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A short note on higher Mordell integrals

par Joshua MALES

Résumé. On sait que les formes modulaires fausses classiques et les formes modulaires quantiques sont intimement liées aux intégrales de Mordell grâce à la thèse de doctorat révolutionnaire de Zwegers. Plus récemment, certaines généralisations des formes modulaires fausses/quantiques, appelées formes de profondeur supérieure (« higher depth »), ont été étudiées de manière intensive. En gros, une forme modulaire fausse/quantique de profondeur d est celle dont l'erreur de modularité se transforme comme une forme modulaire fausse/quantique de profondeur $d-1$. Dans cette courte note, nous utilisons des techniques de Bringmann, Kaszian et Milas pour montrer que les intégrales doubles d'Eichler d'une famille de formes modulaires quantiques de profondeur deux et de poids un, précédemment étudiées par l'auteur, peuvent être reliées à certaines intégrales de Mordell supérieures, ce qui signifie qu'elles peuvent être écrites comme une certaine intégrale double, à la Zwegers.

Abstract. Classical mock modular and quantum modular forms are known to have an intimate relationship with Mordell integrals thanks to Zwegers groundbreaking Ph.D. thesis. More recently, generalisations of mock/quantum modular forms to so-called "higher depth" versions have been intensively studied. In essence, a mock/quantum modular form of depth d is such that the error of modularity transforms as another mock/quantum modular form of depth $d-1$. In this short note we use techniques of Bringmann, Kaszian, and Milas to show that the double Eichler integrals of a family of depth two quantum modular forms of weight one previously studied by the author can be related to certain "higher" Mordell integrals, meaning it may be written as a certain double integral, à la Zwegers.

1. Introduction

The Mordell integral

$$h(z) = h(z;\tau) := \int_{\mathbb{R}} \frac{\cosh(2\pi z w)}{\cosh(\pi w)} e^{\pi i \tau w^2} \mathrm{d}w, \tag{1.1}$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$, is intricately linked to various areas of number theory. In particular, classical results show the connection between specialisations of (1.1) are connected to the Riemann zeta function [17], Gauss sums [10, 11], and class number formulas [13, 14].

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More recently, Zwegers used Mordell integrals to describe the completion of Lerch sums in his celebrated thesis [20]. In particular, Zwegers observed that we can relate (1.1) to an Eichler integral in the following way

$$h(a\tau - b) = -e^{-2\pi i a\left(b+\frac{1}{2}\right)} q^{\frac{a^2}{2}} \int_0^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(w)}{\sqrt{-i(w+\tau)}} \mathrm{d}w. \tag{1.2}$$

Here, $g_{a,b}$ is the weight $\frac{3}{2}$ unary theta function given by $(a, b \in \mathbb{R})$

$$g_{a,b}(\tau) := \sum_{n \in a+\mathbb{Z}} n e^{2\pi i b n} q^{\frac{n^2}{2}}.$$

Zwegers then showed that a modular completion of Lerch sums may be found. To do so, he found that the error of modularity $h(a\tau - b)$ also appears when considering integrals of the same form as (1.2) with lower integration boundary $-\bar{\tau}$ instead of 0.

Furthermore, Eichler integrals of the form

$$\int_{-\bar{\tau}}^{i\infty} \frac{g(w)}{(-i(w+\tau))^{\frac{3}{2}}} \mathrm{d}w$$

with g a cuspidal theta function have been studied by many authors in recent times, perhaps most notably in relation to quantum modular forms, e.g. [5, 7, 16]. Quantum modular forms were introduced by Zagier in [18, 19] and are essentially functions $f\colon \mathcal{Q} \to \mathbb{C}$ for some fixed $\mathcal{Q} \subseteq \mathbb{Q}$, whose errors of modularity (for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^k f(M\tau)$$

are in some sense "nicer" than the original function. Often, for example, the original function f is defined only on $\mathbb{Q}$, but the "errors of modularity" can be defined on some open subset of $\mathbb{R}$. Quantum modular forms have been the topic of much interest in the past decade, for example there is a fascinating connection between them and mock modular forms (surveyed in [15]) which has been investigated in papers such as [2, 5, 6], among others. Interesting examples of quantum modular forms also lie at the interface of physics and knot theory, see e.g. a study of Kashaev invariants of (p, q)-torus knots in [8, 9] and investigations of Zagier into limits of quantum invariants of 3-manifolds and knots [19]. Understanding the error of modularity of quantum modular forms is then clearly an important problem. This paper serves to extend results of Bringmann, Kaszian, and Milas to a certain infinite family of so-called quantum modular forms of depth two (see the sequel for precise definitions).

A certain generalisation of quantum modular forms was introduced in [3]. The authors define so-called higher depth quantum modular forms, and provide two examples of such forms of depth two that arise from characters of vertex operator algebras. In the simplest case, quantum modular forms of depth two are functions that satisfy

$$f(\tau) - (c\tau + d)^k f(M\tau) \in \mathsf{Q}_k(\Gamma)\mathcal{O}(R) + \mathcal{O}(R),$$

where $\mathsf{Q}_k(\Gamma)$ is the space of quantum modular forms of weight k on Γ, and $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$. A crucial step in showing the generalised quantum modularity property of their functions F_1 and F_2 is the appearance of a two-dimensional Eichler integral of the shape

$$\int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{g_1(\omega_1)g_2(\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}} \mathrm{d}\omega_2 \mathrm{d}\omega_1, \tag{1.3}$$

where the g_j lie in the space of vector-valued modular forms on $\mathrm{SL}_2(\mathbb{Z})$. In the next paper in the series of Bringmann, Kaszian, and Milas [4] the connection between such a two-dimensional Eichler integral and higher Mordell integrals is explored, in particular with the example of the function F_1 carried over from [3].

In particular, the higher Mordell integrals investigated in [4] provide the error of modularity of their function F_1, in turn developing the theory of Zwegers to higher dimensions. In the present paper, we take the example of depth two Mordell integrals of [4] and extend this to an infinite family of similar functions, thereby also providing an infinite family of errors of modularity of the relatively new higher depth quantum modular forms. One therefore also sees that by understanding higher Mordell integrals, we already obtain intrinsic information about the higher depth quantum modular form. Furthermore, the construction given in this paper gives hints as to how one could obtain similar results for arbitrary depth. The outline is sketched in the following.

In [12] a family of functions is given as a generalisation of the function F_1. Each function F in this more general family from [12] is of the shape (up to addition by one-dimensional partial theta functions)

$$\sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)},$$

with $Q(x) = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2$ a positive definite integral binary quadratic form, $\mathscr{S}$ a finite set of pairs $\alpha \in \mathbb{Q}^2 \backslash \{(0,0)\}$, and $\varepsilon\colon \mathscr{S} \to \mathbb{R}\backslash\{0\}$. Each F is shown to be vector-valued quantum modular form of depth two and weight one. Similarly to [3], a key compenent is the introduction of the double Eichler integral

$$\mathcal{E}_\alpha(\tau) := -\frac{\sqrt{D}}{4} \int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{\theta_1(\alpha;\omega_1,\omega_2) + \theta_2(\alpha;\omega_1,\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}} \mathrm{d}\omega_2 \mathrm{d}\omega_1, \tag{1.4}$$

where $D \coloneqq 4a_1a_3 - a_2^2 > 0$, and θ_1, θ_2 are given explicitly in Section 3. It is shown in [12] that using Shimura theta functions we may rewrite this in the form (1.3).

This short note serves to show that techniques of Bringmann, Kaszian, and Milas of relating their double Eichler integral to higher Mordell integrals in [4] immediately carry over to the more general setting of [12]. In a similar fashion to [4] we define

$$H_\alpha(\tau) \coloneqq -\sqrt{D}\int_0^\infty \int_{\omega_1}^\infty \frac{\theta_1(\alpha;\omega_1,\omega_2)+\theta_2(\alpha;\omega_1,\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_1+\tau)}}\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

along with the functions

$$\mathcal{F}_\alpha(x) \coloneqq \frac{\sinh(2\pi x)}{\cosh(2\pi x)-\cos(2\pi\alpha)}, \qquad \mathcal{G}_\alpha(x) \coloneqq \frac{\sin(2\pi\alpha)}{\cosh(2\pi x)-\cos(2\pi\alpha)}.$$

Our result is the following theorem (there is also a related expression for $\alpha \in \mathbb{Z}^2$, taking first a limit in α_1 and using the same method as below, and then taking a limit in α_2).

Theorem 1.1. *For $\alpha \notin \mathbb{Z}^2$, we have that*

$$H_\alpha(\tau) = \int_{\mathbb{R}^2} g_\alpha(\tau)\mathrm{d}\omega,$$

where we set

$$g_\alpha(\tau) \coloneqq \begin{cases} 2\mathcal{G}_{\alpha_1}(\omega_1)\mathcal{G}_{\alpha_2}(\omega_2) - 2\mathcal{F}_{\alpha_1}(\omega_1)\mathcal{F}_{\alpha_2}(\omega_2) & \text{if } \alpha_1,\alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_0(\omega_1)\mathcal{F}_{\alpha_2}(\omega_2) + \dfrac{2}{\pi\omega_1}\mathcal{F}_{\alpha_2}\left(\omega_2 + \dfrac{a_2}{2a_3}\omega_1\right) & \text{if } \alpha_1 \in \mathbb{Z}, \alpha_2 \notin \mathbb{Z}, \\ -2\mathcal{F}_{\alpha_1}(\omega_1)\mathcal{F}_0(\omega_2) + \dfrac{2}{\pi\omega_2}\mathcal{F}_{\alpha_1}\left(\omega_1 + \dfrac{a_2}{2a_1}\omega_2\right) & \text{if } \alpha_1 \notin \mathbb{Z}, \alpha_2 \in \mathbb{Z}. \end{cases}$$

2. Preliminaries

Here we recall a few relevant results on double error functions that we need in the rest of this note. We first define a rescaled version of the usual one-dimensional error function. For $u \in \mathbb{R}$ set

$$E(u) \coloneqq 2\int_0^u e^{-\pi\omega^2}\mathrm{d}\omega. \tag{2.1}$$

We also require, for non-zero u, the function

$$M(u) \coloneqq \frac{i}{\pi}\int_{\mathbb{R}-iu} \frac{e^{-\pi\omega^2-2\pi i u\omega}}{\omega}\mathrm{d}\omega.$$

A relation between $M(u)$ and $E(u)$, for non-zero u, is given by

$$M(u) = E(u) - \operatorname{sgn}(u). \tag{2.2}$$

We further need the two-dimensional analogues of the above functions. Following [1] and changing notation slightly, we define $E_2\colon \mathbb{R}\times\mathbb{R}^2\to\mathbb{R}$ by

$$E_2(\kappa;u):=\int_{\mathbb{R}^2}\operatorname{sgn}(\omega_1)\operatorname{sgn}(\omega_2+\kappa\omega_1)e^{-\pi\left((\omega_1-u_1)^2+(\omega_2-u_2)^2\right)}\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

where throughout we denote components of vectors just with subscripts. Again following [1], for $u_2, u_1-\kappa u_2\neq 0$, we define

$$M_2(\kappa;u_1,u_2):=-\frac{1}{\pi^2}\int_{\mathbb{R}-iu_2}\int_{\mathbb{R}-iu_1}\frac{e^{-\pi\omega_1^2-\pi\omega_2^2-2\pi i(u_1\omega_1+u_2\omega_2)}}{\omega_2(\omega_1-\kappa\omega_2)}\mathrm{d}\omega_1\mathrm{d}\omega_2. \tag{2.3}$$

Then we have that

$$\begin{aligned}M_2(\kappa;u_1,u_2)&=E_2(\kappa;u_1,u_2)-\operatorname{sgn}(u_2)M(u_1)\\&\quad-\operatorname{sgn}(u_1-\kappa u_2)M\left(\frac{u_2+\kappa u_1}{\sqrt{1+\kappa^2}}\right)-\operatorname{sgn}(u_1)\operatorname{sgn}(u_2+\kappa u_1).\end{aligned} \tag{2.4}$$

The relation (2.4) extends the definition of $M_2(u)$ to include $u_2=0$ or $u_1=\kappa u_2$ - note however that M_2 is discontinuous across these loci. Further, it is shown in the proof of Lemma 7.1 of [12] that for $u=u(n):=(2\sqrt{a_1}n_1+\frac{a_2}{\sqrt{a_1}}n_2, mn_2)$, along with $\kappa:=\frac{a_2}{\sqrt{D}}$, and $m:=\sqrt{4a_3-\frac{a_2^2}{a_1}}$ we have that

$$\begin{aligned}&M_2(\kappa;\sqrt{v}u)\\&\quad=-\frac{\sqrt{D}n_2(2a_1n_1+a_2n_2)}{2a_1}q^{Q(n)}\\&\qquad\times\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i(2a_1n_1+a_2n_2)^2\omega_1}{2a_1}}}{\sqrt{-i(\omega_1+\tau)}}\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi iDn_2^2\omega_2}{2a_1}}}{\sqrt{-i(\omega_2+\tau)}}\mathrm{d}\omega_2\mathrm{d}\omega_1\\&\qquad-\frac{\sqrt{D}n_1(a_2n_1+2a_3n_2)}{2a_3}q^{Q(n)}\\&\qquad\times\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i(a_2n_1+2a_3n_2)^2\omega_1}{2a_3}}}{\sqrt{-i(\omega_1+\tau)}}\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi iDn_1^2\omega_2}{2a_3}}}{\sqrt{-i(\omega_2+\tau)}}\mathrm{d}\omega_2\mathrm{d}\omega_1.\end{aligned} \tag{2.5}$$

3. Proof of Theorem 1.1

Proof. By analytic continuation it suffices to show that the theorem holds for $\tau=iv$, and we begin by showing that

$$H_\alpha(iv)=2\lim_{r\to\infty}\sum_{\substack{n\in\alpha+\mathbb{Z}^2\\|n_j-\alpha_j|\le r}}M_2\left(\kappa;\sqrt{\frac{v}{2}}u\right)e^{2\pi vQ(n)},$$

We begin with the expression (2.5) evaluated at $\tau = iv$, giving

$$M_2(\kappa;\sqrt{v}u) \tag{3.1}$$
$$= -\frac{\sqrt{D}n_2(2a_1n_1+a_2n_2)}{2a_1}q^{Q(n)} \times \int_{iv}^{i\infty} \frac{e^{\frac{\pi i(2a_1n_1+a_2n_2)^2\omega_1}{2a_1}}}{\sqrt{-i(\omega_1+iv)}} \int_{\omega_1}^{i\infty} \frac{e^{\frac{\pi i Dn_2^2\omega_2}{2a_1}}}{\sqrt{-i(\omega_2+iv)}} d\omega_2 d\omega_1$$
$$- \frac{\sqrt{D}n_1(a_2n_1+2a_3n_2)}{2a_3}q^{Q(n)} \times \int_{iv}^{i\infty} \frac{e^{\frac{\pi i(a_2n_1+2a_3n_2)^2\omega_1}{2a_3}}}{\sqrt{-i(\omega_1+iv)}} \int_{\omega_1}^{i\infty} \frac{e^{\frac{\pi i Dn_1^2\omega_2}{2a_3}}}{\sqrt{-i(\omega_2+iv)}} d\omega_2 d\omega_1.$$

We make the shift $\omega_j \to 2i\omega_j + iv$. The terms in the exponential in the first term on the right-hand side become

$$\frac{\pi i(2a_1n_1+a_2n_2)^2(2i\omega_1+iv)}{2a_1} = -\pi\frac{(2a_1n_1+a_2n_2)^2}{a_1}\omega_1 - \pi v\frac{(2a_1n_1+a_2n_2)^2}{2a_1},$$

along with

$$\frac{\pi i Dn_2^2(2i\omega_2+iv)}{2a_1} = -\pi\frac{Dn_2^2}{a_1}\omega_2 - \pi v\frac{Dn_2^2}{2a_1}.$$

Pulling out the above two terms dependent on v gives $-2\pi vQ(n)$. Then we see that the first term on the right-hand side of (3.1) is equal to

$$\frac{\sqrt{D}n_2(2a_1n_1+a_2n_2)}{a_1}e^{-4\pi vQ(n)} \times \int_0^\infty \frac{e^{-\frac{\pi(2a_1n_1+a_2n_2)^2\omega_1}{a_1}}}{\sqrt{\omega_1+v}} \int_{\omega_1}^\infty \frac{e^{-\frac{\pi Dn_2^2\omega_2}{a_1}}}{\sqrt{\omega_2+v}} d\omega_2 d\omega_1.$$

A similar expression holds for the second term, and thus we can write $e^{4\pi vQ(n)}M_2(\kappa;\sqrt{v}u)$ as the sum of the two terms

$$\frac{\sqrt{D}n_2(2a_1n_1+a_2n_2)}{a_1}\int_0^\infty\int_{\omega_1}^\infty \frac{e^{-\frac{\pi(2a_1n_1+a_2n_2)^2\omega_1}{a_1}-\frac{\pi Dn_2^2\omega_2}{a_1}}}{\sqrt{\omega_2+v}\sqrt{\omega_1+v}} d\omega_2 d\omega_1,$$

and

$$\frac{\sqrt{D}n_1(a_2n_1+2a_3n_2)}{a_3}\int_0^\infty\int_{\omega_1}^\infty \frac{e^{-\frac{\pi(a_2n_1+2a_3n_2)^2\omega_1}{a_3}-\frac{\pi Dn_1^2\omega_2}{a_3}}}{\sqrt{\omega_2+v}\sqrt{\omega_1+v}} d\omega_2 d\omega_1.$$

Let $v \to \frac{v}{2}$, sum over $n \in \alpha + \mathbb{Z}^2$ such that $|n_j - \alpha_j| \le r$ and let $r \to \infty$. In the same way as [4] we may use Lebesgue's dominated convergence theorem to obtain

$$(3.2)\quad 2 \lim_{r\to\infty} \sum_{\substack{n\in\alpha+\mathbb{Z}^2 \\ |n_j-\alpha_j|\le r}} M_2\left(\kappa; \sqrt{\frac{v}{2}}u\right) e^{2\pi v Q(n)}$$
$$= -\sqrt{D}\int_0^\infty \int_{\omega_1}^\infty \frac{\theta_1(\alpha;\omega)+\theta_2(\alpha;\omega)}{\sqrt{\omega_2+iv}\sqrt{\omega_1+iv}} \mathrm{d}\omega_2 \mathrm{d}\omega_1,$$

where we set

$$\theta_1(\alpha;\omega_1,\omega_2) \coloneqq \frac{1}{a_1}\sum_{n\in\alpha+\mathbb{Z}^2} (2a_1n_1+a_2n_2)n_2 e^{\frac{\pi i(2a_1n_1+a_2n_2)^2\omega_1}{2a_1}+\frac{\pi i D n_2^2\omega_2}{2a_1}}$$

and

$$\theta_2(\alpha;\omega_1,\omega_2) \coloneqq \frac{1}{a_3}\sum_{n\in\alpha+\mathbb{Z}^2} (a_2n_1+2a_3n_2)n_1 e^{\frac{\pi i(a_2n_1+2a_3n_2)^2\omega_1}{2a_3}+\frac{\pi i D n_1^2\omega_2}{2a_3}}.$$

Further, it is clear by definition that the right-hand side of (3.2) is equal to $H_\alpha(iv)$, and so we have shown the first claim.

Remark 3.1. We note that these theta functions are exactly those appearing in the double Eichler integral associated to the family of quantum modular forms of depth two given in [12].

Now we concentrate on $M_2(\kappa; \sqrt{v}u)$. Assuming that $u_2, u_1 - \kappa u_2 \neq 0$ (which happens precisely when $\alpha_1, \alpha_2 \notin \mathbb{Z}$), rewriting (2.3) implies that

$$M_2(\kappa;\sqrt{v}u)$$
$$= -\frac{1}{\pi^2} e^{-\pi v(u_1^2+u_2^2)} \int_{\mathbb{R}^2} \frac{e^{-\pi v\omega_1^2 - \pi v\omega_2^2}}{(\omega_2 - iu_2)(\omega_1-\kappa\omega_2 - i(u_1-\kappa u_2))} \mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Plugging in our definition of $u(n)$ we thus find that

$$M_2\left(\kappa;\sqrt{\frac{v}{2}}u\right)$$
$$= M_2\left(\kappa; \sqrt{\frac{v}{2}}\left(2\sqrt{a_1}n_1 + \frac{a_2}{\sqrt{a_1}}n_2\right), mn_2\right)$$
$$= -\frac{1}{\pi^2} e^{-2\pi v Q(n)} \int_{\mathbb{R}^2} \frac{e^{\frac{-\pi v\omega_1^2-\pi v\omega_2^2}{2}}}{(\omega_2 - imn_2)(\omega_1 - \kappa\omega_2 - 2\sqrt{a_1}in_1)} \mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Letting $\omega_1 \to 2\sqrt{a_1}\omega_1 + \kappa\omega_2$ yields the integral as

$$-\frac{1}{\pi^2} e^{-2\pi v Q(n)} \int_{\mathbb{R}^2} \frac{e^{\frac{-\pi v(2\sqrt{a_1}\omega_1+\kappa\omega_2)^2-\pi v\omega_2^2}{2}}}{(\omega_2 - imn_2)(\omega_1 - in_1)} \mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Then shifting $\omega_2 \to m\omega_2$ gives

$$-\frac{1}{\pi^2}e^{-2\pi vQ(n)}\int_{\mathbb{R}^2}\frac{e^{\frac{-\pi v(2\sqrt{a_1}\omega_1+\kappa m\omega_2)^2-\pi v m^2\omega_2^2}{2}}}{(\omega_2-in_2)(\omega_1-in_1)}\mathrm{d}\omega_1\mathrm{d}\omega_2$$
$$=-\frac{1}{\pi^2}e^{-2\pi vQ(n)}\int_{\mathbb{R}^2}\frac{e^{-2\pi vQ(\omega)}}{(\omega_2-in_2)(\omega_1-in_1)}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Therefore we have that

$$\lim_{r\to\infty}\sum_{\substack{n\in\alpha+\mathbb{Z}^2\\|n_j-\alpha_j|\leq r}} M_2\left(\kappa;\sqrt{\frac{v}{2}}u\right)e^{2\pi vQ(n)}$$
$$=\lim_{r\to\infty}\sum_{\substack{n\in\alpha+\mathbb{Z}^2\\|n_j-\alpha_j|\leq r}}-\frac{1}{\pi^2}\int_{\mathbb{R}^2}\frac{e^{-2\pi vQ(\omega)}}{(\omega_2-in_2)(\omega_1-in_1)}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

In exactly the same fashion as [4] we use that

$$\pi\cot(\pi x)=\lim_{r\to\infty}\sum_{k=-r}^{r}\frac{1}{x+k}$$

to rewrite

$$-\lim_{r\to\infty}\sum_{\substack{n\in\mathbb{Z}^2\\|n_j|\leq r}}\frac{1}{(i\omega_1+\alpha_1+n_1)(i\omega_2+\alpha_2+n_2)}$$
$$=-\pi^2\cot(\pi(i\omega_1+\alpha_1))\cot(\pi(i\omega_2+\alpha_2)).$$

We therefore have (using Lebesgue's theorem of dominated convergence) that

$$\lim_{r\to\infty}\sum_{\substack{n\in\alpha+\mathbb{Z}^2\\|n_j-\alpha_j|\leq r}} M_2\left(\kappa;\sqrt{\frac{v}{2}}u\right)e^{2\pi vQ(n)}$$
$$=\int_{\mathbb{R}^2}\cot(\pi i\omega_1+\pi\alpha_1)\cot(\pi i\omega_2+\pi\alpha_2)e^{-2\pi vQ(\omega)}.$$

We may then use simple trigonometric rules to split the cotangent functions into sine and cosine (and their hyperbolic counterpart) functions by use of the formula

$$\cot(x+iy)=-\frac{\sin(2x)}{\cos(2x)-\cosh(2y)}+i\frac{\sinh(2y)}{\cos(2x)-\cosh(2y)}.$$

This gives the integral as

$$\int_{\mathbb{R}^2}\left(\mathcal{G}_{\alpha_1}(\omega_1)\mathcal{G}_{\alpha_2}-\mathcal{F}_{\alpha_1}(\omega_1)\mathcal{F}_{\alpha_2}(\omega_2)\right)e^{-2\pi vQ(\omega)}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Next, as in [4], we turn to the situation when $\alpha_1 \in \mathbb{Z}$ and $\alpha_2 \notin \mathbb{Z}$. Then there is a term in the summation where $u_1 - \kappa u_2 = 0$. However, in view of (2.4) it still makes sense to consider our function towards this locus of discontinuity of M_2. We are free to assume $\alpha_1 = 0$, since it is clear that the Mordell integral is invariant under $\alpha \to \alpha_1 + 1$. Then we consider the integral

$$\begin{aligned}(3.3)\quad & -2\lim_{\alpha_1\to 0}\int_{\mathbb{R}^2}\mathcal{F}_{\alpha_1}(\omega_1)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau Q(\omega)}\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &= -\int_{\mathbb{R}^2}\mathcal{F}_0(\omega_1)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau\left(a_1\omega_1^2+a_3\omega_2^2\right)}\sum_{\pm}\pm e^{\pm 2\pi i\tau a_2\omega_1\omega_2}\mathrm{d}\omega_1\mathrm{d}\omega_2,\end{aligned}$$

where by $\sum_\pm$ we mean the sum over possible choices of $+$ and $-$. We see that

$$\mathcal{F}_0(\omega_1) = \frac{\sinh(2\pi\omega_1)}{\cosh(2\pi\omega_1)-1}$$

has a pole at $\omega_1 = 0$. Therefore, we write

$$\mathcal{F}_0(\omega_1) = \left(\mathcal{F}_0(\omega_1) - \frac{1}{\pi\omega_1}\right) + \frac{1}{\pi\omega_1}.$$

The contribution of the first term of the left-hand side to (3.3) is then seen to be

$$\begin{aligned}&-\int_{\mathbb{R}^2}\left(\mathcal{F}_0(\omega_1)-\frac{1}{\pi\omega_1}\right)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau\left(a_1\omega_1^2+a_3\omega_2^2\right)}\sum_{\pm}\pm e^{\pm 2\pi i\tau a_2\omega_1\omega_2}\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &\quad= -\int_{\mathbb{R}^2}\left(\mathcal{F}_0(\omega_1)-\frac{1}{\pi\omega_1}\right)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau\left(a_1\omega_1^2+a_3\omega_2^2\right)}e^{2\pi i\tau a_2\omega_1\omega_2}\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &\qquad -\int_{\mathbb{R}^2}\left(\mathcal{F}_0(\omega_1)-\frac{1}{\pi\omega_1}\right)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau\left(a_1\omega_1^2+a_3\omega_2^2\right)}e^{-2\pi i\tau a_2\omega_1\omega_2}\mathrm{d}\omega_1\mathrm{d}\omega_2.\end{aligned}$$

Changing $\omega_1 \to -\omega_1$ in the second integral gives overall

$$= -2\int_{\mathbb{R}^2}\left(\mathcal{F}_0(\omega_1)-\frac{1}{\pi\omega_1}\right)\mathcal{F}_{\alpha_2}(\omega_2)e^{2\pi i\tau Q(\omega)}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

We are left to investigate the contribution arising from $\frac{1}{\pi\omega_1}$ to (3.3). For this, we write

$$(3.4)\quad \mathcal{F}_{\alpha_2}(\omega_2) = \left(\mathcal{F}_{\alpha_2}(\omega_2) - \mathcal{F}_{\alpha_2}\left(\omega_2 \pm \frac{a_2}{2a_3}\omega_1\right)\right) + \mathcal{F}_{\alpha_2}\left(\omega_2 \pm \frac{a_2}{2a_3}\omega_1\right).$$

Note in particular that we introduce the arguments in the $\mathcal{F}_{\alpha_2}$ functions coming from the diagonalisation of the quadratic form

$$a_3\omega_2^2 \pm a_2\omega_1\omega_2 + a_1\omega_1^2 = a_3\left(\omega_2 \pm \frac{a_2}{2a_3}\omega_1\right)^2 + \left(a_1 - \frac{a_2^2}{4a_3}\right)\omega_1^2.$$

The first term of (3.4) yields the contribution

$$-\frac{2}{\pi}\int_{\mathbb{R}^2}\frac{1}{\omega_1}\left(\mathcal{F}_{\alpha_2}(\omega_2)-\mathcal{F}_{\alpha_2}\left(\omega_2\pm\frac{a_2}{2a_3}\omega_1\right)\right)e^{2\pi i\tau Q(\omega)}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

The contribution of the final term is seen to be

$$-\int_{\mathbb{R}}\frac{e^{2\pi i\tau\left(a_1-\frac{a_2^2}{4a_3}\right)\omega_1^2}}{\omega_1}\int_{\mathbb{R}}\sum_{\pm}\pm\mathcal{F}_{\alpha_2}\left(\omega_2\pm\frac{a_2}{2a_3}\omega_1\right)\times e^{2\pi i\tau a_3\left(\omega_2\pm\frac{a_2}{2a_3}\omega_1\right)^2}\mathrm{d}\omega_1\mathrm{d}\omega_2.$$

Inspecting the inner integral, the term with a minus sign under the change of variables $\omega_2\to\omega_2+\frac{a_2}{a_3}\omega_1$ is seen to cancel with the term with positive sign, thus giving overall no contribution.

The argument when $\alpha_1\notin\mathbb{Z}$ and $\alpha_2\in\mathbb{Z}$ runs in a similar way, and this completes the proof. □

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