Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 18, n° 2 (2006), p. 445-470.

<http://jtnb.cedram.org/item?id=JTNB_2006__18_2_445_0>

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A study of the mean value of the error term in the mean square formula of the Riemann zeta-function in the critical strip $3/4 \leq \sigma < 1$

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Résumé. Pour σ dans la bandre critique $1/2 < \sigma < 1$, on note $E_\sigma(T)$ le terme d'erreur de la formule asymptotique de $\int_1^T |\zeta(\sigma+it)|^2\,dt$ (pour T grand). C'est un analogue du terme d'erreur classique $E(T)$ $(= E_{1/2}(T))$. L'étude de $E(T)$ a une longue histoire, mais celle de $E_\sigma(T)$ est assez récente. En particulier, lorsque $3/4 < \sigma < 1$, on connaît peu d'informations sur $E_\sigma(T)$. Pour en gagner, nous étudions la moyenne $\int_1^T E_\sigma(u)\,du$. Dans cet article, nous donnons une expression en série de type Atkinson et explorons quelques une des propriétés de la moyenne comme fonction en T.

Abstract. Let $E_\sigma(T)$ be the error term in the mean square formula of the Riemann zeta-function in the critical strip $1/2 < \sigma < 1$. It is an analogue of the classical error term $E(T)$. The research of $E(T)$ has a long history but the investigation of $E_\sigma(T)$ is quite new. In particular there is only a few information known about $E_\sigma(T)$ for $3/4 < \sigma < 1$. As an exploration, we study its mean value $\int_1^T E_\sigma(u)\,du$. In this paper, we give it an Atkinson-type series expansion and explore many of its properties as a function of T.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function, and let

$$E(T) = \int_0^T |\zeta(1/2+it)|^2\,dt - T(\log\frac{T}{2\pi} + 2\gamma - 1)$$

denote the error term in the mean-square formula for $\zeta(s)$ (on the critical line). The behaviour of $E(T)$ is interesting and many papers are devoted

Manuscrit reçu le 20 septembre 2004.

to study this function. Analogously, it is defined for $1/2<\sigma<1$,

$$E_\sigma(T)=\int_0^T|\zeta(\sigma+it)|^2\,dt-\left(\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}\right).$$

The behaviour of $E_\sigma(T)$ is very interesting too, and in fact, more delicate analysis is required to explore its properties such as the Atkinson-type series expansion and mean square formula, see ([17]-[20]). Excellent surveys are given in [11] and [18].

In the critical strip $1/2<\sigma<1$, our knowledge of $E_\sigma(T)$ is not 'uniform', for example, an asymptotic formula for the mean square is available for $1/2<\sigma\le 3/4$ but not for the other part. In fact, not much is known for the case $3/4<\sigma<1$, except perhaps some upper bound estimates and

$$T\ll\int_1^T E_\sigma(t)^2\,dt\ll T\qquad(3/4<\sigma<1).\tag{1.1}$$

(See [7], [20] and [14].) To furnish this part, we look at the mean value $\int_1^T E_\sigma(u)\,du$. The mean values of $E(T)$ and $E_\sigma(T)$ $(1/2<\sigma<3/4)$ are respectively studied in [2] and [6], each of which gives an Atkinson-type expansion. Correspondingly, we prove an analogous formula with a good error term in the case $3/4\le\sigma<1$. Actually, the tight lower bound in (1.1) is shown in [14] based on this formula. The proof of the asymptotic formula relies on the argument of [2] and uses the tools available in [2] and [19]. But there is a difficulty which we need to get around. In [2], Hafner and Ivić used a result of Jutila [9] on transformation of Dirichlet Polynomials, which depends on the formula

$$\textstyle\sum'_{a\le n\le b}d(n)f(n)=\int_a^b(\log x+2\gamma)f(x)\,dx+\sum_{n=1}^{\infty}d(n)\int_a^b f(x)\alpha(nx)\,dx,$$

where $\alpha(x)=4K_0(4\pi\sqrt{x})-2\pi Y_0(4\pi\sqrt{x})$ is a combination of the Bessel functions K_0 and Y_0. It is not available in our case but this can be avoided by using an idea in [19].

In addition, we shall regard the mean value as a function of T and study its behaviour; more precisely, we consider

$$G_\sigma(T)=\int_1^T(E_\sigma(t)+2\pi\zeta(2\sigma-1))\,dt.\tag{1.2}$$

(The remark below Corollary 1 explains the inclusion of $2\pi\zeta(2\sigma-1)T$.) Unlike the case $1/2\le\sigma<3/4$, the function $G_\sigma(T)$ is now more fluctuating. Nevertheless we can still explore many interesting properties, including some power moments, $\Omega_\pm$-results, gaps between sign-changes and limiting distribution functions, by using the tools in [19], [23], [4], [3], [1] and [13]. Particularly, we can determine the exact order of magnitude of the gaps of sign-changes (see Theorems 5 and 6). The limiting distribution function is not computed in the case $1/2 \le \sigma < 3/4$, perhaps because it is less interesting in the sense that the exact order of magnitude of $G_\sigma(t)$ $(1/2 \le \sigma < 3/4)$ is known; therefore, the limiting distribution is 'compactly supported'. Here, a limiting distribution $P(u)$ is said to be compactly supported if $P(u) = 0$ for all $u \le a$ and $P(u) = 1$ for all $u \ge b$, for some constants $a < b$. (Note that a distribution function is non-decreasing.) However, in our case the distribution never vanishes (i.e. never equal to 0 or 1), and we evaluate the rate of decay.

2. Statement of results

Throughout the paper, we assume $3/4 \le \sigma < 1$ to be fixed and use c, c' and c'' to denote some constants which may differ at each occurrence. The implied constants in $\ll$- or O-symbols and the unspecified positive constants c_i $(i = 1, 2, \ldots)$ may depend on σ.

Let $\sigma_a(n) = \sum_{d|n} d^a$ and $\operatorname{arsinh} x = \log(x + \sqrt{x^2+1})$. We define

$$\Sigma_1(t, X) = \sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \sum_{n \le X} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(t, n) \sin f(t, n),$$

$$\Sigma_2(t, X) = 2\left(\frac{t}{2\pi}\right)^{1/2-\sigma} \sum_{n \le B(t,\sqrt{X})} \frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}} \left(\log \frac{t}{2\pi n}\right)^{-2} \sin g(t, n),$$

where

$$e_2(t, n) = \left(1 + \frac{\pi n}{2t}\right)^{-1/4} \left(\sqrt{\frac{2t}{\pi n}}\, \operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2},$$

$$f(t, n) = 2t\, \operatorname{arsinh}\sqrt{\frac{\pi n}{2t}} + (2\pi n t + \pi^2 n^2)^{1/2} - \frac{\pi}{4},$$

$$g(t, n) = t \log \frac{t}{2\pi n} - t + \frac{\pi}{4},$$

$$B(t, \sqrt{X}) = \frac{t}{2\pi} + \frac{X}{2} - \sqrt{X}\Big(\frac{t}{2\pi} + \frac{X}{4}\Big)^{1/2} = \Big(\Big(\frac{t}{2\pi} + \frac{X}{4}\Big)^{1/2} - \frac{\sqrt{X}}{2}\Big)^2.$$

Theorem 1. *Let $\sigma \in [3/4, 1)$, $T \ge 1$ and $N \asymp T$. We have*

$$\int_1^T E_\sigma(t)\, dt = -2\pi\zeta(2\sigma - 1)T + \Sigma_1(T, N) - \Sigma_2(T, N) + O(\log^2 T).$$

The next result follows with the trivial bounds on $\Sigma_1(T, N)$ and $\Sigma_2(T, N)$.

Corollary 1. *For all $T \ge 3$, we have*

$$\int_1^T E_\sigma(t)\, dt = -2\pi\zeta(2\sigma - 1)T + O(\sqrt{T}) \qquad (3/4 < \sigma < 1),$$

and

$$\int_1^T E_{3/4}(t)\,dt = -2\pi\zeta(1/2)T + O(\sqrt{T}\log T).$$

Remark. It suggests that $E_\sigma(t)$ is a superimposition of the constant $-2\pi\zeta(2\sigma-1)$ and an oscillatory function, say, $E^*_\sigma(t)$. (Indeed this viewpoint had appeared in [18].)

Define $G_\sigma(t) = \int_1^t E^*_\sigma(t)\,dt$, which is (1.2). Then,

$$G_\sigma(t) \ll t^{1/2} \quad (3/4 < \sigma < 1) \quad \text{and} \quad G_{3/4}(t) \ll t^{1/2}\log T.$$

Integrating termwisely with partial integrations, one gets

$$\int_T^{2T} G_\sigma(t)\,dt = o(T^{1+(5/4-\sigma)}) \qquad (3/4 \le \sigma < 1). \tag{2.1}$$

In addition, we have the following higher power moments.

Theorem 2. *Let $\sigma \in [3/4, 1)$ and $T \ge 1$. We have*

(1) $\displaystyle\int_T^{2T} G_\sigma(t)^2\,dt = B(\sigma)\int_T^{2T} (t/(2\pi))^{5/2-2\sigma}\,dt + O(T^{3-2\sigma})$,

(2) $\displaystyle\int_T^{2T} G_\sigma(t)^3\,dt = -C(\sigma)\int_T^{2T} (t/(2\pi))^{15/4-3\sigma}\,dt + O(T^{(13-8\sigma)/3})$,

where $B(\sigma)$ and $C(\sigma)$ are defined by

$$B(\sigma) = \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n)^2 n^{2\sigma-7/2} = \zeta(7/2-2\sigma)\zeta(3/2+2\sigma)\zeta(5/2)^2\zeta(5)^{-1},$$

$$C(\sigma) = \frac{3}{2}\sum_{s=1}^{\infty}\frac{\mu(s)^2}{s^{21/4-3\sigma}}\sum_{a,b=1}^{\infty}\frac{\sigma_{1-2\sigma}(sa^2)}{a^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(sb^2)}{b^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(s(a+b)^2)}{(a+b)^{7/2-2\sigma}}.$$

(3) *for any real $k \in [0, A_0)$ and any odd integer $l \in [0, A_0)$ where $A_0 = (\sigma - 3/4)^{-1}$,*

$$\int_1^T |G_\sigma(t)|^k\,dt \sim \alpha_k(\sigma)T^{1+k(5/4-\sigma)}$$

and

$$\int_1^T G_\sigma(t)^l\,dt \sim \beta_l(\sigma)T^{1+l(5/4-\sigma)}.$$

for some constants $\alpha_k(\sigma) > 0$ and $\beta_l(\sigma)$ depending on σ. (A_0 denotes ∞ when $\sigma = 3/4$.)

Remark. We have no information about the value of $\beta_l(\sigma)$, which may be positive, negative or even zero. (See [16] for possible peculiar properties of series of this type.)

It is expected that $G_\sigma(t)$ is oscillatory and its order of magnitude of $G_\sigma(t)$ is about $t^{5/4-\sigma}$. (2.1) shows a big cancellation between the positive and negative parts, but Theorem 2 (2) suggests that it skews towards negative. This phenomenon also appears in the case $1/2 \le \sigma < 3/4$. Now, we look at its distribution of values from the statistical viewpoint.

Theorem 3. *For $3/4 \le \sigma < 1$, the limiting distribution $D_\sigma(u)$ of the function $t^{\sigma-5/4}G_\sigma(t)$ exists, and is equal to the distribution of the random series $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ where*

$$a_n(t) = \sqrt{2}\frac{\mu(n)^2}{n^{7/4-\sigma}}\sum_{r=1}^{\infty}(-1)^{nr}\frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}}\sin(2\pi rt - \pi/4)$$

and t_n's are independent random variables uniformly distributed on $[0,1]$. Define $\mathrm{tail}(D_\sigma(u)) = 1 - D_\sigma(u)$ for $u \ge 0$ and $D_\sigma(u)$ for $u < 0$. Then

$$\begin{gathered}\exp(-c_1\exp(|u|)) \ll \mathrm{tail}(D_{3/4}(u)) \ll \exp(-c_2\exp(|u|)),\\ \exp(-c_3|u|^{4/(4\sigma-3)}) \ll \mathrm{tail}(D_\sigma(u)) \ll \exp(-c_4|u|^{4/(4\sigma-3)})\end{gathered} \tag{2.2}$$

for $3/4 < \sigma < 1$.

Remark. $D_\sigma(u)$ is non-symmetric and skews towards the negative side because of Theorem 2 (2). Again it is true for $1/2 \le \sigma < 3/4$. But in the case $1/2 \le \sigma < 3/4$, the closure of the set $\{u \in \mathbb{R} : 0 < D_\sigma(u) < 1\}$ is compact and it differs from our case.

To investigate the oscillatory nature, we consider the extreme values of $G_\sigma(t)$ and the frequency of occurrence of large values. These are revealed in the following three results.

Theorem 4. *We have*

$$G_{3/4}(T) = \Omega_-(\sqrt{T}\log\log T) \quad \textit{and} \quad G_{3/4}(T) = \Omega_+(\sqrt{T}\log\log\log T).$$

For $3/4 < \sigma < 1$,

$$G_\sigma(T) = \Omega_-(T^{5/4-\sigma}(\log T)^{\sigma-3/4})$$

and

$$G_\sigma(T) = \Omega_+\Big(T^{5/4-\sigma}\exp\big(c_5\frac{(\log\log T)^{\sigma-3/4}}{(\log\log\log T)^{7/4-\sigma}}\big)\Big).$$

Theorem 5. *For $\sigma \in [3/4, 1)$ and for every sufficiently large T, there exist t_1, $t_2 \in [T, T+c_6\sqrt{T}]$ such that $G_\sigma(t_1) \geq c_7 t_1^{5/4-\sigma}$ and $G_\sigma(t_2) \leq -c_7 t_2^{5/4-\sigma}$. In particular, $G_\sigma(t)$ has (at least) one sign change in every interval of the form $[T, T+c_8\sqrt{T}]$.*

Theorem 6. *Let $\sigma \in [3/4, 1)$ and $\delta > 0$ be a fixed small number. Then for all sufficiently large $T \geq T_0(\delta)$, there are two sets S^+ and S^- of disjoint intervals in $[T, 2T]$ such that*

1. *every interval in $S^\pm$ is of length $c_9\delta\sqrt{T}$,*
2. *the cardinality of $S^\pm \geq c_{10}\delta^{4(1-\sigma)}\sqrt{T}$,*
3. *$\pm G_\sigma(t) \geq (c_{11} - \delta^{5/2-2\sigma})t^{5/4-\sigma}$ for all $t \in I$ with $I \in S^\pm$ respectively.*

Remark. Theorems 5 and 6 determine the order of magnitude of the gaps between sign-changes.

3. Series representation

This section is to prove Theorem 1 and we need two lemmas, which come from [2, Lemma 3] and [19, Lemma 1] with [22] respectively.

Lemma 3.1. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < 1/2$, $a < T/(8\pi k)$, $b \geq T$, $k \geq 1$, $T \geq 1$,*

$$U(t) = \left(\frac{t}{2\pi k} + \frac{1}{4}\right)^{1/2}, V(t) = 2\mathrm{arsinh}\sqrt{\frac{\pi k}{2t}},$$

$$L_k(t) = (2ki\sqrt{\pi})^{-1} t^{1/2} V(t)^{-\gamma-1} U(t)^{-1/2} \left(U(t) - \frac{1}{2}\right)^{-\alpha} \left(U(t) + \frac{1}{2}\right)^{-\beta} \times \exp\left(itV(t) + 2\pi ikU(t) - \pi ik + \frac{\pi i}{4}\right),$$

and

$$J(T) = \int_T^{2T} \int_a^b y^{-\alpha}(1+y)^{-\beta} \left(\log\frac{1+y}{y}\right)^{-\gamma} \times \exp(it\log(1+1/y) + 2\pi iky)\,dy\,dt.$$

Then uniformly for $|\alpha - 1| \geq \epsilon$, $1 \leq k \leq T+1$, we have

$$J(T) = L_k(2T) - L_k(T) + O(a^{1-\alpha}) + O(Tk^{-1}b^{\gamma-\alpha-\beta}) + O((T/k)^{(\gamma+1-\alpha-\beta)/2} T^{-1/4} k^{-5/4}).$$

In the case $-k$ in place of k, the result holds without $L_k(2T) - L_k(T)$ for the corresponding integral.

Lemma 3.2. *Let*

$$\Delta_{1-2\sigma}(t) = \textstyle\sum'_{n\le t}\sigma_{1-2\sigma}(n) - \left(\zeta(2\sigma)t + \dfrac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma} - \dfrac{1}{2}\zeta(2\sigma-1)\right)$$

where the sum $\sum'_{n\le t}$ *counts half of the last term only when* t *is an integer. Define* $\tilde{\Delta}_{1-2\sigma}(\xi) = \int_0^\xi \Delta_{1-2\sigma}(t)\,dt - \zeta(2\sigma-2)/12$. *Assuming* $3/4 \le \sigma < 1$, *we have for* $\xi \ge 1$,

$$\begin{aligned}\tilde{\Delta}_{1-2\sigma}(\xi) = C_1\xi^{5/4-\sigma}&\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\cos(4\pi\sqrt{n\xi}+\pi/4)\\ &+ C_2\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-9/4}\cos(4\pi\sqrt{n\xi}-\pi/4)\\ &+ O(\xi^{1/4-\sigma})\end{aligned}$$

where the two infinite series on the right-hand side are uniformly convergent on any finite closed subinterval in $(0,\infty)$, *and the values of the constants are* $C_1 = -1/(2\sqrt{2}\pi^2)$, $C_2 = (5-4\sigma)(7-4\sigma)/(64\sqrt{2}\pi^3)$. *In addition, we have for* $3/4 \le \sigma < 1$,

$$\Delta_{1-2\sigma}(v) \ll v^{1-\sigma}, \qquad \int_1^x \Delta_{1-2\sigma}(v)^2\,dv \ll x\log x,$$

$$\tilde{\Delta}_{1-2\sigma}(\xi) \ll \xi^r\log\xi, \qquad \int_1^x \tilde{\Delta}_{1-2\sigma}(v)^2\,dv \ll x^{7/2-2\sigma}$$

where $0 < r = -(4\sigma^2 - 7\sigma + 2)/(4\sigma - 1) \le 1/2$.

Proof of Theorem 1. From [17, (3.4)] and [20, (3.1)], we have

$$\begin{aligned}\int_{-t}^{t}|\zeta(\sigma+iu)|^2\,du = 2\zeta(2\sigma)t + 2\zeta(2\sigma-1)\Gamma(2\sigma-1)\frac{\sin(\pi\sigma)}{1-\sigma}t^{2-2\sigma}\\ - 2i\int_{\sigma-it}^{\sigma+it}g(u,2\sigma-u)\,du + O(\min(1,|t|^{-2\sigma})).\end{aligned}$$

(Note that the value of c_3 in [20, (3.1)] is zero.) Hence, we have

$$E_\sigma(t) = -i\int_{\sigma-it}^{\sigma+it}g(u,2\sigma-u)\,du + O(\min(1,t^{-2\sigma})).$$

Define

$$h(u,\xi) = 2\int_0^\infty y^{-u}(1+y)^{u-2\sigma}\cos(2\pi\xi y)\,dy. \tag{3.1}$$

Assume $AT \le X \le T$ and X is not an integer where $0 < A < 1$ is a constant. Then, following [19, p.364-365], we define

$$
\begin{aligned}
\mathrm{G}_1(t) &= \sum_{n\le X} \sigma_{1-2\sigma}(n) \int_{\sigma-it}^{\sigma+it} h(u,n)\,du,\\
\mathrm{G}_2(t) &= \Delta_{1-2\sigma}(X) \int_{\sigma-it}^{\sigma+it} h(u,X)\,du,\\
\mathrm{G}_3(t) &= \int_{\sigma-it}^{\sigma+it} \int_X^\infty (\zeta(2\sigma)+\zeta(2-2\sigma)\xi^{1-2\sigma}) h(u,\xi)\,d\xi\,du,\\
\mathrm{G}_4^*(t) &= \tilde{\Delta}_{1-2\sigma}(X) \int_{\sigma-it}^{\sigma+it} \frac{\partial h}{\partial \xi}(u,X)\,du,\\
\mathrm{G}_4^{**}(t) &= \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi) \int_{\sigma-it}^{\sigma+it} \frac{\partial^2 h}{\partial \xi^2}(u,\xi)\,du\,d\xi.
\end{aligned}
\tag{3.2}
$$

Then, we have

$$
\begin{aligned}
\int_T^{2T} E_\sigma(t)\,dt = -i\int_T^{2T} \mathrm{G}_1(t)\,dt + i\int_T^{2T} \mathrm{G}_2(t)\,dt - i\int_T^{2T} \mathrm{G}_3(t)\,dt\\
- i\int_T^{2T} \mathrm{G}_4^*(t)\,dt - i\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt + O(1)
\end{aligned}
\tag{3.3}
$$

1) Evaluation of $\int_T^{2T} \mathrm{G}_1(t)\,dt$. By Lemma 3.1 with $\gamma = 1$, $\alpha = \beta = \sigma$, we have from (3.1),

$$
\begin{aligned}
\int_T^{2T} \int_{\sigma-it}^{\sigma+it} h(u,n)\,du\,dt &= 4i \int_T^{2T} \int_0^\infty (y(1+y))^{-\sigma} (\log(1+1/y))^{-1}\\
&\qquad \times \sin(t\log((1+y)/y)) \cos(2\pi n y)\,dy\,dt\\
&= 2i\ \mathrm{Im} \int_T^{2T} \int_0^\infty (y(1+y))^{-\sigma} (\log(1+1/y))^{-1}\\
&\qquad \times \Big\{ \exp(i(t\log((1+y)/y) + 2\pi n y))\\
&\qquad + \exp(i(t\log((1+y)/y) - 2\pi n y)) \Big\}\,dy\,dt\\
&= 2i\ \mathrm{Im}\ (L_n(2T) - L_n(T)) + O(T^{3/4-\sigma} n^{\sigma-9/4})
\end{aligned}
$$

Noting that

$$
L_n(t) = (i\sqrt{2})^{-1} (t/(2\pi))^{5/4-\sigma} (-1)^n n^{\sigma-7/4} e_2(t,n) \exp(i(f(t,n)+\pi/2)),
$$

we get with (3.2) that

$$\int_T^{2T} \mathrm{G}_1(t)\,dt = \sqrt{2}i\left(\frac{t}{2\pi}\right)^{5/4-\sigma}\sum_{n\le X}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(t,n)\sin f(t,n)\Big|_T^{2T} + O(T^{3/4-\sigma}). \tag{3.4}$$

2) Evaluation of $\int_T^{2T}\mathrm{G}_2(t)\,dt$. The treatment is similar to G_1. From (3.2) and Lemma 3.1,

$$\int_T^{2T}\int_{\sigma-it}^{\sigma+it} h(u,X)\,du\,dt = 2i\ \mathrm{Im}(L_X(2T)-L_X(T)) + O(T^{3/4-\sigma}X^{\sigma-9/4}).$$

Since $L_X(t)\ll t^{5/4-\sigma}X^{\sigma-7/4}\ll T^{-1/2}$ for $t=T$ or $2T$, we have

$$\int_T^{2T}\mathrm{G}_2(t)\,dt \ll \Delta_{1-2\sigma}(X)T^{-1/2}\ll T^{1/2-\sigma}. \tag{3.5}$$

3) Evaluation of $\int_T^{2T}\mathrm{G}_3(t)\,dt$. Using [17, (4.6)], we have

$$\begin{aligned}\mathrm{G}_3(t) = {}& -2i\pi^{-1}(\zeta(2\sigma) \\ &+ \zeta(2-2\sigma)X^{1-2\sigma})\int_0^\infty y^{-\sigma-1}(1+y)^{-\sigma}(\log(1+1/y))^{-1} \\ &\times \sin(2\pi Xy)\sin(t\log(1+1/y))\,dy \\ &+ (1-2\sigma)\pi^{-1}\zeta(2-2\sigma)X^{1-2\sigma}\int_0^\infty y^{-1}(1+y)^{1-2\sigma}\sin(2\pi Xy) \\ &\times\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(\frac{1+y}{y}\right)^u du\,dy.\end{aligned}$$

Direct computation shows that for $y>0$,

$$\begin{aligned}\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}(1+1/y)^u\,du = {}& 2\pi i\left(\frac{1+y}{y}\right)^{2\sigma-1} \\ &+ \Big(\int_{-\infty+it}^{\sigma+it}+\int_{\sigma-it}^{-\infty-it}\Big)(1+1/y)^u(u+1-2\sigma)^{-1}\,du.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_T^{2T}\mathrm{G}_3(t)\,dt = {}& 2i(1-2\sigma)\zeta(2-2\sigma)TX^{1-2\sigma}I_1 \\ &- 2i\pi^{-1}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})I_2 \\ &+ \pi^{-1}(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}I_3\end{aligned} \tag{3.6}$$

where

$$I_1 = \int_0^\infty y^{-2\sigma} \sin(2\pi Xy)\, dy$$

$$I_2 = \int_T^{2T} \int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}(\log(1+1/y))^{-1} \times \sin(2\pi Xy) \sin(t\log(1+1/y))\, dy\, dt,$$

$$I_3 = \int_T^{2T} \int_0^\infty y^{-1}(1+y)^{1-2\sigma} \sin(2\pi Xy) \times \left(\int_{-\infty+it}^{\sigma+it} + \int_{\sigma-it}^{-\infty-it}\right)(1+1/y)^u(u+1-2\sigma)^{-1}\, du\, dy\, dt.$$

Then, $I_1 = 2^{2\sigma-2}\pi^{2\sigma}X^{2\sigma-1}/(\Gamma(2\sigma)\sin(\pi\sigma))$ which is the main contribution. Interchanging the integrals, we have

$$I_2 = -\int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}\big(\log(1+1/y)\big)^{-2} \times \sin(2\pi Xy)\cos(t\log(1+1/y))\Big|_{t=T}^{t=2T}\, dy.$$

We split the integral into two parts $\int_0^c + \int_c^\infty$ for some large constant $c > 0$. Expressing the product $\sin(\cdots)\cos(\cdots)$ as a combination of $\exp(i(t\log(1+1/y) \pm 2\pi Xy))$, since $(d/dy)(t\log(1+1/y) \pm 2\pi Xy) = \pm 2\pi X - t/(y(1+y)) \gg X$ for $y \geq c$ (recall $t = T$ or $2T$), the integral $\int_c^\infty$ is $\ll X^{-1}$ by the first derivative test. Applying the mean value theorem for integrals, we have

$$\int_0^c \ll \left| \int_{c'}^{c''} y^{-1-\sigma}(1+y)^{-1}\sin(2\pi Xy)\cos(t\log(1+1/y))\, dy \right|.$$

Integration by parts yields that the last integral $\int_{c'}^{c''}$ equals

$$t^{-1}\Big(y^{-\sigma}\sin(2\pi Xy)\sin(t\log(1+1/y))\Big|_{c'}^{c''} - \int_{c'}^{c''} O(y^{-\sigma-1}|\sin(2\pi Xy)| + y^{-\sigma}X)\, dy\Big) \ll 1. \tag{3.7}$$

Hence $I_2 \ll 1$. For I_3, the extra integration over t is in fact not necessary to yield our bound. Thus, we write $I_3 = \int_T^{2T}(I_{31} + I_{32})\, dt$, separated according to the integrals over u. I_{31} and I_{32} are treated in the same way, so we work out I_{31} only. Using integration by parts over u,

$$I_{31} = \int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y)) \times\left\{\frac{(1+1/y)^\alpha}{\alpha+1-2\sigma+it}\bigg|_{\alpha=-\infty}^{\alpha=\sigma} + \int_{-\infty}^{\sigma}(1+1/y)^\alpha\frac{d\alpha}{(\alpha+1-2\sigma+it)^2}\right\}dy.$$

Then we consider

$$\int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))(1+1/y)^\alpha dy.$$

Again, we split the integral into $\int_0^c + \int_c^\infty$. Then $\int_c^\infty \ll X^{-1}$. If $\alpha \le -2$, then $\int_0^c \ll 1$ trivially; otherwise, we have (see (3.7))

$$\int_0^c \ll \left|\int_{c'}^{c''} y^{-1-\alpha}(1+y)^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))\,dy\right| \ll 1.$$

Therefore, $I_{31} \ll T^{-1}$ and so $I_3 \ll 1$. Putting these estimates into (3.6), we get

$$\begin{aligned}\int_T^{2T} \mathrm{G}_3(t)\,dt &= i2^{2\sigma-1}\pi^{2\sigma}\frac{(1-2\sigma)\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\pi\sigma)}T + O(1)\\ &= -2\pi i\zeta(2\sigma-1)T + O(1).\end{aligned}\tag{3.8}$$

4) Evaluation of $\int_T^{2T}\mathrm{G}_4^*(t)\,dt$. From [19, Section 4], we obtain

$$\int_T^{2T}\mathrm{G}_4^*(t)\,dt = 4i\tilde{\Delta}_{1-2\sigma}(X)((2\sigma-1)I_1 + I_2 - \sigma I_3 - I_4)$$

where by Lemma 3.1, (recall $L_X(t) \ll T^{-1/2} \ll X^{-1/2}$ for $t = T$ or $2T$)

$$\begin{aligned}I_1 &= X^{2\sigma-2}\int_T^{2T}\int_0^\infty\frac{\cos(2\pi y)\sin(t\log(1+X/y))}{y^\sigma(X+y)^\sigma\log(1+X/y)}\,dy\,dt\\ &= X^{-1}\int_T^{2T}\int_0^\infty\frac{\cos(2\pi Xy)\sin(t\log(1+1/y))}{y^\sigma(1+y)^\sigma\log(1+1/y)}\,dy\,dt \ll X^{-3/2}\\ I_2 &= X^{2\sigma-1}\int_T^{2T}t\int_0^\infty\frac{\cos(2\pi y)\cos(t\log(1+X/y))}{y^\sigma(X+y)^{\sigma+1}\log(1+X/y)}\,dy\,dt\\ &\ll X^{-1}T\sup_{T\le T_1\le T_2\le 2T}\left|\int_{T_1}^{T_2}\int_0^\infty\frac{\cos(2\pi Xy)\cos(t\log(1+1/y))}{y^\sigma(1+y)^{\sigma+1}\log(1+1/y)}\,dy\,dt\right|\\ &\ll X^{-1/2}\end{aligned}$$

and similarly I_3, $I_4 \ll X^{-3/2}$. With Lemma 3.2,

$$\int_T^{2T}\mathrm{G}_4^*(t)\,dt \ll T^{r-1/2}\log T \ll \log T.\tag{3.9}$$

5) Evaluation of $\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt$. [19, (3.6) and Section 5] gives

$$\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt = -4iI_1 + 4iI_2 + 4iI_3. \tag{3.10}$$

I_1, I_2 and I_3 are defined as follows: write

$$w(\xi, y) = \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2}y^{-\sigma}(1+y)^{-\sigma-2}(\log(1+1/y))^{-1}\cos(2\pi\xi y), \tag{3.11}$$

then

$$\begin{aligned} I_1 &= \int_X^\infty \int_T^{2T} t^2 \int_0^\infty w(\xi, y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi \\ I_2 &= \int_X^\infty \int_T^{2T} t \int_0^\infty w(\xi, y)H_1(y)\cos(t\log(1+1/y))\,dy\,dt\,d\xi \\ I_3 &= \int_X^\infty \int_T^{2T} \int_0^\infty w(\xi, y)H_0(y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi \end{aligned}$$

where $H_0(y)$ and $H_1(y)$ are linear combinations of $y^\mu(\log(1+1/y))^{-\nu}$ with $\mu+\nu \le 2$ and $\mu+\nu \le 1$ respectively. (Remark: It is stated in [19] $\mu+\nu \le 2$ only for both $H_0(y)$ and $H_1(y)$.)

When $\xi \ge X \asymp T \asymp t$ and $\mu+\nu \le 2$, we have

$$\int_T^{2T} \int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy\,dt \ll 1, \tag{3.12}$$

$$\int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy \ll T^{-1/2}. \tag{3.13}$$

The estimate (3.13) can be seen from [19, p.368]. To see (3.12), we split the inner integral into $\int_0^c + \int_c^\infty$. First derivative test gives $\int_c^\infty \ll \xi^{-1}$. For $\int_0^c$, we integrate over t first and plainly $\int_0^c \int_T^{2T} \ll 1$.

Using (3.12) and Lemma 3.2, we have $I_3 \ll \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2}\,d\xi \ll T^{1/4-\sigma}$. Applying integration by parts to the t-integral, we find that $I_2 \ll T^{3/4-\sigma}$ with (3.12) and (3.13). (Here we have used $\mu+\nu \le 1$ for $H_1(y)$.) Since

$$\begin{aligned} \int_T^{2T} t^2\sin(t\log(1+1/y))\,dt &= -t^2(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\Big|_T^{2T} \\ &\quad + 2t(\log(1+1/y))^{-2}\sin(t\log(1+1/y))\Big|_T^{2T} \\ &\quad - 2(\log(1+1/y))^{-2}\int_T^{2T}\sin(t\log(1+1/y))\,dt, \end{aligned}$$

the last two terms contribute $T^{3/4-\sigma}$ and $T^{1/4-\sigma}$ in I_1 respectively by using (3.13) and (3.12). Substituting into (3.10), we get with [17, Lemma 3] (or

[5, Lemma 15.1]) and (3.11)

$$
\begin{aligned}
\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt
&= 4it^2 \int_X^\infty \int_0^\infty w(\xi,y)(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\,dy\,d\xi\Big|_{t=T}^{t=2T} \\
&\quad + O(T^{3/4-\sigma}) \\
&= i\pi^{-1/2}t^{5/2}\int_X^\infty \frac{\tilde{\Delta}_{1-2\sigma}(\xi)\cos(tV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3V^2U^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi\Big|_T^{2T} \\
&\quad + O(T^{3/4-\sigma})
\end{aligned}
\tag{3.14}
$$

where U and V are defined as in Lemma 3.1 with k replaced by ξ. Applying the argument in [19, Section 6] to (3.14), we get

$$
\begin{aligned}
\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt
&= -2i\left(\frac{t}{2\pi}\right)^{1/2-\sigma} \sum_{n\le B(t,\sqrt{X})} \frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}}\left(\log\frac{t}{2\pi n}\right)^{-2}\sin g(t,n)\Big|_{t=T}^{t=2T} \\
&\quad + O(\log T).
\end{aligned}
\tag{3.15}
$$

(Remark: The σ in [19, Lemma 4] should be omitted, as mentioned in [18].)

Inserting (3.4), (3.5), (3.8), (3.9), (3.15) into (3.3), we obtain

$$
\begin{aligned}
\int_T^{2T} E_\sigma(t)\,dt
&= -2\pi\zeta(2\sigma-1)T + \Sigma_1(t,X)\big|_T^{2T} - \Sigma_2(t,X)\big|_T^{2T} + O(\log T).
\end{aligned}
\tag{3.16}
$$

6) Transformation of Dirichlet Polynomial. Let $X_1,\ X_2 \asymp T$ (both are not integers) and denote $B_1 = B(T,\sqrt{X_1})$ and $B_2 = B(T,\sqrt{X_2})$. Assume $X_1 < X_2$. Write

$$
F(x) = x^{\sigma-1}\left(\log\frac{T}{2\pi x}\right)^{-2}\exp(i(T\log\frac{T}{2\pi x}+2\pi x-T+\frac{\pi}{4})),
$$

then we have

$$
\begin{aligned}
\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})} \sigma_{1-2\sigma}(n)n^{\sigma-1}(\log(T/(2\pi n)))^{-2}\sin g(T,n) \\
= \mathrm{Im}\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n).
\end{aligned}
\tag{3.17}
$$

Stieltjes integration gives

$$\begin{aligned}\sum_{B_2<n\le B_1}&\sigma_{1-2\sigma}(n)F(n)\\ &= \int_{B_2}^{B_1} F(t)(\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})\,dt + \Delta_{1-2\sigma}(t)F(t)\Big|_{B_2}^{B_1}\\ &\quad - \int_{B_2}^{B_1}\Delta_{1-2\sigma}(t)F'(t)\,dt\\ &= I_1+I_2-I_3,\ \text{say.}\end{aligned}\tag{3.18}$$

Now, since $(d/dt)(g(T,t)+2\pi t)=2\pi-T/t<-c$ when $B_2<t<B_1$, we have

$$\begin{aligned}I_1 &= \int_{B_2}^{B_1}(\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})\\ &\qquad\times t^{\sigma-1}(\log(T/(2\pi t)))^{-2}\exp(i(g(T,t)+2\pi t))\,dt\\ &\ll T^{\sigma-1}.\end{aligned}$$

By Lemma 3.2, $I_2\ll 1$. Direct computation gives

$$F'(t)=i(2\pi-\frac{T}{t})t^{\sigma-1}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(T\log\frac{T}{2\pi t}+2\pi t-T+\frac{\pi}{4}))+O(t^{\sigma-2})$$

where $B_2\le t\le B_1$. As $\int_{B_2}^{B_1}|\Delta_{1-2\sigma}(t)|t^{\sigma-2}\,dt\ll T^{\sigma-1}\sqrt{\log T}$, we have by (3.18) that

$$\begin{aligned}\sum_{B_2<n\le B_1}&\sigma_{1-2\sigma}(n)F(n)\\ &= -i\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\int_{B_2}^{B_1}\Delta_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}\\ &\qquad\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\,dt+O(1).\end{aligned}\tag{3.19}$$

The integral $\int_{B_1}^{B_2}$ in (3.19) is, after by parts,

$$\begin{aligned}&\tilde\Delta_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\Bigg|_{B_1}^{B_2}\\ &-\int_{B_2}^{B_1}\tilde\Delta_{1-2\sigma}(t)\frac{d}{dt}\left\{(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\right\}dt\end{aligned}$$

The first term is $\ll T^{\sigma-1/2}\log T$ by Lemma 3.2. Besides, computing directly shows that for $B_2 \le t \le B_1$,

$$\frac{d}{dt}\{\cdots\} = i(2\pi t - T)^2 t^{\sigma-3}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t - T\log t)) + O(t^{\sigma-2}).$$

Treating the O-term with Lemma 3.2, (3.19) becomes

$$\begin{aligned}\sum_{B_2<n\le B_1}&\sigma_{1-2\sigma}(n)F(n)\\ &= -\exp(i(T\log\frac{T}{2\pi} - T + \frac{\pi}{4}))\int_{B_2}^{B_1}\tilde{\Delta}_{1-2\sigma}(t)(2\pi t - T)^2 t^{\sigma-3}\\ &\quad\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t - T\log t))\,dt + O(T^{\sigma-1/2}\log T).\end{aligned}\tag{3.20}$$

Inserting the Voronoi-type series of $\tilde{\Delta}_{1-2\sigma}(t)$ (see Lemma 3.2) into (3.20), we get

$$\begin{aligned}\sum_{B_2<n\le B_1}&\sigma_{1-2\sigma}(n)F(n)\\ &= -\exp(i(T\log\frac{T}{2\pi} - T + \frac{\pi}{4}))\\ &\quad\times\left\{C_1\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}J_1(n) + C_2\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{9/4-\sigma}}J_2(n)\right\}\\ &\quad + O(T^{\sigma-1/2}\log T)\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}J_1(n) &= \int_{B_2}^{B_1}(2\pi t - T)^2 t^{-7/4}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(2\pi t - T\log t))\\ &\quad\times\cos(4\pi\sqrt{nt} + \frac{\pi}{4})\,dt\\ J_2(n) &= \int_{B_2}^{B_1}(2\pi t - T)^2 t^{-9/4}\Big(\log\frac{T}{2\pi t}\Big)^{-2}\exp(i(2\pi t - T\log t))\\ &\quad\times\cos(4\pi\sqrt{nt} - \frac{\pi}{4})\,dt.\end{aligned}$$

Applying the first derivative test or bounding trivially, we have $J_2(n) \ll T^{-1/4}$ for $n \le cT$, $J_2(n) \ll T^{3/4}$ for $cT < n < c'T$ and $\ll T^{1/4}n^{-1/2}$ for

$n \geq c'T$. Thus, the second sum in (3.21) is

$$\ll \left(T^{-1/4} \sum_{n\leq cT} + T^{3/4} \sum_{cT<n<c'T} \right) \sigma_{1-2\sigma}(n) n^{\sigma-9/4} + T^{1/4} \sum_{n\geq c'T} \sigma_{1-2\sigma}(n) n^{\sigma-11/4}$$

$$\ll T^{\sigma-1/2}. \tag{3.22}$$

After a change of variable $t = x^2$,

$$J_1(n) = \int_{\sqrt{B_2}}^{\sqrt{B_1}} (2\pi x^2 - T)^2 x^{-5/2} \left(\log \frac{T}{2\pi x^2}\right)^{-2} \times \Big\{ \exp\big(i(2\pi x^2 - 2T\log x + 4\pi\sqrt{n}x + \frac{\pi}{4})\big) + \exp\big(i(2\pi x^2 - 2T\log x - 4\pi\sqrt{n}x - \frac{\pi}{4})\big)\Big\}\, dx.$$

Then we use [5, Theorem 2.2], with $f(x) = x^2 - \pi^{-1}T\log x$, $\Phi(x) = x^{3/2}$, $F(x) = T$, $\mu(x) = x/2$ and $k = \pm 2\sqrt{n}$. Thus,

$$\begin{aligned} J_1(n) = {}& \delta_n 2\pi^2 \left(\frac{T}{2\pi}\right)^{3/4} e_2(T,n)\exp(i(f(T,n) - T\log\frac{T}{2\pi} + T - \pi n + \frac{3\pi}{4})) \\ &+ O(\delta_n T^{-1/4}) + O(T^{3/4}\exp(-c\sqrt{nT} - cT)) \\ &+ O(T^{3/4}\min(1, |\sqrt{X_1} \pm \sqrt{n}|^{-1})) \\ &+ O(T^{3/4}\min(1, |\sqrt{X_2} \pm \sqrt{n}|^{-1})) \end{aligned}$$

where $\delta_n = 1$ if $B_2 < x_0 < B_1$ and $k > 0$, or $\delta_n = 0$ otherwise. ($x_0 = \sqrt{T/(2\pi) + n/4} - \sqrt{n}/2$ is the saddle point.) Note that $B_2 < x_0 < B_1$ is equivalent to $X_1 < n < X_2$. Thus, for the first term in (3.21), we have

$$\begin{aligned} &-C_1 \exp\big(i(T\log\frac{T}{2\pi} - T + \frac{\pi}{4})\big) \sum_{n=1}^{\infty} \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} J_1(n) \\ &\quad = \frac{1}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{3/4} \sum_{X_1<n<X_2} \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(T,n)\exp(i(f(T,n) - \pi n + \pi)) \\ &\quad\quad + O(T^{\sigma-1/2}\log T) \end{aligned}$$

Together with (3.22), (3.21) and (3.17), we obtain

$$\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})} \sigma_{1-2\sigma}(n)n^{\sigma-1}(\log T/(2\pi n))^{-2}\sin g(T,n)$$
$$= -\frac{1}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{3/4} \sum_{X_1<n<X_2}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(T,n)\sin f(T,n)$$
$$+ O(T^{\sigma-1/2}\log T). \tag{3.23}$$

We can complete our proof now. Taking $X=[T]-1/2$ in (3.16), we have $\Sigma_i(t,X)-\Sigma_i(t,T)\ll \log T$ for $i=1,2$ and $t=T,2T$; hence

$$\int_T^{2T} E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T+\Sigma_1(t,t)\big|_T^{2T}-\Sigma_2(t,t)\big|_T^{2T}$$
$$-\Big(\big(\Sigma_1(2T,2T)-\Sigma(2T,T)\big)+\big(\Sigma_2(2T,T)-\Sigma_2(2T,2T)\big)\Big)$$
$$+O(\log T).$$

Choosing X_1 and X_2 in (3.23) to be half-integers closest to T and $2T$ respectively, then $(\Sigma_1(2T,2T)-\Sigma(2T,T))+(\Sigma_2(2T,T)-\Sigma_2(2T,2T))\ll \log T$. Hence,

$$\int_1^T E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T+\Sigma_1(T,T)-\Sigma_2(T,T)+O(\log^2 T).$$

The extra $\log T$ in the O-term comes from the number of dyadic intervals. Suppose $N\asymp T$. We apply (3.23) again with $X_1=[N]+1/2$ and $X_2=[T]+1/2$ to yield our theorem.

4. The second and third power moments

The proof of the second moment is quite standard, see [19], [21] or [5] for example. Part (1) of Theorem 2 follows from that for $N\asymp T$,

$$\int_T^{2T}\Sigma_2(t,N)^2\,dt\ll T,\qquad \int_T^{2T}\Sigma_1(t,N)\Sigma_2(t,N)\,dt\ll T\log T,$$
$$\int_T^{2T}\Sigma_1(t,N)^2\,dt = B(\sigma)\int_T^{2T}\left(\frac{t}{2\pi}\right)^{5/2-2\sigma}dt+O(T^{3-2\sigma}).$$

Moreover, one can show

Lemma 4.1. *Define* $\Sigma_{M,M'}(t)=\Sigma_{1,M'}(t)-\Sigma_{1,M}(t)$ *for* $1\le M\le M'\ll T$. *Then, we have*

$$\int_T^{2T}\Sigma_{M,M'}(t)^2\,dt\ll T^{7/2-2\sigma}M^{2\sigma-5/2}.$$

The next result is of its own interest and will be used in the proof of the third moment.

Proposition 4.1. *Let* $0 \le A < (\sigma - 3/4)^{-1}$. *Then, we have*

$$\int_T^{2T} |G_\sigma(t)|^A \, dt \ll T^{1+A(5/4-\sigma)}.$$

Proof. The case $0 \le A \le 2$ is proved by Hölder's inequality and part (1) of Theorem 2. Consider the situation $2 < A < (\sigma - 3/4)^{-1}$. Then, for $T \le t \le 2T$ and $N \asymp T$, we have $\Sigma_2(t, N) \ll T^{1/2}$ and hence $\int_T^{2T} |\Sigma_2(t,N)|^A \, dt \ll T^{A/2}$. We take $N = 2^R - 1 \asymp T$ and write $M = 2^r$. Then $\Sigma_1(t, N) \le \sum_{r \le R} |\Sigma_{M,2M}(t)|$. By Hölder's inequality, we have

$$|\Sigma_1(t, N)|^A \ll \left(\sum_{r \le R} \alpha_r^A |\Sigma_{M,2M}(t)|^A \right) \left(\sum_{r \le R} \alpha_r^{-A/(A-1)} \right)^{A-1}.$$

Taking $\alpha_r = M^{(1-A(\sigma-3/4))/(2A)}$ with the trivial bound $\Sigma_{M,2M}(t) \ll T^{5/4-\sigma} M^{\sigma-3/4}$, we have

$$\begin{aligned} &\int_T^{2T} |\Sigma_1(t, N)|^A \, dt \\ &\quad \ll_A T^{(5/4-\sigma)(A-2)} \sum_{r \le R} \alpha_r^A M^{(\sigma-3/4)(A-2)} \int_T^{2T} \Sigma_{M,2M}(t)^2 \, dt \\ &\quad \ll_A T^{1+A(5/4-\sigma)} \end{aligned} \tag{4.1}$$

by Lemma 4.1.

Proof of Theorem 2 (2). We have, with $M = [\delta T^{1/3}]$ for some small constant $\delta > 0$,

$$\begin{aligned} &\int_T^{2T} G_\sigma(t)^3 \, dt \\ &\quad = \int_T^{2T} \Sigma_{1,M}(t)^3 \, dt + O\Big(\int_T^{2T} |\Sigma_{M,T}| (G_\sigma(t)^2 + \Sigma_{1,M}^2(t)) \, dt\Big). \end{aligned} \tag{4.2}$$

Proposition 4.1 and (4.1) yields that the O-term is $O(T^{(13-8\sigma)/3})$. The integral on the right-sided of (4.2) is treated by the argument in [23]. Then the result follows.

5. Limiting distribution functions

We first quote some results from [1, Theorem 4.1] and [3, Theorem 6].

Let F be a real-valued function defined on $[1, \infty)$, and let $a_1(t)$, $a_2(t), \ldots$ be real-valued, continuous and of period 1 such that $\int_0^1 a_n(t) \, dt = 0$ and $\sum_{n=1}^\infty \int_0^1 a_n(t)^2 \, dt < \infty$. Suppose that there are positive constants γ_1,

$\gamma_2, \ldots$ which are linearly independent over $\mathbf{Q}$, such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \min(1, |F(t) - \sum_{n\le N} a_n(\gamma_n t)|)\, dt = 0.$$

Fact I. For every continuous bounded function g on $\mathbb{R}$, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(F(t))\, dt = \int_{-\infty}^{\infty} g(x)\nu(dx),$$

where $\nu(dx)$ is the distribution of the random series $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ and t_n are independent random variables uniformly distributed on $[0, 1]$. Equivalently, the distribution function of F, $P_T(u) = T^{-1}\mu\{t \in [1, T] : F(t) \le u\}$, converges weakly to a function $P(u)$, called the limiting distribution, as $T \to \infty$.

Fact II. If $\int_1^T |F(t)|^A\, dt \ll T$, then for any real $k \in [0, A)$ and integral $l \in [0, A)$, the following limits exist:

$$\lim_{T\to\infty} T^{-1}\int_1^T |F(t)|^k\, dt \text{ and } \lim_{T\to\infty} T^{-1}\int_1^T F(t)^l\, dt.$$

Now, let us take $F(t) = t^{2\sigma-5/2}G_\sigma(2\pi t^2)$, $\gamma_n = 2\sqrt{n}$ and

$$a_n(t) = \sqrt{2}\frac{\mu(n)^2}{n^{7/4-\sigma}}\sum_{r=1}^{\infty}(-1)^{nr}\frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}}\sin(2\pi rt - \pi/4). \tag{5.1}$$

Following the computation in [3, p.402] with Lemma 4.1, we get

$$\int_T^{2T}(F(t) - \sum_{n\le N} a_n(2\sqrt{n}t))^2\, dt \ll TN^{2\sigma-5/2} \qquad (N \le \sqrt{T}).$$

Then Theorem 2 (c) and the first part of Theorem 3 are immediate consequence of Facts I and II with Proposition 4.1.

We proceed to prove the lower bounds in (2.2) with the idea in [1, Section V].

Lemma 5.1. *Let n be squarefree. Define*

$$A_n = \{t \in [0, 1] : a_n(t) > B^{-1}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\}$$

where $B = 4A(\sum_{r=1}^{\infty} r^{4\sigma-7})^{-1}$ and $A = \sqrt{2}\sum_{r=1}^{\infty}\sigma_{1-2\sigma}(r^2)r^{2\sigma-7/2}$. Then, we have $\mu(A_n) \ge 1/(AB)$ where μ is the Lebesgue measure.

The proof makes use of the fact that $\int_0^1 a_n^+(t)\, dt = \int_0^1 a_n^-(t)\, dt$ where $a_n^{\pm}(t) = \max(0, \pm a_n(t))$, and

$$\int_0^1 a_n^+(t)^2\, dt + \int_0^1 a_n^-(t)^2\, dt = \frac{1}{n^{7/2-2\sigma}}\sum_{r=1}^{\infty}\frac{\sigma_{1-2\sigma}(nr^2)^2}{r^{7-4\sigma}}.$$

The readers are referred to [1] for details.

Proof of lower bounds in Theorem 3. By Markov's inequality, we have

$$\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}) \ge 1 - \frac{1}{4K}\sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt \ge \frac{3}{4}$$

where $\Pr(\#)$ denotes the probability of the event $\#$ and

$$K = \sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt < +\infty.$$

Consider the set

$$E_n = \Big\{(t_1, t_2, \dots) : t_m \in A_m \text{ for } 1 \le m \le n \text{ and } |\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}\Big\}$$

where $A_m = [0, 1]$ if m is not squarefree. Then,

$$\Pr(E_n) = \prod_{m=1}^{n}\Pr(A_m)\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}) \ge \frac{3}{4(AB)^n}$$

due to $\Pr(A_m) = \mu(A_m)$ and Lemma 5.1. When $(t_1, t_2, \dots) \in E_n$, we have

$$\begin{aligned}\sum_{m=1}^{\infty} a_m(t_m) &\ge \frac{1}{B}\sum_{\substack{m\le n\\ m \text{ squarefree}}}\frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} - 2\sqrt{K}\\ &\gg \begin{cases}\log n & \text{if } \sigma = 3/4,\\ n^{\sigma-3/4} & \text{if } 3/4 < \sigma < 1.\end{cases}\end{aligned}$$

Our result for $1 - D_\sigma(u)$ follows after we replace n by $[e^u]$ if $\sigma = 3/4$ and by $[u^{4/(4\sigma-3)}]$ if $3/4 < \sigma < 1$. The case of $D_\sigma(-u)$ can be proved in the same way.

To derive the upper estimates, we need a result on the Laplace transform of limiting distribution functions [13, Lemma 3.1].

Lemma 5.2. *Let X be a real random variable with the probability distribution $D(x)$. Suppose $D(x) > 0$ for any $x > 0$. For the two cases: (i) $\psi(x) = x\log x$ and $\phi(x) = \log x$, or (ii) $\psi(x) = x^{4/(7-4\sigma)}$ and $\phi(x) = x^{(4\sigma-3)/4}$, there exist two positive numbers L and L' such that*

(a) *if* $\limsup_{\lambda\to\infty}\psi(\lambda)^{-1}\log E(\exp(\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} x^{-1}\log(1 - D(\phi(x))) \le -L',$$

(b) *if* $\limsup_{\lambda\to\infty}\psi(\lambda)^{-1}\log E(\exp(-\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} x^{-1}\log D(-\phi(x)) \le -L'.$$

Proof of upper bounds in Theorem 3. We take $N = \lambda$ if $\sigma = 3/4$, and $N = \lambda^{4/(7-4\sigma)}$ if $3/4 < \sigma < 1$. When $n \le N$, we use

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp(\lambda A \frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}}).$$

Recall that $A = \sqrt{2}\sum_{r=1}^{\infty}\sigma_{1-2\sigma}(r^2)r^{7/2-2\sigma}$. Now consider $n > N$. If $\lambda A\sigma_{1-2\sigma}(n) < n^{7/4-\sigma}$, then by the inequality $e^x \le 1+x+x^2$ for $x \le 1$, and $\int_0^1 a_n(t)\,dt = 0$, we have

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp((\lambda A)^2 \frac{\sigma_{1-2\sigma}(n)^2\mu(n)^2}{n^{7/2-2\sigma}}).$$

Otherwise, $\lambda A\sigma_{1-2\sigma}(n) \ge n^{7/4-\sigma}$, it is obvious that

$$\begin{aligned}\int_0^1 \exp(\pm\lambda a_n(t))\,dt &\le \exp(\lambda A \frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}})\\ &\le \exp((\lambda A)^2 \frac{\sigma_{1-2\sigma}(n)^2\mu(n)^2}{n^{7/2-2\sigma}}).\end{aligned}$$

Therefore, $\log E(\exp(\pm\lambda X))$ is

$$\begin{aligned}&\le \lambda A \sum_{n\le N}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} + (\lambda A)^2\sum_{n>N}\frac{\sigma_{1-2\sigma}(n)^2}{n^{7/2-2\sigma}}\\ &\ll \begin{cases}\lambda\log\lambda & \text{if } \sigma = 3/4,\\ \lambda^{4/(7-4\sigma)} & \text{if } 3/4 < \sigma < 1.\end{cases}\end{aligned}$$

The proof is complete with Lemma 5.2.

6. $\Omega_\pm$-results

This section is devoted to prove Theorem 4. We apply the methods in [2] or [7], but beforehand, we transform $G_\sigma(t)$ into a simple finite series by convolution with the kernel

$$K(u) = 2B\left(\frac{\sin 2\pi Bu}{2\pi Bu}\right)^2.$$

Similarly to [15], we have, for $1 \ll B \ll L^{1/4} \ll T^{1/16}$,

$$t^{2\sigma-5/2}\int_{-L}^{L} G_\sigma(2\pi(t+u)^2)K(u)\,du = S_B(t) + O(B^{4\sigma-5}) \tag{6.1}$$

where

$$S_B(t) = \sqrt{2}\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(4\pi\sqrt{n}t - \frac{\pi}{4}). \tag{6.2}$$

To prove the Ω_--result, we use Dirichlet's Theorem to align the angles. More specifically, for any small $\delta > 0$, we can find $l \in [T^{1/10},$

$(1+\delta^{-B^2})T^{1/10}]$ such that $\|l\sqrt{n}\| < \delta$. Taking $B \ll \delta\sqrt{\log T}$, we have $l \in [T^{1/10}, T^{1/5}]$ and

$$S_B(l) = -\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} + O\Big(\delta\sum_{n\le B^2}(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\Big). \tag{6.3}$$

A simple calculation shows that

$$2^{2\sigma}\cdot\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} = \sum_{n\le B^2}(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}.$$

We thus infer $S_B(t) = \Omega_-(\log\log t)$ if $\sigma = 3/4$, and $\Omega_-((\log t)^{\sigma-3/4})$ if $3/4 < \sigma < 1$.

We proceed to prove the Ω_+-result with the method in [2]. Take $x = \delta\log\log T\log\log\log T$ and $B = T^{1/100}$ $(L = B^4)$ for a small number $\delta > 0$. We consider the convolution of $S_B(t)$ with a kernel involving the function

$$T_x(u) = \prod_{q\in\mathbf{Q}_x}(1+\cos(4\pi\sqrt{q}u)) = \prod_{q\in\mathbf{Q}_x}\left(1+\frac{e^{4\pi i\sqrt{q}u}+e^{-4\pi i\sqrt{q}u}}{2}\right)$$

where $\mathbf{Q}_x$ is the set of positive squarefree integers whose prime factors are odd and smaller than x. The convolution will pick out terms with the desired frequencies,

$$\epsilon\int_{-\infty}^{\infty}S_B(t+u)T_x(u)\left(\frac{\sin\epsilon\pi u}{\epsilon\pi u}\right)^2 du = \sqrt{2}\sum_{\substack{n\le B^2\\ n\in\mathbf{Q}_x}}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(4\pi\sqrt{n}t-\frac{\pi}{4}).$$

To maximize the right-hand side, we apply Dirichlet's theorem again to find a number $l \in [T^{1/10}, (1+\delta^{-|\mathbf{Q}_x|})T^{1/10}]$ so that the right-side is

$$\gg \prod_{p\in\mathbf{Q}_x}(1+\frac{\sigma_{1-2\sigma}(p)}{p^{7/4-\sigma}}) \gg \begin{cases}\log x & \text{if } \sigma = 3/4,\\ \exp(cx^{\sigma-3/4}/(\log x)) & \text{if } \sigma > 3/4.\end{cases}$$

This follows from the the estimation of $\sum_{p\le x}p^{\sigma-7/4}$ for $\sigma = 3/4$ and $\sigma > 3/4$ respectively. The cardinality of $\mathbf{Q}_x$ is $O(\exp(cx/\log x))$ for some positive constant c. Our choice of x ensures that l is of a size of a small power of T. Consequently, we obtain

$$\sup_{T^{1/10}\ll u\ll T^{1/4}} S_B(u) \gg \begin{cases} \log\log\log T & \text{if } \sigma = 3/4, \\ \exp(c(\log\log T)^{\sigma-3/4}(\log\log\log T)^{\sigma-7/4}) & \text{if } \sigma > 3/4. \end{cases}$$

7. Occurrence of large values

Proof of Theorem 5. Define $K_\tau(u) = (1-|u|)(1+\tau\sin(4\pi\alpha u))$ where $\tau = -1$ or $+1$ and α is a large constant. Following the argument in [4], we derive that

$$\int_{-1}^{1}(t+u)^{2\sigma-5/2}\int_{-L}^{L} G_\sigma(2\pi(t+u+v)^2)K(v)\,dvK_\tau(u)\,du = \frac{\tau}{2}(1-B^{-1})\cos(4\pi t-\pi/4)+O(\alpha^{-2})+O(B^{4\sigma-5}).$$

where $\delta_{1,n} = 1$ if $n = 1$ and 0 otherwise. Our assertion follows by choosing B and α ($L = B^4$) sufficiently large, and $\|4t\| \le 1/8$ with $t \in [\sqrt{T}, \sqrt{T}+1]$. (Note that τ can be $+1$ or -1 at our disposal.)

To prove Theorem 6, we need the next lemma which is the key.

Lemma 7.1. *For $T^{5/12} \le H \le T^{1/2}$,*

$$\int_{T}^{2T} \max_{0\le h\le H}(G_\sigma(t+h)-G_\sigma(t))^2\,dt \ll TH^{5-4\sigma}$$

where the implied constant depends on σ.

Proof. Following the arguments in [8], we have

$$(7.1)\quad \int_{T}^{2T}(G_\sigma(t+h)-G_\sigma(t))^2\,dt \ll Th^{5-4\sigma}\min((\sigma-3/4)^{-1}, \log(T/h^2))$$

where $\log^2 T \le h \le \sqrt{T}$. Let $b = T^{1/24}$ and $H = 2^\lambda b$. Then, as in [4], we can show

$$\max_{0\le h\le H}|G_\sigma(t+h)-G_\sigma(t)| \le \max_{1\le j\le 2^\lambda}|G_\sigma(t+jb)-G_\sigma(t)| + O(T^{2(1-\sigma)/3+\epsilon}b$$

for any fixed t. Let us take $1 \le j_0 = j_0(t) \le 2^\lambda$ such that

$$|G_\sigma(t+j_0b)-G_\sigma(t)| = \max_{1\le j\le 2^\lambda}|G_\sigma(t+jb)-G_\sigma(t)|.$$

Then we can express $j_0 = 2^\lambda\sum_{\mu\in S_t}2^{-\mu}$ for some set S_t of non-negative integers. Hence,

$$G_\sigma(t+j_0b)-G_\sigma(t) = \sum_{\mu\in S_t} G_\sigma(t+(\nu+1)2^{\lambda-\mu}b)-G_\sigma(t+\nu 2^{\lambda-\mu}b)$$

where $0 \le \nu = \nu_{t,\mu} < 2^\mu$ is an integer. By Cauchy-Schwarz's inequality and inserting the remaining ν's, we get

$$\begin{aligned}
&(G_\sigma(t+j_0b) - G_\sigma(t))^2 \\
&\le \Big(\sum_{\mu\in S_t} 2^{-(1-\sigma)\mu}\Big) \sum_{\mu\in S_t} 2^{(1-\sigma)\mu}(G_\sigma(t+(\nu+1)2^{\lambda-\mu}b) - G_\sigma(t+\nu 2^{\lambda-\mu}b))^2 \\
&\ll \sum_{\mu\in S_t}\sum_{0\le\nu<2^\mu} 2^{(1-\sigma)\mu}(G_\sigma(t+(\nu+1)2^{\lambda-\mu}b) - G_\sigma(t+\nu 2^{\lambda-\mu}b))^2
\end{aligned}$$

as $\sum_{\mu\in S_t} 2^{-(1-\sigma)\mu} \ll 1$. Integrating over $[T,2T]$ and using (7.1), we see that

$$\begin{aligned}
&\int_T^{2T} \max_{0\le h\le H} (G_\sigma(t+h) - G_\sigma(t))^2\, dt \\
&\ll \sum_{\mu\in S_t}\sum_{0\le\nu<2^\mu} 2^{(1-\sigma)\mu} \int_{T+\nu 2^{\lambda-\mu}b}^{2T+\nu 2^{\lambda-\mu}b} (G_\sigma(t+2^{\lambda-\mu}b) - G_\sigma(t))^2\, dt + T^{17/12+\epsilon} \\
&\ll TH^{5-4\sigma} \sum_{\mu\in S_t}\sum_{0\le\nu<2^\mu} 2^{-(4-3\sigma)\mu} \\
&\ll TH^{5-4\sigma}.
\end{aligned}$$

This complete the proof of Lemma 7.1.

Proof of Theorem 6. Define $G_\sigma^\pm(t) = \max(0, \pm G_\sigma(t))$. By Theorem 2 (c), we have $\int_T^{2T} |G_\sigma(t)|^3\, dt \ll T^{1+3(5/4-\sigma)}$. Hence, Cauchy-Schwarz inequality gives

$$\left(\int_T^{2T} G_\sigma(t)^2\, dt\right)^2 \le \int_T^{2T} |G_\sigma(t)|\, dt \int_T^{2T} |G_\sigma(t)|^3\, dt.$$

we have $\int_T^{2T} |G_\sigma(t)|\, dt \gg T^{1+(5/4-\sigma)}$. Together with (2.1), $\int_T^{2T} G_\sigma^\pm(t)\, dt \ge c_{12}\int_T^{2T} t^{5/4-\sigma}\, dt$.

Consider $K^\pm(t) = G_\sigma^\pm(t) - (c_{12}-\epsilon)t^{5/4-\sigma}$ where $\epsilon = \delta^{5/2-2\sigma}$, we have

$$\int_T^{2T} K^\pm(t)\, dt \ge \epsilon \int_T^{2T} t^{5/4-\sigma}\, dt$$

and $K^\pm(t+h) - K^\pm(t) = G_\sigma^\pm(t+h) - G_\sigma^\pm(t) + O(T^{1/4-\sigma}h)$. Since $|G_\sigma^\pm(t+h) - G_\sigma^\pm(t)| \le |G_\sigma(t+h) - G_\sigma(t)|$, it follows that together with Lemma 7.1,

$$\int_T^{2T} \max_{h\le H} |K^\pm(t+h) - K^\pm(t)|\, dt \ll TH^{5/2-2\sigma} + T^{5/4-\sigma}H.$$

Define $\omega^\pm(t) = K^\pm(t) - \max_{h\le H}|K^\pm(t+h) - K^\pm(t)|$. Taking $H = c'\epsilon^{2/(5-4\sigma)}\sqrt{T}$ $(= c'\delta\sqrt{T})$ for some sufficiently small constant $c' > 0$, we have

$$\int_T^{2T} \omega^{\pm}(t)\,dt \geq \epsilon \int_T^{2T} t^{5/4-\sigma}\,dt - \int_T^{2T} \max_{h\leq H} |K^{\pm}(t+h) - K^{\pm}(t)|\,dt$$
$$\gg \epsilon T^{1+(5/4-\sigma)}.$$

Let $\mathcal{I}^{\pm} = \{t \in [T, 2T] : \omega^{\pm}(t) > 0\}$. Then

$$\int_T^{2T} \omega^{\pm}(t)\,dt \leq \int_{\mathcal{I}^{\pm}} \omega^{\pm}(t)\,dt \leq \int_{\mathcal{I}^{\pm}} K^{\pm}(t)\,dt$$
$$\leq \left(\int_{\mathcal{I}^{\pm}} dt\right)^{1/2} \left(\int_T^{2T} K^{\pm}(t)^2\,dt\right)^{1/2}.$$

We infer $|\mathcal{I}^{\pm}| \gg \epsilon^2 T$ as $\int_T^{2T} K^{\pm}(t)^2\,dt \ll \int_T^{2T} G_\sigma(t)^2\,dt + T^{7/2-2\sigma}$. When $t \in \mathcal{I}^{\pm}$, we have $K^{\pm}(t) \geq \max_{h\leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \geq 0$. Hence, $K^{\pm}(u) \geq 0$ for all $u \in [t, t+H]$, i.e. $G^{\pm}_\sigma(t) \geq (c_{12} - \epsilon)t^{5/4-\sigma}$. The number of such intervals is not less than $|\mathcal{I}^{\pm}|/H \gg c_{13}\delta^{4(1-\sigma)}\sqrt{T}$.

Acknowledgement

The author would thank the referee for the valuable comments. Also, he wishes to express his sincere gratitude to Professor Kohji Matsumoto for his warm encouragement. Hearty thanks are due to Charlies Tu for unfailing supports.

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