta = \begin{cases} \frac{2d}{e} & \text{if } p > 3 \\ \frac{2d}{e} + 1 & \text{if } p = 3 \\ max(\frac{2d}{e}, 1 + \frac{3d}{2e}) & \text{if } p = 2, \end{cases} \tag{1.8}$$

in terms of explicit values of $G_{P^\gamma}(\psi)$ for characters ψ normalized in the sense of Mauclaire (chiefly (2.1) and (3.1) in sections 2 and 3). Specifically,

$G_{P^\gamma}(\psi)$ is found to be (chiefly Theorem 1 and 2 in [13] and their corollaries) of the form

$$G_{P^\gamma}(\psi) = p^{f(\gamma - \frac{re-d}{2})}\zeta_q^n \zeta_8^\kappa, \tag{1.9}$$

where κ is explicitly determined (mod 8) depending on the parity of $re-d$.

Saliè [18] explicitly determined $R(\eta, z)$ in the classical case $n = 1$ (so $K = \mathbf{Q}_p$) for $q = p^r$ with $r > 1$ and $o(\eta) = 1$ or 2. Recently R. Evans [6] determined $R(\eta, z)$ when K is totally and tamely ramified over $\mathbf{Q}_p$. My aim here is to generalize these results for the general case $n > 1$ using the relation (1.7) and relying on the author's recent explicit evaluation of the Gauss sums $G_{P^\gamma}(\chi)$ [13] and exponential sums of the form $\sum \chi(x)^{ax}\zeta_q^{bx}$ [10]. To best describe this analog, I write $n+1 = p^b y$ for $b \geq 0$ and $p \nmid y$, and set $l = gcd(n+1, p-1)$ if p is odd or $l = gcd(n+1, 2)$ if $p = 2$. For odd primes p, let H denote the group of l-roots of unity modulo p^{r-b} (or just modulo p when $r \leq b$). I show in section 2 that for $r > b+1$ with $s' \geq \delta$, $\gamma \geq \gamma_0$ and $o(\eta)|p-1$ that $R(\eta, z)$ vanishes if $z^{\frac{p-1}{l}} \not\equiv 1 (\text{mod } p^{b+1})$, else up to a $4l$-root of unity is a conjugate of

$$p^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \sum_{x \in H} (\frac{x}{p})^{b+\epsilon+f(re-d)} \bar{\eta}(x) \zeta_{p^{r-b}}^x. \tag{1.10}$$

Under the same hypotheses, I also determine $R(\eta, z)$ for smaller powers $r \leq b+1$. However, the important case when $q = p$, an odd prime, and $K/\mathbf{Q}_p$ is tamely ramified remains unresolved.

In section 3 I treat the case $p = 2$. I show that for $r > b+4$ with n odd, $s' \geq \delta$, $\gamma \geq \gamma_0$ and $o(\eta)|l$, that $R(\eta, z)$ vanishes if $z \not\equiv 1 (\text{mod } 2^{b+2})$, else up to sign is a conjugate of

$$2^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \cdot 2 \cos(\frac{2\pi}{2^{r-b}}). \tag{1.11}$$

I separately determine $R(\eta, z)$ for smaller powers $r \leq b+4$ under the same hypotheses.

I wish to mention some consequences and related results regarding the explicit values for $R(\eta, z)$ found here. Expressions (1.10) and (1,11) lead to a bound

$$|R(\eta, z)| \leq l p^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}}$$

for $r > b+1$ ($r > b+4$ if $p = 2$), that is a modest improvement of the customary Deligne [5] bound $|R(\eta, z)| \leq (n+1)p^{f(\gamma - \frac{re-d}{2})}$ when $b > 0$. Moreover, from such expressions (1.10) and (1.11), the non-vanishing sums $R(\eta, z)$ are seen to be integer multiples of ordinary Gauss periods for p^{r-b} or a twist of such by the character η . Thus additional improvement in the upper bound $|R(\eta, z)|$ may be obtained using recent estimates for Gauss periods obtained by Bourgain and Chang (chiefly, Theorem 4.7 in

[4]; see also [3]). When $l > 1$ one may essentially replace l by $lp^{-\omega}$ in the bound above, where $\omega > 0$ depends on l and p^{r-b}. These details and the extent to which ω can be effectively determined are beyond the scope of the presentation here, but will be discussed elsewhere.

The author has recently studied Gauss periods and quadratic twists of such for prime powers [9] to obtain formulas for the beginning coefficients of their minimal polynomials and associated power sums of zeros. When $|H| = 2$ in (1.10) and $o(\eta) = 1$ or 2, a closed form expression for the minimal polynomial and the associated power sums is actually obtained [7]. Those results can be applied with these here to describe the polynomial satisfied by the Kloosterman sums $R(\eta, z)$ for $z \in \mathbf{Z}/q\mathbf{Z}^*$ when $o(\eta) = 1$ or 2. This determination, which generalizes the author's previous results [8,11] for Kloosterman polynomials and hyper-Kloosterman polynomials, will appear elsewhere.

2. Kloosterman sums for odd prime powers p^r, $r > 1$

Here I evaluate the twisted Kloosterman sums (1.3) for odd prime powers when $o(\eta)|p-1$. As in the introduction I write $n+1 = p^b y$, for $b \geq 0$ and $p \nmid y$, and set $l = \gcd(n+1, p-1)$. I consider $q = p^r$, $r > 1$, with $r = 2s$ or $2s+1$ where $s' = s + \epsilon$ with $\epsilon = 0$ or 1 according as r is even or odd. For any $w \not\equiv 0 (\text{mod } p)$, let $\bar{w}$ denote the multiplicative inverse of w mod p^r. Let H denote the group of l-roots of unity modulo p^{r-b}, or just modulo p when $r \leq b$. Choose a numerical character ψ modulo q which generates the numerical characters modulo q and is *normalized* so that

$$\psi(1+p^s) = \zeta_{p^s}^{-1} \ \text{ for } \alpha = 2s \geq 2 \text{ even, or} \tag{2.1}$$

$$\psi(1+p^s+(\frac{p+1}{2})p^{2s}) = \zeta_{p^{s+1}}^{-1} \ \text{ for } \alpha = 2s+1 \geq 3 \ \text{ odd.}$$

Note that $\chi = \psi^v$ is primitive whenever $p \nmid v$. Now $\eta = \psi^w$ for some integer w, $1 \leq w \leq \phi(q)$, and it follows from (1.7) that

$$R(\eta, z) = \frac{1}{\phi(q)} \sum_{j=1, p\nmid j}^{\phi(q)} \bar{\psi}^j(z) G_q(\psi^j) G_{P^\gamma}(\psi^{j+w}) \tag{2.2}$$

for any $z \in \mathbf{Z}/q\mathbf{Z}^*$ since $G_q(\chi) = 0$ if χ is imprimitive. The Gauss sums $G_{P^\gamma}(\chi)$ have been determined in [13].

Proposition 2.1. *For any $v \in \mathbf{Z}/q\mathbf{Z}^*$, $q = p^r$ odd with $s' \geq \delta$ and $\gamma \geq \gamma_0$,*

$$G_{P^\gamma}(\psi^v) = \begin{cases} p^{f(\gamma - \frac{re-d}{2})} \psi^{nv}(v) \zeta_q^{nv} \\ V(\frac{v}{p})^f (\frac{N_{T/Q_p}(Tr_{K/T} \Pi^{e-d-1} u^{1-\epsilon})}{p}) \psi^{nv}(v) \zeta_q^{nv} \end{cases}$$

where $V = (-1)^{f-1} \zeta_8^{(1-p)f} p^{f(\gamma - \frac{re-d}{2})}$, according as $re-d$ is even or odd.

Proof. For any $v \in \mathbf{Z}/q\mathbf{Z}^*$ write $v \equiv tl (\bmod q(p-1))$ with $t = 1+(v-1)q \equiv 1 (\bmod q)$ and $l = v-(v-1)q^2 \equiv 1(\bmod p-1)$. Then $\chi = \psi^t$ is normalized modulo q, so $G_{P^\gamma}(\psi^v) = G_{P^\gamma}(\chi^l)$ is easily seen to equal $\chi^{ln}(l)\sigma_l(G_{P^\gamma}(\chi))$, where σ_l is the automorphism given by $\sigma_l(\zeta_{q(p-1)}) = \zeta_{q(p-1)}^l$. Thus from Theorem 1 and Corollary 1 in [13]

$$G_{P^\gamma}(\psi^v) = \begin{cases} \psi^{vn}(l)p^{f(\gamma-\frac{re-d}{2})}\zeta_q^{ln} \\ V(\frac{l}{p})^f(\frac{N_{T/Q_p}(Tr_{K/T}\Pi^{e-d-1}u^{1-\epsilon})}{p})\psi^{vn}(l)\zeta_q^{ln} \end{cases}$$

according as $re-d$ is even or odd. Since $t \equiv 1(\bmod q)$, the result as stated in the proposition follows. $\square$

When $K = \mathbf{Q}_p$ with $\gamma = r > 1$, Proposition 2.1 reduces to Mauclaire's evaluation of the ordinary Gauss sums $G_q(\chi)$ for primitive characters χ modulo q. (See also Evans [6]).

Corollary 2.1. *For any $v \in \mathbf{Z}/q\mathbf{Z}^*$, $q = p^r$ odd with $r > 1$,*

$$G_q(\psi^v) = \begin{cases} p^{r/2}\psi^v(v)\zeta_q^v & \text{if } r \text{ even} \\ (\frac{v}{p})\zeta_8^{1-p}p^{r/2}\psi^v(v)\zeta_q^v & \text{if } r \text{ odd.} \end{cases}$$

The following lemmas will prove crucial in the evaluation of $R(\eta, z)$ here.

Lemma 2.1. *If ψ is normalized modulo p^r for $r > 2$, then ψ^p is normalized modulo p^{r-1}. The sole exception occurs for $r = 4$ with $p = 3$ where $\psi^3(22) = \zeta_9^{-4}$ not ζ_9^{-1} in (2.1).*

Proof. First note that for $s > 0$ that $(1 + p^s + \frac{p+1}{2}p^{2s})^p \equiv (1 + p^s)^p \equiv 1+p^{s+1}(\bmod p^{2s+1})$. Thus for odd $r = 2s+1 \geq 3$, $\psi^p(1+p^s) = \psi(1+p^{s+1}) = \zeta_{p^s}^{-1}$ from (2.1) so ψ^p is normalized modulo p^{r-1}. In addition, one readily sees that $(1 + p^s + \frac{p+1}{2}p^{2s})^p \equiv 1 + p^{s+1} \pmod{p^{2s+2}}$, except for $s = 1$ with $p = 3$ where $22^3 \equiv 37 (\bmod 81)$. Thus for even $r = 2s + 2 > 2$, $\psi^p(1 + p^s + \frac{p+1}{2}p^{2s}) = \psi(1 + p^{s+1}) = \zeta_{p^{s+1}}^{-1}$ from (2.1) so ψ^p is normalized modulo p^{r-1}, though in the exceptional case $s = 1$ and $p = 3$, $\psi^3(22) = \psi(37) = \zeta_9^{-4}$ since $37 \equiv 10^4 (\bmod 81)$. The proof of the lemma is now complete. $\square$

Lemma 2.2. *Let χ be any numerical character modulo p^r for $r > 1$ and normalized as in (2.1). Then for any integer w*

$$\chi(1 + wp^{s'}) = \zeta_{p^s}^{-w}.$$

Proof. The proof of Lemma 2.2 follows routinely from the Binomial Theorem and the observation made at the outset of the proof of Lemma 2.1. $\square$

Before proceeding to the statement of the main results here I require a fact from [10] concerning certain incomplete exponential sums.

Proposition 2.2. *Let χ be a normalized generator for the group of numerical characters modulo p^α, $\alpha > 1$, and $y \not\equiv 0 (\mathrm{mod}\ p)$. For any character η defined modulo q but of order dividing $p-1$,*

$$\sum_{v=1, p\nmid v}^{\phi(p^\alpha)/(y,p-1)} \chi(v)^{yv}\eta(v)\zeta_{p^\alpha}^{yv} = \begin{cases} \frac{p-1}{(y,p-1)} p^{\frac{\alpha-2}{2}} \sum_{x\in H'} \eta(x)\zeta_{p^\alpha}^{yx} \\ \frac{p-1}{(y,p-1)} (\frac{y}{p}) p^{\frac{\alpha-3}{2}} \zeta_8^{p-1} \sqrt{p} \sum_{x\in H'} \eta(x) (\frac{x}{p}) \zeta_{p^\alpha}^{yx}, \end{cases}$$

according as α is even or odd. (Here H' is the group of $(y, p-1)$-roots of unity modulo p^α.)

The above proposition is readily demonstrated by the argument used in the proof of Theorem 3 and Corollary 10 in [10]. Since that argument required only minor modification, I shall omit the proof here. Additionally, since $(y, p-1)$ and y have the same parity, a re-examination of the proof there shows that the modified sum $\sum_{v=1,p\nmid v}^{\phi(p^\alpha)/(y,p-1)} (\frac{v}{p})^y \chi(v)^{yv}\eta(v)\zeta_{p^\alpha}^{yv}$ has the same value as above.

I am now ready to determine the values $R(\eta, z)$ for odd prime powers. The computation of $R(\eta, z)$ naturally breaks into two cases $r > b+1$ and $1 < r \le b+1$. I consider the case $r > b+1$ first, with $re - d$ even.

Theorem 2.1. *Let $q = p^r$ with $r > b+1$, $s' \ge \delta$, $\gamma \ge \gamma_0$ and $o(\eta)|p-1$, where $re-d$ is even. Then*

$$R(\eta, 1) = \zeta_8^{(p-1)b(b+2\epsilon)} (\frac{y}{p})^{b+\epsilon} p^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \sum_{x\in H} (\frac{x}{p})^b \bar{\eta}(x) \zeta_{p^{r-b}}^{yx}.$$

The sole exception when $p = 3$ and $r = b+3 > 3$ is

$$R(\eta, 1) = i^{b(b+2\epsilon)} (\frac{y}{3}) 3^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \sum_{x\in H} (\frac{x}{3})^b \bar{\eta}(x) \zeta_{27}^{19yx}.$$

Furthermore, $R(\eta, z) = 0$ if $z^{\frac{p-1}{l}} \not\equiv 1 (\mathrm{mod}\ p^{b+1})$ else $R(\eta, z)$ is determined from $R(\eta^{\bar{v}}, 1)$ by (1.5) where $z \equiv v^{n+1} (\mathrm{mod}\ q)$.

Proof. One may write $\eta = \psi^w$ with $p^{r-1}|w$. Then from (1.7) and Proposition 2.1, one finds for any $z \in \mathbf{Z}/q\mathbf{Z}^*$ that

$$R(\eta, z) = \frac{1}{\phi(q)} \sum_{j=1, p\nmid j}^{\phi(q)} \bar{\psi}^j(z) G_q(\psi^j) G_{P^\gamma}(\psi^{j+w})$$

$$= \frac{p^{f(\gamma - \frac{re-d}{2}) + \frac{r}{2}}}{\phi(q)} \zeta_8^{(1-p)\epsilon} \sum_{j=1, p\nmid j}^{\phi(q)} (\frac{j}{p})^\epsilon \bar{\psi}^j(z) \psi^j(j) \psi^{(j+w)n}(j+w) \zeta_q^{j(n+1)+nw}$$

when $re - d$ is even. One may write the expression for $R(\eta, z)$ above as

$$\frac{p^{f(\gamma - \frac{re-d}{2}) + \frac{r}{2}}}{\phi(q)} \zeta_8^{(1-p)\epsilon} \sum_{v=1, p\nmid v}^{h} \sum_{i=0}^{lp^b - 1} (\frac{v}{p})^\epsilon \bar{\psi}^{v+hi}(z) \psi^{v+hi}(v+hi) \zeta_q^{v+hi} L,$$

where $L = \psi^{(v+hi+w)n}(v + hi + w)\zeta_q^{nv+nhi+nw}$ and each j is uniquely expressed $j = v + hi$ with $1 \leq v \leq h = \phi(p^{r-b})/l$, $0 \leq i < lp^b$ and $p \nmid v$, or equivalently as

$$\text{(2.3)}\qquad \frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{\phi(q)}\zeta_8^{(1-p)\epsilon}\sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^\epsilon\bar{\psi}^v(z)\psi^{v(n+1)}(v)\psi^{wn}(v)\zeta_q^{v(n+1)}$$

$$\cdot\sum_{i=0}^{lp^b-1}\bar{\psi}^{hi}(z)\psi^{v(n+1)}(1+\bar{v}hi)\psi^{hi}(v+hi)\psi^{vn}(1+\bar{v}w)M,$$

where $M = \psi^{hin}(v+hi+w)\psi^{nw}(1+\bar{v}(hi+w))\zeta_q^{hi(n+1)+nw}$, since $(1+\bar{v}hi)(1+\bar{v}w) \equiv 1+\bar{v}(hi+w) \bmod q$ as $r > b+1$. But $\psi^{v(n+1)}(1+\bar{v}hi) = \zeta_q^{-(n+1)hi}$ and $\psi^{vn}(1 + \bar{v}w) = \zeta_q^{-nw}$ from (2.1) and the fact $\psi^{hin}(v + hi + w)\psi^{hi}(v + hi) = \psi^{hi(n+1)}(v + hi) = 1$ as $\phi(q)|h(n + 1)$. In addition $\psi^{nw}(1 + \bar{v}(hi + w)) = 1$, so the inner sum reduces to

$$\sum_{i=0}^{lp^b-1}\bar{\psi}^{hi}(z) = \begin{cases} lp^b & \text{if } z \in (\mathbf{Z}/q\mathbf{Z})^{*n+1} \\ 0 & \text{otherwise,}\end{cases}$$

since ψ^h has order lp^b. Thus $R(\eta, z) = 0$ for $r > b + 1$ whenever $z \notin (\mathbf{Z}/q\mathbf{Z})^{*n+1}$. But $z \in (\mathbf{Z}/q\mathbf{Z})^{*n+1}$ if and only if $z^{\frac{p-1}{l}} \equiv 1(\text{mod } p^{b+1})$ from Euler's criterion, so the last statement of the theorem follows. In view of (1.5) it is enough now to compute $R(\eta, 1)$. From (2.3) one obtains

$$\text{(2.4)}\qquad R(\eta,1) = \frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{h}\zeta_8^{(1-p)\epsilon}\sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^\epsilon\psi^{v(n+1)}(v)\eta^n(v)\zeta_q^{v(n+1)}$$

$$= \frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{h}\zeta_8^{(1-p)\epsilon}\sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^\epsilon\psi^{p^byv}(v)\eta^n(v)\zeta_{p^{r-b}}^{yv}.$$

Now ψ^{p^b} generates the group of numerical characters modulo p^{r-b}, and is normalized modulo p^{r-b} (except when $p = 3$ and $r = b + 3$, $\psi^{3^b} = \chi^w$ for some normalized character χ modulo 27 with $w \equiv 4(\text{mod } 9)$), by repeatedly applying Lemma 2.1. By applying Proposition 2.2 with character choice $\eta^n(v)(\frac{v}{p})^\epsilon$ if l is even, and in view of the comment that immediately followed

$$\sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^\epsilon\psi^{p^byv}(v)\eta^n(v)\zeta_{p^{r-b}}^{yv} =$$

$$\frac{p-1}{l}p^{\frac{r-b-2}{2}}\zeta_8^{(p-1)(b-\epsilon)^2}(\frac{y}{p})^{b+\epsilon}\sum_{x\in H}(\frac{x}{p})^b\eta^n(x)\zeta_{p^{r-b}}^{yx}$$

since $l = (y, p-1)$ (but in the exceptional case $p = 3$ and $r = b + 3$ direct calculation here finds $\sum_{v=1,3\nmid v}^{18/l} (\frac{v}{3})^{\epsilon} \psi^{3^b yv}(v) \eta^n(v) \zeta_{27}^{yv} = \frac{2}{l} i\sqrt{3} (\frac{y}{3}) \cdot \sum_{x \in H} (\frac{x}{3}) \eta^n(x) \zeta_{27}^{19yx}$). This together with (2.4) yields the expressions for $R(\eta, 1)$ as stated in the theorem. □

I next consider the case $re-d$ is odd. Set $\tau = N_{T/Q_p}(Tr_{K/T}\, \Pi^{e-d-1} u^{1-\epsilon})$.

Theorem 2.2. *Let $q = p^r$ with $r > b+1$, $s' \geq \delta$, $\gamma \geq \gamma_0$ and $o(\eta) | p-1$, where $re-d$ is odd. Then*

$$R(\eta, 1) = \\ (-1)^{f-1} \zeta_8^{(p-1)(b^2+2b\epsilon-f)} (\frac{y}{p})^{b+\epsilon} (\frac{\tau}{p}) p^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \sum_{x \in H} (\frac{x}{p})^{b+f} \bar{\eta}(x) \zeta_{p^{r-b}}^{yx}.$$

The sole exception when $p = 3$ and $r = b+3 > 3$ is

$$R(\eta, 1) = i^{(b^2+2b\epsilon-f)} (\frac{y\tau}{3}) 3^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} \sum_{x \in H} (\frac{x}{3})^{b+f} \bar{\eta}(x) \zeta_{27}^{19yx}.$$

Furthermore, $R(\eta, z) = 0$ if $z^{\frac{p-1}{l}} \not\equiv 1 (\mathrm{mod}\ p^{b+1})$ else $R(\eta, z)$ is determined from $R(\eta^{\bar{v}}, 1)$ by (1.5) where $z \equiv v^{n+1} (\mathrm{mod}\ q)$.

Proof. The argument is similar to the case $re-d$ is even. Writing $\eta = \psi^w$ with $p^{r-1} | w$, one again finds from (1.7) and Proposition 2.1 that for any $z \in \mathbf{Z}/q\mathbf{Z}^*$,

$$R(\eta, z) = \frac{(-1)^{f-1}}{\phi(q)} \zeta_8^{(1-p)(f+\epsilon)} (\frac{\tau}{p}) p^{f(\gamma - \frac{re-d}{2}) + \frac{r}{2}} \\ \cdot \sum_{j=1, p\nmid j}^{\phi(q)} (\frac{j}{p})^{f+\epsilon} \bar{\psi}^j(z) \psi^j(j) \psi^{(j+w)n}(j+w) \zeta_q^{j(n+1)+nw}$$

when $re-d$ is odd. One may rewrite the expression for $R(\eta, z)$ above as

$$\frac{(-1)^{f-1}}{\phi(q)} \zeta_8^{(1-p)(f+\epsilon)} (\frac{\tau}{p}) p^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2}} \sum_{v=1, p\nmid v}^{h}$$

$$\cdot \sum_{i}^{lp^b-1} (\frac{v}{p})^{f+\epsilon} \bar{\psi}^{v+hi}(z) \psi^{v+hi}(v+hi) \psi^{(v+hi+w)n}(v+hi+w) \zeta_q^{(n+1)(v+hi)+nw},$$

where each $j = v + hi$ with $1 \le v \le h$, $0 \le i < lp^b$ and $p \nmid v$, or equivalently as

$$\frac{(-1)^{f-1}}{\phi(q)}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})p^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}}\cdot \sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^{f+\epsilon}\bar{\psi}^v(z)\psi^{v(n+1)}(v)\psi^{wn}(v)\zeta_q^{v(n+1)}W,$$

where W is the inner sum in (2.3). Arguing as before one finds that $R(\eta, z) = 0$ for $r > b+1$ whenever $z \notin (\mathbf{Z}/q\mathbf{Z})^{*n+1}$ and otherwise that $R(\eta, z)$ is determined from $R(\eta^{\bar{v}}, 1)$ by (1.5) where $z \equiv v^{n+1} (\mathrm{mod}\ q)$. This time

$$R(\eta, 1) = \frac{(-1)^{f-1}}{h}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}} \cdot \sum_{v=1,p\nmid v}^{h}(\frac{v}{p})^{f+\epsilon}\psi^{v(n+1)}(v)\eta^n(v)\zeta_q^{v(n+1)}.$$

Applying Proposition 2.2 with character choice $\eta^n(v)(\frac{v}{p})^{f+\epsilon}$ in place of $\eta^n(v)$ if l is even yields the expression for $R(\eta, 1)$ as stated. Direct computation yields the result in the exceptional case when $p = 3$ with $r = b+3 > 3$. The proof of the theorem is now complete. $\square$

Next I consider the case $1 < r \le b+1$, where H is just the group of l-roots of unity modulo p and $h = (p-1)/l$.

Theorem 2.3. *Let $q = p^r$ with $1 < r \le b+1$, $s' \ge \delta$, $\gamma \ge \gamma_0$ and $o(\eta)|p-1$. Then for $z \equiv 1 (\mathrm{mod}\ p^{r-1})$,*

$$R(\eta, z) = \begin{cases} \zeta_8^{(1-p)\epsilon}p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}\sum_{x\in H}\zeta_p^{(n+z)x/p^{r-1}}(\frac{x}{p})^{\epsilon}\bar{\eta}(x) \\ (-1)^{f-1}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1} \\ \quad \cdot \sum_{x\in H}\zeta_p^{(n+z)x/p^{r-1}}(\frac{x}{p})^{f+\epsilon}\bar{\eta}(x) \end{cases}$$

according as $re - d$ is even or odd.

Furthermore, $R(\eta, z) = 0$ if $z^{(p-1)/l} \not\equiv 1 (\mathrm{mod}\ p^{r-1})$ else $R(\eta, z)$ is determined from some $R(\eta^{\bar{v}}, z')$ above with $z' \equiv 1 (\mathrm{mod}\ p^{r-1})$ by (1.5).

Proof. First note that ψ^{n+1} has order dividing $p-1$, and write $\eta = \psi^w$ with $p^{r-1}|w$ as before. For the case $re - d$ even, one has from (1.7) and Proposition 2.1 again

$$R(\eta, z) = \frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{\phi(q)}\zeta_8^{(1-p)\epsilon}\sum_{j=1,p\nmid j}^{\phi(q)}(\frac{j}{p})^{\epsilon}\bar{\psi}^j(z)\psi^j(j)\psi^{(j+w)n}(j+w)\zeta_q^{j(n+1)+wn},$$

which equals

$$(2.5)\quad \frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{\phi(q)}\zeta_8^{(1-p)\epsilon}\sum_{x=1}^{p-1}(\frac{x}{p})^{\epsilon}\bar{\psi}^{x}(z)\psi^{x(n+1)+wn}(x)\zeta_q^{x(n+1)}$$
$$\cdot\sum_{t=0}^{\phi(p^{r-1})-1}\bar{\psi}^{pt}(z)\psi^{pt}(x+pt)\psi^{x}(1+\bar{x}pt)\psi^{(x+w)n}(1+\bar{x}(w+pt))V,$$

where $V=\psi^{ptn}(x+pt+w)\zeta_q^{(n+1)pt+wn}$, and each j is uniquely expressed $j=x+pt$ with $1\le x<p$ and $0\le t<\phi(p^{r-1})$. Since ψ^{n+1} and ψ^{w} both can be defined modulo p, the inner sum above is easily seen to become

$$\sum_{t=0}^{\phi(p^{r-1})-1}\bar{\psi}^{pt}(z)\psi^{pt(n+1)}(x)\psi^{x}(1+\bar{x}pt)\psi^{xn}(1+\bar{x}(w+pt))\zeta_q^{wn}.$$

But $\psi^{xn}(1-\bar{x}w)=\zeta_q^{wn}$ from (2.1) and

$$\psi^{x}(1+\bar{x}pt)\psi^{xn}(1+\bar{x}(w+pt))\psi^{xn}(1-\bar{x}w)=\psi^{x}(1+\bar{x}pt)\psi^{xn}(1+\bar{x}pt)=1,$$

so the inner sum in turn reduces to

$$(2.6)\qquad \sum_{t=0}^{\phi(p^{r-1})-1}\bar{\psi}^{pt}(z)\psi^{pt(n+1)}(x)=\sum_{t=0}^{\phi(p^{\alpha-1})-1}\psi^{pt}(\bar{z}x^{n+1})$$
$$=\begin{cases}p^{\alpha-2}(p-1)\\ 0\end{cases}$$

according as $x^{n+1}\equiv z(\mathrm{mod}\ p^{r-1})$ or not. The last equality above holds since ψ^{p} has conductor p^{r-1} and generates the numerical characters modulo p^{r-1}. Thus $R(\eta,z)=0$ unless $z\in(\mathbf{Z}/p^{r-1}\mathbf{Z})^{*n+1}$ in this case. But when $z\equiv v^{n+1}(\mathrm{mod}\ p^{r-1})$ then $R(\eta,z)$ is determined from some $R(\eta,z')$ for $z'=z\bar{v}^{n+1}\equiv 1(\mathrm{mod}\ p^{r-1})$ using (1.5). Thus, it suffices to consider $z\equiv 1(\mathrm{mod}\ p^{r-1})$, say $z=1+\lambda p^{r-1}$ for some integer λ. Then from (2.5) and Lemma 2.2

$$\begin{aligned}R(\eta,z)&=\zeta_8^{(1-p)\epsilon}p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}\sum_{x\in H}(\frac{x}{p})^{\epsilon}\bar{\psi}^{x}(1+\lambda p^{r-1})\psi^{x(n+1)+wn}(x)\zeta_q^{x(n+1)}\\ &=p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}\zeta_8^{(1-p)\epsilon}\sum_{x\in H}\zeta_p^{\lambda x+(n+1)x/p^{r-1}}(\frac{x}{p})^{\epsilon}\eta^{n}(x)\\ &=p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}\zeta_8^{(1-p)\epsilon}\sum_{x\in H}\zeta_p^{(n+z)x/p^{r-1}}(\frac{x}{p})^{\epsilon}\bar{\eta}(x)\end{aligned}$$

since $\psi(x^{n+1})=1$ for $x\in H$ here as $lp^{r-1}|(n+1)$.

Arguing similarly when $1 < r \leq b+1$ when $re-d$ is odd, one obtains the expression

$$(-1)^{f-1}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})\frac{p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}}}{\phi(q)}\sum_{x=1}^{p-1}(\frac{x}{p})^{f+\epsilon}\bar{\psi}^x(z)\psi^{x(n+1)}(x)\zeta_q^{x(n+1)}\cdot V$$

for $R(\eta, z)$, where V is the same inner sum in (2.5). Again $R(\eta, z) = 0$ unless $z \in (\mathbf{Z}/p^{r-1}\mathbf{Z})^{*n+1}$, limiting consideration to $z = 1+\lambda p^{r-1}$ for some integer λ as before. This time for such z,

$$R(\eta, z) = (-1)^{f-1}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}$$

$$\cdot\sum_{x\in H}(\frac{x}{p})^{f+\epsilon}\bar{\psi}^x(1+\lambda p^{r-1})\psi^{x(n+1)+wn}(x)\zeta_q^{(n+1)x}$$

$$= (-1)^{f-1}\zeta_8^{(1-p)(f+\epsilon)}(\frac{\tau}{p})p^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-1}\sum_{x\in H}(\frac{x}{p})^{f+\epsilon}\bar{\eta}(x)\zeta_p^{(n+z)x/p^{r-1}}$$

as before. The proof is now complete. □

The following corollary is the special case $K = \mathbf{Q}_p$ in Theorems 2.1 and 2.2. It includes the results of Saliè [18] when $o(\eta) = 1$ or 2.

Corollary 2.2. *For any $\gamma \geq r > 1$ with $o(\eta)|p-1$,*

$$R(\eta, 1) = (\frac{2}{p})^r p^{\gamma-\frac{r}{2}}\sum_{x\in H}\bar{\eta}(x)\zeta_q^{2x}.$$

In particular, $R(\eta, z) = 0$ if $(\frac{z}{p}) = -1$ else $R(\eta, z)$ is determined from $R(\eta^{\bar{v}}, 1)$ by (1.5) where $v^2 \equiv z(\bmod q)$ with $(v, p-1) = 1$.

Example. To illustrate the results above consider the tamely ramified extension $K = \mathbf{Q}_5(\sqrt{-3}, \sqrt{5})$ with $e = f = 2$ and $d = 1$. One has $l = (n+1, p-1) = 1$ here, $b = 1$ and $y = 1$ with $\delta = 1+\epsilon$ and $\gamma_0 = re-d = 2r-1$.

For $q = 25$, one obtains from Theorem 2.3 for $\gamma \geq \gamma_0 = 3$

$$R(\eta, z) = -5^{2\gamma-3}\zeta_5^{(4+z)/5}$$

for $z \equiv 1(\bmod 5)$ and any character η with $f(\eta)|5$.

For $r > 2$, one may choose $\Pi = \sqrt{5}$ as uniformizant to find $u = 1$ so $\tau = N_{T/\mathbf{Q}_p}(Tr_{K/T}\,\Pi^{e-d-1}u^{1-\epsilon}) = 4$ in Theorem 2.2. Then for $\gamma \geq \gamma_0$,

$$R(\eta, 1) = 5^{2\gamma-2r+\frac{3}{2}}\zeta_q^1$$

for any character η with $f(\eta)|5$. The value $R(\eta, z)$ for $z \in (\mathbf{Z}/q\mathbf{Z})^{*n+1}$ are found from (1.5) to satisfy

$$R(\eta, z) = (\frac{v}{5})5^{2\gamma-2r+\frac{3}{2}}\zeta_q^v,$$

where v satisfies $v^{n+1} \equiv z(\text{mod } q)$ since $o(\eta)|4$. These are in agreement with values of $R(\eta, z)$ computed directly using (1.3) for several small values of r.

I conclude this section with a comment concerning the case $r = 1$. While the determination of $R(\eta, z)$ is largely unresolved when $K/\mathbf{Q}_p$ is tamely ramified, one finds when $K/\mathbf{Q}_p$ is wildly ramified that $R(\eta, z)$ is a multiple of a twisted ordinary Gauss period with $\frac{p-1}{(e,p-1)}$ terms. To be more precise, let ψ be a generator for the group of numerical characters modulo p with $\eta = \psi^w$ for some integer $0 \le w < p-1$. The Gauss sums $G_{P^\gamma}(\chi)$ in (1.6) for $q = p$ have been determined for any numerical character χ modulo p (chiefly, Proposition 5 in [13]). When $K/\mathbf{Q}_p$ is wildly ramified

$$G_{P^\gamma}(\chi) = \begin{cases} p^{f(\gamma-1)}(p^f - 1) & \text{if } o(\chi)|e \\ 0 & \text{otherwise.} \end{cases}$$

Thus one finds from (1.7) with $t = (e, p-1)$ that

$$\begin{aligned} R(\eta, z) &= \frac{1}{p-1} \sum_{i=1}^{p-1} \bar{\psi}^i(z) G_p(\psi^i) G_{P^\gamma}(\psi^{w+i}) \\ &= p^{f(\gamma-1)} \frac{p^f - 1}{p-1} \sum_{j=1}^{t} \psi^{w - \frac{(p-1)j}{t}}(z) G_p(\psi^{-w + \frac{(p-1)j}{t}}) \\ &= p^{f(\gamma-1)} \frac{p^f - 1}{p-1} \sum_{j=1}^{t} \psi^{w - \frac{(p-1)j}{t}}(z) \sum_{x=1}^{p-1} \psi^{-w + \frac{(p-1)j}{t}}(x) \zeta_p^x \\ &= p^{f(\gamma-1)} \frac{p^f - 1}{p-1} \sum_{x=1}^{p-1} \bar{\psi}^w(\bar{z}x) \zeta_p^x \sum_{j=1}^{t} \psi^{\frac{(p-1)j}{t}}(\bar{z}x). \end{aligned}$$

Hence

$$R(\eta, z) = t p^{f(\gamma-1)} \frac{p^f - 1}{p-1} \sum_{v \in (Z/pZ)^{*t}} \bar{\eta}(v) \zeta_p^{vz},$$

an integer multiple of a Gauss period twisted by the character $\bar{\eta}$.

3. Kloosterman sums for $q = 2^r$

Here I evaluate the Kloosterman sums (1.3) when $p = 2$. When $r = 1$ direct computation using (1.3) yields

$$R(1, z) = \begin{cases} 2^{f(\gamma-1)} & \text{if } K/\mathbf{Q}_2 \text{ is wildly ramified} \\ 2^{f(\gamma-1)}(2^f - 1) & \text{if } K/\mathbf{Q}_2 \text{ tamely ramified} \end{cases}$$

for any $\gamma \ge \gamma_0$. For $r > 1$, I consider $q = 2^r$, $r > 1$, with $r = 2s$ or $r = 2s + 1$ where $s' = s + \epsilon$ with $\epsilon = 0$ or 1, according as r is even or odd. For any odd integer w, let $\bar{w}$ denote the multiplicative inverse of w

modulo 2^r, and $(\frac{2}{w}) = (-1)^{\frac{w^2-1}{8}}$ and $(\frac{-2}{w}) = (-1)^{\frac{w-1}{2}}(\frac{2}{w})$ be the primitive quadratic characters modulo 8. As before I write $n+1 = 2^b y$ for $b \geq 0$ and y odd, and set $l = gcd(n+1, 2)$. Choose a numerical character ψ modulo q which generates the *even* numerical characters modulo q and is *normalized* so

$$\begin{aligned} (3.1) \qquad \psi(1+2^s) &= \zeta_{2^s}^{-1} && \text{for } r = 2s > 2 \text{ even,} \\ \psi(1+2^s+2^{2s-1}) &= \zeta_{2^{s+1}}^{-1} && \text{for } r = 2s+1 > 3 \quad \text{odd, or} \\ \psi(5) &= -1 && \text{for } r = 3. \end{aligned}$$

Let $\xi(x)$ denote the quadratic character modulo q given by $\xi(x) = (-1)^{\frac{x-1}{2}}$. Any primitive numerical character modulo q has the form $\chi = \psi^v$ or $\xi \cdot \psi^v$ for some odd integer v, $1 \leq v \leq 2^{r-2}$. So it follows from (1.7) that

(3.2)

$$R(\eta, z) = \frac{1}{\phi(q)} \sum_{j=1, j \; odd}^{2^{r-2}} \bar{\psi}^j(z) G_q(\psi^j) G_{P^\gamma}(\psi^j \eta) + \xi \cdot \bar{\psi}^j(z) G_q(\xi \cdot \psi^j) G_{P^\gamma}(\xi \cdot \psi^j \eta)$$

for any $z \in \mathbf{Z}/q\mathbf{Z}^*$ since $G_q(\chi) = 0$ if χ is imprimitive.

The Gauss sums $G_{P^\gamma}(\chi)$ have been determined in [13]. To state the result it will be necessary to consider, when e is even, the unique solution $x_0, x_1, ..., x_{e/2-1}$ modulo $2O_T$ of the following triangular system of linear congruences:

$$\begin{aligned} (3.3) \qquad & u_1 x_0 \equiv z_1 \\ & u_2 x_0 + u_1 x_1 \equiv z_2 \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \\ & u_{e/2} x_0 + u_{e/2-1} x_1 + \cdots + u_1 x_{e/2-1} \equiv z_{e/2} \end{aligned}$$

where for $1 \leq i \leq e/2$, $u_i \equiv 2Tr_{K/T}\, \Pi^{-d-i} u^\epsilon$ and z_i uniquely satisfy modulo $2O_T$ the congruence

$$z_i^2 \equiv \begin{cases} Tr_{K/T}\, \Pi^{-w-2i} u & \text{if } r \text{ odd} \\ Tr_{K/T}\, \Pi^{-w-2i} + (Tr_{K/T}\, \Pi^{-\frac{w}{2}-i})^2 & \text{if } r \text{ even,} \end{cases}$$

with $w = max(2[d/2] - e, 0)$. One may also uniquely express

$$(3.4) \qquad u_1 \equiv \omega_0 + 2\omega_1 \ \text{ modulo } \ 4O_T \ \text{ with } \ \omega_0 \in U \ \text{ and } \ \omega_1 \in U \cup \{0\},$$

where U denotes the group of $2^f - 1$ roots of unity lying in T.

The Gauss sums $G_{P^\gamma}(\chi)$ are given by

Proposition 3.1. *For any odd integer v, $q = 2^r \neq 8$ with $s' \geq \delta$ and $\gamma \geq \gamma_0$,*

$$G_{P^\gamma}(\psi^v) = 2^{f(\gamma-\frac{re-d}{2})}\zeta_q^{nv}\zeta_8^{\kappa v}(\frac{2}{v})^{f(re-d)}\psi^{vn}(v)$$

and

$$G_{P^\gamma}(\xi\cdot\psi^v) = 2^{f(\gamma-\frac{re-d}{2})}\zeta_q^{nv}\zeta_8^{\kappa v}(\frac{2}{v})^{f(re-d)}\xi^n(v)\psi^{vn}(v),$$

where for tamely ramified extensions

$$\kappa = \begin{cases} n+(4+\frac{e^2-1}{2})f-4 & \text{if } r=2s+1 \\ 0 & \text{if } r=2s. \end{cases}$$

Otherwise for wildly ramified extensions if d is even,

$$\kappa = \begin{cases} 2Tr(\Pi^{-d}R^2) & \text{if } r \text{ odd} \\ 4Tr(\Pi^{-(e+d)}R^2-\Pi^{-\frac{e+d}{2}}R) & \text{if } r \text{ even }, \end{cases}$$

and if d is odd,

$$\kappa = \begin{cases} 5f-4+2Tr_{T/Q_2}(Tr_{K/T}\Pi^{-d-1}R^2 & \\ \qquad -(Tr_{K/T}\Pi^{\frac{e}{2}-d-1}R)^2/u_1) & \text{if } r \text{ odd} \\ 5f-4+4Tr(\Pi^{-(e+d+1)}R^2+\Pi^{-\frac{e+d+1}{2}}R) & \\ \qquad -2Tr_{T/Q_2}(v/u_1+\omega_1/\omega_0) & \text{if } r \text{ even.} \end{cases}$$

Here $R = x_0 + x_1\Pi + \cdots + x_{e/2-1}\Pi^{e/2-1}$ where x_0, x_1, ... ,$x_{e/2-1}$ uniquely solve (3.3) modulo $2O_T$, $u_1 = \omega_0 + 2\omega_1 \pmod{4O_T}$ as in (3.4) and

$$v = (Tr_{K/T}\Pi^{-\frac{d+1}{2}})^2 + (2Tr_{K/T}\,\Pi^{-d-\frac{e}{2}-1}R)^2$$

when d is odd.

For $q = 8$ with $K/\mathbf{Q}_2$ tamely ramified and $\chi = \xi^\nu\psi$ ($\nu = 0$ or 1),

$$G_{P^\gamma}(\chi) = (-1)^{f-1}(\frac{2}{e})^f 2^{f(\gamma-\frac{2e+1}{2})}\zeta_8^{n(1-\chi(-1))} \text{ for } \gamma \geq \gamma_0.$$

The result follows immediately from Theorem 2 and Corollary 3 in [13] since $G_{P^\gamma}(\chi^v) = \chi^{vn}(v)\sigma_v(G_{P^\gamma}(\chi))$ for any numerical character χ defined modulo q. When $K = \mathbf{Q}_2$ and $r > 1$, Proposition 3.1 reduces to Mauclaire's evaluation of the ordinary Gauss sums $G_q(\chi)$ for primitive characters modulo q.

Corollary 3.1. *For any odd v, where $q = 2^r$ with $r > 1, r \neq 3$*

$$G_q(\psi^v) = 2^{r/2}(\frac{2}{v})^\epsilon\zeta_8^{\epsilon v}\psi^v(v)\zeta_q^v$$

and

$$G_q(\xi\cdot\psi^v) = 2^{r/2}(\frac{2}{v})^\epsilon\zeta_8^{\epsilon v}\xi(v)\psi^v(v)\zeta_q^v.$$

For $r=3$,

$$G_8(\chi)=\sqrt{8}\zeta_8^{1-\chi(-1)} \text{ if } \chi(5)=-1.$$

The next result is the analog of Lemma 2.2 in section 2 and follows routinely from the Binomial Theorem.

Lemma 3.1. *Let χ be any numerical character modulo 2^r normalized as in (3.1) with $r\geq 3$. Then for any integer w*

$$\chi(1+w2^{s'})=\zeta_{2^s}^{-w}.$$

Before stating the main result of this section I require some facts about imcomplete exponential sums of the form

$$\sum_{v=1,\ v\equiv 1(\text{mod } 4)}^{2^{\alpha-2}} \chi^{\lambda v}(v)\zeta_{2^\alpha}^v, \qquad \lambda\equiv 1(\text{mod } 4),$$

where χ is a primitive character modulo 2^α ($\alpha\geq 4$) normalized as in (3.1). It is shown in [10] that

$$\sum_{v=1,\ v\equiv 1(\text{mod } 4)}^{2^{\alpha-2}} \chi(v)^{\lambda v}\zeta_{2^\alpha}^v = 2^{\frac{\alpha-4}{2}}\zeta_{2^\alpha}^{t(\lambda)}, \tag{3.5}$$

where $t(\lambda)$ is a certain integer-valued function of λ (chiefly from Proposition 5 and Theorem 2 in [10] in view of (27) there.).

For r odd one has (chiefly, Corollaries 3 and 4 in [10])

$$t(\lambda)=\begin{cases}1-2^{\alpha-3} & \text{if } \lambda\equiv 1(\text{mod } 2^{\frac{\alpha+1}{2}})\\ 1+2^{\alpha-3} & \text{if } \lambda\equiv 1+2^{\frac{\alpha-1}{2}}(\text{mod } 2^{\frac{\alpha+1}{2}}),\end{cases} \tag{3.6}$$

independent of the choice of normalized character χ modulo 2^α. For r even one has (chiefly, Corollary 3 in [10])

$$t(\lambda)=1 \text{ if } \lambda\equiv 1(\text{mod } 2^{\alpha/2}). \tag{3.7}$$

I am now ready to determine the sums $R(\eta,z)$. The following result treats the case $b=0$ where $l=1$, so η must be trivial.

Proposition 3.2. *For n even with $r>3$, $s'\geq\delta$ and $\gamma\geq\gamma_0$,*

$$R(1,1)=(-1)^{\epsilon[\frac{n+2}{4}]}2^{f(\gamma-\frac{re-d}{2})}\zeta_8^{\kappa-\epsilon n}\zeta_q^{n+1}.$$

The sole exception occurs for $r=5$ when $f(5e-d)+\kappa$ is odd, where

$$R(1,1)=(-1)^{[\frac{n+2}{4}]}2^{f(\gamma-\frac{5e-d}{2})}\zeta_8^{\kappa+n+2}\zeta_q^{n+1}.$$

In general for any odd z, $R(1,z)$ is determined from $R(1,1)$ by (1.5).

Proof. From Proposition 3.1, Corollary 3.1 and (3.2) above,

$$\begin{aligned}
R(1,1) &= \frac{1}{\phi(q)} \sum_{j=1, j\ odd}^{2^{r-2}} (1+\xi(j)) G_q(\psi^j) G_{P^\gamma}(\psi^j) \\
&= \frac{2}{\phi(q)} \sum_{j=1, j\equiv 1 (\mathrm{mod}\ 4)}^{2^{r-2}} G_q(\psi^j) G_{P^\gamma}(\psi^j) \\
&= 2^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2} + 2} \sum_{j=1, j\equiv 1 (\mathrm{mod}\ 4)}^{2^{r-2}} (\frac{2}{j})^{f(re-d)+\epsilon} \zeta_8^{(\epsilon+\kappa)j} \psi^{j(n+1)}(j) \zeta_q^{j(n+1)} \\
&= 2^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2} + 2} \zeta_8^{\epsilon+\kappa} \sum_{j=1, j\equiv 1 (\mathrm{mod}\ 4)}^{2^{r-2}} (\frac{2}{j})^{f(re-d)+\kappa} \psi^{j(n+1)}(j) \zeta_q^{j(n+1)}
\end{aligned}$$

since $\zeta_8^j(\frac{2}{j}) = \zeta_8$ for any $j \equiv 1 (\mathrm{mod}\ 4)$. Since $(\frac{2}{j})\psi^j(j) = \psi^{j(1+2^{r-3})}(j)$, it follows from (3.5), (3.6) and (3.7) that the sum

$$\sum_{j=1, j\equiv 1 (\mathrm{mod}\ 4)}^{2^{r-2}} (\frac{2}{j})^{f(re-d)+\kappa} \psi^{j(n+1)}(j) \zeta_q^{j(n+1)} = 2^{\frac{r-4}{2}} \zeta_q^{n+1} \zeta_8^{-\epsilon(n+1)}$$

for $r > 5$ and for $r = 4$ or 5 when $f(re-d)+\kappa$ is even. The exceptional case occurs for $r = 5$ when $f(re-d)+\kappa$ is odd. The case $r = 4$ with $f(re-d)+\kappa$ odd cannot occur here since from Proposition 3.1, κ is odd if and only if d is odd. In particular, for r even, $f(re-d)+\kappa$ is odd only if f even, $d \geq e+1$ is odd and κ is odd, so $\delta > 2$ and consequently $r > 4$. As $(\mathbf{Z}/q\mathbf{Z})^{*n+1} = \mathbf{Z}/q\mathbf{Z}^*$, $R(1,z)$ is determined from $R(1,1)$ for any odd z by (1.5). This completes the proof of the proposition. $\square$

I now assume that $b > 0$ (so n is odd) throughout the remainder of the section and $r > 3$. The computation naturally breaks into the cases $r > b+4$ and $3 < r \leq b+4$. I consider the case $r > b+4$ first, where character sums of the form $\sum_{x\in X} (\frac{2}{x})^\beta \psi^{2^b yx}(x) \zeta_{2^{r-b}}^{yx}$ naturally arise, where $\beta = 0$ or 1 and $X = \{1, 5, 9, ..., 2^{r-b-2}-3\}$. These exponential sums have been evaluated in [10] in terms of the normalized characters ψ_α modulo 2^α in Proposition 6 there. Specifically, let $\{k_\alpha\}$ $(\alpha > 3)$ be given modulo $2^{\alpha-2}$ by

$$k_\alpha = \begin{cases} -R(1-2^{\frac{\alpha}{2}-1}) & \text{if } \alpha \geq 4 \text{ even} \\ -R & \text{if } \alpha \geq 5 \text{ odd,} \end{cases} \tag{3.8}$$

where R is the 2-adic unit $R = \frac{1}{4} log\ 5$. Then the normalized characters ψ_α modulo 2^α are given by

$$\psi_\alpha(5) = \zeta_{2^{\alpha-2}}^{k_\alpha}, \psi_\alpha(-1) = 1 \quad (\alpha > 3). \tag{3.9}$$

and satisfy

Proposition 3.3. *For $r > b+4$ with $b > 0$ and normalized characters ψ_r as given in (3.9) above, the sum*

$$\sum_{x\in X} (\frac{2}{j})^{\beta} \psi_r^{2^b x}(x) \zeta_{2^{r-b}}^{x} = 2^{\frac{r-b-4}{2}} \zeta_{2^{r-b}}^{1} \zeta_8^{-1} \quad (\beta = 0 \text{ or } 1), \tag{3.10}$$

with the following exceptions. The sum (3.10) equals $2^{\frac{r-b-4}{2}} \zeta_{2^{r-b}}^{1} \zeta_8^{1}$ if $r = b+5 > 6$ with $\beta = 1$, or if $b = 1$ and r even *with $r > 6$ or $\beta = 0$. It equals $-2^{\frac{r-b-4}{2}} \zeta_{2^{r-b}}^{1} \zeta_8^{-1}$ if $r = b+6 \neq 8$ with $\beta = 0$, or if $b = 2$ and r* even *with $r > 8$ or $\beta = 1$.*

Proof. I consider the case r is even first. For b odd, one finds from (3.8) that

$$\psi_r^{2^b} = \psi_{r-b}^{\frac{k_r}{k_{r-b}}} \quad \text{where} \quad \frac{k_r}{k_{r-b}} \equiv \begin{cases} 1 \bmod 2^{\frac{r-b+1}{2}} & \text{if } b > 1 \\ 1 - 2^{\frac{r-2}{2}} \bmod 2^{\frac{r}{2}} & \text{if } b = 1, \end{cases}$$

so

$$\frac{k_r}{k_{r-b}} + \beta 2^{r-b-3} \equiv \begin{cases} 1 \bmod 2^{\frac{r-b+1}{2}} & \text{if } b > 1 \text{ and } r > b+6 \\ 1 + \beta 2^2 \bmod 2^3 & \text{if } b > 1 \text{ and } r = b+5 \\ 1 - 2^{\frac{r-2}{2}} \bmod 2^{\frac{r}{2}} & \text{if } b = 1 \text{ and } r > 6 \\ 1 + 2^2 \bmod 2^3 & \text{if } b = 1, \beta = 0 \text{ and } r = 6 \\ 1 \bmod 2^3 & \text{if } b = 1, \beta = 0 \text{ and } r = 6. \end{cases}$$

For $b \geq 2$ even, one similarly has from (3.8) that

$$\psi_r^{2^b} = \psi_{r-b}^{\frac{k_r}{k_{r-b}}} \quad \text{where} \quad \frac{k_r}{k_{r-b}} \equiv \begin{cases} 1 + 2^{\frac{r-b-2}{2}} \bmod 2^{\frac{r-b+2}{2}} & \text{if } b > 2 \\ 1 - 2^{\frac{r-4}{2}} \bmod 2^{\frac{r}{2}} & \text{if } b = 2, \end{cases}$$

so

$$\frac{k_r}{k_{r-b}} + \beta 2^{r-b-3} \equiv \begin{cases} 1 + 2\frac{r-b-2}{2} \bmod 2^{\frac{r-b+2}{2}} & \text{if } b > 2 \text{ and } r > b+6 \\ 1 + 2^2 + \beta 2^3 \bmod 2^4 & \text{if } b > 2 \text{ and } r = b+6 \\ 1 - 2^{\frac{r-4}{2}} \bmod 2^{\frac{r}{2}} & \text{if } b = 2 \text{ and } r > 8 \\ 1 - 2^2 + \beta 2^3 \bmod 2^4 & \text{if } b = 2 \text{ and } r = 8. \end{cases}$$

The corresponding values of (3.10) with r even when $b > 0$ follows from (3.6) and (3.7) or Corollaries 4 and 5 in [10] in view of formula (27) there.

It remains to consider the case r is odd. For $b > 0$ even, $\psi^{2^b} = \psi_{r-b}$ so

$$1 + \beta 2^{r-b-3} \equiv \begin{cases} 1 \bmod 2^{\frac{r-b+1}{2}} & \text{if } r > b+5 \\ 1 + \beta 2^2 \bmod 2^3 & \text{if } r = b+5. \end{cases}$$

For b odd, one finds from (3.8) that

$$\psi_r^{2^b} = \psi_{r-b}^{\frac{k_r}{k_{r-b}}} \quad \text{where} \quad \frac{k_r}{k_{r-b}} \equiv 1 + 2^{\frac{r-b-2}{2}} \bmod 2^{\frac{r-b+2}{2}}$$

so

$$\frac{k_r}{k_{r-b}} + \beta 2^{r-b-3} \equiv \begin{cases} 1 + 2^{\frac{r-b-2}{2}} \bmod 2^{\frac{r-b+2}{2}} & \text{if } r > b+6 \\ 1 + 2^2 + \beta 2^3 \bmod 2^4 & \text{if } r = b+6. \end{cases}$$

The corresponding values of (3.10) with r odd when $b > 0$ now follow as before. □

I am ready to state the main result when $r > b+4$ with $b > 0$ (so $l = 2$), and η of order 1 or 2 so of the form $\eta = \xi^\nu \psi^w$ with $\nu = 0$ or 1, $w = 0$ or 2^{r-3}. For convenience I set $\rho = f(re-d) + \kappa + w/2^{r-3}$.

Theorem 3.1. *Let $q = 2^r$ with $r > b+4 \geq 5$, $s' \geq \delta$, $\lambda \geq \lambda_0$ and $\eta = \xi^\nu \psi^w$, where $\nu = 0$ or 1 and $w = 0$ or 2^{r-3} with ψ normalized as in (3.1). Then*

$$R(\eta,1) = (\frac{2}{y})^{r-b} 2^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} (\zeta_8^{\epsilon+\kappa-y} \zeta_q^{n+1} + (-1)^\nu \zeta_8^{-\epsilon-\kappa+y} \zeta_q^{-n-1}),$$

except when $r = b+6$ with $b \neq 2$ or $r > 8$ even with $b = 2$

$$R(\eta,1) = -(\frac{2}{y})^{r-b} 2^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} (\zeta_8^{\epsilon+\kappa-y} \zeta_q^{n+1} + (-1)^\nu \zeta_8^{-\epsilon-\kappa+y} \zeta_q^{-n-1})$$

or when $r = b+5 > 6$ or $r > 6$ even with $b = 1$

$$R(\eta,1) = (\frac{2}{y}) 2^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} (\zeta_8^{\epsilon+\kappa+y} \zeta_q^{n+1} + (-1)^\nu \zeta_8^{-\epsilon-\kappa-y} \zeta_q^{-n-1}).$$

Furthermore, $R(\eta,z) = 0$ if $z \not\equiv 1 (\mathrm{mod}\ 2^{b+2})$ else $R(\eta,z)$ is determined from $R(\eta^{\bar{v}},1)$ by (1.5) where $z \equiv v^{n+1}\ (\mathrm{mod}\ q)$.

Proof. From Proposition 3.1, Corollary 3.1 and (3.2) one finds for the choice of η here that

$$R(\eta,z) = \frac{1}{\phi(q)} \sum_{j=1,\, j\ odd}^{2^{r-2}} (1+\xi(z)) \xi^\nu(j) \bar{\psi}^j(z) G_q(\psi^j) G_{P^\gamma}(\psi^{j+w})$$

vanishes for $z \equiv 3 (\mathrm{mod}\ 4)$; while for $z \equiv 1 (\mathrm{mod}\ 4)$,

(3.11)

$$R(\eta,z) = 2^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2} + 2} \sum_{j=1,\, j\ odd}^{2^{r-2}} \xi^\nu(j) \bar{\psi}^j(z) (\frac{2}{j})^{f(re-d)+\epsilon} \zeta_8^{(\epsilon+\kappa)j} \psi^j(j) V,$$

where $V = \psi^{n(j+w)}(j+w) \zeta_q^{j(n+1)+nw}$. Writing $j = v + hi$ where $h = 2^{r-b-2}$ uniquely with $0 < v < h$ odd and $0 \leq i < 2^b$, one may express the sum

(3.11) above as

$$2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+2}\sum_{v=1, v\ odd}^{h}\sum_{i=0}^{2^b-1}\xi^{\nu}(v)\bar{\psi}^{v+hi}(z)(\frac{2}{v})^{f(re-d)+\epsilon}\zeta_8^{(\epsilon+\kappa)v}$$
$$\cdot\psi^{v+hi}(v+hi)\psi^{n(v+hi+w)}(v+hi+w)\zeta_q^{(v+hi)(n+1)+nw}$$

or equivalently as

$$2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+2}$$
$$\cdot\sum_{v=1,\ v\ odd}^{h}\xi^{\nu}(v)\bar{\psi}^{v}(z)(\frac{2}{v})^{f(re-d)+\epsilon}\zeta_8^{(\epsilon+\kappa)v}\psi^{v(n+1)}(v)\psi^{nw}(v)\zeta_q^{v(n+1)}$$
$$\cdot\sum_{i=0}^{2^{\beta}-1}\bar{\psi}^{hi}(z)\psi^{v}(1+\bar{v}hi)\psi^{hi}(v+hi)\psi^{vn}(1+\bar{v}(hi+w))Y,$$

where $Y=\psi^{nw}(1+\bar{v}(hi+w))\psi^{hin}(v+hi+w)\zeta_q^{hi(n+1)+nw}$. Since $(1+\bar{v}hi)$ $(1+\bar{v}w)\equiv 1+\bar{v}(hi+w) \pmod q$ as $r>b+4$, $\psi^{v}(1+\bar{v}hi)\psi^{vn}(1+\bar{v}(hi+w))=\psi^{v(n+1)}(1+\bar{v}hi)\psi^{vn}(1+\bar{v}w)=\zeta_q^{-(n+1)hi}\cdot\zeta_q^{-nw}$ from Lemma 3.1. Also $\psi^{hin}(v+hi+w)\psi^{hi}(v+hi)=\psi^{hi(n+1)}(v+hi)=1$ since $2^{r-2}|h(n+1)$. In addition $\psi^{nw}(1+\bar{v}(hi+w))=1$, so the inner sum above reduces to

$$\sum_{i=0}^{2^b-1}\bar{\psi}^{hi}(z)=\begin{cases}2^b & \text{if } z\in(\mathbf{Z}/q\mathbf{Z})^{*n+1}\\ 0 & \text{otherwise,}\end{cases}$$

since ψ^h has order 2^b. Thus $R(\eta,z)=0$ for $r>b+4$ whenever $z\notin(\mathbf{Z}/q\mathbf{Z})^{*n+1}$. But if $z\in(\mathbf{Z}/q\mathbf{Z})^{*n+1}$ then $z\equiv 1(\bmod\ 2^{b+2})$ and setting $B=\psi^{v(n+1)}(v)\zeta_q^{v(n+1)}$,

(3.12)

$$R(\eta,z)=2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+b+2}\sum_{v=1,\ v\ odd}^{h}\bar{\psi}^{v}(z)\xi^{\nu}(v)(\frac{2}{v})^{\epsilon+f(re-d)}\zeta_8^{(\epsilon+\kappa)v}\psi^{wn}(v)B$$

is determined from $R(\eta^{\bar{v}},1)$ by (1.5) where $z\equiv v^{n+1} \pmod q$, so it suffices to compute $R(\eta,1)$. Now

$$\psi^{(v-h)(n+1)}(v-h)\zeta_q^{(v-h)(n+1)}=\psi^{v(n+1)}(v)\zeta_q^{v(n+1)}\psi^{v(n+1)}(1-\bar{v}h)\zeta_4^{-y}=B$$

since ψ is normalized of order 2^{r-2} and $\psi(1-\bar{v}h)^{v(n+1)}=\zeta_4^{y}$ from (3.1) and Lemma 3.1. Consequently, since $\psi^{wn}(v)=(\frac{2}{v})^{w/2^{r-3}}$, one may write

$R(\eta,1)$ from (3.12) as

$$2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+b+2} \cdot \sum_{v=1,v\equiv 1(\mathrm{mod}\ 4)}^{h} (\frac{2}{v})^{\epsilon+f(re-d)+\frac{w}{2^{r-3}}} (\zeta_8^{(\epsilon+\kappa)v}\psi^{v(n+1)}(v)\zeta_q^{v(n+1)} + (-1)^{\nu}\zeta_8^{-(\epsilon+\kappa)v}\psi^{-v(n+1)}(v)\zeta_q^{-v(n+1)})$$

or equivalently

$$2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+b+2}(\zeta_8^{\epsilon+\kappa}\sum_{x\in X}(\frac{2}{x})^{\rho}\psi^{2^b yx}(x)\zeta_{2^{r-b}}^{yx} +(-1)^{\nu}\zeta_8^{-\epsilon-\kappa}\sum_{x\in X}(\frac{2}{x})^{\rho}\psi^{-2^b yx}(x)\zeta_{2^{r-b}}^{-yx}) \tag{3.13}$$

where $X=\{1,5,9,\cdots,2^{r-b-2}-3\}$. It follows now from Proposition 3.3 that if ρ is even then

$$R(\eta,1)=2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\frac{2}{y})^{r-b}(\zeta_8^{\epsilon+\kappa-y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa+y}\zeta_q^{-n-1})$$

except when $r=b+6$ with $b\neq 2$ or $r>8$ even with $b=2$

$$R(\eta,1)=-2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\zeta_8^{\epsilon+\kappa-y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa+y}\zeta_q^{-n-1}),$$

or when $r\geq 6$ is even with $b=1$

$$R(\eta,1)=2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\frac{2}{y})(\zeta_8^{\epsilon+\kappa+y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa-y}\zeta_q^{-n-1}).$$

On the other hand, if ρ is odd then

$$R(\eta,1)=2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\frac{2}{y})^{r-b}(\zeta_8^{\epsilon+\kappa-y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa+y}\zeta_q^{-n-1}),$$

except when r is even with $b=2$

$$R(\eta,1)=-2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\zeta_8^{\epsilon+\kappa-y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa+y}\zeta_q^{-n-1}),$$

or when $r=b+5>6$ or $r>6$ is even with $b=1$

$$R(\eta,1)=2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\frac{2}{y})(\zeta_8^{\epsilon+\kappa+y}\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa-y}\zeta_q^{-n-1}).$$

This completes the proof of the theorem. □

Next consider the case $4<r\leq b+4$ with $b>0$. One finds here that

Theorem 3.2. *Let $q = 2^r$ with $r > 4$, $s' \geq \delta$, $\gamma \geq \gamma_0$ and $\eta = \xi^\nu \psi^w$, where $\nu = 0$ or 1 and $w = 0$ or 2^{r-3} with ψ normalized as in (3.1). For $r < b+4$, if $z \equiv 1 + \rho 2^{r-3} (\mathrm{mod}\ 2^{r-2})$ then*

$$R(\eta, z) = 2^{f(\gamma - \frac{re-d}{2}) + \frac{r}{2} - 2} (\zeta_8^{\epsilon+\kappa} \zeta_q^{n+z} + (-1)^\nu \zeta_8^{-\epsilon-\kappa} \zeta_q^{-n-z})$$

and otherwise is 0. For $r = b+4$, if $z \equiv 1 + (\rho+1) 2^{r-3} (\mathrm{mod}\ 2^{r-2})$ then

$$R(\eta, z) = 2^{f(\gamma - \frac{re-d}{2}) + \frac{b}{2}} (\zeta_8^{\epsilon+\kappa} \zeta_q^{n+z} + (-1)^\nu \zeta_8^{-\epsilon-\kappa} \zeta_q^{-n-z})$$

and otherwise is 0.

The sole exceptions occur for $r = 5$ where the above formulas hold with κ replaced by $\kappa - 2$ if $z \equiv 5 (\mathrm{mod}\ 8)$ and with ϵ replaced by $\epsilon - 2n$ if $w = 4$.

Proof. As before one finds $R(\eta, z) = 0$ if $z \equiv 3 (\mathrm{mod}\ 4)$, so assume $z \equiv 1 (\mathrm{mod}\ 4)$. From (3.2) and Proposition 3.1 one again has that

$$R(\eta, z) = \frac{1}{2^{r-2}} \sum_{j=1,\ j\ odd}^{2^{r-2}} \xi^\nu(j) \bar{\psi}^j(z) G_q(\psi^j) G_{P^\gamma}(\psi^{j+w}) = 2^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2} + 2}$$

$$\cdot \sum_{j=1,\ j\ odd}^{2^{r-2}} \xi^\nu(j) \bar{\psi}^j(z) (\frac{2}{j})^{\epsilon + f(re-d)} \zeta_8^{(\epsilon+\kappa)j} \psi^j(j) \psi^{n(j+w)}(j+w) \zeta_q^{j(n+1)+nw}.$$

I consider the case $r > 5$ or $w = 0$ first, leaving a discussion of the exceptional cases until the end of the proof. In this case the above expression becomes

$$2^{f(\gamma - \frac{re-d}{2}) - \frac{r}{2} + 2} \sum_{j=1, j\equiv 1 (\mathrm{mod}\ 4)}^{2^{r-2}} (\bar{\psi}^j(z) \zeta_8^{\epsilon+\kappa} (\frac{2}{j})^\rho \psi^{j(n+1)}(j) \zeta_q^{j(n+1)} +$$

$$(-1)^\nu \psi^j(z) \zeta_8^{-\epsilon-\kappa} (\frac{2}{j})^\rho \psi^{-j(n+1)}(j) \zeta_q^{-j(n+1)})$$

upon noting that

$$\psi^j(j) \psi^{n(j+w)}(j+w) \zeta_q^{nw} = (\frac{2}{j})^{\frac{w}{2^{r-3}}} \psi^{j(n+1)}(j) \quad \text{for } j \equiv 1 (\mathrm{mod}\ 4) \tag{3.14}$$

and that

$$\bar{\psi}^{j-2^{r-2}}(z)\psi^{(j-2^{r-2})(n+1)}(j-2^{r-2})\zeta_q^{(j-2^{r-2})(n+1)} = \bar{\psi}^j(z)\psi^{j(n+1)}(j)\zeta_q^{j(n+1)}$$

since $\psi^{j(n+1)}(1-\bar{j}2^{r-2})\zeta_q^{-2^{r-2}(n+1)} = 1$ by (3.1) and Lemma 3.1. Thus

(3.15)

$$R(\eta,z) = 2^{f(\gamma-\frac{re-d}{2})-\frac{r}{2}+2}(\zeta_8^{\epsilon+\kappa}\sum_{j=1,j\equiv 1(\mathrm{mod}\ 4)}^{2^{r-2}}(\frac{2}{j})^{\rho}\bar{\psi}^j(z)\psi^{j(n+1)}(j)\zeta_q^{j(n+1)}$$
$$+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa}\sum_{j=1,j\equiv 1(\mathrm{mod}\ 4)}^{2^{r-2}}(\frac{2}{j})^{\rho}\psi^j(z)\psi^{-j(n+1)}(j)\zeta_q^{-j(n+1)}).$$

In particular, for $r < b+4$,

$$\sum_{j=1,\ j\equiv 1(\mathrm{mod}\ 4)}^{2^{r-2}}\bar{\psi}^j(z)(\frac{2}{j})^{\rho}\psi^{j(n+1)}(j)\zeta_q^{j(n+1)} = \bar{\psi}(z)\zeta_q^{n+1}\sum_{t=0}^{2^{r-4}-1}(-1)^{t\rho}\bar{\psi}^{4t}(z)$$

since $\psi^{j(n+1)}(j)\zeta_q^{j(n+1)} = \zeta_q^{n+1}$ again from (3.1) and Lemma 3.1 for $j \equiv 1(\mathrm{mod}\ 4)$. Now

$$\sum_{t=0}^{2^{r-4}-1}(-1)^{t\rho}\bar{\psi}^{4t}(z) = \begin{cases} 2^{r-4} & \text{if } z\equiv 1+\rho 2^{r-3}(\mathrm{mod}\ 2^{r-2}) \\ 0 & \text{otherwise,} \end{cases}$$

since $\bar{\psi}^4$ has order 2^{r-4}. Thus from (3.14) with $r < b+4$, one finds for $z \equiv 1+\rho 2^{r-3}(\mathrm{mod}\ 2^{r-2})$ that

(3.16) $$R(\eta,z) = 2^{f(\gamma-\frac{re-d}{2})+\frac{r}{2}-2}(\zeta_8^{\epsilon+\kappa}\bar{\psi}(z)\zeta_q^{n+1}+(-1)^{\nu}\zeta_8^{-\epsilon-\kappa}\psi(z)\zeta_q^{-n-1}),$$

and otherwise $R(\eta,z)=0$.

When $r = b+4$ one sees that $\psi^{j(n+1)}(j)\zeta_q^{j(n+1)} = \zeta_q^{n+1}$ or $-\zeta_q^{n+1}$ according as $j \equiv 1$ or $5(\mathrm{mod}\ 8)$ by (3.1) and Lemma 3.1, in view of the expansion

$$j^{j(n+1)} = (1+4\cdot\frac{j-1}{4})^{2^{r-4}jy} \equiv 1+2^{r-2}jy\frac{j-1}{4}+2^{r-1}\frac{j-1}{4}\ \ (\mathrm{mod}\ 2^r).$$

Now

$$\sum_{j=1,j\equiv 1(\text{mod }4)}^{2^{r-2}} \bar{\psi}^j(z)(\frac{2}{j})^\rho \psi^{j(n+1)}(j)\zeta_q^{j(n+1)}$$
$$= \zeta_q^{n+1}(\sum_{j=1,j\equiv 1(\text{mod }8)}^{2^{r-2}} \bar{\psi}^j(z) - (-1)^\rho \sum_{j=1,j\equiv 5(\text{mod }8)}^{2^{r-2}} \bar{\psi}^j(z))$$
$$= \zeta_q^{n+1}\bar{\psi}(z)(\sum_{t=0,t\ even}^{2^{r-4}-1} \bar{\psi}^{4t}(z) - (-1)^\rho \sum_{t=0,t\ odd}^{2^{r-4}-1} \bar{\psi}^{4t}(z))$$
$$= \zeta_q^{n+1}\bar{\psi}(z)(1-(-1)^\rho\bar{\psi}^4(z)) \sum_{t=0}^{2^{r-5}-1} \bar{\psi}^{8t}(z).$$

But

$$\sum_{t=0}^{2^{r-5}-1} \bar{\psi}^{8t}(z) = \begin{cases} 2^{r-5} & \text{if } z \equiv 1(\text{mod } 2^{r-3}) \\ 0 & \text{otherwise,} \end{cases}$$

as $\bar{\psi}^8$ has order 2^{r-5} and $\bar{\psi}^4(z) = 1$ for $z \equiv 1(\text{mod } 2^{r-2})$ or -1 for $z \equiv 1+2^{r-3}(\text{mod } 2^{r-2})$. Thus one finds for $z \equiv 1+(\rho+1)2^{r-3}(\text{mod } 2^{r-2})$ when $r = b+4$ that

$$R(\eta, z) = 2^{f(\gamma-\frac{re-d}{2})+\frac{b}{2}}(\zeta_8^{\epsilon+\kappa}\bar{\psi}(z)\zeta_q^{n+1} + (-1)^\nu \zeta_8^{-\epsilon-\kappa}\psi(z)\zeta_q^{-n-1}), \tag{3.17}$$

and otherwise $R(\eta, z) = 0$. Since $\psi(z) = \zeta_q^{1-z}$ for $z \equiv 1(\text{mod } 4)$ here, one obtains the expressions as stated from (3.16) and (3.17).

It remains to consider the exceptional cases when $r = 5$. If $w = 4$ the right side of the expression in (3.14) is found from direct computation to equal $-(\frac{2}{j})\psi^{j(n+1)}(j)\zeta_4^n$, leading to the formulas for $R(\eta, z)$ in (3.16) and (3.17), but with ϵ replaced by $\epsilon - 2n$. If $z \equiv 5(\text{mod } 8)$ one finds that $\psi(z) = \zeta_4\zeta_q^{1-z}$ instead of $\psi(z) = \zeta_q^{1-z}$ in formulas (3.16) and (3.17). These changes result in the modifications as stated for the exceptional cases.

This concludes the proof of the theorem. $\square$

For completeness I include the evaluation of $R(\eta, z)$ for the small values of r, $1 < r \leq 4$, for which the Gauss sums are given in Proposition 3.1, but $R(\eta, z)$ is not considered in Proposition 3.2 and Theorems 3.1 and 3.2. For these cases $K/\mathbf{Q}_2$ is tamely ramified with $\delta = 1+\epsilon$. For $r > 2$ write $\eta = \psi^w\xi^\nu$ for $w = 0$ or 2^{r-3} and $\nu = 0$ or 1, where ψ is an even character modulo q normalized as in (3.1). The values $R(\eta, z)$ are readily computed from Proposition 3.1 using (1.7) (see also [Proposition 3, 13]).

Proposition 3.4. *Let $K/\mathbf{Q}_2$ be tamely ramified with η as above and $\gamma \geq \gamma_0$.*

$$\text{For } q = 4, \quad R(\eta, z) = \begin{cases} 2^{f(\gamma - \frac{e+1}{2})}\zeta_4^{n+z} & \textit{if } \eta = 1 \\ 0 & \textit{if } \eta = \xi. \end{cases}$$

$$\text{For } q = 8, R(\eta, z) = \begin{cases} (-1)^{f-1}(\frac{2}{e})^f(\frac{2}{z})i^{\nu z}(1 + i^{n+(-1)^\nu z}) & \\ \quad \cdot 2^{f(\gamma - \frac{2e+1}{2}) - \frac{1}{2}} & \textit{if } w = 0 \\ 0 & \textit{if } w = 1. \end{cases}$$

For $q = 16$ with n odd,

$$R(\eta, z) = \begin{cases} 2^{f(\gamma - \frac{3e+1}{2})} & \\ \quad \cdot(\zeta_{16}^{(-1)^{w/2}z+n} + (-1)^\nu \zeta_{16}^{-(-1)^{w/2}z-n}) & \textit{if } z \equiv 1 (\mathrm{mod}\ 4) \\ 0 & \textit{if } z \equiv 3 (\mathrm{mod}\ 4). \end{cases}$$

I note that Proposition 3.2 already includes the case for $q = 16$ with n even, so there is no need to repeat it above.

Example. To illustrate Theorems 3.1 and 3.2 and Proposition 3.4 above consider the field $K = \mathbf{Q}_2(2^{1/3})$ where $f = 1$, $e = 3$ and $d = 2$. $K/\mathbf{Q}_2$ is tamely ramified with $\kappa = 0$ or -1 according as $r > 3$ is even or odd from Proposition 3.1, $\gamma_0 = re - d = 3r - 2$, $b = 2$ and $y = 1$. Choosing $\Pi = 2^{1/3}$ as a uniformizant, one also has $u = 1$. Using (1.3) directly to compute $R(\eta, z)$ for several small values of r yields:

For $q = 4$,

$$R(\eta, z) = \begin{cases} 2^{\gamma-2}\zeta_4^{3+z} & \text{if } \eta = 1 \\ 0 & \text{if } \eta = \xi. \end{cases}$$

For $q = 8$,

$$R(\eta, z) = \begin{cases} 2^{\gamma-4}(\frac{2}{z})(1 + i^{3+z}) & \text{if } \eta = 1 \\ 2^{\gamma-4}(\frac{2}{z})(i^z - i) & \text{if } \eta = \xi \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 16$ with $z \equiv 1 (\mathrm{mod}\ 4)$

$$R(\eta, z) = \begin{cases} 2^{\gamma-5}(\zeta_{16}^{z+3} + \zeta_{16}^{-z-3}) & \text{if } \eta = 1 \\ 2^{\gamma-5}(\zeta_{16}^{z+3} - \zeta_{16}^{-z-3}) & \text{if } \eta = \xi \\ 2^{\gamma-5}(\zeta_{16}^{-z+3} + \zeta_{16}^{z-3}) & \text{if } \eta = (\frac{2}{p}) \\ 2^{\gamma-5}(\zeta_{16}^{-z+3} - \zeta_{16}^{z-3}) & \text{if } \eta = (\frac{-2}{p}), \end{cases}$$

else for $z \equiv 3 (\mathrm{mod}\ 4)$, $R(\eta, z) = 0$.

For $q = 32$,

$$R(\eta, z) = \begin{cases} 2^{\gamma-6}(\zeta_{32}^{z+3} + \zeta_{32}^{-z-3}) & \text{for } z \equiv 1 (\text{mod } 8) \text{ with } \eta = 1 \\ 2^{\gamma-6}(\zeta_{32}^{z+3} - \zeta_{32}^{-z-3}) & \text{for } z \equiv 1 (\text{mod } 8) \text{ with } \eta = \xi \\ 2^{\gamma-6}(\zeta_{32}^{z+3} + \zeta_{32}^{-z-3}) & \text{for } z \equiv 5 (\text{mod } 8) \text{ with } \eta = (\frac{2}{p}) \\ 2^{\gamma-6}(\zeta_{32}^{z+3} - \zeta_{32}^{-z-3}) & \text{for } z \equiv 5 (\text{mod } 8) \text{ with } \eta = (\frac{-2}{p}), \end{cases}$$

else $R(\eta, z) = 0$.

For $q = 64$,

$$R(\eta, z) = \begin{cases} 2^{\gamma-7}(\zeta_{64}^{z+3} + \zeta_{64}^{-z-3}) & \text{for } z \equiv 9 (\text{mod } 16) \text{ with } \eta = 1 \\ 2^{\gamma-7}(\zeta_{64}^{z+3} - \zeta_{64}^{-z-3}) & \text{for } z \equiv 9 (\text{mod } 16) \text{ with } \eta = \xi \\ 2^{\gamma-7}(\zeta_{64}^{z+3} + \zeta_{64}^{-z-3}) & \text{for } z \equiv 1 (\text{mod } 16) \text{ with } \eta = (\frac{2}{p}) \\ 2^{\gamma-7}(\zeta_{64}^{z+3} - \zeta_{64}^{-z-3}) & \text{for } z \equiv 1 (\text{mod } 16) \text{ with } \eta = (\frac{-2}{p}), \end{cases}$$

else $R(\eta, z) = 0$.

The values above agree with those given from Proposition 3.4 and Theorem 3.2, where ρ is seen to satisfy $\rho \equiv \frac{w}{2^{r-3}}$ (mod 2). For $r > 6$, $R(\eta, z)$ is determined from Theorem 3.1.

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