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Nicolas GOUILLON

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Résumé. Nous donnons une minoration explicite pour les formes linéaires en deux logarithmes. Pour cela nous spécialisons la méthode de Schneider avec multiplicité décrite dans [10]. Nous améliorons substantiellement les constantes numériques intervenant dans les énoncés existants pour le cas de deux logarithmes, obtenus avec la méthode de Baker ou bien celle de Schneider avec multiplicité. Notre constante est de l'ordre de 5.10^4 au lieu de 10^8.

Abstract. We give an explicit lower bound for linear forms in two logarithms. For this we specialize the so-called Schneider method with multiplicity described in [10]. We substantially improve the numerical constants involved in existing statements for linear forms in two logarithms, obtained from Baker's method or Schneider's method with multiplicity. Our constant is around 5.10^4 instead of 10^8.

1. Introduction

Our goal is to give explicit lower bounds for linear forms in two logarithms with numerical coefficients as small as possible. To this end we intend to specify in the case of linear forms in two logarithms the most general works of [9] and [10] on linear forms in n logarithms. Our method combines the use of interpolation determinants with that of Shneider's method with multiplicity. This last method can be seen as the dual of Baker's method (see [9] for duality). Our work is situated between those of [5] which study linear forms in two logarithms with Schneider's classical method and those of [10] which use Schneider's method with multiplicity. The interest of the method used here is to provide the same type of lower bound as Baker's method [1] but with smaller numerical coefficients. We improve the existing results : for example the essential constant of Corollary 9.22 in [10] is greater than 5.10^8 in the case of two logarithms. Here we reduce this value to under 6.10^4. Let us note that the works using Baker's method furnish a constant around 10^8 in the case of two logarithms (Corollary 3 of

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[7]). The improvements obtained in this paper result from two important points. First we use a multiplicity estimate whose proof is reminiscent of the original method by D.W. Masser [6] and which appears in our case to be more efficient than the general statements previously employed. Secondly we have studied precisely the numerical constraints connecting the parameters of Theorem 2.1 below.

The plan of this paper is the following. We give our results in section 2. All of them follow from Theorem 2.1 which is our main result. The zero estimate used in the proof of main theorem is proved in section 3. The section 4 is devoted to the proof of Theorem 2.1. Finally we prove the corollaries in section 5.

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2. Statements of the results

Let α_1 and α_2 be two non zero complex algebraic numbers and let $\log \alpha_1$ and $\log \alpha_2$ be any nonzero determinations of their logarithms. Our aim is to obtain lower bounds for the absolute value of the linear form :

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2,$$

with b_1 and b_2 two nonzero relative integers.

For any algebraic number α of degree d over $\mathbb{Q}$ and whose minimal polynomial over $\mathbb{Z}$ is written as $a \prod_{i=1}^{d}(X-\alpha^{(i)})$ where the roots $\alpha^{(i)}$ are complex numbers, let us denote by

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log \max(1, |\alpha^{(i)}|)\right)$$

the usual absolute logarithmic height of α. We put

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Our main result is the following.

Theorem 2.1. *Let K and L be integers ≥ 1, let T_1, T_2, T_3 , R_1, R_2, R_3, S_1, S_2 and S_3 be integers ≥ 0. Let E be a real number $\geq e$. We set*

$$R = R_1 + R_2 + R_3, \quad S = S_1 + S_2 + S_3, \quad T = T_1 + T_2 + T_3.$$

So we put

$$N = \frac{(K+1)(K+2)}{2}(L+1), \quad B = \frac{R|b_2| + S|b_1|}{2K}$$

and denote by g, ω, ω_0 real numbers which satisfy the lower bounds :

$$g \geq \frac{1}{4} - \frac{N}{12(R+1)(S+1)(T+1)}, \quad \omega \geq 1 - \frac{N}{2(R+1)(S+1)(T+1)},$$

$$\omega_0 \geq \frac{2(R+1)(S+1)(T+1)}{N}.$$

Let a_1, a_2 *be positive real numbers so that*

$$a_i \geq E|\log \alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i), \quad i = 1, 2.$$

Suppose that

$$T_1 \geq K,$$

$$\text{Card}\{rb_2 + sb_1; 0 \leq r \leq R_1, 0 \leq s \leq S_1\} \geq K + 1,$$

$$(T_1 + 1)\text{Card}\{\alpha_1^r \alpha_2^s; 0 \leq r \leq R_1, 0 \leq s \leq S_1\} \geq L + 1,$$

$$(T_2 + 1)\text{Card}\{\alpha_1^r \alpha_2^s; 0 \leq r \leq R_2, 0 \leq s \leq S_2\} \geq 2KL + 1, \tag{1}$$

$$(T_2 + 1)\text{Card}\{rb_2 + sb_1; 0 \leq r \leq R_2, 0 \leq s \leq S_2\} \geq K^2 + 1,$$

$$(T_3 + 1)\text{Card}\{(rb_2 + sb_1, \alpha_1^r \alpha_2^s); 0 \leq r \leq R_3, 0 \leq s \leq S_3\} \geq 3K^2L + 1.$$

Suppose moreover that

$$\begin{aligned}\frac{V}{2} > D \Bigg[&\log(\frac{N}{2}) + \frac{K}{3}\log\left(\frac{Rb_2 + Sb_1}{2K}\right) + \frac{1454K}{927} + \frac{K}{3}\log\left(\frac{T}{KL}\right) \\ &+1 + (\omega T + \omega_0)\log\left(\frac{107(K+3)L}{309\omega T}\right) + 2\omega T + \omega_0\Bigg] \\ &+T\log E + \frac{K}{3}\log E + \log 2 + g\frac{L+1}{2}((R+1)a_1 + (S+1)a_2),\end{aligned} \tag{2}$$

where

$$V = \frac{1}{4}\left(1 - \frac{1}{L+1} + \sqrt{1 - \frac{2}{L+1}}\right)(K+2)(L+1)\log E.$$

Then we have

$$|\Lambda'| \geq e^{-V}, \quad \textit{with} \quad \Lambda' = \Lambda \cdot \max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

Now we give three corollaries of Theorem 2.1 in the case where α_1 and α_2 are multiplicatively independent. We put

$$b = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1},$$

with A_1, A_2 real numbers > 1 so that

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \quad (i = 1, 2).$$

Corollary 2.2. *Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -9400\left(3.317 + \frac{1.888}{D} + 0.946\log D\right) D^4 h \log A_1 \log A_2,$$

with

$$h = \max\left\{\log b + 3.1, \frac{1000}{D}, 498 + \frac{284}{D} + 142\log D\right\}.$$

Corollary 2.3. *Suppose moreover that α_1 and α_2 are real numbers > 0. Then*

$$\log|\Lambda| \geq -7200\left(3.409 + \frac{1.705}{D} + 0.946\log D\right) D^4 h \log A_1 \log A_2,$$

with

$$h = \max\left\{\log b + 3.1, \frac{1000}{D}, 512 + \frac{256}{D} + 142\log D\right\}.$$

Corollary 2.4. *Suppose that the determinations chosen for $\log \alpha_1$ and $\log \alpha_2$ are positive real numbers and are linearly independent over $\mathbb{Q}$. Put*

$$E = 1 + \min\left\{\frac{D\log A_1}{\log \alpha_1}, \frac{D\log A_2}{\log \alpha_2}\right\} \geq 2,$$

$$\log E^* = \max\left\{\frac{\log E}{D}, \frac{\log E}{D} + 0.946\log\frac{D}{\log\log E} + 3.965\right\},$$

$$h = \max\left\{\log b + \log(E) - \log\log(E) - 2.27, \frac{265\log(E)}{D}, 150\log E^*\right\}.$$

Suppose moreover that $E \leq \min\{A_1^{D/2}, A_2^{D/3}\}$, then

$$\log|\Lambda| \geq -8550 D^4 h \log A_1 \log A_2 \log(E^*)(\log E)^{-3}.$$

The choice of the above constant 1000 (respectively 265) in the definition of the parameter h in Corollaries 2.2 and 2.3 (respectively 2.4) is arbitrary. The other numerical constants in these corollaries depend on this choice. We note that the multiplicative constants are decreasing functions in the variable b. Asymptoticaly (when b tends to infinity) the multiplicative constants of Corollaries 2.2, 2.3 and 2.4 are respectively around of 8800, 6800 and 8450. If we compare our results with those obtained from Baker's method as in [7] Corollary 2, or from Schneider's method with multiplicity as in [10] Corollary 9.22, we notice that they are better. Our constants are roughly equal to the square root of corresponding constants in [7] and [10]. We can also compare Corollaries 2.2 and 2.3 (resp. 2.4) with Corollaries 1 and 2 (resp. 3) in [5]. In this case we note that the lower bounds given in Corollaries 2.2 and 2.3 (resp. 2.4) are more efficient only if we have approximatively $\log b \geq 3000 \log eD$ (resp. $\log b \geq 3200 \log E^*$).

3. Multiplicity estimate

An important point of the proof is an improvement of multiplicity estimate used. In our case we work with the product group $\mathbb{C}^m \times \mathbb{C}^\times$ whose group law is written additively. For any element w in $\mathbb{C}^\times$ and any element $(v_0, \ldots, v_{m-1})$ in $\mathbb{C}^m$ we denote briefly $(\underline{v}, w) = (v_0, \ldots, v_{m-1}, w) \in \mathbb{C}^m \times \mathbb{C}^\times$. Let $\mathfrak{D} := \frac{\partial}{\partial X_0} + Y\frac{\partial}{\partial Y}$ a derivation operating on the polynomial's ring $\mathbb{C}[\underline{X}, Y]$. Let T be an integer ≥ 0, we say that a polynomial $P \in \mathbb{C}[\underline{X}, Y]$ vanishes to order $> T$ with respect to $\mathfrak{D}$ on the set $\Sigma \subseteq \mathbb{C}^m \times \mathbb{C}^\times$, if for any integer $0 \leq t \leq T$, $\mathfrak{D}^t P$ vanishes identically on Σ. This condition meaning that $P \equiv 0$ on Σ when $t = 0$. We give here a refinement of the zero estimate of [2] by replacing in condition (2) the term $\mathrm{Card}\left(\frac{\Sigma_j}{W\times\{\mu\}}\right)$ by $\mathrm{Card}\left(\frac{\Sigma_j}{W\times\{1\}}\right)$.

Theorem 3.1. *Let K, L, m be integers ≥ 1, let $T_1, \ldots, T_{m+1}$ be integers ≥ 0 and let $\Sigma_1, \ldots, \Sigma_{m+1}$ be nonempty finite sets of $\mathbb{C}^m \times \mathbb{C}^\times$. Assume that the following conditions hold.*
(1) For all $j = 1, \ldots, m$, and any vector subspace W of $\mathbb{C}^m$ with dimension $\leq m - j$, we have

$$\binom{T_j+1}{\varepsilon_j}\mathrm{Card}\left(\frac{\Sigma_j}{W\times\mathbb{C}^\times}\right) > K^j, \quad \textit{where } \varepsilon_j = \begin{cases} 1 \textit{ if } (1,0,\ldots,0) \notin W \\ 0 \textit{ otherwise.}\end{cases}$$

(2) For all $j = 1, \ldots, m+1$, and any vector subspace W of $\mathbb{C}^m$ with dimension $\leq m + 1 - j$, we have

$$(T_j+1)\mathrm{Card}\left(\frac{\Sigma_j}{W\times\{1\}}\right) > jK^{j-1}L.$$

Then any polynomial $P \in \mathbb{C}[\underline{X}, Y]$ of total degree $\leq K$ in $\underline{X}$ and of degree $\leq L$ in Y which vanishes on $\Sigma_1 + \cdots + \Sigma_{m+1}$ to order $> T_1 + \cdots + T_{m+1}$ with respect to $\mathfrak{D}$ is identically zero.

Proof. We argue as in the proof of Theorem 1 in [2] with a refinement in case *iii*. In this case we use the inequality $\delta_{r-1,1}(V) \geq \mathrm{Card}\,(\mu)$ instead of $\delta_{r-1,1}(V) \geq 1$. Indeed since $H_V = W \times \mu$ where $\mu \subseteq \mathbb{C}^\times$ is finite we have

$$\mathrm{Card}\,(\overline{V} \cap (L_{r-1} \times \mathbb{P}^1)) \geq \mathrm{Card}\,((\tilde{V} \cap L_{r-1}) \times (\mu \cap \mathbb{P}^1)) \geq \mathrm{Card}\,(\mu).$$

Then we deduce

$$\mathrm{Card}\,(\mu)\mathrm{Card}\left(\frac{\Sigma_j}{H_V}\right)(T_j+1) \leq jK^{j-1}L.$$

Observe finally that

$$\mathrm{Card}\left(\frac{\Sigma_j}{W\times\mu}\right) \geq \mathrm{Card}\left(\frac{\Sigma_j}{W\times\{1\}}\right)/\mathrm{Card}\,(\mu),$$

we obtain the inequality

$$\mathrm{Card}\,(\mu)\frac{\mathrm{Card}\left(\frac{\Sigma_j}{W\times\{1\}}\right)}{\mathrm{Card}\,(\mu)} \leq jK^{j-1}L$$

which contradicts condition (2) of Theorem 3.1. □

4. Proof of Theorem 2.1

Without loss of generality we may assume $|\alpha_1| \geq 1$, $|\alpha_2| \geq 1$ and $b_1 \geq 1$, $b_2 \geq 1$ (see [5] for more details). The proof combines the approachs of [5] and [10] using interpolation determinants as in [4]. For this we introduce the matrix M whose coefficients are

$$\gamma^{r,s,t}_{k_0,k_1,l} = (rb_2 + sb_1)^{k_1} l^{t-k_0} \binom{t}{k_0} \alpha_1^{rl} \alpha_2^{sl},$$

where (r, s, t) with $(0 \leq t \leq T,\ 0 \leq r \leq R,\ 0 \leq s \leq S)$ is the column index, while (k_0, k_1, l) with $(k_0 + k_1 \leq K,\ 0 \leq l \leq L)$ is the row index. The sketch of proof is the usual one : we use a zero estimate to show that the matrix M has maximal rank, we take a maximal square submatrix with non-vanishing determinant Δ, we produce a lower bound for $|\Delta|$ by means of Liouville's estimate and an upper bound by means of Schwarz's Lemma, and the conclusion follows.

4.1. Rank of M. The following lemma implies that M has maximal rank.

Lemma 4.1. *Under condition* (1) *of Theorem 2.1 the matrix M has maximal rank equal to N.*

Proof. The coefficients of M are the values of the monomials

$$\mathfrak{D}^t\left(\frac{X_0^{k_0}}{k_0!} X_1^{k_1} Y^l\right)$$

evaluated at the points $(0, rb_2 + sb_1, \alpha_1^r\alpha_2^s)$, $(0 \leq r \leq R, 0 \leq s \leq S)$. So if the N rows of M are lineary dependent there exists a nonzero polynomial $P \in \mathbb{C}[X_0, X_1, Y]$ of total degree $\leq K$ in $\underline{X}$ and of degree $\leq L$ in Y which vanishes at the points $(0, rb_2 + sb_1, \alpha_1^r\alpha_2^s)$, $0 \leq r \leq R, 0 \leq s \leq S$, to order $> T$ with respect to $\mathfrak{D}$. Then Theorem 3.1 furnishes a contradiction. □

4.2. Transformation of M. To obtain the lower bound announced we must modify the matrix M. Indeed if we work directly with M we obtain an extra term $\log\log b$ in the lower bound of corollaries. To this end we define for any $z \in \mathbb{C}$ and any $n \in \mathbb{N}$ the function $\triangle$ by

$$\triangle\,(z; n) = \frac{z(z-1)\cdots(z-n+1)}{n!},$$

with for $n = 0$

$$\triangle\ (z; 0) = 1.$$

For any $a \in \mathbb{N}$ and any $b \in \mathbb{N}^*$ we define the polynomial $\delta_b(z; a) \in \mathbb{Q}[z]$ of degree a by

$$\delta_b(z; a) = \triangle\ (z; b)^q \triangle\ (z; r),$$

where by Euclidean division $a = bq + r$. For any integer $c \geq 0$ we denote furthermore

$$\delta_b(z; a, c) = \left(\frac{d}{dz}\right)^c \delta_b(z; a).$$

For any positive integer n, let us denote by $\nu(n)$ the least common multiple of $1, 2, \ldots, n$. Let now $\tilde{M}$ be the matrix with the coefficients

$$\tilde{\gamma}^{r,s,t}_{k_0,k_1,l} = \triangle\ (rb_2 + sb_1; k_1)\nu(T')^{k_0} \frac{\delta_{T'}(l; t, k_0)}{k_0!} \alpha_1^{rl} \alpha_2^{sl},$$

where T' is a parameter which will be chosen later. We deduce $\tilde{M}$ from M by linear combinations on rows and columns. The difficulty is to prove that if we replace l^{t-k_0} by $\delta_{T'}(l; t, k_0)$ we do not change the rank of M. This is achevied by

Lemma 4.2. *Let $T \in \mathbb{N}^*$ and let $\{\delta(z; t); 0 \leq t \leq T\}$ be a basis in $\mathbb{C}[z]$ of the space of the polynomials of degree $\leq T$. Let $Q \in GL_{T+1}(\mathbb{C})$ be the matrix defined by*

$$(1, z, \ldots, z^T)Q = (\delta(z; 0), \ldots, \delta(z; T)).$$

We recall furthermore

$$\delta(l; t, i) = \left(\left(\frac{d}{dz}\right)^i \delta(z; t)\right)_{z=l}.$$

Then for any $l \in \mathbb{N}$, any $k \in \mathbb{N}$ and any $0 \leq t \leq T$, we have

$$\frac{\delta(l; t, k)}{k!} = \sum_{\nu=0}^{T} q_{\nu,t} \binom{\nu}{k} l^{\nu-k}, \tag{4.1}$$

where the $q_{\nu,t}$ are the coefficients of Q.

Proof. Notice that we have for all $k \in \mathbb{N}$

$$\left(\frac{d}{dz}\right)^k (1, z, \ldots, z^T)Q = \left(\frac{d}{dz}\right)^k (\delta(z; 0), \ldots, \delta(z; T)).$$

□

Then as in paragraph 9.2.2 of [10] it is easy to prove that M and $\tilde{M}$ have the same rank.

4.3. Arithmetical lower bounds for the minors of $\tilde{M}$. To prove the main Lemma 4.6 below, we give here three technical Lemmas. First we state Lemmas 4.3 and 4.4 whose proofs are omitted (see the appendix of [3] for details).

Lemma 4.3. *Let K be an integer ≥ 1. We have the upper bound*

$$\log\left(\prod_{\substack{(k_0,k_1)\in\mathbb{N}^2\\ k_0+k_1\leq K}} \frac{1}{k_0!}\right) \leq \Big(-\frac{K}{3}\frac{(K+1)(K+2)}{2}\Big)\log(K) + \frac{11}{18}K\frac{(K+1)(K+2)}{2}.$$

Lemma 4.4. *Let N be an integer ≥ 1 and let R, S, T be integers ≥ 0 verifying $(R+1)(S+1)(T+1) \geq N$. Let $(t_1,\dots,t_N)$ be a sequence of integers between 0 and T with each value appearing at most $(R+1)(S+1)$ times. Then we have*

$$\sum_{i=1}^{N} t_i \leq NT - \frac{N^2}{2(R+1)(S+1)} + 2T(R+1)(S+1) - \frac{N}{2}.$$

Lemma 4.5. *Let T and T' be two integers so that $0 < T' < T$. Let $(t_{k_0,k_1,l})$ be a sequence of N integers between 0 and T which is indexed by the triplets (k_0,k_1,l) where $0 \leq k_0+k_1 \leq K$ and $0 \leq l \leq L$. Assume that each $t_{k_0,k_1,l}$ appears at most $(R+1)(S+1)$ times. Then we have*

$$\log\left(\prod_{l=0}^{L}\prod_{k_0+k_1\leq K}\left|\frac{1}{k_0!}\delta_{T'}(l;t_{k_0,k_1,l},k_0)\right|\right) \leq \frac{KN}{3}\log\frac{T}{KL} + \frac{11}{18}KN + (\omega T+\omega_0+T')N + (\omega T+\omega_0)N\log\frac{\max\{L,T'-1\}}{T'}$$

with ω and ω_0 defined in Theorem 2.1.

Proof. Using the estimate

$$\frac{1}{b!^q r!} \leq \frac{1}{b^a}e^{a+b},$$

we obtain for any $l\in\mathbb{N}$,

$$|\delta_b(l;a,c)| \leq c!\binom{a}{c}|(l-b+1)|^{a-c}\frac{1}{b!^q r!} \leq c!\binom{a}{c}\frac{\max\{l,b-1\}^{a-c}}{b^a}e^{a+b}.$$

Then we have

$$\left|\frac{1}{k_0!}\delta_{T'}(l;t_{k_0,k_1,l},k_0)\right| \leq \binom{t_{k_0,k_1,l}}{k_0}\frac{\max\{l,T'-1\}^{t_{k_0,k_1,l}-k_0}}{T'^{t_{k_0,k_1,l}}}e^{t_{k_0,k_1,l}+T'}. \tag{4.3}$$

Then we bound trivialy in the right hand-side of (4.3) : $\binom{t_{k_0,k_1,l}}{k_0} \leq \frac{T^{k_0}}{k_0!}$ and $l \leq L$. Thus we use Lemmas 4.3 and 4.4 to conclude. □

We give now the main lemma of this section. Let Δ be a nonzero minor of order N extracted of $\tilde{M}$. For a suitable ordering of rows and columns in Δ we can write

$$\Delta = \det\left(\triangle\ (r_jb_2+s_jb_1;k_{1,i})\frac{\nu(T')^{k_{0,i}}}{k_{0,i}!}\delta_{T'}(l_i;t_j,k_{0,i})\alpha_1^{r_jl_i}\alpha_2^{s_jl_i}\right)_{1\leq i,j\leq N}.$$

Lemma 4.6. *Put*

$$g = \frac{1}{4} - \frac{N}{12(R+1)(S+1)(T+1)},$$

$$G_1 = \frac{N(L+1)(R+1)g}{2} \quad G_2 = \frac{N(L+1)(S+1)g}{2},$$

$$M_1 = \frac{L(r_1+\cdots+r_N)}{2}, \quad M_2 = \frac{L(s_1+\cdots+s_N)}{2}.$$

Then we have the lower bound

$$\begin{aligned}\log|\Delta| \geq -(D-1)\Big[&\log(N!) + \frac{KN}{3}\log\left(\frac{Rb_2+Sb_1}{2K}\right) + \frac{22KN}{18}\\ &+ \frac{KN}{3}\log\left(\frac{T}{KL}\right) + (\omega T+\omega_0)N\log\left(\frac{\max\{L,T'-1\}}{T'}\right)\\ &+\frac{107KNT'}{309} + (\omega T+\omega_0+T')N\Big] + (M_1+G_1)\log(|\alpha_1|)\\ &+ (M_2+G_2)\log(|\alpha_2|) - 2DG_1h(\alpha_1) - 2DG_2h(\alpha_2).\end{aligned}$$

Proof. We proceed along the same lines as Lemma 6 of [5]. Consider the polynomial

$$\begin{aligned}P(X,Y) = \sum_\sigma \operatorname{sgn}(\sigma)\prod_{i=1}^{N}\triangle\ (b_2r_{\sigma(i)}+b_1s_{\sigma(i)};k_{1,i})\nu(T')^{k_{0,i}}\\ \times\frac{\delta_{T'}(l_i;t_{\sigma(i)},k_{0,i})}{k_{0,i}!}X^{l_ir_{\sigma(i)}}Y^{l_is_{\sigma(i)}},\end{aligned}$$

where σ runs over all permutations $\sigma \in \mathfrak{S}_N$ and where $\operatorname{sgn}(\sigma)$ means the signature of the permutation σ. By expanding the determinant Δ, we get

$\Delta = P(\alpha_1, \alpha_2)$. By multilinearity of determinant we can write for any $\eta \in \mathbb{C}$:

$$P(z_1, z_2) = \det\left(\frac{(b_2 r_j + b_1 s_j - \eta)^{k_{1,i}}}{k_{1,i}!}\frac{\nu(T')^{k_{0,i}}}{k_{0,i}!}\delta_{T'}(l_i; t_j, k_{0,i}) z_1^{l_i r_j} z_2^{l_i s_j}\right).$$

Choose $\eta = (Rb_2 + Sb_1)/2$. We bound $\nu(n) \le \exp(\frac{107n}{103})$ (see [11], Lemma 2.3 p. 127). Then Lemma 4.5 implies the upper bound

$$L(P) \le N! \left(\frac{Rb_2 + Sb_1}{2K}\right)^{\frac{KN}{3}} \left(\frac{T}{KL}\right)^{\frac{KN}{3}} \left(\frac{\max\{L, T'-1\}}{T'}\right)^{(\omega T + \omega_0)N}$$

$$\times \exp\left[\frac{22KN}{18} + \frac{107KNT'}{309} + (\omega T + \omega_0 + T')N\right]. \tag{4.2}$$

To get a good lower bound for $|\Delta|$ we have to notice that P is divisible by a large power of X and Y. More precisely we use the estimates

$$M_1 - G_1 \le \sum_{\nu=1}^{N} l_\nu r_\nu \le G_1 + M_1, \quad M_2 - G_2 \le \sum_{\nu=1}^{N} l_\nu s_\nu \le G_2 + M_2,$$

which follow from Lemma 4 of [5] where to obtain G_1 and M_1 we replace K by $(K+1)(K+2)/2$, L by $L+1$, R by $R+1$ and S by $(S+1)(T+1)$. Then we conclude in the same way as in the proof of Lemma 6 of [5]. □

4.4. Analytic upper bound for $|\Delta|$. As in [4] here is the crucial point where the smallness of $|\Lambda|$ is used.

Lemma 4.7. *Let E be a real number $\ge e$. Assume $|\Lambda'| \le e^{-V}$ and recall*

$$V = \frac{1}{4}\left(1 - \frac{1}{L+1} + \sqrt{1 - \frac{2}{L+1}}\right)(K+2)(L+1)\log E.$$

Then we have

$$\begin{aligned}\log|\Delta| \le\ & M_1 \log|\alpha_1| + M_2 \log|\alpha_2| + N\log 2 - \frac{NV}{2} + TN\log E \\ & + \log(N!) + \frac{KN}{3}\log\left(\frac{Rb_2 + Sb_1}{2K}\right) + \frac{22KN}{18} + \frac{KN}{3}\log E \\ & + \frac{KN}{3}\log\left(\frac{T}{KL}\right) + (\omega T + \omega_0)N\log\left(\frac{\max\{L, T'-1\}}{T'}\right) \\ & + \frac{107KNT'}{309} + (\omega T + \omega_0 + T')N + E(G_1|\log\alpha_1| + G_2|\log\alpha_2|).\end{aligned}$$

Proof. We proceed in the same way as in Lemma 6 of [4]. First we center the exponents l_i around their average value $L/2$. Next without loss of generality we may assume

$$b_1 \log\alpha_1 \le b_2 \log\alpha_2,$$

so that $\Lambda \geq 0$. Set $\beta = b_1/b_2$, then

$$\log \alpha_2 = \beta \log \alpha_1 + \frac{\Lambda}{b_2}.$$

Therefore using the above equality and (4.1) we expand Δ to obtain :

$$\Delta = \alpha_1^{M_1} \alpha_2^{M_2} \sum_{\substack{(\nu_1,\dots,\nu_N)\in\mathbb{N}^N \\ \nu_j \leq T, 1\leq j\leq N}} \prod_{j=1}^{N} q_{\nu_j,t_j} \sum_{I\subseteq\{1,\dots,N\}} (\Lambda')^{N-|I|} \Delta_{I,\underline{\nu}},$$

where $|I|$ is the cardinality of I and

$$\Delta_{I,\underline{\nu}} = \det \begin{pmatrix} c_{i,1} & \dots & c_{i,N} \\ \\ \theta_{i,1} c_{i,1} & \dots & \theta_{i,N} c_{i,N} \end{pmatrix} \begin{matrix} \} \, i \in I \\ \\ \} \, i \notin I \end{matrix}$$

with

$$c_{i,j} = \frac{b_2^{k_{1,i}}}{k_{1,i}!} (r_j + s_j \beta - \eta)^{k_{1,i}} \nu(T')^{k_{0,i}} \binom{\nu_j}{k_{0,i}} l_i^{\nu_j - k_{0,i}} \alpha_1^{\lambda_i (r_j + s_j \beta - \eta)}$$

$$\theta_{i,j} = \frac{e^{\lambda_i s_j \Lambda / b_2} - 1}{\Lambda'}, \quad \lambda_i = l_i - \frac{L}{2}.$$

We give now an upper bound for $|\Delta_I|$. Let us consider the entire function Φ_I of the complex variable z defined by

$$\Phi_I(z) = \sum_{\substack{(\nu_1,\dots,\nu_N)\in\mathbb{N}^N \\ \nu_j \leq T, 1\leq j\leq N}} \prod_{k=1}^{N} q_{\nu_j,t_j} \Phi_{I,\underline{\nu}}(z),$$

where

$$\Phi_{I,\underline{\nu}}(z) = \det \begin{pmatrix} \left(\frac{\partial}{\partial z_0}\right)^{\nu_1} \varphi_i(z\underline{\xi}_1), \dots, \left(\frac{\partial}{\partial z_0}\right)^{\nu_N} \varphi_i(z\underline{\xi}_N) \\ \\ \theta_{i,1} \left(\frac{\partial}{\partial z_0}\right)^{\nu_1} \varphi_i(z\underline{\xi}_1), \dots, \theta_{i,N} \left(\frac{\partial}{\partial z_0}\right)^{\nu_N} \varphi_i(z\underline{\xi}_N) \end{pmatrix} \begin{matrix} \} \, i \in I \\ \\ \} \, i \notin I \end{matrix}$$

with

$$\varphi_i(z_0, z_1) = \frac{b_2^{k_{1,i}}}{k_{1,i}!} z_1^{k_{1,i}} \frac{\nu(T')^{k_{0,i}}}{k_{0,i}!} z_0^{k_{0,i}} e^{z_0 l_i} \alpha_1^{\lambda_i z_1}$$

and

$$\underline{\xi}_j = (\xi_{0,j}, \xi_{1,j}) = (0, r_j + s_j \beta - \eta).$$

Notice that $\Phi_I(1) = \Delta_I$. Here is the key point of our argument.

Lemma 4.8. *For any set $I \subseteq \{1,\dots,N\}$ of cardinality $|I|$ and all N-tuples $(\nu_1,\dots,\nu_N)$, $\nu_i \le T$ $(i=1,\dots,N)$, the function $\Phi_{I,\underline{\nu}}(z)$ has a zero at the origin with multiplicity*

$$T_I \ge \frac{|I|}{2}\left(\frac{|I|+1}{K+1} - \frac{K}{2} - 1\right) - TN.$$

Proof. We can write

$$\varphi_i(z_0,z_1) = p_i(z_0,z_1)e^{l_i(z_0+z_1\log\alpha_1)}e^{-(L/2)z_1},$$

where

$$p_i(z_0,z_1) = \frac{b_2^{k_{1,i}}}{k_{1,i}}\frac{\nu(T')^{k_{0,i}}}{k_{0,i}} z_0^{k_{0,i}} z_1^{k_{1,i}},$$

is a monomial of total degree $\le K$. By multilinearity we obtain

$$\Phi_{I,\underline{\nu}}(z) = \exp\left(-\frac{Lz}{2}\sum_{j=1}^{N}\xi_{1,j}\right)\tilde{\Phi}_{I,\underline{\nu}}(z),$$

where

$$\tilde{\Phi}_{I,\underline{\nu}}(z) = \det\begin{pmatrix} \left(\frac{\partial}{\partial z_0}\right)^{\nu_1}\phi_i(z\underline{\xi}_1),\dots,\left(\frac{\partial}{\partial z_0}\right)^{\nu_N}\phi_i(z\underline{\xi}_N) \\ \theta_{i,1}\left(\frac{\partial}{\partial z_0}\right)^{\nu_1}\phi_i(z\underline{\xi}_1),\dots,\theta_{i,N}\left(\frac{\partial}{\partial z_0}\right)^{\nu_N}\phi_i(z\underline{\xi}_N)\end{pmatrix}\begin{matrix}\}\, i\in I \\ \\ \}\, i\notin I\end{matrix}$$

with

$$\phi_i = p_i(z_0,z_1)e^{l_i(z_0+z_1\log\alpha_1)}, \quad i=1,\dots,N.$$

We apply Lemma 9.14 of [10] and next Lemma 7.3 of [10] to the function $\tilde{\Phi}_{I,\underline{\mu}}(z)$ to conclude. □

Then it follows that the function $\Phi_I(z)$ has a zero at the origin with multiplicity $\ge T_I$. Hence usual Schwarz Lemma implies

$$|\Delta_I|_1 \le E^{-T_I}|\Delta_I|_E. \tag{4.4}$$

Now we give an upper bound for $|\Phi_I(z)|$.

Lemma 4.9. *For any set $I \subseteq \{1,\dots,N\}$ and any complex number z so that $|z|>1$ we have*

$$\begin{aligned}|\Phi_I(z)| \le N!\left(\frac{Rb_2+Sb_1}{2K}\right)^{\frac{KN}{3}}\left(\frac{T}{KL}\right)^{\frac{KN}{3}}\left(\frac{\max\{L,T'-1\}}{T'}\right)^{(\omega T+\omega_0)N} \\ \times\exp\left[\frac{22KN}{18} + (\omega T+\omega_0+T')N + \frac{107KNT'}{309}\right] \\ \times|z|^{\frac{KN}{3}}\exp(|z|(G_1|\log\alpha_1| + G_2|\log\alpha_2|)).\end{aligned}$$

Proof. From equality (4.1) we have for all $1 \leq j \leq N$ and all $i \in I$

$$\sum_{\nu=0}^{T} q_{\nu,t_j} \left(\frac{\partial}{\partial z_0}\right)^{\nu} \varphi_i(z\underline{\xi}_j) = \frac{b_2^{k_{1,i}}}{k_{1,i}!}(z\xi_{1,j})^{k_{1,i}} \nu(T')^{k_{0,i}} \frac{\delta_{T'}(l_i; t_j, k_{0,i})}{k_{0,i}!} \alpha_1^{\lambda_i(z\xi_{1,j})}.$$

Then since $|\theta_{i,j}| \leq 1$, for all $1 \leq i,j \leq N$, it follows that

$$|\Phi_I(z)| \leq \sum_{\sigma} sg(\sigma) \left| \prod_{i=1}^{N} \alpha_1^{z\sum \lambda_i \xi_{1,\sigma(i)}} \frac{(zb_2\xi_{1,\sigma(i)})^{k_{1,i}}}{k_{1,i}!} \frac{\nu(T')^{k_{0,i}}}{k_{0,i}!} \delta_{T'}(l_i; t_{\sigma(i)}, k_{0,i}) \right|,$$

where σ runs over all permutations $\sigma \in \mathfrak{S}_N$. Therefore

$$|\Delta_I(z)| \leq L(P) \left| \alpha_1^{z\sum \lambda_i \xi_{1,\sigma(i)}} \right| |z|^{\frac{KN}{3}}. \tag{4.5}$$

Hence the same arguments as in Lemma 8 of [4] implie that

$$\left| \alpha_1^{\sum \lambda_i \xi_{1,\sigma(i)}} \right| \leq \exp(|z|(G_1|\log \alpha_1| + G_2|\log \alpha_2|)). \tag{4.6}$$

Combining (4.5), (4.6) and (4.2) we conclude immediately. □

Now we restart from (4.4) and we combine Lemma 4.8 and Lemma 4.9 applied with $|z| \leq E$ to obtain

$$|\Delta| \leq \left|\alpha_1^{M_1}\right| \left|\alpha_2^{M_2}\right| 2^N L(P) E^{\frac{K}{3}} e^{E(G_1|\log \alpha_1| + G_2|\log \alpha_2|)} \max_I \left\{ |\Lambda'|^{N-|I|} E^{-T_I} \right\}.$$

We recall that $|\Lambda'| \leq e^{-V}$. So we maximize in function of $|I|$ the expression

$$-(N - |I|)V - T_I \log E.$$

This maximum over $\mathbb{R}$ is reached for

$$|I| = \left(\frac{V}{\log E} + \frac{K+2}{4} - \frac{1}{2(K+1)}\right)(K+1).$$

Then since

$$V = \frac{1}{4}\left(1 - \frac{1}{(L+1)} + \sqrt{1 - \frac{2}{(L+1)}}\right)(K+2)(L+1)\log E$$

we have

$$\log \max_I \{-(N - |I|)V - T_I \log E\} \leq -\frac{NV}{2} + TN \log E.$$

This finishes the proof. □

4.5. Conclusion. Choose

$$T' = \left[\frac{309\omega T}{(309+107K)}\right] + 1$$

and observe that $N! \le \left(\frac{N}{2}\right)^N$ for $N \ge 6$ and that

$$\log\left(\frac{L}{T'}\right) \le \log\left(\frac{107(K+3)L}{309\omega T}\right).$$

Then from Lemma 4.6 and 4.7 we obtain

$$\begin{aligned}\frac{V}{2} \le D\Bigg[&\log(\frac{N}{2}) + \frac{K}{3}\log\left(\frac{Rb_2+Sb_1}{2K}\right) + \frac{22K}{18} + \frac{107K}{309} + 1\\ &+\frac{K}{3}\log\left(\frac{T}{KL}\right) + (\omega T+\omega_0)\log\left(\frac{107(K+3)L}{309\omega T}\right) + 2\omega T + \omega_0\Bigg]\\ &+T\log E + \frac{K}{3}\log E + \log 2 + g\frac{L+1}{2}((R+1)a_1+(S+1)a_2).\end{aligned}$$

This inequality contradicts hypothesis (2) of Theorem 2.1 therefore $|\Lambda'| \ge e^{-V}$.

5. Proof of corollaries

In this section we assume that α_1 and α_2 are multiplicatively independent. Our goal is to give a lower bound for $|\Lambda|$. To this end we specialize, as in [5], a part of the parameters involved in Theorem 2.1. We recall the notations $\log E = \lambda$ and $N = \frac{(K+2)(K+1)(L+1)}{2}$. Fix

$$g = 0.241 \quad , \quad \omega = 0.946, \quad \omega_0 = 20 \quad \text{and} \quad \gamma = 1.309.$$

5.1. Choices of parameters. Let c_0 and c_1 be positive real numbers which will be specified later. Set :

$$K = \left[c_0 a_1 a_2 D \log E^* \lambda^{-3}\right], \tag{5.1}$$

$$L = \left[c_1 D h \lambda^{-1}\right], \tag{5.2}$$

$$R_1 = \left[\sqrt{K+1}\sqrt{\frac{a_2}{a_1}}\right], \tag{5.3a}$$

$$R_2 = \left[\max\left\{\frac{2^{1/3}}{(K+1)^{1/3}}, \frac{1}{(L+1)^{1/3}}\right\}(K+1)^{2/3}\frac{(2\gamma D\log E^*)^{1/3}3^{1/3}}{g^{1/3}(a_1a_2)^{1/6}}\sqrt{\frac{a_2}{a_1}}\right], \tag{5.3b}$$

$$R_3 = \left[(K+1)^{2/3}\frac{(2\gamma D\log E^*)^{1/3}3^{1/3}}{g^{1/3}(a_1a_2)^{1/6}}\sqrt{\frac{a_2}{a_1}}\right], \tag{5.3c}$$

$$S_1 = \left[\sqrt{K+1}\sqrt{\frac{a_1}{a_2}}\right], \tag{5.4a}$$

$$S_2=\left[\max\left\{\frac{2^{1/3}}{(K+1)^{1/3}},\frac{1}{(L+1)^{1/3}}\right\}(K+1)^{2/3}\frac{(2\gamma D\log E^*)^{1/3}3^{1/3}}{g^{1/3}(a_1a_2)^{1/6}}\sqrt{\frac{a_1}{a_2}}\right], \tag{5.4b}$$

$$S_3=\left[(K+1)^{2/3}\frac{(2\gamma D\log E^*)^{1/3}3^{1/3}}{g^{1/3}(a_1a_2)^{1/6}}\sqrt{\frac{a_1}{a_2}}\right], \tag{5.4c}$$

$$T_1=\max\left\{\left[\frac{L+1}{K+1}\right],K\right\}, \tag{5.5a}$$

$$T_2=\left[\frac{g^{2/3}(L+1)(a_1a_2)^{1/3}(K+1)^{2/3}}{(2\gamma D\log E^*)^{2/3}}\max\left\{\frac{2^{1/3}}{(K+1)^{1/3}},\frac{1}{(L+1)^{1/3}}\right\}\right], \tag{5.5b}$$

$$T_3=\left[\frac{g^{2/3}(L+1)(a_1a_2)^{1/3}3^{1/3}(K+1)^{2/3}}{(2\gamma D\log E^*)^{2/3}}\right]. \tag{5.5c}$$

We give now some conditions on the parameters c_0, c_1, λ, a_1, a_2, h and E^*. We need these to obtain Corollaries 2.2, 2.3 et 2.4 :

$$\lambda\geq 1, \tag{5.6}$$

$$a_i\geq\max\{3,3\lambda,E|\log\alpha_i|-\log|\alpha_i|+2Dh(\alpha_i),\qquad (i=1,2). \tag{5.7}$$

$$\log E^*\geq\max\left\{\frac{\lambda}{D},\frac{\lambda}{D}+2.101-\omega\log\lambda+\frac{\omega}{3}\log c_0+\omega\log D\right\}, \tag{5.8}$$

$$h\geq\max\left\{4,\frac{265\lambda}{D},150\log E^*\right\}, \tag{5.9}$$

$$h\geq\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log(\lambda)+\frac{\lambda}{D}+2.72, \tag{5.10}$$

$$5000\geq c_0\geq 300, \tag{5.11}$$

$$c_1\geq 5.1. \tag{5.12}$$

We deduce from (5.7), (5.8), (5.9), (5.11) and (5.12) that :

$$K+1\geq 2700\quad\text{ and }\quad L+1\geq 1350. \tag{5.13}$$

From (5.8) and (5.11) we obtain

$$\log E^*\geq\frac{\lambda}{D}+3.898-\omega\log\lambda+\omega\log D\geq 4.84. \tag{5.14}$$

These choices enable us to show that the values chosen for g, ω, ω_0 in this section are upper bound for respectively $\frac{1}{4}-\frac{N}{12(R+1)(S+1)(T+1)}$, $1-\frac{N}{2(R+1)(S+1)(T+1)}$ and $\frac{2(R+1)(S+1)(T+1)}{N}$. The computations are tedious but elementary (see [3] for details).

Remark. *The significant hypotheses are* (5.8) *and* (5.10).

Remark. *The numerical constants contained in the above conditions implie the three corollaries and can be changed to other applications.*

5.2. Lower bound for $|\Lambda|$. We assume here that the numbers $rb_2 + sb_1$ $(0 \le r \le R, 0 \le s \le S)$ are pairwise distinct. If this last condition is not satisfied, Liouville's inequality furnishes a much better lower bound for $|\Lambda|$ than the one which is required.

5.2.1. *Study of condition* (1) *of Theorem 2.1.* Since all the numbers $rb_1 + sb_2$, $(0 \le r \le R, 0 \le s \le S)$ are pairwise distinct, condition (1) of Theorem 2.1 can be written

$$T_1 \ge K, \tag{5.15a}$$

$$(R_1+1)(S_1+1) \ge \max\left\{K+1, \frac{L+1}{T_1+1}\right\} = K+1, \tag{5.15b}$$

$$(R_2+1)(S_2+1) \ge \max\left\{\frac{K^2+1}{T_2+1}, \frac{2KL+1}{T_2+1}\right\}, \tag{5.15c}$$

$$(R_3+1)(S_3+1) \ge \frac{3K^2L+1}{T_3+1}, \qquad (5.15d).$$

This inequality are clearly verified with our above choices.

5.2.2. *Study of condition (2).* In this section we show, under conditions (5.6) – (5.17) that condition (2) of Theorem 2.1 is implied by condition (2′) given at the end of paragraph. To prove this we use the following Lemmas 5.1 and 5.2. Put

$$B = \frac{Rb_2 + Sb_1}{2K}.$$

Lemma 5.1. *Under the hypotheses* (5.7) – (5.13) *we have*

$$\log B + \frac{1454}{309} + \frac{\lambda}{D} + \frac{6}{K}\log\left(\frac{K+2}{2}\right) + \log\left(\frac{T}{KL}\right) \le h.$$

Proof. First we deduce from (5.1) and (5.3) that

$$\begin{aligned}\frac{R}{K} &= \frac{R}{K+1}(1+\frac{1}{K})\\ &\le \frac{\lambda}{a_1}(1+\frac{1}{K})\left(\frac{1}{\sqrt{c_0 D \log E^*\lambda^{-1}}} + \frac{(2\gamma)^{1/3}}{c_0^{1/3} g^{1/3}}\left(3^{1/3} + \frac{1}{\Gamma^{1/3}}\right)\right),\end{aligned}$$

where

$$\Gamma = \min\left\{\frac{K+1}{2}, L+1\right\}.$$

Then (5.8), (5.11) and (5.13) implie :

$$\frac{R}{K} \le 0.566\frac{\lambda}{a_1}, \quad \text{et} \quad \frac{S}{K} \le 0.566\frac{\lambda}{a_2},$$

hence that

$$\log B \le \log\left(\frac{b_2}{a_1} + \frac{b_1}{a_2}\right) + \log\lambda - 1.26. \tag{5.16}$$

Now from (5.5), (5.6), (5.11) and (5.13) we obtain the upper bound

$$\frac{T}{KL} \le 0.475. \tag{5.17}$$

Then combining (5.16), (5.17) and (5.13) we deduce our claim.

□

Lemma 5.2. *Under the same hypotheses* (5.7) – (5.13) *we have the upper bound*

$$T\left(\lambda + D\omega\left(2 + \log\frac{107(K+3)L}{309\omega T}\right)\right) + \frac{g(L+1)}{2}((R+1)a_1 + (S+1)a_2) \le \Phi + \gamma D \log E^* \frac{L+1}{K+1}$$

where

$$\Phi = D\gamma\log E^* K + g(L+1)\sqrt{(K+1)a_1a_2} + \frac{g(L+1)(a_1+a_2)}{2} + \frac{3g^{2/3}(\gamma a_1 a_2 D\log E^*)^{1/3}}{2^{2/3}}(L+1)(K+1)^{2/3}\left(3^{1/3} + \frac{1}{\Gamma^{1/3}}\right).$$

Proof. First we notice that we have by (5.5), (5.11) and (5.13)

$$\frac{(K+3)L}{T} \le \frac{K+3}{K+1}\frac{(K+1)(L+1)}{T} \le 3.404 c_0^{1/3}\frac{D\log E^*}{\lambda}.$$

Therefore combining this upper bound and (5.8), (5.14) we deduce

$$\frac{\lambda}{D} + 2\omega + \omega\log\left(\frac{107(K+3)L}{309\omega T}\right) \le \left(1 + \omega\frac{\log\log E^*}{\log E^*}\right)\log E^* \le \gamma\log E^*.$$

Hence Lemma 5.2 follows easily from the estimate

$$DT\gamma\log E^* + \frac{g(L+1)}{2}((R+1)a_1 + (S+1)a_2) \le \Phi + \gamma D\log E^*\frac{L+1}{K+1}$$

which is obtained by (5.3) – (5.5) (see [3] for details). □

Let us denote

$$\theta = \frac{1}{8}\left(1 - \frac{1}{L+1} + \sqrt{1 - \frac{2}{L+1}}\right),$$

so that

$$\frac{V}{2} = \theta(K+2)(L+1)\lambda.$$

Notice that $\log(N/2) \leq 2\log((K+2)/2) + \log(L+1)$, then Lemmas 5.1 and 5.2 show that condition (2) of Theorem 2.1 is implied by

$$\theta(K+1)(L+1)\lambda \geq \frac{D(K+1)h}{3} + \Phi + \Omega, \tag{5.18}$$

where

$$\Omega = -\theta(L+1)\lambda - \frac{Dh}{3} + D\omega_0\left(1 + \log\frac{107(K+3)L}{309\omega T}\right) + D\log(L+1) + D\gamma\log E^*\frac{L+1}{K+1} + \log 2 + D.$$

First we give an upper bound for Ω. Put

$$\Omega_1 = -\theta(L+1)\lambda + \log 2 + D + \frac{\gamma(L+1)\lambda}{2700} + D\log(c_1h) + 8\cdot 10^{-4}D,$$
$$\Omega_2 = -\frac{Dh}{3} + D\omega_0\left(1 + \log\frac{107(K+3)L}{309\omega T}\right) + D\log D.$$

It is easily seen that $\Omega \leq \Omega_1 + \Omega_2$. Remark $\theta \geq 0.249$ we easily verify that Ω_1 is a decreasing function in the variable c_1h. Then we deduce $\Omega_1 < 0$ from (5.9) and (5.12). To bound Ω_2 we proceed as in Lemma 5.2 and we obtain

$$\Omega_2 \leq -\frac{Dh}{3} + \frac{D(\omega_0+1)}{\omega}(\log E^* + \log\log E^*).$$

Therefore $\Omega_2 \leq 0$ by (5.9). Now we establish the inequality

$$\theta(K+1)(L+1)\lambda \geq \frac{DKh}{3} + \Phi.$$

Since $L+1 \geq c_1Dh\lambda^{-1}$ it suffices to prove that

$$\theta - \frac{1}{3c_1} - \frac{\Phi}{(K+1)(L+1)\lambda} \geq 0. \tag{5.19}$$

By the definition of Φ we have

$$\frac{\Phi}{(K+1)(L+1)\lambda} \leq \frac{\gamma\log E^*}{c_1h} + \frac{3\gamma^{1/3}g^{2/3}}{2^{2/3}c_0^{1/3}}\left(\frac{1}{\tilde{\Gamma}^{1/3}} + 3^{1/3}\right) + \frac{g}{\sqrt{c_0D\log E^*\lambda^{-1}}} + \frac{g\lambda^2}{c_0\min\{a_1,a_2\}D\log E^*},$$

where

$$\tilde{\Gamma} = \min\{c_1Dh\lambda^{-1}, c_0a_1a_2D\log E^*\lambda^{-3}/2\}.$$

Moreover by (5.2) and (5.12) we have

$$\theta \geq \frac{1}{8}\left(1 - \frac{1}{5.1Dh\lambda^{-1}} + \sqrt{1 - \frac{2}{5.1Dh\lambda^{-1}}}\right).$$

Therefore we deduce that (5.19) holds if

$$
\begin{aligned}
\frac{1}{3c_1} + \frac{\gamma \log E^*}{hc_1} \leq \frac{1}{8} &\left(1 - \frac{1}{5.1Dh\lambda^{-1}} + \sqrt{1 - \frac{2}{5.1Dh\lambda^{-1}}}\right) \\
&- \frac{g}{\sqrt{c_0(D\log E^*)\lambda^{-1}}} - \frac{g}{c_0 \min\{a_1, a_2\} D \log E^* \lambda^{-2}} \\
&- \frac{3\gamma^{1/3} g^{2/3}}{2^{2/3} c_0^{1/3}} \left(\frac{1}{\tilde{\Gamma}^{1/3}} + 3^{1/3}\right). \qquad (2')
\end{aligned}
$$

In conclusion we have shown that ($2'$) implies (2). Hence to apply Theorem 2.1 in the proof of corollaries we shall verify ($2'$) instead of (2).

5.3. Obtention of numerical values in corollaries. First notice that if conditions (1) and (2) of Theorem 2.1 are verified the conclusion of this theorem implies that

$$\log |\Lambda'| \geq -2\theta(K+2)(L+1)\lambda.$$

Then in the same way as in [5] we obtain

$$\log |\Lambda| \geq -2\theta(1 + 2.10^{-5})(K+2)(L+1)\lambda. \qquad (5.20)$$

Therefore we have (5.20) if ($2'$) and (5.7) – (5.13) are satisfied. Now we continue the specialization of the parameters to obtain the statements of Corollaries 2.2, 2.3 and 2.4. These corollaries are deduced from the lower bound

$$\log |\Lambda| \geq -\frac{1}{2}(1 + \frac{2}{K})(1 + \frac{1}{L})(1 + 2.10^{-5}) c_0 c_1 D^2 h a_1 a_2 \log E^* \lambda^{-3}, \quad (5.23)$$

which follows from (5.22), (5.1) and (5.2).

For Corollary 2.2 we put

$$E = 6.6, \quad c_0 = 317, \quad c_1 = 5.378, \quad a_i = (E+2)D \log A_i, \quad (i = 1, 2),$$

$$\log E^* = 3.317 + \frac{1.888}{D} + 0.946 \log D,$$

$$h = \max\left\{\log b + 3.1, \frac{1000}{D}, 498 + \frac{284}{D} + 142 \log D\right\}.$$

Then (5.8) and (5.9) hold. Next we check (5.7) and (5.10) which follow respectively from

$$
\begin{aligned}
E|\log \alpha_i| - \log |\alpha_i| + 2Dh(\alpha_i) &\leq E|\log \alpha_i| + 2Dh(\alpha_i) \\
&\leq (E+2)D \log A_i, \quad (i = 1, 2)
\end{aligned}
$$

and

$$\log\left(\frac{b_2}{a_1} + \frac{b_1}{a_2}\right) = \log b - \log(E+2).$$

Finally we must prove that $(2')$ holds. To this end we remark that $a_i \geq 8.6$, $(i = 1, 2)$, and that $\log E^* \geq 4.84$ by (5.14). Moreover we have $\frac{\log E^*}{h} \leq \frac{1}{150}$ by (5.9) and $\frac{1}{Dh} \leq \frac{1}{1000}$ by our choice of h. Then replacing the parameters c_0, c_1 and E by their numerical above choices we get $(2')$.

The proof of Corollary 2.3 is similar. Without loss of generality we may assume that $\log \alpha_1$ and $\log \alpha_2$ are two real positive numbers . Then we choose

$$E = 5.5, \quad c_0 = 313, \quad c_1 = 5.386, \quad a_i = (E+1)D \log A_i, \quad (i = 1, 2),$$

$$\log E^* = 3.409 + \frac{1.705}{D} + 0.946 \log D,$$

$$h = \max \left\{ \log b + 3.1, \frac{1000}{D}, 512 + \frac{256}{D} + 142 \log D \right\}.$$

In this case (5.7) and (5.10) follow respectively from

$$\begin{aligned} E|\log \alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i) &\leq (E-1)|\log \alpha_i| + 2Dh(\alpha_i) \\ &\leq (E+1)D \log A_i, \quad (i = 1, 2) \end{aligned}$$

and

$$\log \left(\frac{b_2}{a_1} + \frac{b_1}{a_2} \right) = \log b - \log(E+1).$$

The condition $(2')$ is shown as in Corollary 2.2.

For Corollary 2.4, we choose

$$c_0 = 368, \quad c_1 = 5.141, \quad a_i = 3D \log A_i, \quad (i = 1, 2),$$

$$E = 1 + \min \left\{ \frac{D \log A_1}{\log \alpha_1}, \frac{D \log A_2}{\log \alpha_2} \right\},$$

$$\log E^* = \max \left\{ \frac{\lambda}{D}, \frac{\lambda}{D} + 0.946 \log \frac{D}{\log \lambda} + 3.965 \right\},$$

$$h = \max \left\{ \log b + \log \lambda + \frac{\lambda}{D} + 1.622, 150 \log E^*, \frac{500\lambda}{D} \right\}.$$

With these choices it is obvious to prove (5.8) – (5.10). To etablish (5.7) we use the following upper bound :

$$\begin{aligned} (E-1) \log \alpha_i + 2Dh(\alpha_i) &= \min \left\{ \frac{D \log A_1}{\log \alpha_1}, \frac{D \log A_2}{\log \alpha_2} \right\} \log \alpha_i + 2Dh(\alpha_i) \\ &\leq 3D \log A_i = a_i. \end{aligned}$$

Finally $(2')$ is proved in the same way as Corollaries 2.2 and 2.3 with $a_i \geq 3\lambda$.

Remark. *For D fixed, we can refine our computations to improve the statements of corollaries.*

5.4. Numerical appendix. This appendix containes numerical tables which complete Corollaries 2.2, 2.3 and 2.4.

TABLE 1

h_1	600	800	1500	2000	2500	3000
E	6.55	6.55	6.6	6.6	6.6	6.6
c_0	325	320	311	308	306	304
c_1	5.378	5.381	5.387	5.381	5.374	5.368
C_1	9650	9490	9230	9130	9050	9000

This first table refers to Corollary 2.2. We give different choices for the constant h_1 involved in the definition of h :

$$h = \max\left\{\log b + 3.4, \frac{h_1}{D}, \frac{h_1}{4}\log E^*\right\}.$$

For any value $600 \leq h_1 \leq 3000$ we have the upper bound

$$\log E^* \leq 3.328 + \frac{1.888}{D} + 0.946\log D.$$

For the values of table 1 the lower bound of Corollary 2.2 is

$$\log|\Lambda| \geq -C_1\left(3.328 + \frac{1.888}{D} + 0.946\log D\right) D^4 h \log A_1 \log A_2.$$

We remark that for $h_1 \geq 3000$ the constant C_1 closed to 9000. This follows from the fact that in this case $L + 1 \geq \frac{K+1}{2}$.

TABLE 2

h_2	600	800	1500	1750	2000	2100
E	5.5	5.5	5.55	5.55	5.55	5.55
c_0	321	316	308	306	305	304
c_1	5.383	5.389	5.382	5.389	5.381	5.389
C_2	7380	7270	7080	7030	7000	6990

This second table refers to Corollary 2.3. As above we give different choices for the constant h_2 which is in the definition of h :

$$h = \max\left\{\log b + 3.4, \frac{h_2}{D}, \frac{h_2}{4}\ logE^*\right\}.$$

We have in any case the upper bound

$$\log E^* \leq 3.417 + \frac{1.714}{D} + 0.946\log D.$$

The lower bound in Corollary 2.3 is here :

$$\log|\Lambda| \geq -C_2\left(3.417 + \frac{1.714}{D} + 0.946\log D\right) D^4 h \log A_1 \log A_2.$$

For the same reasons as before we only consider the value of h_2 up to 2100.

In the case of corollary 2.4 the variations of the main constant are too small to construct an interesting table.

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Nicolas GOUILLON
Institut de Mathématiques de Luminy
163, Avenue de Luminy, case 907
13288 Marseille Cedex 9, France
E-mail: gouillon@iml.univ-mrs.fr