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Elio JOSEPH

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Upper bounds and spectrum for approximation exponents for subspaces of $\mathbb{R}^n$

par Elio JOSEPH

Résumé. Cet article reprend l'idée énoncée par W. M. Schmidt en 1967, généralisant l'approximation diophantienne classique à des sous-espaces vectoriels de $\mathbb{R}^n$. Étant donné deux sous-espaces vectoriels de $\mathbb{R}^n$, A et B de dimensions respectives d et e, avec $d+e \leqslant n$, la proximité entre A et B est mesurée par $t = \min(d,e)$ angles canoniques $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_t \leqslant \pi/2$; on pose $\psi_j(A,B) = \sin\theta_j$. Si B est un sous-espace vectoriel rationnel, sa complexité est mesurée par sa hauteur $H(B) = \operatorname{covol}(B \cap \mathbb{Z}^n)$. On note $\mu_n(A|e)_j$ l'exposant d'approximation défini comme la borne supérieure (éventuellement égale à $+\infty$) de l'ensemble des $\beta > 0$ tels que l'inégalité $\psi_j(A,B) \leqslant H(B)^{-\beta}$ soit vérifiée pour une infinité de sous-espaces vectoriels rationnels B de dimension e. On s'intéresse à la valeur minimale $\mathring{\mu}_n(d|e)_j$ que prend $\mu_n(A|e)_j$ quand A décrit l'ensemble des sous-espaces vectoriels de dimension d de $\mathbb{R}^n$ tels pour tout sous-espace vectoriel rationnel B de dimension e, on ait $\dim(A\cap B) < j$. On montre que si A est inclus dans un sous-espace vectoriel rationnel F de dimension k, son exposant dans $\mathbb{R}^n$ est le même que son exposant dans $\mathbb{R}^k$ via un isomorphisme rationnel $F \to \mathbb{R}^k$. Ceci permet de déduire de nouvelles majorations de $\mathring{\mu}_n(d|e)_j$. On étudie aussi les valeurs prises par $\mu_n(A|e)_e$ quand A est un sous-espace vectoriel de $\mathbb{R}^n$ vérifiant $\dim(A \cap B) < e$ pour tout sous-espace rationnel B de dimension e.

Abstract. This paper uses W. M. Schmidt's idea formulated in 1967 to generalise the classical theory of Diophantine approximation to subspaces of $\mathbb{R}^n$. Given two subspaces of $\mathbb{R}^n$, A and B of respective dimensions d and e, with $d+e \leqslant n$, the proximity between A and B is measured by $t = \min(d,e)$ canonical angles $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_t \leqslant \pi/2$; we set $\psi_j(A,B) = \sin\theta_j$. If B is a rational subspace, its complexity is measured by its height $H(B) = \operatorname{covol}(B \cap \mathbb{Z}^n)$. We denote by $\mu_n(A|e)_j$ the exponent of approximation defined as the upper bound (possibly equal to $+\infty$) of the set of $\beta > 0$ such that for infinitely many rational subspaces B of dimension e, the inequality $\psi_j(A,B) \leqslant H(B)^{-\beta}$ holds. We are interested in the minimal value $\mathring{\mu}_n(d|e)_j$ taken by $\mu_n(A|e)_j$ when A ranges through the set of subspaces of dimension d of $\mathbb{R}^n$ such that for all rational subspaces B of dimension e one has $\dim(A\cap B) < j$. We show that if A is included in a rational subspace F of dimension k, its exponent in $\mathbb{R}^n$ is the same as its exponent in $\mathbb{R}^k$ via a rational isomorphism $F \to \mathbb{R}^k$. This

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allows us to deduce new upper bounds for $\mathring{\mu}_n(d|e)_j$. We also study the values taken by $\mu_n(A|e)_e$ when A is a subspace of $\mathbb{R}^n$ satisfying $\dim(A \cap B) < e$ for all rational subspaces B of dimension e.

1. Introduction

Diophantine approximation in its classical sense studies how well points of $\mathbb{R}^n$ can be approximated by rational points. We will focus here on a different but related problem, stated by W. M. Schmidt in 1967 (see [12]), which studies the approximation of subspaces of $\mathbb{R}^n$ by rational subspaces. The results exposed here can be found with extended details in my Ph.D. thesis (see [5, Chapters 5 and 6]).

Let us say that a subspace of $\mathbb{R}^n$ is *rational* whenever it admits a basis of vectors of $\mathbb{Q}^n$; let us denote by $\mathfrak{R}_n(e)$ the set of rational subspaces of dimension e of $\mathbb{R}^n$. A subspace A of $\mathbb{R}^n$ is said to be (e,j)*-irrational* whenever for all $B \in \mathfrak{R}_n(e)$, $\dim(A \cap B) < j$; let us denote by $\mathfrak{I}_n(d,e)_j$ the set of all (e,j)-irrational subspaces of dimension d of $\mathbb{R}^n$. Notice that $A \in \mathfrak{I}_n(d,e)_1$ if, and only if, for any $B \in \mathfrak{R}_n(e)$: $A \cap B = \{0\}$.

In order to formulate the problems we will consider, we need a notion of *complexity* for a rational subspace and a notion of *proximity* between two subspaces of $\mathbb{R}^n$.

Let $B \in \mathfrak{R}_n(e)$ and $\Xi = (\xi_1, \ldots, \xi_N) \in \mathbb{Z}^N$, with $N = \binom{n}{e}$, be a vector in the class of Plücker coordinates of B. Let us define the *height* of B to be:

$$H(B) = \|\Xi\| / \gcd(\xi_1, \ldots, \xi_N)$$

where $\|\cdot\|$ stands for the Euclidean norm. In particular, when Ξ has setwise coprime coordinates: $H(B) = \|\Xi\|$.

We will also make use of an equivalent definition of the height of a rational subspace. Given vectors $X_1, \ldots, X_e \in \mathbb{R}^n$, let us denote by $M \in \mathrm{M}_{n,e}(\mathbb{R})$ the matrix whose j-th column is X_j for $j \in \{1, \ldots, e\}$. The *generalised determinant* of the family $(X_1, \ldots, X_e)$ is defined as $D(X_1, \ldots, X_e) = \sqrt{\det({}^tMM)}$. The following result establishes a link between the generalised determinant and the height of a rational subspace (see Theorem 1 of [12]).

Theorem 1.1 (Schmidt, 1967)**.** *Let $B \in \mathfrak{R}_n(e)$ and $(X_1, \ldots, X_e)$ be a basis of $B \cap \mathbb{Z}^n$. Then*

$$H(B) = D(X_1, \ldots, X_e).$$

For $X, Y \in \mathbb{R}^n \setminus \{0\}$, let us define a measure of the distance between these two vectors by $\psi(X,Y) = \sin\widehat{(X,Y)} = \|X \wedge Y\| \cdot \|X\|^{-1} \cdot \|Y\|^{-1}$, where $\mathbb{R}^n$ is endowed with the standard Euclidean norm $\|\cdot\|$, $\wedge\colon \mathbb{R}^n \times \mathbb{R}^n \to \Lambda^2(\mathbb{R}^n)$ stands for the exterior product on $\mathbb{R}^n$, and the Euclidean norm is naturally extended to $\Lambda^2(\mathbb{R}^n)$ so that $\|X \wedge Y\|$ is the area of the parallelogram spanned by X and Y. Let us define by induction $t = \min(d,e)$ angles between two subspaces A and B of $\mathbb{R}^n$ of respective dimensions d and e. The first one is defined as

$$\psi_1(A,B) = \min_{\substack{X\in A\setminus\{0\}\\ Y\in B\setminus\{0\}}} \psi(X,Y)$$

and let X_1 and Y_1 be two unitary vectors such that $\psi_1(A,B) = \psi(X_1,Y_1)$. Let $j\in\{1,\dots,t-1\}$ and assume that the first j angles $\psi_1(A,B),\dots,\psi_j(A,B)$ have been constructed together with pairs of vectors $(X_1,Y_1),\dots,(X_j,Y_j)\in A\times B$ such that $\psi_\ell(A,B) = \psi(X_\ell,Y_\ell)$ for $\ell\in\{1,\dots,j\}$. If S is a set of vectors of $\mathbb{R}^n$, let us denote by $\mathrm{Span}(S)$ the intersection of all subspaces V containing S. Let A_j and B_j be two subspaces of A and B respectively, such that $A = \mathrm{Span}(X_1,\dots,X_j)\overset{\perp}{\oplus} A_j$ and $B = \mathrm{Span}(Y_1,\dots,Y_j)\overset{\perp}{\oplus} B_j$. The $(j+1)$-th angle is then defined as

$$\psi_{j+1}(A,B) = \min_{\substack{X\in A_j\setminus\{0\}\\ Y\in B_j\setminus\{0\}}} \psi(X,Y),$$

and let X_{j+1} and Y_{j+1} be two unitary vectors such that $\psi_{j+1}(A,B) = \psi(X_{j+1},Y_{j+1})$.

According to [12, Theorem 4], there exist orthonormal bases $(X_1,\dots,X_d)$ and $(Y_1,\dots,Y_e)$ of A and B respectively, such that for all $(i,j)\in\{1,\dots,d\}\times\{1,\dots,e\}$, $X_i\cdot Y_j = \delta_{i,j}\cos\theta_i$, where δ is the Kronecker delta, the θ_ℓ are real numbers such that $0\leqslant\theta_t\leqslant\dots\leqslant\theta_1\leqslant 1$, and $\cdot$ is the canonical scalar product on $\mathbb{R}^n$; notice that $\psi_j(A,B) = \sin\theta_j$. The angles defined between A and B are canonical since the numbers $\theta_1,\dots,\theta_t$ does not depend on the choice of the bases $(X_1,\dots,X_d)$ and $(Y_1,\dots,Y_e)$ and are invariant under the application of an orthogonal transformation on A and B simultaneously.

Let us now formulate the generalisation of the classical Diophantine approximation problem. Let $n\geqslant 2$, $d,e\in\{1,\dots,n-1\}$ be such that $d+e\leqslant n$, $j\in\{1,\dots,\min(d,e)\}$. For $A\in\mathfrak{I}_n(d,e)_j$, let $\mu_n(A|e)_j$ be the upper bound in $[0,+\infty]$ of all $\beta>0$ such that

$$\psi_j(A,B)\leqslant\frac{1}{H(B)^\beta}$$

holds for infinitely many $B\in\mathfrak{R}_n(e)$. Let

$$\mathring{\mu}_n(d|e)_j = \inf_{A\in\mathfrak{I}_n(d,e)_j}\mu_n(A|e)_j.$$

The determination of $\mathring{\mu}_n(d|e)_j$ in terms of n, d, e and j is still an open problem. Some partial results are known (see [12, Theorems 12, 13, 15, 16 and 17]; [9, Satz 2]; [11, Theorem 9.3.2]). In [6], it was shown that the exponent for the approximation of a plane in $\mathbb{R}^4$ by rational planes to the first angle is exactly 3. It was also established a new upper bound for the approximation of a 3-subspace of $\mathbb{R}^5$ by rational planes to the first angle, and new lower and upper bounds in the general case. The new lower bound implies that $\mathring{\mu}_n(d|d)_d \xrightarrow[n\to+\infty]{} 1/d$. All of this can be found with extended details in [6, see Theorems 1.5, 1.6, 1.7, 1.8 and Corollary 1.1]. In this paper, new upper bounds on $\mathring{\mu}_n(d|e)_j$ will be proved in Propositions 3.1 and 3.2.

Some other topics are quite related with the subject studied by this paper. For instance, to study Diophantine approximation in projective spaces, the reader can look at [3] and [4]. To learn more about Diophantine approximation on Grassmannians, see for instance [7] and [1].

Another problem tackled by this paper is the determination of the set $\mu_n(\mathfrak{I}_n(d,e)_j|e)_j$ in terms of (n,d,e,j), i.e. the set of values taken by $\mu_n(A|e)_j$ for $A \in \mathfrak{I}_n(d,e)_j$. The partial answer provided can be found in Theorem 1.4 below.

One of the main results of the present paper, which allows us to improve on several known upper bounds for $\mathring{\mu}_n(d|e)_j$, is the following theorem and its corollary below.

Theorem 1.2. *Let $n \geqslant 2$ and $k \in \{2,\dots,n\}$. Let $d,e \in \{1,\dots,k-1\}$ be such that $d+e \leqslant k$, and $j \in \{1,\dots,\min(d,e)\}$. Let A be a subspace of dimension d of $\mathbb{R}^n$, such that there exists a subspace $F \in \mathfrak{R}_n(k)$ such that $A \subset F$. Let us denote by $\varphi\colon F \to \mathbb{R}^k$ a rational isomorphism and let $\widetilde{A} = \varphi(A)$, which is a subspace of dimension d of $\mathbb{R}^k$. Let us assume that for any rational subspace B' of dimension e contained in F, one has*

$$\dim(A \cap B') < j. \tag{1.1}$$

Then $A \in \mathfrak{I}_n(d,e)_j$, $\widetilde{A} \in \mathfrak{I}_k(d,e)_j$ and

$$\mu_n(A|e)_j = \mu_k(\widetilde{A}|e)_j.$$

One can notice that Hypothesis (1.1) of Theorem 1.2 is *a priori* a weak version of the hypothesis $A \in \mathfrak{I}_n(d,e)_j$ (i.e. $\dim(A \cap B) < j$ for all $B \in \mathfrak{R}_n(e)$); Theorem 1.2 shows that these two hypotheses are in fact equivalent.

Corollary 1.3. *Let $n \geqslant 2$ and $k \in \{2,\dots,n\}$. Let $d,e \in \{1,\dots,k-1\}$ be such that $d+e \leqslant k$, and $j \in \{1,\dots,\min(d,e)\}$. Then one has*

$$\mathring{\mu}_n(d|e)_j \leqslant \mathring{\mu}_k(d|e)_j.$$

This corollary leads to new upper bounds in Subsection 3.1; for instance Proposition 3.2 gives if $n \geqslant 6$, $d \in \{3,\dots,\lfloor n/2\rfloor\}$ and $\ell \in \{1,\dots,d\}$: $\mathring{\mu}_n(d|\ell)_1 \leqslant 2d^2/(2d-\ell)$, improving on several known upper bounds.

The other main result of this paper deals with the spectrum of $\mu_n(\bullet|e)_j$ when $d=e=j$.

Theorem 1.4. *Let $n \geqslant 2$ and $\ell \in \{1,\dots,\lfloor n/2\rfloor\}$, one has*

$$\left[1+\frac{1}{2\ell}+\sqrt{1+\frac{1}{4\ell^2}},+\infty\right] \subset \left\{\mu_n(A|\ell)_\ell,\ A \in \mathfrak{I}_n(\ell,\ell)_\ell\right\}.$$

In Section 2 we state some lemmas on the height and the proximity, which will find use in the other sections. In Section 3, we will prove and use Theorem 1.2 to deduce new upper bounds on $\mathring{\mu}_n(d|e)_j$. Section 4 is dedicated to prove Theorem 1.4, which brings a partial answer to the problem of the determination of the set $\mu_n(\mathfrak{I}_n(d,e)_j|e)_j$; the main theorem of Section 3 is also used in this section.

2. Some results about the height and the proximity

The first lemma is proved in [12, Lemma 13].

Lemma 2.1 (Schmidt, 1967). *Let A and B be two subspaces of $\mathbb{R}^n$ of dimensions d and e respectively, let φ be a non-singular linear transformation of $\mathbb{R}^n$. There exists a constant $c(\varphi) > 0$ such that for all $j \in \{1,\dots,\min(d,e)\}$, $\psi_j(\varphi(A),\varphi(B)) \leqslant c(\varphi)\psi_j(A,B)$.*

We will make use of the brief Lemma 2.3 below, but first, we require a lemma of Schmidt (see [12, Lemma 12]), which will also find use in the proofs of Lemma 3.3 and Theorem 1.2.

Lemma 2.2. *Let A and B be two subspaces of $\mathbb{R}^n$ of dimensions d and e respectively, let $j \in \{1,\dots,\min(d,e)\}$. Then $\psi_j(A,B)$ is the smallest number λ so there is a subspace $A_j \subset A$ so that for every $X \in A_j \setminus \{0\}$, there exists $Y \in B \setminus \{0\}$ such that $\psi(X,Y) \leqslant \lambda$.*

Lemma 2.3. *Let A and B be two non-trivial subspaces of $\mathbb{R}^n$ such that* $\dim A \leqslant \dim B$. *Then*

$$\forall\, X \in A \setminus \{0\}, \quad \psi_1(\mathrm{Span}(X),B) \leqslant \psi_{\dim A}(A,B).$$

Proof. Let $X \in A \setminus \{0\}$. One has

$$\begin{aligned}\psi_1(\mathrm{Span}(X),B) &= \min_{Y\in B\setminus\{0\}} \psi(X,Y)\\ &\leqslant \max_{Z\in A\setminus\{0\}}\ \min_{Y\in B\setminus\{0\}} \psi(Z,Y)\\ &= \min\{\varphi,\ \forall\, Z \in A\setminus\{0\},\ \exists\, Y \in B\setminus\{0\},\ \psi(Z,Y)\leqslant\varphi\}\\ &= \psi_{\dim A}(A,B)\end{aligned}$$

using Lemma 2.2. □

Now, we prove a result on the behaviour of the height of a rational subspace when applying a rational morphism.

Lemma 2.4. *Let $n \geqslant 3$ and $e,p \in \{1,\dots,n\}$; let $B \in \mathfrak{R}_e(n)$ and F be two rational subspaces of $\mathbb{R}^n$ such that $B \subset F$; let $\varphi\colon F \to \mathbb{R}^p$ be a rational morphism such that* $\dim\varphi(B) = \dim B$. *There exists a constant $c(\varphi) > 0$, depending only on φ, such that*

$$H(\varphi(B)) \leqslant c(\varphi)H(B).$$

Proof. Let us extend φ to a rational endomorphism of $\mathbb{R}^n$ by extending its codomain from $\mathbb{R}^p$ to $\mathbb{R}^n$, and by letting $\varphi(x) = x$ for all $x \in F^{\perp}$.

First, assume that the subspace B is a rational line L. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$ be such that $\gcd(\xi_1, \dots, \xi_n) = 1$ and $L = \mathrm{Span}(\xi)$. One has $\varphi(L) = \mathrm{Span}(\varphi(\xi))$, and there exists $c_1(\varphi) > 0$ independent of ξ such that $\|\varphi(\xi)\| \leqslant c_1(\varphi)\|\xi\|$. Let $(\zeta_1, \dots, \zeta_n) \in \mathbb{Q}^n$ be the coordinates of $\varphi(\xi)$. Since $\varphi \in \mathrm{M}_n(\mathbb{Q})$, there exists $k \in \mathbb{Z} \setminus \{0\}$ such that $k\varphi \in \mathrm{M}_n(\mathbb{Z})$, so $k\varphi(\xi) \in \mathbb{Z}^n$, and then for all $i \in \{1, \dots, n\}$, $k\zeta_i \in \mathbb{Z}$. Let $\mathfrak{b}$ be the fractional ideal spanned by the ζ_i, one has $k\mathfrak{b} = k(\zeta_1\mathbb{Z} + \cdots + \zeta_n\mathbb{Z}) \subset \mathbb{Z}$, thus $kN(\mathfrak{b}) = N(k\mathfrak{b}) \geqslant 1$. Therefore, using the generalised definition of the height of a rational subspace (see [12, Equation (1) p. 432]):

$$H(\varphi(L)) = N(\mathfrak{b})^{-1}\|\varphi(\xi)\| \leqslant kc_1(\varphi)\|\xi\| = c(\varphi)\|\xi\| = c(\varphi)H(L)$$

with $c(\varphi) = kc_1(\varphi)$.

Let us extend the result to a rational subspace B of dimension e. Let $N = \binom{n}{e}$ and B^* be the rational line of $\mathbb{R}^N$ spanned by the Plücker coordinates of B. The hypothesis of the lemma gives $\dim \varphi(B) = \dim B$, so the rational line $\varphi(B)^*$ spanned by the Plücker coordinates of $\varphi(B)$ also belongs to $\mathbb{R}^N$. Notice that $H(B) = H(B^*)$ and $H(\varphi(B)) = H(\varphi(B)^*)$. Let us denote by $S \in \mathrm{M}_n(\mathbb{Q})$ the matrix of φ in the canonical basis of $\mathbb{R}^n$. Then $\Lambda^e(S) \in \mathrm{M}_N(\mathbb{Q})$, the matrix formed with all $e \times e$ minors of S in lexicographic order, is the matrix of $\varphi^{(e)}$, the e-th compound of φ, in the canonical basis of $\mathbb{R}^N$. One has $\varphi(B)^* = \varphi^{(e)}(B^*)$ (see [12, p. 433]), so $H(\varphi(B)^*) = H(\varphi^{(e)}(B^*))$. This falls into the case of dimension 1 in $\mathbb{R}^N$, therefore the first part of the proof concludes and gives a constant $c^{(e)}(\varphi)$.

Notice that the constant $c(\varphi)$ does not depend on e by taking $c(\varphi) = \max_{1 \leqslant e \leqslant n} c^{(e)}(\varphi)$. □

3. Inclusion in a rational subspace

Here, we will focus on the case where the subspace we are trying to approach is included in a rational subspace. This will lead to several improvements on the known upper bounds for $\mathring{\mu}_n(d|e)_j$.

First, we will state in Subsection 3.1 the new results that can be deduced from Corollary 1.3 of Theorem 1.2. Then, we will establish two lemmas in Subsection 3.2 which will find use in the proof of the main result, Theorem 1.2, in Subsection 3.3.

3.1. Improvements on some upper bounds. First, the upper bound $\mathring{\mu}_5(2|2)_1 \leqslant 4$ given by Theorem 16 of [12] is improved (Theorem 12 of [12] gives $\mathring{\mu}_5(2|2)_1 \geqslant 20/9$, so an equality is not obtained here).

Proposition 3.1. *One has*

$$\mathring{\mu}_5(2|2)_1 \leqslant 3.$$

The proof of Proposition 3.1 requires Theorem 1.5 of [6]: $\mathring{\mu}_4(2|2)_1 = 3$.

In a very similar fashion, the following proposition is deduced from Theorem 1.7 of [6]: $\mathring{\mu}_{2d}(d|\ell)_1 \leqslant 2d^2/(2d-\ell)$ for $d \geqslant 2$ and $\ell \in \{1,\dots,d\}$.

Proposition 3.2. *Let $n \geqslant 6$, $d \in \{3,\dots,\lfloor n/2 \rfloor\}$ and $\ell \in \{1,\dots,d\}$, one has*

$$\mathring{\mu}_n(d|\ell)_1 \leqslant \frac{2d^2}{2d-\ell}.$$

This proposition improves on several upper bounds when n is close to $2d$ and ℓ is close to d; for instance, the upper bounds $\mathring{\mu}_9(4|3)_1 \leqslant 7$ and $\mathring{\mu}_{30}(10|8)_1 \leqslant 18$ proved by Schmidt (see [12, Theorem 16]) are improved respectively to $32/5$ and $50/3$.

3.2. Two useful lemmas to prove Theorem 1.2. Let us prove two lemmas which will find use in the proof of Theorem 1.2 in Subsection 3.3. The first lemma studies how the proximity between two subspaces behaves when applying a projection. This proof follows the ideas of Lemma 13 of [12], though the endomorphism is not assumed to be invertible here.

Lemma 3.3. *Let A and F be two subspaces as in Theorem 1.2. Let $\mathcal{R}$ be a non-empty subset of $\mathbb{R}^n$ such that $\mathcal{R} \cap F^\perp = \emptyset$ and such that there exists a constant $c > 0$ satisfying*

$$\forall\, X \in \mathcal{R}, \quad \|p_F^\perp(X)\| \geqslant c\|X\| \tag{3.1}$$

where $p_F^\perp$ is the orthogonal projection onto F. Let D be a subspace of $\mathbb{R}^n$ such that $\dim D \geqslant j$ and $D \subset \mathcal{R} \cup \{0\}$. Then there exists a constant $c' > 0$ depending only on c such that

$$\psi_j(A,D) \geqslant c'\psi_j(A, p_F(D)).$$

Proof. Hypothesis (3.1) gives a constant $c > 0$ such that for all $X \in \mathcal{R}$, $c\|X\| \leqslant \|p_F^\perp(X)\|$. In particular $c \leqslant 1$ since $p_F^\perp$ is an orthogonal projection, therefore we may assume that $F \setminus \{0\} \subset \mathcal{R}$ since $\|p_F^\perp(X)\| = \|X\|$ for any $X \in F$. Our first goal is to show that there exists a constant $c_1 > 0$ (depending only on $c > 0$), such that

$$\forall\, X \in F \setminus \{0\}, \quad \forall\, Y \in \mathcal{R}, \quad \psi(X, p_F^\perp(Y)) \leqslant c_1\psi(X,Y). \tag{3.2}$$

Let $X \in F \setminus \{0\}$ and $Y \in \mathcal{R}$. Without loss of generality, assume that $\|X\| = \|Y\| = 1$ and $X \cdot Y \geqslant 0$. One has $\psi^2(X, p_F^\perp(Y)) = (\|p_F^\perp(Y)\|^2 - (X \cdot p_F^\perp(Y))^2)/\|p_F^\perp(Y)\|^2$. Let $\lambda = \|p_F^\perp(Y)\|$ and notice that $0 \leqslant (X \cdot p_F^\perp(Y) - \lambda)^2$ leads to $\lambda^2 - (X \cdot p_F^\perp(Y))^2 \leqslant 2(\lambda^2 - \lambda(X \cdot p_F^\perp(Y)))$. Thus,

$$\psi^2(X, p_F^\perp(Y)) \leqslant 2\frac{\|p_F^\perp(Y)\| - X \cdot p_F^\perp(Y)}{\|p_F^\perp(Y)\|}.$$

Using Hypothesis (3.1), $\psi^2(X, p_F^\perp(Y)) \leqslant \frac{2}{c}(\|p_F^\perp(Y)\| - X \cdot p_F^\perp(Y))$, but

$$\frac{1}{2}\left(\|X - p_F^\perp(Y)\|^2 - 1 - \|p_F^\perp(Y)\|^2\right) = -X \cdot p_F^\perp(Y),$$

so

$$\psi^2(X, p_F^\perp(Y)) \leqslant \frac{1}{c}\|p_F^\perp(X-Y)\|^2 - \frac{1}{c}\left(\|p_F^\perp(Y)\| - 1\right)^2 \leqslant \frac{1}{c}\|X-Y\|^2. \tag{3.3}$$

Let us mention an elementary geometric claim.

Claim 3.1. Let U and V be unitary vectors such that $U \cdot V \geqslant 0$. One has $\psi(U,V) \geqslant \frac{\sqrt{2}}{2}\|U - V\|$.

Proof of Claim 3.1. Let $p_{\mathrm{Span}(V)}^\perp$ be the orthogonal projection onto $\mathrm{Span}(V)$, $\alpha = \|U - p_{\mathrm{Span}(V)}^\perp(U)\|$ and $\beta = \|V - p_{\mathrm{Span}(V)}^\perp(U)\|$. One has $\|U-V\|^2 = \alpha^2 + \beta^2$, and since U is unitary: $\psi(U,V) = \psi(U, p_{\mathrm{Span}(V)}^\perp(U)) = \|U - p_{\mathrm{Span}(V)}^\perp(U)\| = \alpha$. Moreover, $U \cdot V \geqslant 0$, so $1 = \|U\|^2 = (1-\beta)^2 + \alpha^2$, hence there exists $\theta \in [0, \pi/2]$ such that $1 - \beta = \cos\theta$ and $\alpha = \sin\theta$. Since $1 - \cos\theta \leqslant \sin\theta$, we obtain $\beta \leqslant \alpha$, and finally $\|U - V\|^2 \leqslant 2\alpha^2 = 2\psi(U,V)^2$. □

Since $X \cdot Y \geqslant 0$ here, Claim 3.1 gives $\psi(X,Y) \geqslant \frac{\sqrt{2}}{2}\|X - Y\|$, so with Inequality (3.3), it yields

$$\psi(X, p_F^\perp(Y)) \leqslant c_1 \psi(X,Y)$$

which is the desired Inequality (3.2), with $c_1 = \sqrt{2/c}$.

For the second part of the proof, Lemma 2.2 tells us that there exists a subspace $A_j \subset A$ of dimension j, such that for all $X \in A_j \setminus \{0\}$, there exists $Y \in D \setminus \{0\}$ such that $\psi(X,Y) \leqslant \psi_j(A,D)$. Let $X \in A_j \setminus \{0\}$ and $Y \in D \setminus \{0\}$ be such that $\psi(X,Y) \leqslant \psi_j(A,D)$. Since $X \in A_j \subset A \subset F$ and $Y \in D \setminus \{0\} \subset \mathcal{R}$, one can use Inequality (3.2) to get $\psi(X, p_F^\perp(Y)) \leqslant c_1\psi(X,Y) \leqslant c_1\psi_j(A,D)$. Thus, $Y' = p_F^\perp(Y) \in p_F^\perp(D)$ is a non-zero vector such that $\psi(X,Y') \leqslant c_1\psi_j(A,D)$; therefore

$$\forall\, X \in A \setminus \{0\}, \quad \exists\, Y' \in p_F^\perp(D) \setminus \{0\}, \quad \psi(X,Y') \leqslant c_1\psi_j(A,D).$$

According to Lemma 2.2, $\psi_j(A, p_F^\perp(D))$ is the smallest number having this property, so

$$\psi_j(A, p_F^\perp(D)) \leqslant c_1\psi_j(A,D). \qquad \square$$

The second lemma shows that one can choose rational subspaces approaching A which intersect $F^\perp$ trivially.

Lemma 3.4. *Under the hypothesis of Theorem 1.2, for all $\alpha < \mu_n(A|e)_j$, there exists a sequence $(B_N)_{N\in\mathbb{N}}$ of rational subspaces of $\mathbb{R}^n$ of dimension e, pairwise distinct, such that for any N large enough: $B_N \cap F^\perp = \{0\}$ and $\psi_j(A, B_N) \leqslant H(B_N)^{-\alpha}$.*

Proof of Lemma 3.4. Let α' be such that $\alpha < \alpha' < \mu_n(A|e)_j$. By definition of $\mu_n(A|e)_j$, there exists a sequence $(B_N)_{N\in\mathbb{N}}$ of rational subspaces of $\mathbb{R}^n$ of dimension e, pairwise distinct, such that for any N large enough: $\psi_j(A, B_N) \leqslant H(B_N)^{-\alpha'}$.

Let $(g_1, \ldots, g_{n-k})$ be a linearly independent family of vectors of $\mathbb{Q}^n$ such that $\mathbb{R}^n = F \oplus \mathrm{Span}(g_1, \ldots, g_{n-k})$. For $\ell \in \mathbb{N}^*$, let $\mathcal{P}_m(\ell)$ be the set of subsets with m elements of $\{1, \ldots, \ell\}$. Let us show by induction on $\ell \in \{n-k, \ldots, n\}$ that there exist vectors $g_{n-k+1}, \ldots, g_\ell$ of $\mathbb{Q}^n$ such that

$$\begin{cases} \dim \mathrm{Span}(g_1, \ldots, g_\ell) = \ell, \\ \forall\, I \in \mathcal{P}_{n-k}(\ell), \quad G_I \cap F = \{0\}; \end{cases} \tag{3.4}$$

here and below, for $I \subset \{1, \ldots, \ell\}$ we let $G_I = \mathrm{Span}\{g_i,\ i \in I\}$. Since $\dim F = k$, the initial case $\ell = n-k$ holds by definition of the $g_1, \ldots, g_{n-k}$. Let $\ell \in \{n-k, \ldots, n-1\}$, assume that the g_i for $i \in \{n-k+1, \ldots, \ell\}$ have been constructed satisfying (3.4). Let

$$G = \mathrm{Span}(g_1, \ldots, g_\ell) \cup \bigcup_{K \in \mathcal{P}_{n-k-1}(\ell)} (F \oplus G_K).$$

The set G is a union of a finite number of subspaces of dimensions at most $n-1$, thus there exists a vector $g_{\ell+1} \in \mathbb{Q}^n \setminus G$. Notice that $\dim \mathrm{Span}(g_1, \ldots, g_{\ell+1}) = \ell+1$. Let us assume that there exists $I \in \mathcal{P}_{n-k}(\ell+1)$ such that $G_I \cap F \neq \{0\}$. Let $u \in G_I \cap F \setminus \{0\}$; by the induction hypothesis, $I \notin \mathcal{P}_{n-k}(\ell)$, so $\ell + 1 \in I$. Let us write I under the form $I = K \cup \{\ell+1\}$ with $K \in \mathcal{P}_{n-k-1}(\ell)$. Since $u \notin F \cap G_K = \{0\}$, there exist $\alpha_{\ell+1} \neq 0$ and some $\alpha_i \in \mathbb{R}$ (for $i \in K$) such that $u = \alpha_{\ell+1} g_{\ell+1} + \sum_{i\in K} \alpha_i g_i$, so

$$g_{\ell+1} = \frac{1}{\alpha_{\ell+1}} \left(u - \sum_{i\in K} \alpha_i g_i \right) \in F \oplus G_K,$$

which can not be by definition of G, because $g_{\ell+1} \notin G$. Thus, $g_{\ell+1}$ satisfies (3.4) and therefore the induction is complete; it is now established that there exist vectors $g_1, \ldots, g_n$ satisfying (3.4).

For $I \in \mathcal{P}_{n-k}(n)$, one has $\dim G_I = n - k = \dim F^\perp$. So $G_I \oplus F = \mathbb{R}^n$: there exists a rational isomorphism $\rho_I \in \mathrm{GL}_n(\mathbb{Q})$ such that $\rho_{I|F} = \mathrm{id}_F$ and $\rho_I(G_I) = F^\perp$. Let $N \in \mathbb{N}$ and let us assume that

$$\forall\, I \in \mathcal{P}_{n-k}(n), \quad \rho_I(B_N) \cap F^\perp \neq \{0\}, \tag{3.5}$$

which is equivalent by definition of ρ_I to $B_N \cap G_I \neq \{0\}$ for any $I \in \mathcal{P}_{n-k}(n)$. Let

$$J = \left\{ i \in \{1, \ldots, n\},\ \exists\, \alpha \neq 0,\ \ \exists\, \lambda_1, \ldots, \lambda_{i-1} \in \mathbb{R},\ \ \alpha g_i + \sum_{\ell=1}^{i-1} \lambda_\ell g_\ell \in B_N \right\}.$$

First, assume that $|J| \leqslant k$. Thus, there exists $I \in \mathcal{P}_{n-k}(n)$ such that $I \cap J = \emptyset$. Since $B_N \cap G_I \neq \{0\}$, there exists a non-zero vector $(\beta_i)_{i\in I} \in \mathbb{R}^I$ such that $\sum_{i\in I} \beta_i g_i \in B_N$. Let i_0 be the largest $i \in I$ such that $\beta_i \neq 0$. Then

$$\sum_{i\in I} \beta_i g_i = \alpha g_{i_0} + \sum_{i=1}^{i_0-1} \beta_i g_i$$

where $\beta_i = 0$ if $i \notin I$ and $\alpha = \beta_{i_0} \neq 0$. So $i_0 \in I \cap J$ which can not be, therefore $|J| > k$. The elements of J give at least $k+1$ linearly independent vectors of B_N, which can not be since $\dim B_N \leqslant k$. Thus, (3.5) is established.

Let $I \in \mathcal{P}_{n-k}(n)$ be such that $\rho_I(B_N) \cap F^\perp = \{0\}$. Since ρ_I is invertible, Lemma 2.1 gives a constant $c(\rho_I) > 0$ such that

$$\psi_j(A, \rho_I(B_N)) = \psi_j(\rho_I(A), \rho_I(B_N)) \leqslant c(\rho_I)\psi_j(A, B_N).$$

Let $c_2 = \max_{I\in\mathcal{P}_{n-k}(n)} c(\rho_I) > 0$, which is a constant independent of B_N; then we have $\psi_j(A, \rho_I(B_N)) \leqslant c_2\psi_j(A, B_N)$. Moreover, since ρ_I is an isomorphism, $\dim(\rho_I(B_N)) = \dim(B_N) = e$, so Lemma 2.4 gives a constant $c'(\rho_I) > 0$ such that $H(\rho_I(B_N)) \leqslant c'(\rho_I)H(B_N)$. Let $c_3 = \max_{I\in\mathcal{P}_{n-k}(n)} c'(\rho_I) > 0$ which is a constant independent of B_N such that $H(\rho_I(B_N)) \leqslant c_3 H(B_N)$. Therefore, for all N large enough:

$$\begin{aligned}\psi_j(A, \rho_I(B_N)) &\leqslant c_2\psi_j(A, B_N)\\ &\leqslant \frac{c_2}{H(B_N)^{\alpha'}} \leqslant \frac{c_2 c_3^{-\alpha'}}{H(\rho_I(B_N))^{\alpha'}} \leqslant \frac{1}{H(\rho_I(B_N))^{\alpha}}.\end{aligned}$$ □

3.3. Proof of Theorem 1.2.

Let us provide a proof of the main theorem.

Proof of Theorem 1.2. First, let us prove that $A \in \mathfrak{I}_n(d,e)_j$ and $\widetilde{A} \in \mathfrak{I}_k(d,e)_j$.

Let $B \in \mathfrak{R}_n(e)$. Notice that $B' = B \cap F$ is a rational subspace of dimension $e' \leqslant e \leqslant \dim F$. Thus, there exists a rational subspace $B'' \subset F$, containing B', and such that $\dim B'' = e$. Hypothesis (1.1) of Theorem 1.2 gives $\dim(A\cap B'') < j$, therefore $\dim(A\cap B') < j$. Since $A\cap B = A\cap F\cap B = A\cap B'$ because $A \subset F$, one has $\dim(A \cap B) < j$, i.e. $A \in \mathfrak{I}_n(d,e)_j$.

Let $\widetilde{B} \in \mathfrak{R}_k(e)$ and $B = \varphi^{-1}(\widetilde{B}) \in \mathfrak{R}_n(e)$. Since φ is an isomorphism, one has $\dim(\widetilde{A} \cap \widetilde{B}) = \dim(\varphi(A) \cap \varphi(B)) = \dim(\varphi(A \cap B)) = \dim(A \cap B) < j$ because $B \in \mathfrak{R}_n(e)$ and $A \in \mathfrak{I}_n(d,e)_j$. This shows that $\widetilde{A} \in \mathfrak{I}_k(d,e)_j$.

Now, let us show that $\mu_n(A|e)_j \geqslant \mu_k(\widetilde{A}|e)_j$. Let $\alpha < \mu_k(\widetilde{A}|e)_j$. There exists a sequence $(\widetilde{B}_N)_{N\geqslant 0}$ of rational subspaces of $\mathbb{R}^k$ of dimension e, pairwise distinct, such that for all N large enough: $\psi_j(\widetilde{A}, \widetilde{B}_N) \leqslant H(\widetilde{B}_N)^{-\alpha}$. For all $N \in \mathbb{N}$, let $B_N = \varphi^{-1}(\widetilde{B}_N) \in \mathfrak{R}_n(e)$ because φ is a rational isomorphism. According to Lemma 2.4, there exists a constant $c_{\varphi^{-1}}$ such that for all

$N \in \mathbb{N}$, $H(B_N) = H(\varphi^{-1}(\widetilde{B}_N)) \leqslant c_{\varphi^{-1}} H(\widetilde{B}_N)$. Using Lemma 2.1, there exists a constant $c'_{\varphi^{-1}} > 0$ such that $\psi_j(A, B_N) = \psi_j(\varphi^{-1}(\widetilde{A}), \varphi^{-1}(\widetilde{B}_N)) \leqslant c'_{\varphi^{-1}} \psi_j(\widetilde{A}, \widetilde{B}_N)$. Therefore, for N large enough,

$$\psi_j(A, B_N) \leqslant c'_{\varphi^{-1}} \psi_j(\widetilde{A}, \widetilde{B}_N) \leqslant \frac{c'_{\varphi^{-1}}}{H(\widetilde{B}_N)^\alpha} \leqslant \frac{c_1}{H(B_N)^\alpha},$$

with $c_1 > 0$ depending only on φ. Since the B_N are pairwise distinct, $\mu_n(A|e)_j \geqslant \alpha$, and since this is true for all $\alpha < \mu_k(\widetilde{A}|e)_j$, one has

$$\mu_n(A|e)_j \geqslant \mu_k(\widetilde{A}|e)_j.$$

Finally, let us establish that $\mu_n(A|e)_j \leqslant \mu_k(\widetilde{A}|e)_j$. Let $\alpha < \mu_n(A|e)_j$. Lemma 3.4 gives us a sequence $(B_N)_{N \in \mathbb{N}}$ of rational subspaces of $\mathbb{R}^n$ of dimension e, pairwise distinct, such that for all N large enough:

$$B_N \cap F^\perp = \{0\} \quad \text{and} \quad \psi_j(A, B_N) \leqslant \frac{1}{H(B_N)^\alpha}. \tag{3.6}$$

Let $N \in \mathbb{N}$ large enough; let us denote by $p_F^\perp$ the orthogonal projection onto F. Since $F \in \mathfrak{R}_n(k)$, $p_F^\perp$ is a rational endomorphism of $\mathbb{R}^n$. Let $B'_N = p_F^\perp(B_N)$; since B_N is a rational subspace, B'_N also is. Let $\mathcal{R}$ be the set of all non-zero vectors of $\mathbb{R}^n$ which form an angle less than $\pi/4$ with the subspace F:

$$\mathcal{R} = \left\{ X \in \mathbb{R}^n \setminus \{0\},\ \psi_1(F, \operatorname{Span}(X)) < \frac{\sqrt{2}}{2} \right\}.$$

Notice that $\mathcal{R} \cap F^\perp = \emptyset$. Let $X \in \mathcal{R}$ and $Y = X - p_F^\perp(X)$; then $\|p_F^\perp(X)\|^2 + \|Y\|^2 = \|X\|^2$. Since $X \in \mathcal{R}$, one has $\psi(X, F) = \psi(X, p_F^\perp(X)) = \|Y\|/\|X\| < \sqrt{2}/2$, so $\|Y\| \leqslant (\sqrt{2}/2)\|X\|$. Thus, $\|p_F^\perp(X)\|^2 = \|X\|^2 - \|Y\|^2 \geqslant \|X\|^2 - \frac{1}{2}\|X\|^2 = \frac{1}{2}\|X\|^2$, so the set $\mathcal{R}$ satisfies Hypothesis (3.1) of Lemma 3.3 with $c = \sqrt{2}/2$.

According to Lemma 2.2, $\psi_j(A, B_N)$ is the smallest number λ for which there exists a subspace $B_{N,j}$ of dimension j such that for every $Y \in B_{N,j} \setminus \{0\}$, there is a vector $X \in A \setminus \{0\}$ such that $\psi(X, Y) \leqslant \lambda$. Let us fix such a subspace $B_{N,j}$. Since N is assumed to be large enough, $\psi_j(A, B_N) \leqslant 1/2$ can be assumed. Therefore, for all $Y \in B_{N,j} \setminus \{0\}$, there exists a non-zero vector $X \in A \subset F$ such that $\psi(X, Y) \leqslant \psi_j(A, B_N) \leqslant 1/2$, so $\psi_1(F, \operatorname{Span}(Y)) \leqslant 1/2 < \sqrt{2}/2$, hence $Y \in \mathcal{R}$. Thus, for all N large enough: $B_{N,j} \setminus \{0\} \subset \mathcal{R}$.

Applying Lemma 3.3 provides a constant $c_4 > 0$ which depends neither on A nor on B_N, such that

$$\psi_j(A, B_N) = \psi_j(A, B_{N,j}) \geqslant c_4 \psi_j(A, p_F^\perp(B_{N,j})) \geqslant c_4 \psi_j(A, B'_N) \tag{3.7}$$

because $B'_N = p_F^\perp(B_N) \supset p_F^\perp(B_{N,j})$. Since $B_N \cap F^\perp = \{0\}$, $\dim B'_N = e$; since $B'_N \subset F$, let $\widetilde{B}_N = \varphi(B'_N) \in \mathfrak{R}_k(e)$. Using Lemma 2.1, there exists a constant $c_\varphi > 0$ such that $\psi_j(\varphi(A), \varphi(B'_N)) \leqslant c_\varphi \psi_j(A, B'_N)$. Let $\beta > \mu_k(\widetilde{A}|e)_j$; using Inequality (3.7) yields that for all N large enough (in terms of β):

$$\begin{aligned}\psi_j(A, B_N) &\geqslant c_4 \psi_j(A, B'_N)\\ &\geqslant c_4 c_\varphi^{-1} \psi_j(\varphi(A), \varphi(B'_N)) = c_4 c_\varphi^{-1} \psi_j(\widetilde{A}, \widetilde{B}_N) \geqslant \frac{c_5}{H(\widetilde{B}_N)^\beta}\end{aligned}$$

with $c_5 > 0$. According to Lemma 2.4, there exists a constant $c_6 > 0$ such that $H(\widetilde{B}_N) = H(\varphi(B'_N)) \leqslant c_6 H(B'_N)$, so $\psi_j(A, B_N) \geqslant c_7 H(B'_N)^{-\beta}$ with $c_7 > 0$. Since $B_N \cap F^\perp = \{0\}$, $\dim(p_F^\perp(B_N)) = \dim(B_N)$. Therefore, Lemma 2.4 can be used again to obtain a constant $c_8 > 0$ such that $H(B'_N) = H(p_F^\perp(B_N)) \leqslant c_8 H(B_N)$. With Inequality (3.6), there exists a constant $c_9 > 0$ such that

$$\frac{1}{H(B_N)^\alpha} \geqslant \psi_j(A, B_N) \geqslant \frac{c_9}{H(B_N)^\beta}.$$

Finally, since $H(B_N)$ tends to infinity when $N \to +\infty$, $\alpha \leqslant \beta$. Because this is true for all $\alpha < \mu_n(A|e)_j$ and for all $\beta > \mu_k(\widetilde{A}|e)_j$, one has

$$\mu_n(A|e)_j \leqslant \mu_k(\widetilde{A}|e)_j. \qquad \square$$

4. The spectrum of $\mu_n(\bullet|\ell)_\ell$

In this section, progress will be made on the determination of the spectrum of $\mu_n(\bullet|\ell)_\ell$ over $\mathfrak{I}_n(\ell,\ell)_\ell$, i.e. on the set $\mu_n(\mathfrak{I}_n(\ell,\ell)_\ell|\ell)_\ell$. The main result is Theorem 1.4: let $n \geqslant 2$ and $\ell \in \{1, \ldots, \lfloor n/2 \rfloor\}$, one has

$$\left[1 + \frac{1}{2\ell} + \sqrt{1 + \frac{1}{4\ell^2}}, +\infty\right] \subset \left\{\mu_n(A|\ell)_\ell,\ A \in \mathfrak{I}_n(\ell,\ell)_\ell\right\}.$$

Remark 4.1. In [11], N. de Saxcé shows that $\mathring{\mu}_n(\ell|\ell)_\ell \leqslant n/(\ell(n-\ell))$. It would be interesting to establish that

$$\left[\frac{n}{\ell(n-\ell)}, +\infty\right] \subset \left\{\mu_n(A|\ell)_\ell,\ A \in \mathfrak{I}_n(\ell,\ell)_\ell\right\}.$$

To prove Theorem 1.4, it is first assumed that $n = 2\ell$, and that $\beta < +\infty$ is fixed in the interval of Theorem 1.4; a subspace approximated exactly to the order β is constructed. First, we establish that the subspace constructed satisfies $A \in \mathfrak{I}_{2\ell}(\ell,\ell)_\ell$, then we show that $\mu_{2\ell}(A|\ell)_\ell \geqslant \beta$, and finally that $\mu_{2\ell}(A|\ell)_\ell \leqslant \beta$. The result will be finally extended to the case $n > 2\ell$ with Theorem 1.2, and to the case $\beta = +\infty$.

Proof of Theorem 1.4. Let $\ell \geqslant 1$ be an integer and $n = 2\ell$. Let β be a real number such that

$$\beta \geqslant 1 + \frac{1}{2\ell} + \sqrt{1 + \frac{1}{4\ell^2}}. \tag{4.1}$$

The goal is to construct $A \in \mathfrak{I}_n(\ell,\ell)_\ell$, a subspace (ℓ,ℓ)-irrational of $\mathbb{R}^n$ such that $\mu_n(A|\ell)_\ell = \beta$. Let $\alpha = \ell\beta$; for all $(i,j) \in \{1,\dots,\ell\}^2$ let

$$\xi_{i,j} = \sum_{k=0}^{\infty} \frac{e_k^{(i,j)}}{\theta^{\lfloor \alpha^k \rfloor}}$$

where the $(e_k^{(i,j)})_{k\in\mathbb{N}}$ are sequences which are yet to be determined, with values in $\{1,2\}$ if $i \neq j$ and with values in $\{2\ell, 2\ell+1\}$ if $i=j$, and where θ is the smallest prime number such that

$$\theta > (n+1)^{n/2} \left(\frac{n}{2}\right)! = \ell!\,(2\ell+1)^\ell. \tag{4.2}$$

Remark 4.2. The fact that θ is chosen to be the *smallest* such number does not have any other purpose but to allow the constants not to depend on θ. In practice, any prime number θ satisfying Inequality (4.2) would work. Hypothesis (4.2) on θ and the fact that the sequences $(e_k^{(i,j)})_{k\in\mathbb{N}}$ belong to $\{1,2\}$ or $\{2\ell, 2\ell+1\}$ will be used in the proof of Claim 4.4 to establish a lower bound on the height of the rational subspaces constructed below.

Let I_ℓ be the identity matrix of $\mathrm{M}_\ell(\mathbb{R})$, $M_\xi = (\xi_{i,j})_{(i,j)\in\{1,\dots,\ell\}^2} \in \mathrm{M}_\ell(\mathbb{R})$ be the matrix of the $\xi_{i,j}$, and M_A be the block matrix:

$$M_A = \begin{pmatrix} I_\ell \\ M_\xi \end{pmatrix} \in \mathrm{M}_{2\ell,\ell}(\mathbb{R}). \tag{4.3}$$

Let us denote by $Y_1, \dots, Y_\ell \in \mathbb{R}^{2\ell}$ the columns of M_A, and let A be the subspace of $\mathbb{R}^{2\ell}$ spanned by the Y_i: $A = \mathrm{Span}(Y_1,\dots,Y_\ell)$. Notice that $\mathrm{rk}(M_A) = \ell$, so $\dim A = \ell$.

Let us establish that there exist sequences $(e_k^{(i,j)})_{k\in\mathbb{N}}$ with values in $\{1,2\}$ if $i\neq j$ and with values in $\{2\ell, 2\ell+1\}$ if $i=j$, such that $A \in \mathfrak{I}_n(\ell,\ell)_\ell$.

For better clarity, let us reindex the $\xi_{i,j}$ for $(i,j)\in\{1,\dots,\ell\}^2$ as $\xi_1,\dots,\xi_{\ell^2}$ by lexicographic order; the sequences $(e_k^{(i,j)})_{k\in\mathbb{N}}$ for $(i,j)\in\{1,\dots,\ell\}^2$ are also reindexed as $(e_k^{(1)})_{k\in\mathbb{N}},\dots,(e_k^{(\ell^2)})_{k\in\mathbb{N}}$ in the same way.

Let us prove by induction on $t \in \{1,\dots,\ell^2\}$ that the sequences $(e_k^{(1)})_{k\in\mathbb{N}}$, $\dots, (e_k^{(t)})_{k\in\mathbb{N}}$ can be chosen such that $\xi_1,\dots,\xi_t$ are $\mathbb{Q}$-algebraically independent. The irrationality exponent of ξ_1 is at least $\alpha > 2$ (it is even equal to α, see [8]), so by Roth's theorem (see [10]), ξ_1 is transcendental. Let $t \in \{1,\dots,\ell^2-1\}$ and let us assume that the numbers $\xi_1,\dots,\xi_t$ are $\mathbb{Q}$-algebraically independent. The set of real numbers algebraic on $\mathbb{Q}(\xi_1,\dots,\xi_t)$ is countable, whereas the set of the sequences $(e_k^{(t+1)})_{k\in\mathbb{N}}$ is not. Therefore, one can choose a sequence $(e_k^{(t+1)})_{k\in\mathbb{N}}$ with values in $\{1,2\}$ or $\{2\ell,2\ell+1\}$ (depending on if t corresponds to a pair (i,j) with $i\neq j$ or $i=j$), such that $\xi_1, \ldots, \xi_{t+1}$ are $\mathbb{Q}$-algebraically independent, which concludes the induction.

Let us assume that there exists $B \in \mathfrak{R}_{2\ell}(\ell)$ such that $A \cap B \neq \{0\}$. Let M_B be a matrix whose columns form a rational basis of B. Notice that $\det\left(M_A \; M_B \right) = 0$ where M_A is the matrix defined in Equation (4.3). Since $M_B \in \mathrm{M}_{2\ell,\ell}(\mathbb{Q})$, one can compute this determinant using a Laplace expansion on its ℓ first columns to obtain a polynomial $P \in \mathbb{Q}[X_1, \ldots, X_{\ell^2}]$ such that $\det\left(M_A \; M_B \right) = P(\xi_1, \ldots, \xi_{\ell^2}) = 0$. The fact that $\xi_1, \ldots, \xi_{\ell^2}$ are $\mathbb{Q}$-algebraically independent yields $P = 0$. Let us decompose the matrix M_B under the form $M_B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ with $B_1, B_2 \in \mathrm{M}_{\ell,\ell}(\mathbb{R})$. The equality $P = 0$ implies that

$$\forall\, Q \in \mathrm{M}_\ell(\mathbb{R}), \quad \Delta_Q = \det \begin{pmatrix} I_\ell & B_1 \\ Q & B_2 \end{pmatrix} = 0.$$

Let us mention this known claim to compute determinants of 2×2 block matrices (see [13, Theorem 3]).

Claim 4.1. Let $A_1, A_2, A_3, A_4 \in \mathrm{M}_\ell(\mathbb{R})$ such that $A_1 A_2 = A_2 A_1$. Then

$$\det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \det(A_4 A_1 - A_3 A_2).$$

Since I_ℓ commutes with B_1, Claim 4.1 can be used to get

$$\forall\, Q \in \mathrm{M}_\ell(\mathbb{R}), \quad \Delta_Q = \det(B_2 - Q B_1) = 0. \tag{4.4}$$

Let $\lambda \in \mathbb{R}$; with $Q = \lambda I_\ell$, one has $\det(B_2 - \lambda B_1) = 0$. Assume that B_1 is invertible, then

$$0 = \Delta_Q = \det((B_2 B_1^{-1} - \lambda I_\ell) B_1) = \det(B_2 B_1^{-1} - \lambda I_\ell) \det(B_1).$$

Thus, the fact that $\det(B_1) \neq 0$ yields that for all $\lambda \in \mathbb{R}$, $\det(B_2 B_1^{-1} - \lambda I_\ell) = 0$. Therefore, for all $\lambda \in \mathbb{R}$, λ is an eigenvalue of $B_2 B_1^{-1}$, and this can not be, so $\det(B_1) = 0$. Let $r = \mathrm{rk}(B_1) < \ell$, let $U, V \in \mathrm{GL}_\ell(\mathbb{R})$ be two invertible matrices such that

$$U B_1 V = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = (J_r \quad 0) \in \mathrm{M}_\ell(\mathbb{R}) \quad \text{where} \quad J_r = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \in \mathrm{M}_{\ell,r}(\mathbb{R}).$$

Let us decompose UB_2V as $UB_2V = \left(C_1 \; C_2 \right) \in \mathrm{M}_\ell(\mathbb{R})$ where the matrix $C_1 \in \mathrm{M}_{\ell,r}(\mathbb{R})$ is formed with the first r columns of UB_2V, and the matrix $C_2 \in \mathrm{M}_{\ell,\ell-r}(\mathbb{R})$ is formed with the last $\ell - r$ columns of UB_2V. Thus, the matrices $\begin{pmatrix} J_r & 0 \\ C_1 & C_2 \end{pmatrix}$ and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ are equivalent since

$$\begin{pmatrix} J_r & 0 \\ C_1 & C_2 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} V \in \mathrm{M}_{2\ell,\ell}(\mathbb{R}). \tag{4.5}$$

Since I_ℓ commutes with UB_1V, Claim 4.1 implies that for all $Q \in \mathrm{M}_\ell(\mathbb{R})$:

$$\det\begin{pmatrix} I_\ell & UB_1V \\ UQU^{-1} & UB_2V \end{pmatrix} = \det(UB_2V - UQU^{-1}UB_1V) = \det(U)\Delta_Q \det(V) = 0$$

using Equation (4.4). Since this is true for all $Q \in \mathrm{M}_\ell(\mathbb{R})$, let $Q' = UQU^{-1}$ to get

$$\forall\, Q' \in \mathrm{M}_\ell(\mathbb{R}), \quad \Delta'_{Q'} = \det\begin{pmatrix} I_\ell & UB_1V \\ Q' & UB_2V \end{pmatrix} = 0.$$

Let $R \in \mathrm{M}_{\ell,r}(\mathbb{R})$, and let us define a block matrix as $Q' = \begin{pmatrix} C_1 - R & 0 \end{pmatrix} \in \mathrm{M}_\ell(\mathbb{R})$. Since I_ℓ and UB_1V commute, Claim 4.1 implies that

$$0 = \Delta'_{Q'} = \det(UB_2V - Q'UB_1V) = \det\left(\begin{pmatrix} C_1 & C_2 \end{pmatrix} - \begin{pmatrix} C_1 - R & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\right) = \det\begin{pmatrix} R & C_2 \end{pmatrix}.$$

If $\mathrm{rk}(C_2) = \ell - r$, then it would be possible to find R such that $\mathrm{rk}\left(\begin{smallmatrix} R & C_2 \end{smallmatrix} \right) = \ell$, which can not be since its determinant $\Delta'_{Q'}$ would be non-zero. Therefore, $\mathrm{rk}(C_2) < \ell - r$. Equation (4.5) yields

$$\mathrm{rk}(M_B) = \mathrm{rk}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \mathrm{rk}\begin{pmatrix} J_r & 0 \\ C_1 & C_2 \end{pmatrix} = r + \mathrm{rk}(C_2) < r + \ell - r = \ell,$$

which can not be since $\dim B = \ell = \mathrm{rk}(M_B)$; hence $A \cap B = \{0\}$ for all $B \in \mathfrak{R}_{2\ell}(\ell)$, i.e. $A \in \mathfrak{I}_{2\ell}(\ell,\ell)_1 \subset \mathfrak{I}_n(\ell,\ell)_\ell$.

The subspace A having been constructed, let us construct rational subspaces B_N for $N \geqslant 1$ approaching A to its ℓ-th angle to the exponent exactly β. Then, we will show that these subspaces B_N are the ones providing the best approximation of A to its ℓ-th angle, which will finally give $\mu_n(A|\ell)_\ell = \beta$.

For $(i,j) \in \{1,\ldots,\ell\}^2$ and $N \geqslant 1$, let

$$f_N^{(i,j)} = \theta^{\lfloor \alpha^N \rfloor} \sum_{k=0}^{N} \frac{e_k^{(i,j)}}{\theta^{\lfloor \alpha^k \rfloor}} \in \mathbb{Z}$$

and M_{B_N} be the block matrix

$$M_{B_N} = \begin{pmatrix} \theta^{\lfloor \alpha^N \rfloor} I_\ell \\ F_N \end{pmatrix} \in \mathrm{M}_{2\ell,\ell}(\mathbb{Z})$$

where F_N is the matrix $(f_N^{(i,j)})_{(i,j)\in\{1,\ldots,\ell\}^2} \in \mathrm{M}_\ell(\mathbb{Z})$. Let us denote by $X_N^{(1)}, \ldots, X_N^{(\ell)}$ the columns of M_{B_N}, and let

$$B_N = \mathrm{Span}(X_N^{(1)}, \ldots, X_N^{(\ell)}) \in \mathfrak{R}_{2\ell}(\ell).$$

One can notice that for all $(i,j) \in \{1,\dots,\ell\}^2$:

$$\sum_{k=N+1}^{\infty} \frac{e_k^{(i,j)}}{\theta^{\lfloor \alpha^k \rfloor}} \leqslant (2\ell+1) \sum_{j=\lfloor \alpha^{N+1} \rfloor}^{\infty} \frac{1}{\theta^j} < \frac{4\ell+2}{\theta^{\lfloor \alpha^{N+1} \rfloor}}$$

because $\theta > 2$, so

$$0 < \xi_{i,j} - \frac{f_N^{(i,j)}}{\theta^{\lfloor \alpha^N \rfloor}} = \sum_{k=N+1}^{\infty} \frac{e_k^{(i,j)}}{\theta^{\lfloor \alpha^k \rfloor}} < \frac{4\ell+2}{\theta^{\lfloor \alpha^{N+1} \rfloor}}. \tag{4.6}$$

Let us show that there exists a constant $c_1 > 0$ depending only on $Y_1, \dots, Y_\ell$ such that

$$\forall\, N \geqslant 1, \quad \psi_\ell(A, B_N) \leqslant \frac{c_1}{H(B_N)^{\alpha/\ell}},$$

which will imply that $\mu_{2\ell}(A|\ell)_\ell \geqslant \alpha/\ell$. In order to do this, let us establish an upper bound for the height of the B_N. We will see later (Claim 4.4) that this upper bound is in fact optimal up to a multiplicative constant.

Claim 4.2. For all $N \geqslant 1$, one has

$$H(B_N) \leqslant c_2 (\theta^{\lfloor \alpha^N \rfloor})^\ell$$

where $c_2 > 0$ depends only on ℓ.

Proof. Because $\theta \geqslant 2$, one has

$$|f_N^{(i,j)}| \leqslant (2\ell+1)\theta^{\lfloor \alpha^N \rfloor} \sum_{k=0}^{N} \frac{1}{\theta^{\lfloor \alpha^k \rfloor}} \leqslant 2(2\ell+1)\theta^{\lfloor \alpha^N \rfloor}. \tag{4.7}$$

Therefore, because all the 2ℓ coefficients of each $X_N^{(j)}$ are smaller than $2(2\ell+1)\cdot\theta^{\lfloor \alpha^N \rfloor}$:

$$H(B_N) \leqslant \|X_N^{(1)} \wedge \dots \wedge X_N^{(\ell)}\| \leqslant \prod_{j=1}^{\ell} \|X_N^{(j)}\| \leqslant (2(2\ell+1)\cdot\sqrt{2\ell})^\ell (\theta^{\lfloor \alpha^N \rfloor})^\ell. \quad \square$$

Let us state a special case of Lemma 6.1 of [6] which will be used below.

Lemma 4.3. *Let $F_1, \dots, F_\ell, B_1, \dots, B_\ell$ be 2ℓ lines of $\mathbb{R}^{2\ell}$. Assume that the F_i span a subspace of dimension ℓ and so do the B_i. Let $F = F_1 \oplus \dots \oplus F_\ell$ and $B = B_1 \oplus \dots \oplus B_\ell$, then one has*

$$\psi_\ell(F, B) \leqslant c_F \sum_{i=1}^{\ell} \psi_1(F_i, B_i)$$

where $c_F > 0$ is a constant depending only on $F_1, \dots, F_\ell$.

For $i \in \{1, \dots, \ell\}$, let $Z_N^{(i)} = \theta^{-\lfloor \alpha^N \rfloor} X_N^{(i)}$. Notice that the definition of $\psi(X,Y)$ leads to the following elementary claim.

Claim 4.3. If X and Y are non-zero vectors, then $\psi(X,Y) \leqslant \|X - Y\|/\|X\|$.

Here, $\|Y_i\| \geqslant 1$, so Claim 4.3 combined with Inequality (4.6) implies that

$$\psi(X_N^{(i)}, Y_i) = \psi(Z_N^{(i)}, Y_i) \leqslant \frac{\|Z_N^{(i)} - Y_i\|}{\|Y_i\|} \leqslant c_3(\theta^{\lfloor \alpha^N \rfloor})^{-\alpha} \tag{4.8}$$

with $c_3 > 0$ depending only on ℓ. Lemma 4.3 gives a constant $c_4 > 0$ depending only on $Y_1, \dots, Y_\ell$ such that

$$\psi_\ell(A, B_N) \leqslant c_4 \sum_{i=1}^{\ell} \psi(X_N^{(i)}, Y_i) \leqslant \frac{c_5}{(\theta^{\lfloor \alpha^N \rfloor})^\alpha} \tag{4.9}$$

with $c_5 > 0$ depending only on $Y_1, \dots, Y_\ell$. Using Claim 4.2 which asserts that $H(B_N) \leqslant c_2(\theta^{\lfloor \alpha^N \rfloor})^\ell$, this yields $\psi_\ell(A, B_N) \leqslant c_1 H(B_N)^{-\alpha/\ell}$.

Now, we will show that the B_N achieve the best approximation of A to its ℓ-th angle. Concretely, let us prove that if $\varepsilon > 0$ and $C \in \mathfrak{R}_{2\ell}(\ell)$ are such that

$$\psi_\ell(A, C) \leqslant \frac{1}{H(C)^{\alpha/\ell + \varepsilon}} \tag{4.10}$$

and if $H(C)$ is large enough (in terms of ℓ and ε), then there exists $N \geqslant 1$ such that $C = B_N$.

Since C is a rational subspace, there exist $v_1, \dots, v_\ell \in \mathbb{Z}^{2\ell}$ such that $(v_1, \dots, v_\ell)$ is a $\mathbb{Z}$-basis of $C \cap \mathbb{Z}^{2\ell}$. One has $H(C) = \|v_1 \wedge \cdots \wedge v_\ell\|$ with Theorem 1.1. To prove that $C = B_N$ for some integer $N \geqslant 1$, let us show that all the $X_N^{(i)}$ for $i \in \{1, \dots, \ell\}$ are in $C = \mathrm{Span}(v_1, \dots, v_\ell)$. Since $\dim C = \dim B_N$, it will imply that $C = B_N$. Let $N \geqslant 1$ and $i \in \{1, \dots, \ell\}$; let us consider the $\ell + 1$ vectors $X_N^{(i)}, v_1, \dots, v_\ell$, and let $Q = \left(X_N^{(i)}\ v_1 \cdots v_\ell \right) \in \mathrm{M}_{2\ell, \ell+1}(\mathbb{Z})$. Since $v_1, \dots, v_\ell$ are linearly independent, to show that $X_N^{(i)} \in \mathrm{Span}(v_1, \dots, v_\ell)$, it is sufficient to show that $\mathrm{rk}(Q) < \ell + 1$, i.e. that all $(\ell+1) \times (\ell+1)$ minors of Q are zero. For this purpose, let us establish that $D = \|X_N^{(i)} \wedge v_1 \wedge \cdots \wedge v_\ell\| = 0$.

Let $p_C^\perp$ be the orthogonal projection onto C and h the vector $h = p_C^\perp(X_N^{(i)}) - X_N^{(i)}$. There exist $\lambda_1, \dots, \lambda_\ell \in \mathbb{R}$ such that $X_N^{(i)}$ can be written as $X_N^{(i)} = \sum_{j=1}^{\ell} \lambda_j v_j - h$. Because $h \in C^\perp$, one has

$$D = \left\| \left(\sum_{j=1}^{\ell} \lambda_j v_j - h \right) \wedge v_1 \wedge \cdots \wedge v_\ell \right\| = \|h\| \cdot \|v_1 \wedge \cdots \wedge v_\ell\| = \|h\| H(C).$$

Moreover, $\|h\| = \|X_N^{(i)}\| \psi_1(X_N^{(i)}, C)$, so

$$D \leqslant c_6 \theta^{\lfloor \alpha^N \rfloor} (\psi(X_N^{(i)}, Y_i) + \psi_1(\mathrm{Span}(Y_i), C)) H(C)$$

with $c_6 > 0$ depending only on ℓ, using Equation (4.7) and a triangle inequality on ψ (for all non-zero vectors Z_1, Z_2, Z_3, one has $\psi(Z_1, Z_2) \leqslant \psi(Z_1, Z_3) + \psi(Z_3, Z_2)$; see [12, Equation (3) p. 446]). Lemma 2.3 yields

$\psi_1(\mathrm{Span}(Y_i), C) \leqslant \psi_\ell(A, C)$; using the upper bound (4.8) to deal with $\psi(Y_i, X_N^{(i)})$ and the upper bound (4.10) to deal with $\psi_\ell(A, C)$, one gets

$$D \leqslant c_6 \theta^{\lfloor \alpha^N \rfloor} H(C) \left(\frac{c_3}{(\theta^{\lfloor \alpha^N \rfloor})^\alpha} + \frac{1}{H(C)^{\alpha/\ell+\varepsilon}} \right) \leqslant c_7 \left(\frac{H(C)}{\theta^{\lfloor \alpha^N \rfloor (\alpha-1)}} + \frac{\theta^{\lfloor \alpha^N \rfloor}}{H(C)^{\alpha/\ell - 1+\varepsilon}} \right) \tag{4.11}$$

with $c_7 > 0$ depending only on ℓ.

From now on, let us choose a particular N: let N be the largest integer such that $\theta^{\alpha^N} \leqslant H(C)^{\alpha/\ell-1+\varepsilon/2}$. Notice that $\theta^{\lfloor \alpha^N \rfloor} \leqslant H(C)^{\alpha/\ell-1+\varepsilon/2}$. Because N is maximal, one has $(\theta^{\alpha^N})^\alpha = \theta^{\alpha^{N+1}} > H(C)^{\alpha/\ell-1+\varepsilon/2}$, so $\theta^{\alpha^N} > H(C)^{(\alpha/\ell-1+\varepsilon/2)/\alpha}$. Since $\alpha = \ell\beta \geqslant (2\ell+1+\sqrt{1+4\ell^2})/2$ yields $\alpha^2 - (2\ell+1)\alpha + \ell \geqslant 0$, so $(\alpha/\ell - 1)/\alpha \geqslant 1/(\alpha-1)$. Thus, $\theta^{\alpha^N} > H(C)^{1/(\alpha-1)+\varepsilon/(2\alpha)}$, whence $\theta^{\lfloor \alpha^N \rfloor} > \theta^{-1} H(C)^{1/(\alpha-1)+\varepsilon/(2\alpha)}$. Therefore, coming back to Inequality (4.11) gives

$$D \leqslant c_8 \left(\frac{1}{H(C)^{(\alpha-1)\varepsilon/(2\alpha)}} + \frac{1}{H(C)^{\varepsilon/2}} \right) \xrightarrow[H(C)\to+\infty]{} 0$$

with $c_8 > 0$ depending only on ℓ. If $H(C)$ is large enough (in terms of ℓ and ε, because $(\alpha-1)/(2\alpha) \geqslant (\ell-1)/(2\ell)$), one has $D < 1$. But $E = \|X_N^{(i)} \wedge v_1 \cdots \wedge v_\ell\|_\infty$ is a positive integer such that $E \leqslant D$, so $E = 0$, i.e. $X_N^{(i)} \wedge v_1 \wedge \cdots \wedge v_\ell = 0$. Thus, it has been shown that if $H(C)$ is large enough, $C = B_N$ where N is the largest integer such that $\theta^{\alpha^N} \leqslant H(C)^{\alpha/\ell-1+\varepsilon/2}$.

Now that it has been established that if $N \in \mathbb{N}^*$ is large enough, the rational subspaces B_N give the best possible approximation of A to its ℓ-th angle, let us show that they approach the subspace A at most to the exponent α/ℓ. In other words, we shall prove that for N large enough, one has

$$\psi_\ell(A, B_N) \geqslant \frac{c}{H(B_N)^{\alpha/\ell}}$$

with $c > 0$ depending only on $Y_1, \ldots, Y_\ell$. For this, we need to establish a lower bound on the height of the B_N (which will imply that the upper bound in Claim 4.2 is optimal up to a multiplicative constant).

Claim 4.4. For all N large enough, one has

$$H(B_N) \geqslant \widetilde{c}(\theta^{\lfloor \alpha^N \rfloor})^\ell$$

with $\widetilde{c} > 0$ depending only on $Y_1, \ldots, Y_\ell$.

Proof of Claim 4.4. Let $N \geqslant 1$; let us establish that the family $(X_N^{(1)}, \ldots, X_N^{(\ell)})$ is a $\mathbb{Z}$-basis of $B_N \cap \mathbb{Z}^{2\ell}$. For this purpose, let us denote by

P the parallelotope spanned by $X_N^{(1)},\dots,X_N^{(\ell)}$, i.e.

$$P=\left\{\sum_{i=1}^{\ell}\lambda_i X_N^{(i)},\ (\lambda_1,\dots,\lambda_\ell)\in[0,1]^\ell\right\},$$

and let us show that the 2^ℓ vertices of P are its only integer points. Let $\mathcal{S}$ be the set of the 2^ℓ vertices of P, i.e.

$$\mathcal{S}=\left\{\sum_{i=1}^{\ell}\delta_i X_N^{(i)},\ (\delta_1,\dots,\delta_\ell)\in\{0,1\}^\ell\right\}.$$

Assume that there exists $X\in(P\setminus\mathcal{S})\cap\mathbb{Z}^{2\ell}$, and let $(\lambda_1,\dots,\lambda_\ell)\in[0,1]^\ell\setminus\{0,1\}^\ell$ be such that $X=\lambda_1X_N^{(1)}+\dots+\lambda_\ell X_N^{(\ell)}\in\mathbb{Z}^{2\ell}$. The first ℓ coordinates of X give that for all $i\in\{1,\dots,\ell\}$, $\lambda_i\theta^{\lfloor\alpha^N\rfloor}\in\mathbb{Z}$. Thus, there exist integers $\gamma_1,\dots,\gamma_\ell\in\{0,\dots,\theta^{\lfloor\alpha^N\rfloor}\}$ such that for all $i\in\{1,\dots,\ell\}$, $\lambda_i=\gamma_i\theta^{-\lfloor\alpha^N\rfloor}$, because the λ_i are in $[0,1]$. Moreover, the last ℓ coordinates of X give that for all $i\in\{1,\dots,\ell\}$, $\lambda_1f_N^{(i,1)}+\dots+\lambda_\ell f_N^{(i,\ell)}\in\mathbb{Z}$, so

$$\text{(4.12)}\quad \forall\, i\in\{1,\dots,\ell\},\qquad \sum_{j=1}^{\ell}\frac{\gamma_j}{\theta^{\lfloor\alpha^N\rfloor}}\cdot\theta^{\lfloor\alpha^N\rfloor}\sum_{k=0}^{N}\frac{e_k^{(i,j)}}{\theta^{\lfloor\alpha^k\rfloor}}=\sum_{k=0}^{N}\frac{1}{\theta^{\lfloor\alpha^k\rfloor}}\sum_{j=1}^{\ell}\gamma_je_k^{(i,j)}\in\mathbb{Z}.$$

For $k\in\{0,\dots,N\}$, let us denote by E_k the matrix $(e_k^{(i,j)})_{(i,j)\in\{1,\dots,\ell\}^2}\in\mathrm{M}_\ell(\mathbb{Z})$, and by Γ the column vector ${}^t(\gamma_1,\dots,\gamma_\ell)$. Thus, the ℓ equations given by (4.12) can be rewritten using matrices as

$$\begin{pmatrix}\displaystyle\sum_{k=0}^{N}\frac{e_k^{(1,1)}}{\theta^{\lfloor\alpha^k\rfloor}} & \cdots & \displaystyle\sum_{k=0}^{N}\frac{e_k^{(1,\ell)}}{\theta^{\lfloor\alpha^k\rfloor}}\\ \vdots & & \vdots\\ \displaystyle\sum_{k=0}^{N}\frac{e_k^{(\ell,1)}}{\theta^{\lfloor\alpha^k\rfloor}} & \cdots & \displaystyle\sum_{k=0}^{N}\frac{e_k^{(\ell,\ell)}}{\theta^{\lfloor\alpha^k\rfloor}}\end{pmatrix}\begin{pmatrix}\gamma_1\\ \vdots\\ \vdots\\ \gamma_\ell\end{pmatrix}\in\mathbb{Z}^\ell,$$

which becomes

$$\sum_{k=0}^{N}\frac{1}{\theta^{\lfloor\alpha^k\rfloor}}E_k\Gamma\in\mathbb{Z}^\ell,$$

and to highlight the last term of this sum:

$$\sum_{k=0}^{N-1}\theta^{\lfloor\alpha^N\rfloor-\lfloor\alpha^k\rfloor}E_k\Gamma+E_N\Gamma\in\theta^{\lfloor\alpha^N\rfloor}\mathbb{Z}^\ell.$$

Since $E_N \in \mathrm{M}_\ell(\mathbb{Z})$, the transpose of its comatrix also belongs to $\mathrm{M}_\ell(\mathbb{Z})$, so

$$\sum_{k=0}^{N-1} \theta^{\lfloor \alpha^N \rfloor - \lfloor \alpha^k \rfloor}\, {}^t\mathrm{com}(E_N) E_k \Gamma + \det(E_N)\Gamma \in \theta^{\lfloor \alpha^N \rfloor}\mathbb{Z}^\ell.$$

For $i \in \{1, \dots \ell\}$ and $k \in \{0, \dots, N-1\}$, let us denote by $L_{k,i} \in \mathrm{M}_{1,\ell}(\mathbb{Z})$ the i-th row of the product ${}^t\mathrm{com}(E_N)E_k$. Thus,

$$\forall\, i \in \{1, \dots, \ell\}, \quad \sum_{k=0}^{N-1} (L_{k,i}\Gamma)\theta^{\lfloor \alpha^N \rfloor - \lfloor \alpha^k \rfloor} + \det(E_N)\gamma_i \in \theta^{\lfloor \alpha^N \rfloor}\mathbb{Z}. \tag{4.13}$$

Let $j \in \{1, \dots, \ell\}$; notice that $e_N^{(j,j)} \geqslant 2\ell > \sum_{i \neq j} e_N^{(i,j)}$, so E_N is a strictly diagonally dominant matrix (the case $\ell = 1$ being trivial). Therefore E_N is invertible, so $\det(E_N) \neq 0$. Moreover, $|e_N^{(i,j)}| \leqslant 2$ if $i \neq j$ and $|e_N^{(i,i)}| \leqslant 2\ell+1$, so by definition of θ (see Inequality (4.2)):

$$|\det(E_N)| = \left| \sum_{\sigma \in \mathfrak{S}_\ell} \varepsilon(\sigma) \prod_{i=1}^{\ell} e_N^{(i,\sigma(i))} \right| \leqslant \ell!\,(2\ell+1)^\ell < \theta.$$

Thus, since $0 < |\det(E_N)| < \theta$, one has $v_\theta(\det(E_N)) = 0$, so $v_\theta(\det(E_N)\gamma_i) = v_\theta(\gamma_i)$. Let $u \geqslant 0$ and $i_0 \in \{1, \dots, \ell\}$ be such that $u = \min(v_\theta(\gamma_1), \dots, v_\theta(\gamma_\ell)) = v_\theta(\gamma_{i_0})$. Coming back to Equation (4.13) yields

$$\forall\, i \in \{1, \dots, \ell\}, \quad v_\theta\left(\sum_{k=0}^{N-1} (L_{k,i}\Gamma)\theta^{\lfloor \alpha^N \rfloor - \lfloor \alpha^k \rfloor} + \det(E_N)\gamma_i\right) \geqslant \lfloor \alpha^N \rfloor,$$

with the convention $v_\theta(0) = +\infty$. Therefore,

$$\forall\, k \in \{0, \dots, N-1\}, \quad v_\theta(\theta^{\lfloor \alpha^N \rfloor - \lfloor \alpha^k \rfloor}) \geqslant \lfloor \alpha^N \rfloor - \lfloor \alpha^{N-1} \rfloor > 0,$$

and since $L_{k,i}\Gamma$ is a $\mathbb{Z}$-linear combination of the γ_i, $v_\theta(L_{k,i}\Gamma) \geqslant \min(v_\theta(\gamma_1), \dots, v_\theta(\gamma_\ell)) = u$. The particular case $i = i_0$ yields

$$v_\theta\left(\sum_{k=0}^{N-1} (L_{k,i_0}\Gamma)\theta^{\lfloor \alpha^N \rfloor - \lfloor \alpha^k \rfloor} + \det(E_N)\gamma_{i_0}\right) = u \geqslant \lfloor \alpha^N \rfloor.$$

Whence, by definition of u, for all $i \in \{1, \dots, \ell\}$, one has $v_\theta(\gamma_i) \geqslant \lfloor \alpha^N \rfloor$. Since all the γ_i are in $\{0, \dots, \theta^{\lfloor \alpha^N \rfloor}\}$, this implies that for all $i \in \{1, \dots, \ell\}$, $\gamma_i \in \{0, \theta^{\lfloor \alpha^N \rfloor}\}$. Thus, $X \in \mathcal{S}$, which can not be.

It has been shown that the only integer points of P are the ones in $\mathcal{S}$, which implies that the family $(X_N^{(1)}, \dots, X_N^{(\ell)})$ is a $\mathbb{Z}$-basis of $B_N \cap \mathbb{Z}^{2\ell}$. Thus, using Theorem 1.1, $H(B_N) = \|X_N^{(1)} \wedge \cdots \wedge X_N^{(\ell)}\|$. But

$$\|\theta^{-\lfloor \alpha^N \rfloor} X_N^{(1)} \wedge \cdots \wedge \theta^{-\lfloor \alpha^N \rfloor} X_N^{(\ell)}\| \xrightarrow[N \to +\infty]{} \|Y_1 \wedge \cdots \wedge Y_\ell\|,$$

so for N large enough:

$$H(B_N) = \left(\theta^{\lfloor \alpha^N \rfloor}\right)^{\ell} \|\theta^{-\lfloor \alpha^N \rfloor} X_N^{(1)} \wedge \cdots \wedge \theta^{-\lfloor \alpha^N \rfloor} X_N^{(\ell)}\| \geqslant \widetilde{c} \left(\theta^{\lfloor \alpha^N \rfloor}\right)^{\ell}$$

with $\widetilde{c} > 0$ depending only on $Y_1, \dots, Y_\ell$. $\square$

Let $Z_N^{(1)} = \theta^{-\lfloor \alpha^N \rfloor} X_N^{(1)}$ and $p_A^\perp$ be the orthogonal projection onto A. Lemma 2.3 gives

$$\psi_\ell(A, B_N) \geqslant \psi_1(\mathrm{Span}(Z_N^{(1)}), A) = \psi(Z_N^{(1)}, p_A^\perp(Z_N^{(1)})). \tag{4.14}$$

Let $\Delta = p_A^\perp(Z_N^{(1)}) - Y_1$ and $\omega = \|p_A^\perp(Z_N^{(1)}) - Z_N^{(1)}\|$. Let us decompose the vector $p_A^\perp(Z_N^{(1)})$ in the basis $(Y_1, \dots, Y_\ell)$:

$$p_A^\perp(Z_N^{(1)}) = \sum_{i=1}^{\ell} \lambda_i Y_i = {}^t\begin{pmatrix}\lambda_1 & \cdots & \lambda_\ell & \star & \cdots & \star\end{pmatrix}$$

where the $\star$ are unspecified coefficients, because for all $i \in \{1, \dots, \ell\}$, the vector Y_i can be written $Y_i = {}^t\left(\delta_{i,1} \cdots \delta_{i,\ell} \star \cdots \star\right)$ where δ is the Kronecker delta. Moreover, for all $i \in \{1, \dots, \ell\}$, $Z_N^{(i)} = {}^t\left(\delta_{i,1} \cdots \delta_{i,\ell} \star \cdots \star\right)$, so $\omega^2 = \|p_A^\perp(Z_N^{(1)}) - Z_N^{(1)}\|^2 \geqslant (\lambda_1 - 1)^2 + \sum_{i=2}^{\ell} \lambda_i^2$. Thus, $|\lambda_1 - 1| \leqslant \omega$, and for all $i \in \{2, \dots, \ell\}$, $|\lambda_i| \leqslant \omega$. Let $j \in \{1, \dots, \ell\}$. One has $\|Y_j\|^2 = 1 + \sum_{i=1}^{\ell}\left(\sum_{k=0}^{\infty} \frac{e_k^{(i,j)}}{\theta^{\lfloor \alpha^k \rfloor}}\right)^2$, but $\alpha > 2$, $\theta \geqslant 2$ and $2\ell + 1 \leqslant \theta$ with Hypothesis (4.2) on θ, so a simple computation gives $\|Y_j\| \leqslant \sqrt{1 + 4\ell} = c_9$. Notice that $\Delta = p_A^\perp(Z_N^{(1)}) - Y_1 = (\lambda_1 - 1)Y_1 + \sum_{i=2}^{\ell} \lambda_i Y_i$, so $\|\Delta\| \leqslant c_{10}\omega$, with $c_{10} > 0$ depending only on ℓ. One has

$$\begin{aligned}\|Z_N^{(1)} \wedge p_A^\perp(Z_N^{(1)})\| &= \|Z_N^{(1)} \wedge (Y_1 + p_A^\perp(Z_N^{(1)}) - Y_1)\| \\ &\geqslant \|Z_N^{(1)} \wedge Y_1\| - \|Z_N^{(1)} \wedge \Delta\|.\end{aligned} \tag{4.15}$$

Notice that $\begin{pmatrix} Z_N^{(1)} & Y_1 \end{pmatrix} \in \mathrm{M}_{2\ell,2}(\mathbb{R})$, and let us denote by $\eta_{i,j}$ its the 2×2 minor corresponding to its i-th and j-th rows with $i < j$; one has

$$\begin{aligned}\|Z_N^{(1)} \wedge Y_1\| &= \sqrt{\sum_{1 \leqslant i < j \leqslant 2\ell} \eta_{i,j}^2} \geqslant |\eta_{1,\ell+1}| = \eta_{1,\ell+1} \\ &= \begin{vmatrix} 1 & 1 \\ \sum_{k=0}^{N} \frac{e_k^{(1,1)}}{\theta^{\lfloor \alpha^k \rfloor}} & \sum_{k=0}^{\infty} \frac{e_k^{(1,1)}}{\theta^{\lfloor \alpha^k \rfloor}} \end{vmatrix} \geqslant \frac{1}{\theta^{\lfloor \alpha^{N+1} \rfloor}}.\end{aligned}$$

Moreover, in the same fashion that it was shown above that $\|Y_j\| \leqslant c_9$, one can establish that $\|Z_N^{(1)}\| \leqslant c_9$. Since $\|\Delta\| \leqslant c_{10}\omega$, it yields $\|Z_N^{(1)} \wedge \Delta\| \leqslant \|Z_N^{(1)}\| \cdot \|\Delta\| \leqslant c_{11}\omega$ with $c_{11} > 0$ depending only on ℓ. With Inequality (4.15), $\|Z_N^{(1)} \wedge p_A^\perp(Z_N^{(1)})\| \geqslant \theta^{-\lfloor \alpha^{N+1} \rfloor} - c_{11}\omega$. Because $\|p_A^\perp(Z_N^{(1)})\| \leqslant$

$\|Z_N^{(1)}\| \leqslant c_9$, one has

$$\omega = \|p_A^{\perp}(Z_N^{(1)}) - Z_N^{(1)}\| = \|Z_N^{(1)}\| \psi(p_A^{\perp}(Z_N^{(1)}), Z_N^{(1)}) = \|Z_N^{(1)}\| \frac{\|p_A^{\perp}(Z_N^{(1)}) \wedge Z_N^{(1)}\|}{\|Z_N^{(1)}\| \cdot \|p_A^{\perp}(Z_N^{(1)})\|} \geqslant \frac{c_{12}}{\theta^{\lfloor \alpha^{N+1} \rfloor}} - c_{13}\omega$$

with $c_{12}, c_{13} > 0$ depending only on ℓ. Hence, $\omega \geqslant c_{12}(1+c_{13})^{-1}\theta^{-\lfloor \alpha^{N+1} \rfloor} = c_{14}\theta^{-\lfloor \alpha^{N+1} \rfloor}$ with $c_{14} > 0$ depending only on ℓ. Let us use Inequality (4.14) to get:

$$\psi_\ell(A, B_N) \geqslant \psi(Z_N^{(1)}, p_A^{\perp}(Z_N^{(1)})) = \frac{\omega}{\|Z_N^{(1)}\|} \geqslant \frac{c_{15}}{\theta^{\lfloor \alpha^{N+1} \rfloor}} \tag{4.16}$$

with $c_{15} > 0$ depending only on ℓ. Notice that $\lfloor \alpha^{N+1} \rfloor \leqslant \alpha^{N+1} \leqslant \lfloor \alpha^N \rfloor \alpha + \alpha$, so $\theta^{-\lfloor \alpha^{N+1} \rfloor} \geqslant \theta^{-\lfloor \alpha^N \rfloor \alpha - \alpha} = \theta^{-\alpha} \cdot (\theta^{-\lfloor \alpha^N \rfloor})^{\alpha}$. Moreover, Claim 4.4 gives a constant $c_{16} > 0$, depending only on $Y_1, \dots, Y_\ell$, such that $H(B_N) \geqslant c_{16}(\theta^{\lfloor \alpha^N \rfloor})^{\ell}$. Thus, with Inequality (4.16), one has

$$\psi_\ell(A, B_N) \geqslant \frac{c_{15}}{\theta^{\lfloor \alpha^{N+1} \rfloor}} \geqslant \frac{c_{15}}{\theta^{\alpha}} \cdot \frac{1}{\theta^{\alpha \lfloor \alpha^N \rfloor}} \geqslant \frac{c_{17}}{H(B_N)^{\alpha/\ell}}$$

with $c_{17} > 0$ depending only on $Y_1, \dots, Y_\ell$.

Notice that we have just proved: $\mu_{2\ell}(A|\ell)_\ell \leqslant \beta$; therefore A is such that $\mu_{2\ell}(A|\ell)_\ell = \beta$.

Finally, only the cases $\beta = +\infty$ and $n > 2\ell$ remain to prove. Let us start by assuming that $n = 2\ell$. If $\beta = +\infty$, for $(i,j) \in \{1, \dots, \ell\}^2$ let $\xi_{i,j} = \sum_{k=0}^{\infty} e_k^{(i,j)} 3^{-k^k}$ where the $(e_k^{(i,j)})_{k \in \mathbb{N}}$ are sequences yet to be determined, with values in $\{1, 2\}$. With the same notation as before, let $M_\xi = (\xi_{i,j})_{(i,j) \in \{1,\dots,\ell\}^2} \in \mathrm{M}_\ell(\mathbb{R})$ and let us denote by A_∞ the subspace spanned by the columns $Y_1, \dots, Y_\ell$ of the matrix $\begin{pmatrix} I_\ell \\ M_\xi \end{pmatrix} \in \mathrm{M}_{2\ell,\ell}(\mathbb{R})$. In the same way as it was done above, one can choose sequences $(e_k^{(i,j)})_{k \in \mathbb{N}}$, for $(i,j) \in \{1, \dots, \ell\}^2$, such that $A_\infty \in \mathfrak{I}_n(\ell, \ell)_\ell$. For $(i,j) \in \{1, \dots, \ell\}^2$ and $N \geqslant 1$, let $f_N^{(i,j)} = 3^{N^N} \sum_{k=0}^{N} e_k^{(i,j)} 3^{-k^k}$, and let us denote by $B_N \in \mathfrak{R}_{2\ell}(\ell)$ the rational subspace spanned by the columns of $\begin{pmatrix} 3^{N^N} I_\ell \\ F_N \end{pmatrix} \in \mathrm{M}_{2\ell,\ell}(\mathbb{R})$ where $F_N = (f_N^{(i,j)})_{(i,j) \in \{1,\dots,\ell\}^2}$. Again in a similar fashion as before, one can show that there exists a constant $c > 0$ depending only on $Y_1, \dots, Y_\ell$ such that for all $N \geqslant 1$, $\psi_\ell(A_\infty, B_N) \leqslant cH(B_N)^{-N/\ell}$. Thus,

$$\forall\, \kappa > 0, \quad \forall\, N \geqslant \kappa\ell, \quad \psi_\ell(A_\infty, B_N) \leqslant \frac{c}{H(B_N)^{\kappa}}. \tag{4.17}$$

Notice that $\psi_\ell(A_\infty, B_N)$ tends to 0 when N tends to infinity. Therefore, there exist infinitely many pairwise distinct subspaces B_N satisfying Inequality (4.17), so for all $\kappa > 0$, $\mu_n(A_\infty|\ell)_\ell \geqslant \kappa$, therefore $\mu_n(A_\infty|\ell)_\ell = +\infty$.

Let us finally consider the case $n > 2\ell$. Let us denote by ϕ a rational isomorphism from $\mathbb{R}^{2\ell}$ to $\mathbb{R}^{2\ell} \times \{0\}^{n-2\ell}$. Let $A' = \phi(A)$; Theorem 1.2 yields $A' \in \mathfrak{I}_n(\ell,\ell)_\ell$ and

$$\mu_n(A'|\ell)_\ell = \mu_{2\ell}(A|\ell)_\ell = \beta$$

which allows us to extend the result to integers $n > 2\ell$. □

Remark 4.4. Since $\ell \geqslant 1$ and since β satisfies Inequality (4.1), one has $\alpha = \ell\beta \geqslant (3+\sqrt{5})/2$. In the case $\ell = 1$, we fall back on a known result on the irrationality exponent of $\xi_{1,1}$:

$$\mu\left(\sum_{k=0}^{\infty} \frac{e_k^{(1,1)}}{\theta^{\lfloor \alpha^k \rfloor}}\right) = \alpha$$

where $\mu(\cdot)$ stands for the irrationality exponent, $(e_k^{(1,1)})_{k\in\mathbb{N}}$ is a sequence with values in $\{2,3\}$, θ is a prime number strictly greater than 3, and α is a real number greater than $(3+\sqrt{5})/2$. The arguments in [8, Section 8] lead easily to this result, but the method developed here is different (and the case $\theta = 3$ is not covered here). If $2 \leqslant \alpha < (3+\sqrt{5})/2$, one still has $\mu(\xi_{1,1}) = \alpha$ thanks to Theorem 2 of [2].

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Elio JOSEPH
10 rue de Paris
91 400 Saclay, France
E-mail: josephelio@gmail.com